A FAMILY OF MÖBIUS INVARIANT 2-KNOT ENERGIES

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Abstract. After reviewing energy functionals for 1-dimensional knots and links, we define a family of Möbius invariant energy functionals E_s for surfaces embedded in $\mathbb{R}^n$. These functionals are all finite for smoothly embedded compact surfaces and infinite for self-intersecting immersed surfaces. They treat disconnected surfaces and connected sums of surfaces correctly. For sufficiently negative s, E_s is not bounded from below. For sufficiently positive s, the evidence to date suggests that E_s *is* bounded from below, but we have not yet found a proof. We also discuss alternate methods of defining surface energy.

Several years ago Jun O'Hara [O1, O2, O3] defined a functional for embeddings of circles in $\mathbb{R}^n$ that he called the energy of a knot. Bryson, Freedman, He and Wang [BFHW, FHW] proved that this energy is Möbius invariant, and bounded the complexity of knots by their energy. Since then others, notably Doyle and Schramm [D,S], have modified and reformulated O'Hara's energy. The formulations are all essentially equivalent, but each exhibits a different important property of the energy. In this paper we generalize some of these ideas to 2 dimensions and define a family of energy functionals for embedded surfaces. These functionals are different from, and postdate, the surface energy defined by Rob Kusner and John Sullivan, which is discussed elsewhere in these proceedings [KS].

Section 1 is a review of the 1-dimensional energy. We begin with the unregularized energy formula and discuss its properties. We then list several ideas for regularization and show how they are essentially equivalent. Each of these methods suggests a regularization in 2 dimensions, most of which turn out *not* to be equivalent. The material in this section is not original, but rather reflects past work by many authors [O1, O2, O3, BFHW, FHW, D, S].

In section 2 we generalize O'Hara's original regularization to 2 dimensions. There is some freedom in the choice of regularization, yielding a 1-parameter family of Möbius invariant surface energies E_s that differ by multiples of the Willmore functional [W]. We compute the energies of several examples, and establish that these energy functionals have all but one of the desirable properties shared by the 1-dimensional energy. The one uncertain property is boundedness, which is discussed further in section 4.

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In section 3 we consider generalizations of the other 1-dimensional regularizations to 2 dimensions. Some approaches fail, due to a log divergence that has no analog in 1 dimension. One successful approach gives the Kusner-Sullivan energy. Another gives an energy similar to ours. We are presently exploring, with Kusner and Sullivan, this last approach.

In section 4 we consider boundedness. The primary difficulty with our energy functionals E_s is that they are not manifestly bounded from below. Indeed, for s sufficiently negative, E_s is *not* bounded from below. However, for s sufficiently positive, it appears likely that E_s is positive semi-definite, although we have not yet found a proof.

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1. The 1-dimensional energy

Let $\gamma_{1,2}$ be 1-manifolds in $\mathbb{R}^n$, and consider the functional

$$\int\int_{\gamma_1 \times \gamma_2} \frac{dx\ dy}{|x-y|^2}, \tag{1.1}$$

where dx and dy are the arclength measures on $\gamma_{1,2}$. This functional is finite if γ_1 and γ_2 are compact and do not intersect, it diverges if they do intersect, and it is invariant under Möbius transformations.

Möbius invariance is seen in figure 1. Recall that the Möbius group is generated by inversions in spheres. Let $\tilde{x}$ and $\tilde{y}$ be the images of x and y under inversion in the sphere S. Since $|x|/|y| = |\tilde{y}|/|\tilde{x}|$, the triangles Oxy and $O\tilde{y}\tilde{x}$ are similar, so $|\tilde{x} - \tilde{y}|^2/|\tilde{x}||\tilde{y}| = |x-y|^2/|x||y|$. Since $|d\tilde{x}| = (|\tilde{x}|/|x|)|dx|$, the invariance of the integral follows.

Next we consider the case where $K = \gamma_1 = \gamma_2$ is a smooth knot or link, with one component possibly extending to the point at infinity. The unregularized "energy of the knot (link)" is

$$E_0(K) = \int\int_{K\times K} |x-y|^{-2} dx\, dy. \tag{1.2}$$

This obviously diverges for any nonempty K. The problem is to find a regularization of the functional, finite for smoothly embedded K, that preserves the desirable formal properties of the unregularized energy. In particular, the regularized energy should:

1. Be finite for sufficiently smooth knots and links, but infinte for self-intersecting immersions.
2. Be Möbius invariant.
3. Be bounded from below.
4. Satisfy a *connected sum rule.* If K_1 and K_2 are open knots, then, in the limit of large separation, the energy of $K_1 \# K_2$ should approach the sum of the energies of K_1 and K_2.
5. Have the *additive link property* (ALP). The regularization should not affect the force between non-intersecting neighborhoods. The energy of a link should equal the sum of the energy of the components plus the *unregularized* cross terms, and the force between two neighborhoods should be independent of whether they are in the same link component or not.

Möbius invariance, boundedness, and the connected sum rule together imply that the energy is minimized by straight lines, or equivalently by circles. By scale invariance, a line must have energy zero. If any open knot had energy less than a line, the energy of the connected sum of that knot with itself N times could be made arbitrarily negative, violating boundedness. Therefore all knots must have positive or zero energy.

The 1-dimensional regularizations of O'Hara, Schramm, and Doyle and Schramm have all the above properties, up to unimportant additive constants, and are all equivalent. However, in generalizing to two dimensions it is easy to lose one or more of these properties. The Kusner-Sullivan energy has the first four properties, but not the ALP. Our energy, described in section 2 below, has the ALP, but we have not yet managed to prove that it is bounded.

O'Hara's counterterm regularization

O'Hara [O1] was the first to define a sensible regularized energy. Let

$$V(x) = \lim_{\epsilon \to 0} \left(\int_{|x-y|>\epsilon} \frac{dy}{|x-y|^2} \qquad - \frac{2}{\epsilon} \right). \tag{1.3}$$

The *counterterm* $-2/\epsilon$ exactly cancels the divergence of the integral, so the limiting potential is well-defined and finite. We then define the energy of a knot to be

$$E^{(c)}(K) = \int V(x) dx. \tag{1.4}$$

Although O'Hara only defined this energy for closed knots, the same definition works for open knots and for links. Since the counterterm is local, the ALP is immediate. It is also an easy calculation to see that the connected sum rule holds, with

$$E^{(c)}(K_1 \# K_2) = E^{(c)}(K_1) + E^{(c)}(K_2) + O(D^{-2}), \tag{1.5}$$

where D is the separation between K_1 and K_2 in the connected sum. Möbius invariance is, however, more subtle, and was first proven by Bryson et al [BFHW] using a different regularization.

The arclength regularization

O'Hara [O2, O3] also defined an energy for knots by subtracting a term involving arclength,

$$E^{(a)}(K) = \int \int_{K \times K} \left(\|x-y\|^{-2} - D(x,y)^{-2} \right) dx dy, \tag{1.6}$$

where $D(x,y)$ is the arclength between x and y. The beauty of this regularization is that it is manifestly positive semi-definite, and equals zero only for K being a straight line.

For closed knots, this arclength-regularized energy turns out to be 4 greater than the counterterm energy, while for open knots it equals the counterterm energy. To see this, hold x fixed and integrate $dy/D(x,y)^2$ over the region $|x-y| > \epsilon$.

$$\int_{|x-y|>\epsilon} D(x,y)^{-2} dy = \frac{2}{\epsilon} - \frac{4}{l} + O(\epsilon), \tag{1.7}$$

where l is the length of the knot. The $O(\epsilon)$ term comes from the difference between cutting off the integral at Euclidean distance ϵ and arclength ϵ. Taking the $\epsilon \to 0$ limit we get that

$$V(x) = -4/l + \int \left(\|x-y\|^{-2} - D(x,y)^{-2} \right) dy. \tag{1.8}$$

Integrating over x we find that $-4/l$ integrates to -4 if l is finite and zero if l is infinite.

Bryson et al [BFHW] proved that $E^{(a)}$ is Möbius invariant up to this difference of 4. If 2 knots, K_1 and K_2, are related by a Möbius transformation, then $E^{(a)}(K_1) = E^{(a)}(K_2)$ if both knots are open or both are closed, while $E^{(a)}(K_1) = E^{(a)}(K_2) - 4$ if K_1 is open and K_2 is closed. Or to put it another way, $E^{(c)}(K_1) = E^{(c)}(K_2)$ in *all* cases.

For links, Freedman, He and Wang define the link energy to be the sum of the energies of the components plus the unregularized cross-terms. For a link L with n_c closed components, $E^{(a)}(L) = E^{(c)}(L) + 4n_c$. So even though the $E^{(a)}$ energy formula makes a distinction between points on the same or different link components, $E^{(a)}$ does have the ALP.

Schramm's wasted length regularization

Oded Schramm [S] discovered an interesting geometrical interpretation of O'Hara's local potential $V(x)$. Fix x, let S_x be the unit sphere centered at x, let I_x be the inversion in S_x, and let $\tilde{y} = I_x(y)$ be the image of y. Then $|dy|/|x-y|^2 = |I_x^* d\tilde{y}|$, so $\int dy/|x-y|^2$ is just the length of the inverted curve.

This is shown in figure 2. The points on K a distance ϵ from x are p_1 and p_2. The coordinates of $\tilde{p}_{1,2}$ are $x \pm (1/\epsilon)\vec{t} + k\vec{n}/2 + O(\epsilon)$, where $\vec{t}$, $\vec{n}$, and k are the unit tangent, principal normal and curvature of the knot at x. The integral in (1.3) gives the length of the inverted curve from $\tilde{p}_1$ to $\tilde{p}_2$, while $2/\epsilon$ is, to within $O(\epsilon)$, the straight-line distance from $\tilde{p}_1$ to $\tilde{p}_2$. So, to within $O(\epsilon)$, $V_\epsilon(x)$ gives the slack, or "wasted length", in the inverted knot between $\tilde{p}_1$ and $\tilde{p}_2$. $V(x)$ gives the total slack in the inverted knot.

Doyle and Schramm's Möbius invariant approach

Suppose for simplicity that the knot K is tangent to the x_1 axis at x. Then the slack in the inverted knot $\tilde{y}$ can be written as $\int |d\tilde{y}| - d\tilde{y}_1$, or equivalently as $\int |d\tilde{y}|(1-\cos(\alpha))$, where α is the angle between the tangent vector at $\tilde{y}$ and the x_1 axis. Since $|d\tilde{y}|$ can be expressed in terms of $|dy|$, we can rewrite the entire energy functional as

$$E^{(m)}(K) = \int\int \frac{(1-\cos(\alpha))}{|x-y|^2} dxdy. \tag{1.9}$$

The angle α can be understood in terms of the original knot K, without recourse to inversions. Let S_1 be the unique circle, tangent to K at x, that passes through y. $I_x(S_1)$ is the line through $\tilde{y}$ parallel to the x_1 axis, which intersects $I_x(K)$ at angle α. Since Möbius transformations are conformal, the angle between S_1 and K at y is also α. Alternatively, let S_2 be the circle, tangent to K at x that passes through y. The angle between S_1 and S_2, measured at either x or y, is again α.

The functional $E^{(m)}$ is manifestly Möbius invariant, positive semi-definite, and minimized by circles and lines. The ALP is less obvious, but it follows from the equality of $E^{(m)}$ and $E^{(c)}$.

There are a few special cases where the equivalence of $E^{(m)}$ and $E^{(c)}$ breaks down. The two functionals are finite and equal when the knot (or link) is embedded. They are both infinite when there are self-intersections at isolated points. However, they disagree on curves that wrap around themselves several times. For example, let L be two superimposed copies of the knot K. If the two copies are given the same orientation, $E^{(m)}(L) = 4E^{(m)}(K)$, while if the orientations disagree, $E^{(m)}(L) = \infty$. By constrast, $E^{(c)}(L) = \infty$ in both cases.

The reason for the discrepancy is that, at points of self-intersection, the effect of the $\cos(\alpha)$ term is to subtract $2(1+\cos(\beta))/\epsilon$, rather than $2/\epsilon$, where β is the angle between the two intersecting strands. When self-intersection points are absent, or form a set of measure zero, this does not matter, but for multiple copies of the same curve it makes a great difference.

Doyle and Schramm [D,S] have proposed a family of Möbius invariant knot energies. Given any function $f(\alpha)$ that goes to zero sufficiently fast at $\alpha = 0$, they define the functional

$$E^{(f)}(K) = \int\int f(\alpha)|x-y|^{-2}dxdy. \tag{1.10}$$

This is manifestly Möbius invariant, and is positive semi-definite if f is. For $f \neq 1-\cos(\alpha)$ this is NOT equivalent to the other energies, and typically does not have the ALP.

2. Energy of surfaces — counterterm approach

We are now ready to generalize the definition of energy to surfaces embedded in $\mathbb{R}^n$. In this section we present our generalization of O'Hara's counterterm approach, and define a 1-parameter family of surface energies. Generalizations of the other 1-dimensional regularizations will be considered in section 3.

The naive unregularized energy of a k-manifold F is

$$\int_{F\times F} |x-y|^{-2k} d^k x d^k y, \tag{2.1}$$

where $d^k x$ is the k-dimensional volume form on F. Since Möbius transformations are conformal, $d^k x d^k y$ transforms like the k-th power of $dxdy$, so (2.1) is formally Möbius invariant. However, it diverges like ϵ^{-k} at short distance. To get a sensible definition we must remove this divergence.

In 1 dimension the divergence was $2/\epsilon$, regardless of the local geometry, and was easy to subtract off. In 2 dimensions there is a π/ϵ^2 divergence that doesn't depend on local geometry, plus a $\ln(\epsilon)$ divergence that does depend on local geometry. Natural generalizations of some 1-dimensional approaches fail because they do not cancel the log divergence. Kusner and Sullivan [KS] have defined an energy that avoids these problems, but which sacrifices the ALP. The challenge is to remove both divergences while preserving the ALP.

We begin by computing the local divergence more precisely. Let

$$V(\epsilon, x) = \int_{|x-y|>\epsilon} |x-y|^{-4} d^2 y. \tag{2.2}$$

For simplicity, take x to be at the origin, and let the tangent plane $T_x F$ be the x_1-x_2 plane. Let

$$L = \begin{pmatrix} e & f \\ f & g \end{pmatrix} \tag{2.3}$$

be the second fundamental form at x, where e, f, and g are vectors normal to the x_1-x_2 plane. Defining

$$\begin{aligned} A(\theta) =& e\cos^2(\theta)/2 + f\cos(\theta)\sin(\theta) + g\sin^2(\theta)/2 \\ B(\theta) =& |e|^2\cos^2(\theta) + |f|^2 + |g|^2\sin^2(\theta) + 2f\cdot(e+g)\sin(\theta)\cos(\theta) \end{aligned} \tag{2.4}$$

we see that, near x, the surface F takes the form

$$z = A(\theta)r^2 + O(r^3), \tag{2.5}$$

where $z = (y_3, y_4, \dots, y_n)$ and (r, θ) are polar coordinates for T_xF. The area form is

$$d^2y = \sqrt{1 + |\nabla z|^2}\, r dr d\theta = r + B(\theta)r^3/2 + O(r^4)\; dr d\theta, \tag{2.6}$$

while the distance from the origin is

$$\sqrt{r^2 + |z|^2} = r + |A(\theta)|^2 r^3/2 + O(r^4) \tag{2.7}$$

and the distance ϵ cutoff is at

$$r = r_\epsilon(\theta) = \epsilon - |A(\theta)|^2 \epsilon^3/2 + O(\epsilon^4). \tag{2.8}$$

Combining (2.4) through (2.8) we get that

$$\begin{aligned} V(\epsilon, x) =& \int_{r>r_\epsilon} [r^2 + |A(\theta)|^2 r^4 + O(r^5)]^{-2} [r + B(\theta)r^3/2 + O(r^4)] dr d\theta \\ =& \int_{r>r_\epsilon} [r^{-3} + (B(\theta)/2 - 2|A(\theta)|^2)/r + O(1)] dr d\theta \\ =& \int_0^{2\pi} \frac{1}{2} r_\epsilon^{-2} - (B(\theta)/2 - 2|A(\theta)|^2) \ln(r_\epsilon) + O(1) d\theta \\ =& \pi\epsilon^{-2} - (\pi/8)\Delta(x)\ln(\epsilon) + O(1), \end{aligned} \tag{2.9}$$

where

$$\Delta(x) = (e+g)^2 - 4(e \cdot g - |f|^2) = (e-g)^2 + 4|f|^2. \tag{2.10}$$

Δ is a measure of how far a point is from being umbilical. If the surface is in $\mathbb{R}^3$, with principal curvatures k_1 and k_2, then $\Delta = (k_1 - k_2)^2$.

The counterterm regularization

To get a suitable regularization, we must find a local expression that has the form (2.9) and transforms correctly under Möbius transformations, and subtract it from $V(\epsilon, x)$. Note that $\ln(\epsilon)$ does *not* transform correctly, since ϵ is a dimensionful quantity. To get a dimensionless quantity, we must divide ϵ by a locally determined length scale. The natural candidate is $\Delta^{-1/2}$, though other choices (such as $2\Delta^{-1/2}$) are equally valid.

We define the local potential energy

$$V_0(x) = \lim_{\epsilon \to 0} V(\epsilon, x) - \pi\epsilon^{-2} + \frac{\pi\Delta(x)}{16} \ln(\Delta(x)\epsilon^2) + \frac{\pi K(x)}{4}, \tag{2.11}$$

where $K(x) = e \cdot g - |f|^2$ is the Gauss curvature at x, and define the energy of a surface to be

$$E_0(F) = \int_F V_0(x) d^2x. \tag{2.12}$$

This is not the only reasonable choice. Had we taken the length scale to be $2\Delta^{-1/2}$ rather than $\Delta^{-1/2}$, we would have gotten a potential that differs from V by $\pi \ln(2)\Delta/8$. Other length scales give potentials that differ by other factors of Δ. It therefore makes sense to consider the entire 1-parameter family of potentials $V_s(x) = V_0(x) + s\Delta(x)$ and the corresponding energies

$$E_s(F) = \int V_s(x) d^2x = E_0(F) + s \int \Delta(x) d^2x \tag{2.13}$$

The integral of Δ is closely related to the Willmore functional $W(F)$, a well-known conformal invariant [W]. Specifically,

$$W(F) = \int_F H^2 d^2x = \int_F \left(\frac{1}{4}\Delta + K \right) d^2x, \tag{2.14}$$

where $H = (e+g)/2$ is the mean curvature. The 2-form $\Delta\ d^2x$ is in fact invariant under all conformal changes of the ambient metric, not just under Möbius transformations, while $\int K$ is a topological invariant. $W(F)$ is invariant under Möbius transformations that do not change the Euler characteristic, but decreases by 4π when a point is sent to infinity.

The $\Delta \ln(\Delta)$ term in the regularization is an object better known to measure theorists than to topologists. Minus $\int \Delta \ln(\Delta) d^2x$ is the *entropy* of the Möbius invariant measure Δd^2x relative to the area measure d^2x.

The integral (2.2), while well-defined, can be difficult to compute. To aid computation, we define an equivalent energy using a smooth cutoff function. Let $\rho(r)$ be a function that approaches 1 as $r \to \infty$ and approaches zero rapidly as $r \to 0$. Define the regularized potential

$$V^\rho(x) = \lim_{\epsilon \to 0} \left(\int \frac{\rho(|x-y|/\epsilon) d^2y}{|x-y|^4} - c(\rho)\pi\epsilon^{-2} + \frac{\pi\Delta}{16} \ln(\Delta\epsilon^2) + \frac{\pi}{4} K \right), \tag{2.15}$$

where $c(\rho) = \int_0^\infty r^{-2}\rho'(r)dr$. When ρ is the characteristic function of $[1, \infty]$, V^ρ is the same as V_0, while if $\rho = \chi_{[R,\infty]}$, V^ρ differs from V_0 by $\pi\Delta\ln(R)/8$. For a general ρ, $V^\rho = V_s$, where $s = -\frac{\pi}{8}\int_0^\infty \rho'(r)\ln(r)dr$. Varying the cutoff function ρ is equivalent to varying s, and we get exactly the same family of energies as before.

A particularly useful cutoff function is $\rho(r) = r^4/(1+r^2)^2$, which reduces the integral in (2.15) to $\int(\epsilon^2 + |x-y|^2)^{-2}d^2y$. This particular cutoff gives V_s with $s = -\pi/16$.

Examples

The plane.

Before exploring the general properties of the energies E_s, we compute a few examples. Our first and simplest example is the plane. Here $K = \Delta = 0$, and

$$V(\epsilon, x) - \pi/\epsilon^2 = -\pi/\epsilon^2 + \int_0^{2\pi} \int_\epsilon^\infty r^{-4} r\,dr\,d\theta = 0. \tag{2.16}$$

Our energy is zero, as it must be by scale invariance.

The sphere.

The next example is the unit sphere, where $\Delta = 0$ and $K = 1$. The potential $V(x)$ is constant by symmetry, and we compute it at the north pole.

$$V(\epsilon, N) = 2\pi \int_{2\sin^{-1}(\epsilon)}^{\pi} \frac{\sin(\theta)d\theta}{(2\sin(\theta/2))^4} = \pi/\epsilon^2 - \pi/4. \tag{2.17}$$

Subtracting the counterterms we get $V_s(N) = 0$, and hence $E_s(S^2) = 0$, reflecting the Möbius equivalence of the sphere to the plane. Note the effect of the topological $\pi K/4$ counterterm in the definition of V_s. Had this term not been included, the sphere would have had energy $-\pi^2$.

The cylinder.

Our third example is the right circular cylinder of radius 1. In this example the two principal curvatures are unequal, and $\Delta = 1$. All points have the same potential, and this is what we compute. Multiplying this potential by 2π gives the energy per unit length of the cylinder.

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The natural coordinates on the cylinder are (α, z), where α is the angle around the circle and z is height, and we take our reference point x to be $(0,0)$. It is also useful to convert from α to $t = 2\sin(\alpha/2)$, the straight-line distance across the circle, as shown in figure 4. By symmetry, it is sufficient to integrate over the shaded regions in figure 4, and then to multiply the total by 4.

The integral over the lightly shaded region is

$$\int_\epsilon^2 \int_0^\infty \frac{2dzdt}{\sqrt{4-t^2}(t^2+z^2)^2} = \frac{\pi}{2}\int_\epsilon^2 \frac{dt}{t^3\sqrt{4-t^2}} \tag{2.18}$$
$$= \frac{\pi}{32}\left(\frac{2\sqrt{4-\epsilon^2}}{\epsilon^2} + \ln\left(\frac{2+\sqrt{4-\epsilon^2}}{\epsilon}\right)\right) = \frac{\pi}{32}\left(4\epsilon^{-2} + 2\ln(2) - \frac{1}{2} - \ln(\epsilon) + O(\epsilon^2)\right).$$

To integrate over the heavily shaded region we expand $2/\sqrt{4-t^2} = 1 + t^2/8 + O(t^4)$. We also switch to polar coordinates $r^2 = t^2 + z^2$, $\tan(\theta) = z/t$. Our integral is then, to within $O(\epsilon^2)$,

$$\int_{-\pi/2}^0 \int_\epsilon^{\epsilon\sec(\theta)} r^{-4}(1 + r^2\cos^2(\theta)/8) r\,dr\,d\theta = \int_{-\pi/2}^0 \epsilon^{-2}\sin^2(\theta)/2 - \cos^2(\theta)\ln(\cos(\theta))/8 d\theta$$
$$= \pi\epsilon^{-2}/8 + \pi(2\ln(2) - 1)/64. \tag{2.19}$$

Adding these two expressions, multiplying by 4, subtracting the counterterms and taking the $\epsilon \to 0$ limit gives

$$V_s(x) = s + \frac{\pi}{8}(3\ln(2) - 1). \tag{2.20}$$

Note that, for s sufficiently negative, the cylinder has negative energy per unit length. By taking a very long cylinder with smooth caps at the ends, we can create a smooth surface with arbitrarily negative energy. As a result, for $s < -(3\ln(2) - 1)\pi/8$, the energy functional E_s is unbounded from below.

The "dimple".

Our final example shows that, for any s, there exist surfaces with points where V_s is negative. Let d be a small parameter. Consider the surface in $\mathbb{C}^2$ given by

$$z_1 = r\exp(i\theta), \qquad z_2(r,\theta) = \begin{cases} r^2\exp(2i\theta) & r \le d \\ d^2\exp(2i\theta) & r \ge d. \end{cases} \tag{2.21}$$

See figure 5. This, of course, is not a smooth surface, but it can be arbitrarily well approximated by smooth surfaces. We compute the potential $V_s(x)$ at $x = 0$ for $d << 1$. The second fundamental form at 0 is $\begin{pmatrix} 2 & 2i \\ 2i & -2 \end{pmatrix}$ and $\Delta(0) = 32$.

The integral for $V(\epsilon,x)$ breaks into two pieces, one for $r>d$ and one for $r<d$. For $r>d$ we have

$$\int_0^{2\pi}\int_d^\infty \sqrt{1+4d^4/r^2}(r^2+d^4)^{-2}r\,dr\,d\theta = \pi/d^2 + O(d^2). \tag{2.22}$$

For $r<d$ we have

$$\begin{aligned}\int_0^{2\pi}\int_{r_\epsilon}^d \sqrt{1+8r^2}(r^2+r^4)^{-2}r\,dr\,d\theta &= \int_0^{2\pi}\int_{r_\epsilon}^d (r^{-3}+2r^{-1}+O(r))\,dr\,d\theta \\ &= \pi + \pi/\epsilon^2 - \pi/d^2 + 4\pi\ln(d/\epsilon) + O(d^2),\end{aligned} \tag{2.23}$$

where r_ϵ is the solution to $r_\epsilon^2 + r_\epsilon^4 = \epsilon^2$. Summing these contributions, subtracting the counterterms and taking the $\epsilon \to 0$ limit we get

$$V_s(0) = 32s + \pi + 10\pi\ln(2) + 4\pi\ln(d) + O(d^2). \tag{2.24}$$

No matter how large s is, we can get $V_s(0)$ to be negative by choosing d small enough.

Properties of the surface energy

Having defined our family of energies, we now check to see if it has the properties 1–5 described in section 1. Since adding a multiple of $\int \Delta$ leaves properties 1, 2, 4 and 5 unchanged, it suffices to check these properties for E_0.

It is easy to see that E_0 is well-defined and finite for smooth embeddings, and varies continuously with the embedding in the C^k norm, for any $k \geq 3$. When F intersects itself, the potential $V_0(x)$ diverges as r^{-2} as x approaches the self-intersecting set. If the self-intersecting set is discrete the integral $E_0 = \int V_0$ diverges logarithmically, while if it has positive dimension the energy diverges even more rapidly.

E_0 is also Möbius invariant. Indeed, the 2-form $V_0(x)d^2x$ is Möbius invariant. This can be proved directly, and we sketch such a proof in the next paragraph. However, Möbius invariance is more easily understood in section 3, where we relate $V_0(x)$ to a quantity that is manifestly Möbius invariant.

Since $V_0 d^2x$ is manifestly invariant under scale changes and Euclidean transformations, we need only check invariance under Möbius transformations I that leave x fixed and for which I_* is the identity on $T_x(\mathbb{R}^n)$. Such a transformation preserves $e-g$ and f, and hence Δ. Since $|x-y|^{-4}d^2y$ is an invariant form, as is d^2x, the change in $V(x)$ can be reduced to the $\epsilon \to 0$ limit of

$$\left(\int_{|I(y)-x|>\epsilon>|y-x|} - \int_{|y-x|>\epsilon>|I(y)-x|}\right) |x-y|^{-4} d^2y \tag{2.25}$$

plus $\pi/4$ times the change in K. By a power-series expansion, this is seen to equal zero in all cases.

Boundedness is problematic. The example of the cylinder shows that E_s is *not* bounded from below for $s < -(3\ln(2)-1)\pi/8$. Furthermore, the example of the dimple shows that V_s cannot be bounded by local information, so a bound on E_s cannot be achieved by integrating a bound on V_s. However, experimentation suggests that, for large s, E_s *is* positive semi-definite and is minimized by spheres. This conjecture is discussed further in section 4.

The connected sum rule follows from Möbius invariance. First suppose that K_1 and K_2 each contain neighborhoods that are exactly spherical. By inverting about a sphere centered in this neighborhood we can assume that K_1 and K_2 are exactly planar outside of compact regions M_1 and M_2, respectively, with finite areas A_1 and A_2. We take $K_1 \# K_2$ to be a plane outside of two regions, isometric to M_1 and M_2, that are separated by a distance D. An easy calculation then shows that

$$|E_s(K_1 \# K_2) - E_s(K_1) - E_s(K_2)| \leq A_1 A_2 / D^4. \tag{2.26}$$

Taking $D \to \infty$ gives the connected sum rule for surfaces with spherical neighborhoods. But since *any* surface can be given a small spherical neighborhood at an arbitrarily small cost in energy, the connected sum rule holds for all surfaces. (This is essentially the same argument as found in [K], where a connected sum rule for the Willmore functional is proven).

Finally, E_s manifestly has the ALP, for the same reason that O'Hara's $E^{(c)}$ does. Namely, the regularization at x is independent of whether additional link components are present or not.

3. Alternative approaches to surface energy

In this section we consider generalizations of other 1-dimensional regularizations. Several natural ideas simply do not work, thanks to the $\ln(\epsilon)$ divergence in $V(\epsilon, x)$. One approach gives functionals similar to the counterterm energies of section 2. Another approach gives the Kusner-Sullivan "first-order" energy, which is discussed elsewhere in these proceedings [KS].

Arclength regularization

There are several possible ways to generalize the 1-dimensional arclength formula (1.7). The most obvious is to define

$$E(F) = \int_{F\times F} (|x-y|^{-4} - D(x,y)^{-4}) d^2x d^2y, \tag{3.1}$$

but this expression is divergent. At short distance $|x-y|^{-4}$ equals $r^{-4} - 2|A(\theta)|^2 r^{-2} + O(r^{-1})$, while $D(x,y)^{-4} = r^{-4} - 8|A(\theta)|^2 r^{-2}/3 + O(r^{-1})$. Unless the second fundamental form is identically zero, the difference between these two expressions is $O(r^{-2})$, and is not integrable.

A different generalization is

$$E(F) = \int_{F\times F} (|x-y|^{-2} - D(x,y)^{-2})^2 d^2x d^2y. \tag{3.2}$$

This is finite, as $|x-y|^{-2} - D(x,y)^{-2} = O(1)$, but it is not Möbius invariant. Also, as a practical matter, it is nearly impossible to compute, since just finding the shortest path between two distant points is a difficult endeavor.

In one dimension, arclength was an easy quantity to work with, in that there were only two paths between any two points to choose from, and in that arclength served as a canonical parametrization for any curve. In two dimensions, arclength is neither calculable nor canonical, and does not appear to provide a useful regularization.

The closest thing to a canonical parametrization in 2 dimensions is a map that preserves conformal structure. Kusner has suggested an energy based on this structure. If $f : \mathbb{R}^k \to \mathbb{R}^n$ is a conformal embedding, he defines

$$E(f) = \int_{\mathbb{R}^k\times\mathbb{R}^k} \frac{|f'(z)|^k |f'(w)|^k}{|f(z)-f(w)|^{2k}} - \frac{1}{|z-w|^{2k}} d^2z d^2w. \tag{3.3}$$

This energy is an invariant of the surface, since if $g : \mathbb{R}^k \to \mathbb{R}^n$ is a conformal map with the same image as f, then $E(g) = E(f)$.

In one dimension this expression is well-defined, and gives yet another description of O'Hara's energy $E^{(c)}$. Kusner and Sullivan have applied this approach to graphs, giving a relative energy for distinct embeddings of the same graph. Unfortunately, in 2 dimensions this energy has a log divergence at non-umbilical points. For example, consider the map $f : \mathbb{C} \to \mathbb{C}^2$, $f(z) = (z, z^2)$. Holding $w = 0$ fixed and integrating over z, we see that $|f'(z)|^2 |f(z) - f(w)|^{-4} = |z|^{-4} + 2|z|^{-2} + O(1)$. The second term in the integrand cancels the $|z|^{-4}$, but the remaining $2|z|^{-2}$ term is not integrable.

Wasted area regularization

As in the 1-dimensional case, let S_x be the unit sphere centered at x, let I_x be inversion in this sphere, and let $\tilde{y} = I_x(y)$. Then the 2-form $|x - y|^{-4} d^2 y$ is the pullback, under I_x, of the area form $d^2\tilde{y}$. As before, the area of $I_x(F)$ is infinite, as $I_x(F)$ extends to infinity. We try to regularize by taking the excess area of $I_x(F)$ relative to some standard surface. The question is, which surface?

Before considering reference surfaces, we investigate what $I_x(F)$ looks like near infinity. Since F takes the form (2.5) near zero, $I_x(F)$ has the form

$$z = A(\theta) + O(1/r), \qquad \partial z/\partial r = O(1/r^2), \qquad \partial z/\partial\theta = dA/d\theta + O(1/r). \tag{3.4}$$

The area form is

$$\sqrt{1 + |\nabla z|^2}\, r dr d\theta = (r + |A'(\theta)|^2/2r + O(r^{-2})) dr d\theta, \tag{3.5}$$

where A' denotes $dA(\theta)/d\theta$. Integrating out to $r = R$ gives $\pi R^2 + \pi\Delta \ln(R)/8 + O(1)$. These are precisely the divergences we have already seen in (2.9), with $R = 1/\epsilon$.

Our reference surface must have the same asymptotic area as $I_x(F)$. This rules out the natural choice of the plane, whose area is merely πR^2. Another possibility is to take the curve $z = A(\theta), r = R$, fill it in with a minimal surface, and take the area of that. Unfortunately, the area of such a minimal surface is $\pi R^2 + O(1)$, with no $\ln(R)$ term, so this approach also fails.

A reference surface that *does* work is the fan shape $z(r, \theta) = A(\theta)$, which we denote M_x. For $r > d$, this is the same as the "dimple" of figure 5. The area form for M_x is exactly $\sqrt{1 + |A'|^2/r^2}\, r dr d\theta$, which has the form (3.5) for large r. We can therefore define

$$\begin{aligned} V^{(w)}(x) &= \text{Excess area of } I_x(F) \text{ relative to } M_x \\ E^{(w)} &= \int_F V^{(w)}(x) d^2 x \end{aligned} \tag{3.6}$$

This definition is the product, and the subject, of a continuing collaboration with Kusner and Sullivan.

Since M_x is not a minimal surface, it is possible for $I_x(F)$ to have less area than M_x, and hence for $V^{(w)}(x)$ to be negative. Indeed, if $I_x(F)$ is a minimal surface for $r < D$ and has the form (3.4) for $r > D$, then $V^{(w)}(x) = -\pi\Delta \ln(D)/8 + O(1)$. (The "dimple" of section 2 is a surface of precisely this type, with $D = 1/d$). This shows that $V^{(w)}$, like V_s, cannot be pointwise bounded from below.

$E^{(w)}$ is finite for smooth embeddings and infinite for immersions with self-intersections. It is Möbius invariant, obeys a connected sum rule and has the ALP. We will prove Möbius invariance directly, and then show that $E^{(w)} - E_s$ is the integral of a local and Möbius invariant quantity. This then establishes the other properties for $E^{(w)}$, and shows that E_s is Möbius invariant.

We examine what a Möbius transformation does to $I_x(F)$ and M_x. $\tilde{x} = \infty$ is fixed, so the transformation can only act on $I_x(F)$ by rotation, reflection, translation or scale change. We show that in all cases $V^{(w)}(x)d^2x$ is unchanged.

A rotation (or reflection) of $I_x(F)$ corresponds to a rotation (reflection) of F around x, hence a rotation (reflection) of the second fundamental form at x, hence a rotation (reflection) of M_x. $I_x(F)$ and M_x still match asymptotically, and the difference in their areas is unchanged.

A translation of $I_x(F)$ corresponds to a Möbius transformation whose derivative is the identity at x. Such a transformation can change the mean curvature $H = (e+g)/2$, but cannot change $e - g$ or f. A translation of $I_x(F)$ normal to the x_1-x_2 plane corresponds to a change in H, which in turn translates M_x the same amount as $I_x(F)$, leaving $V^{(w)}$ unchanged. A translation in the x_1-x_2 plane does not change H, and so is *not* matched by a translation of M_x. However, the difference in area form between M_x and its translate is absolutely integrable, and integrates to zero, so the shift of $I_x(F)$ relative to M_x does not change $V^{(w)}(x)$.

Finally, rescaling $I_x(F)$ by a factor of λ corresponds to rescaling F by λ^{-1}, rescaling the second fundamental form at x by λ, and so rescaling M_x by λ. $V^{(w)}(x)$ gets rescaled by λ^2, while d^2x gets rescaled by λ^{-1}, leaving $V^{(w)}(x)d^2x$ unchanged. This completes the proof that $V^{(w)}d^2x$, and therefore $E^{(w)}$, is Möbius invariant.

Now we compare $E^{(w)}$ to E_s. We can compute $V^{(w)}$ by taking the area of that part of $I_x(F)$ within distance R of the origin, subtracting the area of the corresponding part of M_x, and taking the limit as $R \to \infty$. The points on $I_x(F)$ of distance less than R from the origin correspond to points on F of distance greater than $\epsilon = 1/R$ from x. This procedure is therefore equivalent to a counterterm regularization, where the counterterm is the area of M_x within distance $R = 1/\epsilon$ of the origin. The distance cutoff is at $r = r_R = R - A(\theta)^2/(2R) + O(R^{-2})$.

The area is

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{r_R} \sqrt{1+|A'(\theta)|^2/r^2}\, r\,dr\,d\theta \\
&\quad = \int_0^{2\pi} \frac{1}{2}\left(r_R\sqrt{r_R^2+|A'|^2} + |A'|^2\ln(r_R+\sqrt{r_R^2+|A'|^2}) - \frac{1}{2}|A'|^2\ln(|A'|^2)\right) d\theta \\
&\quad = \int_0^{2\pi} \frac{1}{2}\left(r_R^2 + |A'|^2/2 + |A'|^2\ln(2R) - \frac{1}{2}|A'|^2\ln(|A'|^2) + O(R^{-2})\right) d\theta \\
&\quad = \int_0^{2\pi} R^2/2 + |A'|^2/4 - |A(\theta)|^2/2 - |A'|^2\ln(|A'|^2/4R^2)/4 + O(R^{-2})d\theta \\
&\quad = \pi R^2 - \pi\Delta\ln(|\Delta/R^2)/16 - \pi K/4 + \pi\Delta(10\ln(2)-1)/32 \\
&\qquad - \frac{\Delta}{32}\int_0^{2\pi} \frac{8|A'|^2}{\Delta}\ln\left(\frac{8|A'|^2}{\Delta}\right) d\theta + O(R^{-2})
\end{aligned} \tag{3.7}$$

The first 3 terms match the V_0 counterterms. $V^{(w)}$ is therefore equal to V_s, with $s = -\pi(10\ln(2)-1)/32$, plus $\Delta/32$ times the final integral, which is the entropy of the S^1 measure $|A'(\theta)|^2 d\theta$ relative to its average, $(\Delta/8)d\theta$. This entropy is a locally defined Möbius invariant quantity that depends on the relative size, and the angle between, $e-g$ and f. In $\mathbb{R}^3$ we can always choose coordinates such that $f=0$, and the integral is $2\pi(1-\ln(2))$. In 4 or more dimensions, the integral can take on any value from 0 to $2\pi(1-\ln(2))$.

The fan shape M_x is not the only reference surface that works. Given any cutoff function $\rho(r)$, we can take the surface

$$z(r,\theta) = A(\theta)\rho(r^2/A_0(\theta)^2), \tag{3.8}$$

where $A_0(\theta) = (e-g)\cos(2\theta)/4 + f\sin(2\theta)/2$ is the oscillatory part of $A(\theta)$. This surface works just as well as M_x, and gives an energy functional that differs from $E^{(w)}$ by local terms such as $\int\Delta$ and the integral of the entropy of $8|A'|^2/\Delta$.

The manifestly Möbius invariant approach

Earlier this year, Kusner and Sullivan [KS] generalized Doyle and Schramms's explicitly Möbius invariant approach to higher dimensions. Let S_x be the sphere, tangent to F at x, that passes through y, and let S_y be the sphere tangent to F at y that passes through x. Let $\alpha(x,y)$ be the angle between S_x and S_y. More precisely, $\cos(\alpha)$ is the inner product of the area form of S_x and the area form of F at y. Then, given any reasonable function $f(\alpha)$ with $f(0)=0$ they write

$$E^{(f)}(F) = \int_{F\times F} f(\alpha)|x-y|^{-4}. \tag{3.9}$$

This is manifestly Möbius invariant, and is finite for embedded surfaces as long as $f \to 0$ sufficiently quickly as $\theta \to 0$. Kusner and Sullivan concentrate in particular on $f(\alpha) = (1-\cos(\alpha))^2$ (or, in k dimensions, $(1-\cos(\alpha))^k$).

This approach has several advantages over the local regularizations E_s, and one major disadvantage. The advantages are that the energy is manifestly bounded from below, that it requires knowledge only of the tangent planes at x and y, not of the curvature, and that it easily generalizes to higher dimensions.

The disadvantage is that it does not have the additive link property. The force between distinct neighborhoods depends not only on their separation but also on their orientation in space. As such, it does not preserve the long-distance behavior of the unregularized funtional (2.1).

To preserve the additive link property, one would have to use $f(\alpha) = 1 - \cos(\alpha)$. This is equivalent to computing the wasted area relative to a plane, or to using the local counterterm π/ϵ^2, and is log divergent at all non-umbilical points. To avoid log divergences, one must either incorporate information about curvature into the regularization (as with E_s) or sacrifice the additive link property (as in [KS]).

It is possible to incorporate information about the second derivative into this approach. The integral

$$\int_{F\times F} |x-y|^{-4}\Big(1 - \cos(\alpha)\sqrt{1 + |A'(\theta)|^2(r^2+|z|^2)^2/r^2}\Big) d^2x d^2y \tag{3.10}$$

is equivalent to $E^{(w)}$. Here (r, θ) are coordinates in the tangent plane T_xF and z is the component of $y - x$ normal to this plane, and A' is related to the second fundamental form at x as before. The complicated second term is just the pullback, under I_x, of the area form of M_x. There is an apparant singularity at points where $r = 0$, $z \neq 0$, but this singularity is integrable. Of course, by introducing the second derivative we lose manifest boundedness, as the integrand may be negative.

4. The question of boundedness

In this section we discuss the boundedness properties of the surface energy functionals E_s defined in section 2. We show that the question may be reduced to evaluating a single number, which we call s_c. We then tell what little is known about this number, and argue that s_c is most likely finite.

The critical parameter

Since the energy of a sphere is zero, for any given s the functional E_s is either

a) Unbounded from below,

b) Bounded from below, but not uniquely minimized by round spheres, or

c) Positive semi-definite, and uniquely minimized by round spheres.

We will prove that there is a (possibly infinite) critical value of s, which we call s_c, such that condition (a) holds for all $s < s_c$ and condition (c) holds for all $s > s_c$. When $s = s_c$, the energy is positive semi-definite, but it is not known whether $E_{s_c}(F) = 0$ implies that F is a round sphere (or a plane, which we consider a sphere of infinite radius).

First consider $W(F) = \int_F \Delta(x) d^2x$. Since $\Delta = |e-g|^2 + 4|f|^2 \geq 0$, $W(F)$ is never negative, and is zero only when F is umbilical everywhere, i.e., a sphere. As a result, if $s_2 > s_1$, then $E_{s_2}(F) \geq E_{s_1}(F)$, with equality if and only if F is a sphere.

Define

$$s_c = \sup\{s \mid E_s \text{ is unbounded from below.}\} \tag{4.1}$$

If $s < s_c$, then there exists an $s_- \in (s, s_c]$ such that E_{s_-} is unbounded from below, by the very definition of s_c. Since $E_s \leq E_{s_-}$, E_s is also unbounded from below.

Now suppose that $s > s_c$. This implies that E_s is bounded from below. By the connected sum rule, if $E_s(F)$ were negative for some F, we could get $E_s(F\#F\#\ldots\#F)$ arbitrarily negative, and E_s would be unbounded. Thus E_s must be positive semi-definite. Similarly, $E_{(s+s_c)/2}$ is positive semi-definite. If F is not a round sphere, $E_s(F) > E_{(s+s_c)/2}(F) \geq 0 = E_s(S^2)$, so E_s is uniquely minimized by round spheres.

Finally, we consider E_{s_c}. For any smooth F, $W(F)$ is finite, so $E_{s_c}(F) = \lim_{s \to s_c^+} E_s(F) \geq 0$. This shows that E_{s_c} is positive semi-definite. However, it is not clear whether $E_{s_c}(F) = 0$ implies that F is a sphere.

The remaining task is to determine whether s_c is finite or infinite, and if finite, to estimate its value.

It is easy to see that $s_c > -\infty$. Let F be any non-spherical surface. Then $W(F) > 0$ and, for any $s < -E_0(F)/W(F)$, $E_s(F) < 0$. This shows that $s_c \geq -E_0(F)/W(F)$. In particular, by looking at the cylinder we see that $s_c \geq (1 - 3\ln(2))\pi/8$.

It is not known whether s_c is finite or positive infinite. The evidence collected to date suggests that s_c is finite. However, no proof has been found, and the evidence is far from conclusive.

Evidence for Boundedness

Suppose that E_s is unbounded for all s. Then it should be possible to find examples whose energies are extremely negative even for large values of s. However, all the wild examples we have constructed have had extremely *positive* energies, as might be expected from a well-behaved and bounded functional. While this is hardly proof of boundedness for large s, it is certainly suggestive.

We present some of these examples here. We do so not so much to convince to reader of boundedness as to build intuition. We hope that these examples will suggest methods by which boundedness might be proved, or by which true counterexamples might be constructed.

Our first example is the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ in $\mathbb{R}^3$. If a/b and a/c are both close to 1, then both W and E_0 will be second order in the eccentricity, and for reasonable values of s our energy will be positive. To get candidates for negative energy we must take extreme values of (a, b, c).

Suppose $a >> b, c$, so that our ellipsoid looks like a cigar. In that case most points will lie along the length of the cigar, which is roughly cylindrical. But we have already calculated the energy of the cylinder, and found it to be positive for s positive. Increasing a only increases the length of the cigar and increases the energy.

Now suppose $a << b, c$, so that our ellipsoid looks like a pancake. Most points lie along a flat face of the pancake, with very high energy due to the vicinity of the other face. Even along the rim, one curvature greatly exceeds the other, making the rim cylinder-like, again with positive energy. Finally, if $a >> b >> c$, we have a cylinder whose cross-section is a highly eccentric ellipse, again with extremely high energy.

A similar analysis applies to the Clifford torus $S^1 \times S^1 \in \mathbb{C}^2$, with two different radii. If the two radii are close, then W and E_0 are comparable, and even for reasonably small s, E_s is positive. If one of the radii is much greater than the other, the neighborhood of each point resembles part of a long cylinder, and E_0 is very large and positive.

Next we consider perturbations of the plane. If such a perturbation has a large natural length scale, it can be rescaled to have length scale 1 and a very small amplitude. Both W and E_0 would be quadratic in this amplitude, and so E_s would be positive for s bigger than the ratio of the leading-order coefficients. To get E_s negative for large s one has to take perturbations with very small length scales, or equivalently large amplitudes.

One such example is the plane wave $(u, v, \cos(ku), \sin(ku)) \in \mathbb{R}^4$ for large k. This is the helix $(u, \cos(ku), \sin(ku)) \in \mathbb{R}^3$ crossed with $\mathbb{R}$. See figure 6. As k increases, the loops of the helix get closer and closer. In the $k \to \infty$ limit, the contributions to $V(p)$ from the single loop highlighted in figure 6, minus the counterterms, approach the (finite) energy of the cylinder, while the other loops give more and more positive energy. As a result, $\lim_{k\to\infty} V_s(p) = +\infty$ for all s.

Perturbations with point peaks do no better. Consider the surface $z = c\exp(-(x^2 + y^2)) \in \mathbb{R}^3$. For large c, the sides of the "mountain" get very steep and resemble the sides of a cylinder, whose potential we know to be positive. The base of the mountain (the region of slope ~ 1) may have negative potential, but its area grows only as $\ln(c)$, while the area of the side grows as c. So again, as $c \to \infty$, the energy of the surface goes to $+\infty$.

This by no means exhausts the supply of potential counterexamples. However, it shows that extreme behavior tends to generate extreme positive energy, not negative energy. In particular, when features get stretched they tend to locally resemble planes or cylinders. Planes get positive energy from the presence of other nearly planes (as in the pancake example), while cylinders have positive energy for $s >> (1 - 3\ln(2))\pi/8$. This limiting behavior suggests that s_c is not only finite, but is fairly small.

There are two more classes of examples that should be noted, which we have not yet computed. One is where the perturbation is peaked at small rings, rather than points or lines. An example of this is the surface $z_2 = cz_1^2\exp(-|z_1|^2) \in \mathbb{C}^2$. The other class is surfaces with many length scales.

5. Conclusions

We have defined a family of Möbius invariant energy functionals E_s for surfaces embedded in $\mathbb{R}^n$. These functionals are all finite for smoothly embedded compact surfaces and infinite for self-intersecting immersed surfaces. They treat disconnected surfaces and connected sums of surfaces correctly.

For a functional to be useful, it must be bounded from below. For $s < (1-3\ln(2))\pi/8$, E_s is not bounded from below. However, for s sufficiently large and positive, the evidence to date suggests that E_s *is* bounded from below, although we have not yet found a proof.

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