Pathologies generated by round-off in dynamical systems

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Abstract. When solving differential equations or other dynamical systems on a computer, the effect of finiteness (round-off) can sometimes be very drastic. We discuss on the one hand rigorous conditions under which statistical (the most rough) properties of the system survive discretization, and on another hand examples where, even in the limit of vanishing perturbation, a "localization" phenomenon takes place: trajectories which should normally be dense remain confined to a small number of points. In some instances, the addition of a small amount of true noise can cure the pathologies. From the point of view of computer theory questions that we shall consider may seem very simple, but often they lead to quite nontrivial mathematical constructions.

Key words: dynamical system, mapping, SBR measure, periodic trajectory, discretization.

0. Introduction

When solving differential equations or other dynamical systems on a computer, we inevitably run into perturbations caused by various types of discretization. The first of them is time discretization, that is, the transition from differential equations or flows to difference equations or maps. The second type is space discretization, that is, the transition from continuous phase space to a discrete lattice. There are also some other types, for example partial time and space discretization when some of the equations describing our system are discretized in space or time and some not. A typical example of that type is a machine-tool with a numerical program control. We shall consider a dynamical system (DS) obtained as a result of discretizations of all types as a perturbed DS. We shall be interested in the questions about the connections between properties of the original and the perturbed system for sufficiently fine discretizations. There are a lot of problems here (see for example [1-9] and references therein) and in this paper we shall restrict ourselves only to the case of space and partial space discretizations.

Let us consider now a difference equation

$$x_{n+1} = f(x_n),$$

acting on some compact subset $X \subset R^d$. What does it mean the space discretization of such systems? Really in the computer simulations we have not a continuous compact phase space, but a discrete lattice of points, and after the calculation of the following point x_{n+1} we in fact have not the precise value of $f(x_n)$ but the nearest point to it in our lattice. So to describe the discretized version of the difference equation above we need some rigorous definitions.

Definition 1. By an ϵ-discretization of a compact set $X \subset R^d$ we mean a choice in the set X of an ordered finite lattice (a collection of points) X_ϵ so that the distances between its neighboring elements do not exceed the value $\epsilon > 0$. By an operator of the ϵ-discretization we mean a map $D_\epsilon : X \to X_\epsilon$ that associates with each point $x \in X$ its closest point from the lattice $x_\epsilon = D_\epsilon(x) \in X_\epsilon$ (if there are several such points we choose the point with the minimal number). The value of the parameter $\epsilon > 0$ we call the diameter of the ϵ-discretization.

Definition 2. By an ϵ-discretized (perturbed) system for the map f with the compact phase space X we mean a pair (f_ϵ, X_ϵ), where $f_\epsilon = D_\epsilon \circ f$.

Note that in the distinction to usual perturbation schemes the discretized (perturbed) system here is given not on the original phase space X but on the lattice X_ϵ.

In the case of round off errors in the computer modeling we deal with the uniform ϵ-discretization and in the case of the unit d-dimensional cube (or torus) phase space X and for $\epsilon = 1/N, N \in Z_+$, the lattice X_ϵ consists of all points $x \in X$ with rational coordinates with the denominator N. Another example can be a random discretization when all the points of the lattice are chosen randomly according to some distribution law (for example, the Poisson distribution).

In many papers (especially of applied type) effects of round off errors in numerical simulations are considered and it seems that these effects may be neglected especially in the case of rough systems (that is stable with respect to smooth perturbations) and sufficiently high computer precision. However we shall show that the effect of finiteness (round-off) can sometimes be very drastic. For example a stable periodic orbit may be replaced by one with a period which is a multiple of the true one or a infinite chaotic orbit may turn into a periodic one with very small period. Understanding such phenomena requires the study of a particular kind of perturbations of dynamical systems. These perturbations are deterministic by their nature but their effect is close in a sense to that of pure random perturbations. It is of interest to remark here that loss of chaoticity for discretized bounded orbits due to the finite number of points in the discretized phase space around them in principle is not an obstacle in the investigation of chaotic properties of dynamical systems (calculation of Lyapunov exponents and so on). On another hand if the discretization is sufficiently fine, usually the periodicity of the trajectory is so long that, in practical applications one couldn't distinguish the result of round-off errors of the action of small random perturbations. Some of the patologies (such as period multiplication and disruption or stabilization of unstable cycles), that we consider, persist for arbitrary fine discretizations for typical dynamical systems. About others our results show that they also persist for an open set in the space of dynamical systems. On another hand numerous numerical simulations (see for example [7-9] and referencies therein) demonstrate that this open set has to be small in a sense.

The paper is organized as follows. Section 2 contains information about properties of periodic orbits under discretization. The most interesting among them are period multiplication of periodic orbits (Theorem 1) with all possible dangers in the study of bifurcations and the disruption or stabilization of unstable cycles - “localization” phenomenon (Theorem 2). In section 3 we consider the discretized chaotic dynamical systems from the statistical point of view. We show that although the phenomena, described in section 2, depend on the the diameter of the discretization in a very irregular way, the statistical approach gives an adequate description (Theorem 4). In section 4 we investigate the case of neutral periodic trajectories, that are nor stable and nor unstable. The most interesting results here are non ergodicity of the discretized irrational rotation on the d-dimensional unit torus (Theorem 7) and pathologies of generalized rotations on the plane (Theorem 9). Contraction of the phase–space and possible methods to avoid this pathology by adding a Markov perturbation are considered in section 5. Partial space discretizations, when some of the variables are discrete and some not, are discussed in section 6. In section 7 we give rigorous sufficient conditions for numerical verification of shadowing of individual trajectories for general (nonhyperbolic) maps (Theorem 12). Section 8 contains the proofs of Theorems 1,7 and 12.

2. Properties of periodic orbits under discretizations.

Remind that an orbit or trajectory of the map f is a sequence of points $\{x_k\}_1^\infty$ such that $x_{k+1} = f(x_k)$ for all k, and in the case of a periodic orbit or or simply a cycle of period n such sequence is formed by a repetition of a finite collection of different points $(x_1, x_2, \cdots, x_n)$ and therefore it is defined by one of these points (for example x_1) and its period. We call this point periodic with the period n. In contrast with the original dynamical system, for every map f the set of non-wandering points (that is the most broad nontrivial invariant set) of the discretized system coincides with the set of periodic points. Therefore, an investigation of asymptotic properties of perturbed systems is reduced to an investigation of solely periodic orbits.

Let us consider at first the most simple case in numerical simulations, which is to find the globally stable periodic orbit of the system. It seems that for sufficiently high precision the only visible effect of perturbations is to deform and shift the periodic orbit. However it turns out that here the "period multiplication" may take place. This phenomenon consists in the emergence in a neighborhood of the original cycle of a new cycle, the period of which is a multiple of that of the parent cycle.

Theorem 1. Suppose that the map f has a cycle of period n in some neighborhood of which, consisting of disjoint neighborhoods of points of this cycle, f is a local homeomorphism, and there is a cycle of the ϵ-discretized system of period k. Then the fraction k/n may take values 1 or 2 in the one-dimensional case and may be arbitrary integer in general multidimensional case.

On Fig. 1 we show the general construction of the period doubling of a cycle of period 2. By solid squares we denote the points of the parent cycle, and by lines the trajectory of the cycle of the discretized system.

Let us consider the multidimensional case. In this case the discretized system may have a cycle of arbitrary period multiplicator with respect to the parent cycle. Geometrically this result means that the orbit of the ϵ-discretized system curls around the original orbit, but in the one-dimensional case the only one variant of rotation around the initial orbit may take place - the rotation through the angle α. However the proof of this statement is not too simple and bases on some properties of homeomorphism of ordered sets. To show that even in the two-dimensional case we may have this phenomenon with arbitrary large multiplicator, let us consider the example of a mapping of the two-dimensional plane, which may be written in polar coordinates as

$$re^{i\phi} \to r\lambda e^{i(\phi+\alpha)}.$$

Here $0 < \lambda < 1$ and the irrational number α are parameters. Each trajectory of this mapping has a spiral shape and tends to the zero point as time tends to infinity. On another hand, if $1-\lambda \ll 1$, then for sufficiently fine uniform ϵ-discretizations any trajectory of the discretized mapping, started from the point (r, ϕ) with $r \gg 1$ will not go to the origin, but will hit into a cycle around the origin. The shape of this cycle is close to a circle with the radius, defined by the following inequality:

$$\epsilon[r\lambda/\epsilon] \geq r,$$

where $[x]$ is the closest to x integer number. Therefore this cycle is a rotation of angle close to α around the origin and we may obtain arbitrary large periods by a suitable choice of parameters λ and α. This example show how "curls" around initial trajectory can appear. The distance between the "curls" has order $\epsilon/ \mid \lambda - 1 \mid$, where λ is the modulus of the cycle multiplier (i.e. the largest eigenvalue of the linearized mapping along the cycle). Hence, far from the bifurcation point (i.e. the point where the cycle became unstable and $\mid \lambda \mid \approx 1$) this effect is hardly noticed. However, it is very difficult to distinguish this effect from a bifurcation of the original system when investigating the bifurcation picture itself (in particular, in the well known Feigenbaum scenario of the transition to chaos by repeated doublings of the period). The unique way to verify that there is no this phenomenon is to repeat the simulation with several different precisions. However further we shall show that this phenomenon depends on the precision in a very complicated way.

We can consider a partition of the set $U \cap X_\epsilon$ to domains, in each of which there are cycles of the discretized system with equal multiplicators. All other points of X_ϵ hit in these sets or go out of the considered neighborhood under the action of f_ϵ.

The problem of the general effects of discretization is of special interest in the case of chaotic dynamical systems (see for example [10]), because all their trajectories are unstable and therefore the influence of perturbations on individual trajectories may grow in time. A typical example is the one-dimensional mapping $x \to \mathrm{Frac}(kx)$ for which the iterates of almost all initial points are uniformly distributed and all features of developed chaos are visible. Here $\mathrm{Frac}(x)$ is the fractional part of the number x. Another important example is so called tent map: $f_{\mathrm{tent}}(x) = \min\{ax, -ax+a\}$. More complex examples occur in the three body problem in celestial mechanics, the Henon mapping $(x, y) \to (1 - ax^2 + y, bx)$ and others (see [10] and references therein for further examples). The theorem above describes the situation also in these cases. Questions about properties of discretized chaotic systems are an active area of mathematical investigation and many problems are still open. Number of questions here is quite larger than number of known answers. We shall consider only some of them. Practically there are only a few results, showing that orbits of the discretized chaotic systems may have a connection to the original orbits. Among these results the main general one shows the connection of the orbit of weakly perturbed system (with arbitrary type of perturbation) to the original orbit. This result is due to D.V.Anosov [11], who showed that for smooth hyperbolic system for each orbit of weakly perturbed system there exists uniformly closed orbit of the initial one. But this shadowing orbit may be not typical in the sense that the distribution of its points (which is the most important feature of chaotic system) differs from the typical one, defined by the Sinai-Bowen-Ruelle measure [10]. The following statement shows that these situations may often happen in case of discretizations.

Theorem 2. If the set of all preimages of a certain cycle is dense in the phase space, then there is a sequence of discretizations with diameters that tend to zero, such that each discretized system has only one cycle that coincides with the original one.

To prove this statement it is sufficient to consider a family of discrete lattices $\{X^{(n)}\}$, points of the n-th of which coincide with the n-th preimage of the points of the considered cycle.

We may consider also the more general approach to this situation. Let us fix a mapping f and a family of discretizations $\{X_\epsilon\}$. Then for each cycle $\bar{x}_\epsilon$ of the discretized mapping f_ϵ we may take into account the number of points in all of its preimages (including the cycle itself), divided by the number of points in the lattice $\{X_\epsilon\}$, and denote this value by $p(\bar{x}_\epsilon)$. This number shows how often we may observe a cycle when initial points are chosen at random. Now, assigning for each discretization $\{X_\epsilon\}$ its cycle $\bar{x}_\epsilon$ with the largest value of $p(\bar{x}_\epsilon)$ and the probabilistic measure μ_ϵ uniformly distributed along this cycle, we may ask a question about the limit points (in the weak topology of measures) of these measures for the typical family of discretizations. It may be shown that for sufficiently "good" mapping f these limit points are invariant measures of the mapping. However the family of invariant measures may be very rich (for example uniform measures on cycles and their closure). We think that the most probable answer is the following: the limit point coincides with one of the measures - Sinai - Bowen - Ruelle measure or the measure of the maximal entropy (see [10] for definitions). Neither of these possibilities can be ruled out at the moment.

It seems that the situation, described in Theorem 2 is unrealistic, but the condition about the density of preimages is typical for chaotic systems, and so the question is only about how often such special discretizations may occur? It is interesting to remark that for the mapping $x \to \mathrm{Frac}(2x)$ and any binary discretization (i.e. computer discretization) the discretized system has only one cycle with the period equal to 1. The same is true also for the mapping $x \to \mathrm{Frac}(kx)$ with natural k and corresponding uniform discretizations. In the general case however discretizations of this type are not uniform and this gives a hope to think that the resonance between the mapping and the discretization in the numerical simulation may appear only randomly.

On another point of view Theorem 2 shows that some sort of a "localization" phenomenon takes place: trajectories which should normally be dense remain confined to a small number of points. This is of special interest, because the most frequently applied numerical method for the calculation of the density of an invariant measure of a chaotic dynamical system on a computer consists in the computation of the histogram of a sufficiently long numerical orbit, and Theorem 2 says that the error here can be very large even in the integral norm. Indeed in the simplest example $x \to \mathrm{Frac}(2x)$ every point is mapped by this mapping in the computer simulation in no more than N steps (where N is the number of binary positions in a computer word) into a zero unstable fixed point of the mapping $x \to \mathrm{Frac}(2x)$ and remains there. For example on a VAX computer it could be verified that the orbit starting from $1/3$ (which is a periodic point of period 2 in the true mapping $\mathrm{Frac}(2x)$) falls to 0 in 57 iterations, even in double precision. One could think that, for computational purposes, there is no need to study asymptotic properties but it is enough to stop the count when the histogram is already stabilized and the accumulated error is not yet large. However, the analysis shows that also this method may not lead to the required result. In the case of the uniform ϵ-discretization the number of steps at which the count should be stopped is of order $-\ln(\epsilon)$, while the guaranteed precision is of order $(-\ln(\epsilon))^{-1/2}$.

Though a number of papers have been devoted to the investigation of the question of convergence of the histograms of trajectories calculated on the computer to the density of a smooth invariant measure, there is actually only one theoretical result due to P.Gora and A.Boyarsky, that confirms the possibility of such convergence.

Theorem 3 [7]. Let f be sufficiently smooth one-dimensional mapping with a smooth invariant measure μ and for all n there is a cycle of length not less than tn, $t>0$, for a uniformly $1/n$-discretized mapping $f_{1/n}$. Then the measures that are uniformly distributed on these cycles converge weakly to μ.

The existence of the required estimate of cycle periods has been obtained in a paper [7] for the the family of piecewise linear mappings $x \to \mathrm{Frac}(3^k x)$. A large amount of numerical data on the influence of uniform ϵ-discretizations on the maximal length of cycles of discretized systems was given in the paper by C.Beck and G.Roepstorff [1], where it was shown numerically, that the maximal cycle length for the family of one-dimensional mappings $f(x) = 1-2\mid x\mid^{\alpha}$, has order $\epsilon^{1/\alpha}$, for $\alpha>0$. This shows that Theorem 3 cannot be applied in this case.

3. Statistical probability under discretizations.

Now let us try to answer the question about how "typical" the behavior of the discretized systems. At first let us consider the question of the existence of a cycle of an ϵ-discretized system for sufficiently small $\epsilon>0$ in small neighborhood of an unstable cycle of the unperturbed system. For different values of ϵ a cycle of an ϵ-discretized system can appear and disappear with arbitrary small changes in ϵ and a choice of X_ϵ. Therefore to answer this question it is necessary to make some additional assumptions about the structure of discretizations. Now we shall restrict ourselves to the most interesting case for applications of uniform discretizations of the d-dimensional unit cube X. In this case ϵ has values of the type $\epsilon = 1/n$, $n \in Z_+$. Even in this case the presence or absence of a cycle for a uniform $1/n$-discretized system depends on n in a very irregular way. The statistical approach seems to be the most natural one for the description of this situation.

Definition 3. By the statistical probability of some event (with respect to a given family of space discretizations) we mean the fraction of that discretizations under which this event takes place.

Thus the statistical probability $p(x_1, x_2, \cdots, x_n)$ of the stabilization of an unstable cycle $(x_1, x_2, \cdots, x_n)$ of a mapping f is the density of the set of those natural numbers N for which $(D_{1/N}x_1, D_{1/N}x_2, \cdots, D_{1/N}x_n)$ is a cycle of a uniformly $1/N$-discretized system.

The meaning of this definition is that the existence in the case of uniform discretizations of a cycle of a discretized system in a small neighborhood of the original system implies the possibility of the actual observation of an unstable trajectory in the computer modeling. It is therefore natural to call this situation the "stabilization of an unstable cycle". Numerical experiments show that in the modeling of chaotic dynamical systems, the cycles of which are all unstable, some of the cycles are observed much more frequently than others [12]. The following statement provides an explanation for this phenomenon.

Theorem 4. Let $\bar{x} = (x_1, x_2, \ldots, x_n)$ be an unstable cycle of the mapping f, which is continuously differentiable in a neighborhood of the cycle, and its coordinates are rationally independent. Then $p(\bar{x})$ is well defined and depends only on the derivatives of f at the points of the cycle: it is equal to the volume of the dn-dimensional polyhedron $\Omega = \Omega(f; \bar{x})$ defined by the system of semi-linear inequalities:

$$\mid f'(x_1)z_1 - z_2 \mid \leq 1/2$$

$$\mid f'(x_2)z_2 - z_3 \mid \leq 1/2$$

$$\cdots$$

$$\mid f'(x_n)z_n - z_1 \mid \leq 1/2$$

$$\mid z_i \mid \leq 1/2, \quad i = 1, 2, \ldots, n$$

where $f'(x)$ is the matrix of partial derivatives of the mapping f at the point x, and $\mid z \mid$ for a vector $z \in R^d$ is the maximal value of moduli of its coordinates.

Proof. To calculate the density of the set of natural numbers it is sufficient to consider only large such numbers, corresponding to sufficiently fine discretizations. Therefore working only in small neiborhoods of the points of the cycle we may assume that the mapping f is linear there. Let us fix a natural number $k \gg 1$. The set of points $\bar{x}_{1/k} = (D_{1/k}x_1, D_{1/k}x_2, \ldots, D_{1/k}x_n)$ is a cycle of the mapping $f_{1/k}$ if and only if the following system of inequalities holds:

$$\mid x_2 + f'(x_1)(D_{1/k}x_1 - x_1) - D_{1/k}x_2 \mid < 1/(2k)$$

$$\mid x_3 + f'(x_2)(D_{1/k}x_2 - x_2) - D_{1/k}x_3 \mid < 1/(2k)$$

$$\cdots$$

$$\mid x_1 + f'(x_n)(D_{1/k}x_n - x_n) - D_{1/k}x_1 \mid < 1/(2k).$$

Indeed, if it is true, then $f_{1/k}(D_{1/k}x_i) = D_{1/k}x_{i+1}$ for all i. On another hand, if one of these inequalities fails, say the i-th inequality, then $f_{1/k}(D_{1/k}x_i) \neq D_{1/k}x_i$. Remind now that $D_{1/k}x_i$ is the best rational approximation of the vector x_i with the rational coordinates with the common denominator k. Therefore, multiplying both hands of these inequalities by k and denoting the difference between a vector $v \in R^d$ and the closest to it vector with all integer coordinates by $\{v\}$, we have:

$$\mid f'(x_1)\{kx_1\} - \{kx_2\} \mid < 1/2$$

$$\mid f'(x_2)\{kx_2\} - \{kx_3\} \mid < 1/2$$

$$\dots$$

$$\mid f'(x_n)\{kx_n\} - \{kx_1\} \mid < 1/2$$

Now let us consider a rotation F_α through the nd-dimensional angle $\alpha = (x_1, x_2, \dots, x_n) \in R^{nd}$ on the dn-dimensional unit torus $T = [-0.5, 0.5)^{nd}$. Then the trajectory of this mapping, starting from the point $\alpha \in T$ hits into the domain Ω, defined in the theorem, if and only if there is a stabilization effect. Thus, $p(\bar{x})$ coincides with the part of the time during which this trajectory spends in Ω and consequently coincides with the F_α-invariant measure of this domain. However, the rotation mapping of the torus through an angle with rationally independent coordinates has a unique invariant measure that is equal to the Lebesgue measure on the torus T. Q.E.D.

To clarify this construction let us consider a close problem which is quite simpler. Let x be arbitrary irrational number in the interval $[0, 1]$ and let us compute how often a rational approximation of this number with a denominator n has precision not worse than some constant $\epsilon \ll 1$ divided by n; which is to calculate the density $p(x)$ of those natural numbers n for which the inequality

$$\mid x - q/n \mid \leq \epsilon/n$$

holds for some natural $q = q(n)$. Multiplying the both parts of this inequality by n we see that it is equal to

$$\mid \mathrm{Frac}(nx) \mid \leq \epsilon$$

Now considering the two intervals on the unit circle defined by the latter inequality and the rotation through the angle x we come to the same situation as in the theorem and can calculate this density precisely: $p(x) = 2\epsilon$. Note that this situation coincides with the stabilization of a fixed point x of a one-dimensional mapping with the derivative at this point $1/(2\epsilon)$.

There is a question about the practical applicability of this statement in a sense which magnitude of k needed to reach the value of $p(\alpha)$ (defined by Theorem 4) with a sufficiently good precision. From the proof above it follows that the answer to this question depends on the rate of convergence of the fraction of the points of a typical trajectory of the torus rotation which happen to be in the specified region. In the "good" case this convergence is of order $\log(k)/k$, so it is sufficiently fast.

It is interesting to remark, that the statistical probability of stabilization obtained above does not depend on the coordinates of the points of the cycle. It should be pointed out that such a uniform estimate is true only for "typical" cycles. If the condition of rational independence of the coordinates (or irrationality of the number x above) does not hold, there are "exceptional" cycles (and even maps for which all the cycles are "exceptional") whose stabilization probability depends on the coordinates of its points. Among the "exceptional" cycles there are cycles with abnormally high probability of stabilization (for example, equal to 1) and also cycles for which this magnitude is abnormally small (in comparison with the "typical" situation described in the theorem).

The method used in the proof of Theorem 4 enables us to study even finer properties of the discretized systems. For example one can calculate by this method the statistical probabilities of all events that can appear in the "period multiplication" phenomenon. The close construction also may be carried out to consider not all uniform discretizations but only binary ones, that is for the case $\epsilon = 2^{-n}$. In this case we can also use the described construction, but the dynamical system appearing in this case on the dn-dimensional unit torus T is $x \to \mathrm{Frac}(2x)$ rather than a rotation. An invariant measure of the latter mapping is not unique and the answer depends on the invariant measure represented by the trajectory of this system, starting from the point $(x_1, x_2, \ldots, x_n)$. However for almost all initial conditions this measure is equal to the Lebesgue measure on the dn-dimensional unit torus T.

Theorem 5. Suppose that the mapping f is continuously differentiable in a neighborhood of its cycle $\bar{x} = (x_1, x_2, \ldots, x_n)$. Then there is a mapping $\hat{f}$ arbitrary C^1-close to f with a cycle $\hat{x} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$ close to initial one, such that the statistical probability of the stabilization of the cycle $\hat{x}$ in the case of binary discretizations of the mapping $\hat{f}$ is equal to 1.

To prove this statement it is sufficiently to note that the preimages of the zero fixed point of the mapping $x \to \mathrm{Frac}(2x)$ (which is the replacement of the rotation mapping for binary discretizations) are dense on the interval $[0, 1]$. Therefore we may choose a mapping $\hat{f}$ such that the cycle $\hat{x}$ on one hand lies in this set of preimages, and on another hand is arbitrary close to the initial one. However the trajectory of the mapping $x \to \mathrm{Frac}(2x)$ starting from the point $\hat{x}$ is far from typical, because it hits into the unstable zero fixed point and stays there. Therefore the statistical probability of the stabilization of the cycle $\hat{x}$ is equal to 1.

For the systems given on the unit torus there is also another approach to investigate how typical are the properties of discretized systems. For a torus (in contrast with a cube) it is natural to fix a discretization up to a shift of the origin and a rotation around it. Therefore, in this case the parameter is not discrete and by the statistical probability of the stabilization of the cycle $(x_1, x_2, \cdots, x_n)$ we mean the value $p(x_1, x_2, \cdots, x_n; \epsilon)$, which is equal to the Lebesgue measure of the set of shifts (≤ 1) of the origin and rotations (through angles from 0 to 2π) around it, divided by 2π for normalization, for which the stabilization effect takes place. Here ϵ is the diameter of the discretization. By a similar definition we can obtain the dependence of the probability of the stabilization on the diameter ϵ. Let us consider the map $\phi : R^1_+ \to R^1_+$, defined by the following relation: $\phi(x) = x/(1+x)$. The following statement shows some variant of the universality of the dependence of this probability of the parameter ϵ.

Theorem 6. Suppose that the mapping f is continuously differentiated in a neighborhood of its cycle $(x_1, x_2, \cdots, x_n)$. Then for any sufficiently small $\epsilon > 0$ the sequence of quantities $\{p(x_1, x_2, \cdots, x_n; \phi^k(\epsilon))\}_{k=1}^{\infty}$ is almost periodic. Here $\phi^k(x)$ is the value of the function ϕ applied k times recursively in the point x.

4. Case of neutral periodic trajectories.

Let us now study the influence of uniform discretizations on the systems, the trajectories of which are neither stable nor unstable, or they have a local invariant component of this type. The main example here is a rotation $f^{(\alpha)}$ of the d-dimensional unit torus T through the angle $\alpha \in R^d$. In this case an investigation of individual trajectories has no sense, and we shall study the influence of discretizations on some global properties of the system, such as the ergodicity property. In the case of discretized system ergodicity means that there exists only one periodic trajectory and therefore the orbit of every point hits into this trajectory after a finite time. Due to the fact that the discretized rotation preserves distances between the points of the uniform lattice, in the ergodic case there is only one periodic trajectory with a period which equal to the number of points in X_ϵ. Since the ergodicity in the case of the rotation does not depend on a shift of origin, or a rotation of our lattice around it, we can give the following definition.

Definition 4. By the statistical probability of ergodicity $p(\alpha)$ we mean the density of the set of those natural numbers N for which uniformly $1/N$-discretized systems $(f^{(\alpha)}_{1/N}, T_{1/N})$ are ergodic.

Theorem 7. If the coordinates of the vector $\alpha \in R^d$ are rationally independent, then $p(\alpha)$ is well defined and is identically equal to zero.

This result shows the qualitative distinction between the effect of perturbations arising from space discretizations and those due to smooth perturbations, since in the latter case the well known KAM theory says that it is the ergodicity which is "typical". This makes clear the qualitative difference of the influence of the round-off errors. However, it will be interesting to precise the link between the chaotic appearance and disappearance of periodic trajectories in the $1/N$-discretized systems, for $N \to \infty$, and the phenomenon of subfurcation [13], arising from smooth perturbations. We note that if the coordinates of the angle α are rationally dependent, that is, the original system is nonergodic, then the statistical probability of ergodicity is also well defined but can be very far from zero (for instance, $p(2\pi * 1/3) = 2/3$.

It is surprising that if we restrict ourselves to the case of the binary discretizations the result would not be the same.

Theorem 8. For almost all vectors $\alpha \in R^d$ the statistical probability of ergodicity with respect to the binary discretizations $p_2(\alpha)$ is well defined and identically equal to $(1/2)^d$.

It is sufficient to prove the one-dimensional version of this statement. Let us fix a natural number n. Then the $1/2^n$-discretized system $(f^{(\alpha)}_{1/2^n}, T_{1/2^n})$ is ergodic if and only if the following inequality holds:

$$\mid 2^n\alpha - (2k+1) \mid < 1/2.$$

This is equivalent to the inequality

$$1/4 < \mathrm{Frac}(2^{n-1}\alpha) < 3/4.$$

Just as we did in the proof of Theorem 4 we may consider now the one-dimensional mapping $x \to \mathrm{Frac}(2x)$ and then, for a typical initial point $x_0 = \alpha$ we have that $p_2(\alpha)$ is equal to the fraction of time that the trajectory started from the point x_0 spends in the segment $(1/4, 3/4)$, that is $1/2$, because the Lebesgue measure is invariant for the considered mapping. Q.E.D.

Another problem arises in the investigation of a rotation $f^{(\alpha)}$ of points in the two-dimensional plane R^2 by a fixed angle α around the origin. It seems that the situation here is just the same as in the previous example, but there is a large difference between these examples. It consists in the fact that although the mapping $f^{(\alpha)}$ preserves distances, the discretized mapping does not preserve them (in distinction to the previous case). Moreover, if we consider a broken line, consisting of consecutive points of some trajectory and straight lines between them, then two such different lines may intersect each other. Without discretization, the trajectories of the mapping $f^{(\alpha)}$ lie on concentric circles. With round-off, it is not known when the trajectories go to the origin and when they go to infinity (neither can be ruled out at the moment). Numerical simulations (see Fig. 2.) show the evidence of a picture similar to the situation in the KAM theory, that is we can see some quasi-circles (invariant tori) with gaps between them. Remark that the distribution of points on these tori may be not ergodic. However there is no analytical proof of this fact.

To show that under the discretization similar neutral systems may behave as dissipative ones, let us consider a generalization of the previous model.

Definition 5. A map $f : R^2 \to R^2$ is called the generalized rotation around some fixed point $x_0 \in R^2$ if $f(x_0) = x_0$ and for any point $x \neq x_0$ the trajectory $\{f^n(x)\}_{n\geq 0}$ started from this point lies on a closed curve, homeomorphic to a circle.

An example of such map is a usual rotation around the origin onto the irrational angle. The latter mapping is Hamiltonian and, as it follows from Kolmogorov - Arnold - Moser (KAM) theory [13, 14], for sufficiently "good" rotation angles and "good" sufficiently small smooth perturbations, the perturbed mapping has invariant curves (homeomorphic to a circle) around the origin and each trajectory of the perturbed system is bounded by this invariant curves. We have noted before, that in the case of perturbations, arising in the computer modeling the behavior of perturbed systems looks similar. However in the case of the generalized rotation this is not longer true.

Let us fix in the two-dimensional plane R^2 a system of orthogonal coordinates (x, y) and a family of "triangle" curves $T_t, t > 0$ the i-th side of which satisfies the equality

$$a_i x + b_i t = y, \quad i = 1, 2, 3.$$

We now define a map $f_\alpha : R^2 \to R^2$ as a generalized rotation around the origin along the curves T_t with the unit rate, i.e. a shift of the current point by means of one of the vectors:

$$\alpha^{(i)} = (1/\sqrt{1 + a_i^2}, a_i/\sqrt{1 + a_i^2}), \quad i = 1, 2, 3.$$

according to the number of the side of the "triangle" curve T_t, on which the current point lies. If during this shift we are going through the corner of the "triangle", then we continue the shift along the next side. The collection of vectors $\alpha^{(i)}$ we denote by α. We show a typical trajectory of this system on Fig. 3.

To study the event that any point of the $1/n$-discretized plane, outside the ball of radius $10/n$ centered in the origin, goes to infinity under the action of the $1/n$-discretized generalized rotation f_α, we may consider the more coarse situation, when after each iteration of the discretized mapping we go to the "triangle" with the larger value of the parameter t. We shall call the latter event the "coarse destabilization".

Theorem 9. Let the vectors $\alpha^{(i)}$ and the vector with all unit coordinates be jointly rationally independent. The density of natural numbers n such that for the $1/n$-discretized generalized rotation f_α the coarse destabilization takes place is equal to $1/2^3$.

Sketch of the **proof**. For a fixed vector $\beta = (\beta_1, \beta_2) \in R^2$ with rationally independent coordinates let us define a map $h_\beta(x) = x + \beta, \quad x \in R^2$ and a function $H_\beta : R^2 \to R^1_+$, which is equl to the distance from the point x to the line $y = \beta_2/\beta_1 x = \gamma_x$. We assume at first that $\gamma > 0$. Let us consider a unit square with sides parallel to coordinate axis and the left lower corner in the origin. We divide this square into four equal squares and consider a set A_+, consisting of three parts: the left upper square, the part of the left lower square lying below the line $y = \gamma x$, and the part of the right upper square lying below the line $y = \gamma x + (1 - \gamma)$. Then the density $p(\beta)$ of natural numbers n such that the $1/n$-discretized mapping $h_{\beta,1/n}$ monotonically increases the distance function H_β (i.e. $H_\beta(x) \le H_\beta(h_{\beta,1/n}(x))$ for any x), is equal to the Lebesgue measure of the set A_+. The explanation is that the $1/n$-discretized mapping h_β increases the distance function if and only if the n-th iterate of the mapping $x \to \mathrm{Frac}(x + \beta)$, beginning at the point $\beta \in R^2$, lies in the set A_+. Therefore $p(\beta)$ is equal to the occurrence time of this set for the rotation mapping above, which is equal to the Lebesgue measure of this set (1/2). The considered set is shown on Fig. 4. The situation with $\gamma < 0$ is considered in the same way. Now applying this result coupled with the rational independency of $\alpha^{(i)}$, we obtain the statement of the theorem, because the statistical probability of the considered event is equal to the multiplication of probabilities of events, related to each $\alpha^{(i)}$. Q.E.D.

Although this example shows that in the general case trajectories of generalized rotations in the computer modeling are not bounded, all numerical simulations with Hamiltonian generalized rotations (for example, with usual rotations onto irrational angles) give another result similar to that in Fig. 2. Therefore we can formulate a conjecture.

Conjecture. Let $f : R^2 \to R^2$ be a Hamiltonian generalized rotation (and thus preserving the Lebesgue measure). Then for any n each trajectory of the $1/n$-discretized mapping is bounded.

If this statement is true, then since each computer trajectory is bounded and lies on the discrete lattice, it hits into a finite cycle. As we mentioned above it looks like an invariant torus in the KAM theory (apart from the distribution of points on such torus).

5. One method of numerical modeling of chaotic systems.

There are two possible ways of computer modeling of chaotic systems. The first one is to compute a good approximation of a typical trajectory of the system under consideration, and then to calculate some statistics along it. The second method is to construct a new dynamical system on the computer so that the statistical properties of the new system are equal in a certain sense to those of the initial one.

Let us define the domain $M_\epsilon(x)$ of the point $x \in X_\epsilon \subset R^d$ under the ϵ-discretization as a union of all points in the genuine phase space X such that under the discretization they coincide with x, that is

$$M_\epsilon(x) = \{y \in X : D_\epsilon(y) = x\}.$$

The main qualitative difference of the discretized system from the parent one is the fact that under the action of the discretized mapping f_ϵ the image of the domain $M_\epsilon(x)$

consists only of one point $f_\epsilon(x)$ and therefore the volume of this element of the phase space decreases dramatically, in distinction to the action of the unperturbed mapping. To cure this pathology we suggest a method of numerical modeling which is based on the replacement of the mapping f by a random Markov process on the lattice X_ϵ, which takes each point $x \in X_\epsilon$ with equal probability to any point of the set $D_\epsilon(f(M_\epsilon(x)))$. We note that for the above example of the mapping $x \to \mathrm{Frac}(kx)$ with the natural k for any uniform ϵ-discretization the constructed random Markov process has a unique invariant measure, which coincides up to a discretization with the Lebesgue measure; the latter is the unique smooth measure of the unperturbed system.

We consider two types of random perturbations. The first is a Markov random process which takes each point $x \in X_\epsilon$ with equal probability to any point of some neighborhood of it on the lattice. Fix a natural number $k \gg 1$ and consider a Markov random process $\pi_{\epsilon,k}$ on X_ϵ with the transition probabilities $q_{\epsilon,k}(x,y)$ of the following form:

$$q_{\epsilon,k}(x,y) = \begin{cases} (2k)^{-d}, & \text{if } \mid x - y \mid \leq k\epsilon; \\ 0, & \text{otherwise.} \end{cases}$$

To define the second type of random perturbations it is more convenient to describe note a random perturbation itself, but a perturbed system in the whole. Let F_ϵ be a Markov random process on X_ϵ with the transition probabilities $p_\epsilon(x,y)$ of the form:

$$p_\epsilon(x,y) = \begin{cases} 1/\mathrm{Card}(D_\epsilon(f(M_\epsilon(x)))), & \text{if } y \in D_\epsilon(f(M_\epsilon(x))); \\ 0, & \text{otherwise.} \end{cases}$$

Now by a stochastically perturbed ϵ-discretized dynamical system we mean a Markov random process $F_{\epsilon,k} = \pi_{\epsilon,k} \circ F_\epsilon$ on X_ϵ, which is the superposition of the random processes mentioned above.

For a sufficiently wide class of chaotic dynamical systems, namely for the so called piecewise expanding systems, the correctness of this method is proved rigorously [3], using the ergodic theorem by Ionescu Tulchi and Marinescu for the Perron - Frobenius operator of the discretized system. Numerical simulations for the Henon mapping, the horseshoe mapping, standard mapping and some others also gave good agreement of statistical properties.

6. Partial space discretizations.

¿From the pure mathematical point of view due to the finiteness of the phase space X_ϵ of the ϵ-discretized dynamical systems, discretization results only in the destruction of chaotic properties. Remark that this does not mean that one could not compute numerically Lyapunov exponents and so on, but only that the discretized dynamical systems are not chaotic (there is no mixing etc.). However this scheme does not describe all possibilities. Let us consider an example. Suppose we have a machine tool with computer control. Then it is natural to consider the part of the system describing the dynamics of the machine tool in a continuous phase space and the part describing the computer control on a discrete lattice. In this way partial discretization arises in applications. Because of lack of a general theory we shall consider only one of the possible approaches to this type of discretizations.

Definition 6. Let the mapping $f = f^{(1)}$ to be decomposed into the sum $f^{(1)}(x) = f^{(2)}(x) + f^{(3)}(x)$; $f^{(i)} : R^d \to R^d$, $i = 1, 2, 3$. We call a "partially ϵ-discretized mapping" the mapping $f_\epsilon(x) = f^{(2)}(x) + D_\epsilon(f^{(3)}(D_\epsilon(x)))$ acting in the initial phase space R^d. Here D_ϵ is the operator of the usual space discretization.

Theorem 10. Suppose that $f^{(i)}$, $i = 1, 2, 3$ are twicely continuously differentiable and the mapping $f^{(1)}$ is dissipative (i.e. has a globally attracting ball) and $f^{(2)}$ is an expanding mapping with a sufficiently large expanding constant. Then the partially ϵ-discretized system (f_ϵ, R^d) has a stochastic attractor with a smooth mixing invariant measure.

Proof of this statement bases on the fact that, for any discretization, the discretized mapping is piecewise expanding. Indeed, let us fix some discretization. Then we have a partition of R^d to cubes X_i, such that for all $x \in X_i$ the function $D_\epsilon(f^{(3)}(D_\epsilon(x)))$ has a common value, say C_i. Therefore for $x \in X_i$, we have $f_\epsilon(x) = f^{(2)}(x) + C_i$. Now, because of the dissipativity of the mapping f and the expanding condition for the mapping $f^{(2)}$, we obtain the statement of the theorem. Q.E.D.

Note that the assumptions of Theorem 10 do not exclude a situation where the initial system for any discretization has a unique globally attracting fixed point. For example let us consider in the one-dimensional case the following functions: $f^{(2)}(x) = 3x; f^{(3)}(x) = -2.5x$. Then the unperturbed mapping $f(x) = 0.5x$ has a unique globally attracting fixed point in the origin, but the partially discretized system has a stochastic attractor in some neighborhood of the origin.

There is also a local variant of the previous result.

Theorem 11. Suppose that $f^{(i)}$, $i = 1, 2, 3$ are twicely continuously differentiable and these mappings have a common periodic trajectory, which is stable with respect to f and is unstable in $f^{(2)}$. Then there is an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, the partially discretized system has a stochastic attractor.

Thus the emergence of chaos under partial discretization seems to be typical, and though the results obtained guarantee only the emergence of local chaotic islands, we believe it is plausible that at least in the case of completely integrable Hamiltonian systems there could appear chaotic domains of a finite size that does not vanish when the discretization diameter tends to zero.

7. Shadowing of individual trajectories.

One of the main questions, arising in the computer modeling of dynamical systems, is to check up the adequacy of the obtained results. That question is especially critical in the case of modeling of chaotic dynamical systems, i.e. systems with statistical laws in the behavior. In computer modeling of discrete time dynamical system, when we calculate a numerical trajectory for the mapping $f : R^d \to R^d$, we may only say that the obtained sequence of points $\{x_i\}$ is an ϵ-trajectory of the mapping f. That is $\mid x_{i+1} - f(x_i) \mid < \epsilon$ for all i. We say that an ϵ-trajectory $\{x_i\}$ is shadowed with the accuracy σ, if there exists a point $x \in R^d$, such that $\mid f^n(x) - x_n \mid < \sigma$ for all n. In other words, if an ϵ-trajectory is shadowed, then there exists a real trajectory of the mapping f lying in its neighborhood.

The first result in which such a possibility is proven to happen for chaotic dynamical systems is the theorem by D.V.Anosov about the shadowing property of ϵ-trajectories in the case of smooth hyperbolic dynamical systems [11]. There are a lot of generalizations of this result (see for example [8,9] and a survey there). Authors in these papers investigate the possibility of the shadowing of any ϵ-trajectory of a mapping f of some class. In the present paper we propose a method of the verification of the shadowing property for a given specific ϵ-trajectory of a mapping for which this property may not hold in the whole (i.e. not all ϵ-trajectories and not for all n are shadowed).

The method consists in the construction of special parallelograms around the points of the ϵ-trajectory with sufficiently small diameters and the verification of the fact that these parallelograms satisfy some property, which we name the local Markov property on the analogy with the definition of the Markov partition of the dynamical system [10]. One can say that this condition is equal to the transversal crossing of the image of the i-th parallelogram (under the action of the mapping f) with the $(i+1)$-st parallelogram. The background of this method is a construction that was proposed in [8] for computer analysis of trajectories of the two dimensional Henon mapping.

Let $f : R^d \to R^d$ be a nonsingular mapping of the Euclidean space R^d with the metric $\mid . \mid$ into itself. In this space we fix an arbitrary sequence of points $\{x_i\}_{i=1}^N$ in the σ-neighborhood ($\sigma \ll 1$) of which the mapping f is continuously differentiable and nondegenerate. For each i we represent the space R^d in the form of the direct sum of two linear subspaces E_i^+ and E_i^- the dimensions of which in the general case depend on the number i. One can take E_i^+ (E_i^-) as the eigenspace of the matrix of derivatives of the mapping f at the point x_i corresponding to eigenvalues of absolute value greater than 1 (smaller or equal to 1). In this paper the subspace E_i^+ and all objects corresponding to it will be called positive and denoted by the sign plus, while E_i^- and corresponding objects will be called negative and denoted by the sign minus. The projection onto E_i^+ parallel to E_i^- we denote by π_i^+, and onto E_i^- parallel to E_i^+, by π_i^-.

For each i we fix three parallelograms: $A_i^+ \subset E_i^+$ and $A_i^- \subset E_i^-$, which are Cartesian products of one-dimensional intervals, and $A_i = A_i^+ * A_i^-$ which is the Cartesian product of the parallelograms A_i^+ and A_i^-. The center of the parallelogram A_i^+ coincides with the point $\pi_i^+(x_i)$, and the center of the parallelogram A_i^- with the point $\pi_i^-(x_i)$. Moreover, we assume that the diameter of the parallelogram A_i is smaller than σ. The boundary of the parallelogram A_i^+ (A_i^-), which is viewed as a region in E_i^+ (E_i^-), will be denoted by ∂A_i^+ (∂A_i^-). Since the boundary of A_i is $\partial A_i^- * \partial A_i^+ \cup \partial A_i^+ * \partial A_i^-$, we introduce the notion of positive and negative boundaries of the parallelogram A_i:

$$\partial^+ A_i = \partial A_i^- * A_i^+, \quad \partial^- A_i = \partial A_i^+ * A_i^-.$$

By A_i^σ we denote the parallelogram with center at the same point, but with positive part $(1+\sigma)$ times smaller and negative $(1+\sigma)$ times greater than in the initial one. The parallelogram $A_i^{-\sigma}$ is defined in the same way, but the value σ is replaced by $-\sigma$.

Definition 7. We shall say that the sequence of parallelograms $\{A_i\}$ satisfies the weak local Markov condition (for the mapping f) if for all i we have:

$$(1) \quad \pi_{i+1}^+(fA_i \cap A_{i+1}) = \pi_{i+1}^+(A_{i+1}),$$

$$(2) \quad \pi_i^-(f^{-1}(f(A_i) \cap A_{i+1})) = \pi_i^-(A_i),$$

$$(3) \quad f(\partial A_i) \cap \partial A_{i+1} = f(\partial^+ A_i) \cap \partial^- A_{i+1},$$

and we will say that it satisfies the local Markov condition if (1) - (3) are still satisfied when we replace A_i by A_i^σ and A_{i+1} by $A_{i+1}^{-\sigma}$.

The meaning of the last operation is that we require the conditions (1) - (3) to be satisfied with some excess.

Theorem 12. There is a $\sigma_0 = \sigma_0(f) > 0$, such that for any $\sigma < \sigma_0$ the local Markov condition for the sequence of parallelograms $\{A_i\}$ implies that the intersection of the preimages of these parallelograms is not empty. Therefore there exists a point $x \in A_0$, such that $f^n(x) \in A_n$ for all natural n.

Conditions (1) - (3) are necessary in a certain sense, because for each $\sigma > 0$ there exists a mapping f and a sequence of parallelograms $\{A_i\}$ such that if only one of the conditions (1) - (3) is not satisfied, then the intersection of the pre-images $f^{-n}(A_n)$ is empty.

Remark that in the Anosovs' case our linear subspaces E_i^+ and E_i^- may be choosen in a such way to coincide with local unstable and stable directions. In the one-dimensional case the weak local Markov condition could be reduced to a very simple form:

$$A_{i+1} \subset fA_i$$

for all i. Therefore if we have a numerical trajectory (i.e. a finite number of points) of a one-dimensional map we can verify the shadowing property in the following way. Choose small intervals around the considered points and if any subsequent interval is a subset of the image (under the action of the map) of the previuos interval, then the shadowing property holds. Another remark concerned the fact that in the proof of this result (see Section 8) we actually do not use the independency of the map f on the number of the point along the trajectory. This gives us a possibility to use the conditions above even in the nonstationary case, when the map f depends on the number of iteration.

8. Main Proofs

Proof of Theorem 1. Let $\bar{x} = (x_1, \ldots, x_n)$ be a cycle of the map f and let $U = \cup U_i$ be its neighborhood, consisting of disjoint neighborhoods of points of the cycle, in which there is a cycle $\bar{y} = (y_1, \ldots, y_k)$ of the ϵ-discretized map.

We start from the one dimensional case ($d = 1$). The map f is a local homeomorphism of a neighborhood U of the cycle, therefore the function $f(x)$ is monotonous on U and hence the function $f_\epsilon(x)$, restricted to the set $U \cap X_\epsilon$, is also monotonous as a superposition of two monotonous functions f and f_ϵ. Let us denote by f_ϵ^* the derivative map [10], constructed for the dynamical system (f_ϵ, X_ϵ) with respect to the set $U_1 \cap X_\epsilon$. Remind that the derivative map f^* for a dynamical system (f, X) with respect to a set U is defined in the following way. For any point $x \in U$ we set $f^*(x) = f^n(x)$, where n is the minimal natural number n such that $f^n(x) \in U$. Let $z_1 < z_2 < \ldots < z_{m_1}$ be points of the cycle $\bar{y}$ lieing in U_1. Due to the fact that the map f is a local homeomorphism, the number m_i of points of the cycle $\bar{y}$ lieing in U_i does not depend on i and is equal to an integer m (the period multiplicator). There are three possibilities:

(a) $f_\epsilon^* z_1 = z_1$. This means that $m = 1$ and $k = n$, i.e. there is no period multiplication.

(b) $f_\epsilon^* z_1 \neq z_1$ and $(f_\epsilon^*)^2 z_1 = z_1$. Then $m = 2$ and $k = 2n$, i.e. the discretized system has a cycle of a double period. Remark, that the necessary condition of the period doubling is a monotonous decrease of $f^n(x)$ in a neighborhood of the cycle.

(c) $f_\epsilon^* z_1 \neq z_1$ and $(f_\epsilon^*)^2 z_1 \neq z_1$. Let us show that such a situation does not exist. At first suppose that the value m is even, i.e. $m = 2l$. Since f_ϵ^* is monotonous, then $(f_\epsilon^*)^2$ is monotonously nondecreasing. Hence we have:

$$z_1 < (f_\epsilon^*)^2 z_1 \leq (f_\epsilon^*)^4 z_1 \leq \ldots \leq (f_\epsilon^*)^{2l} = z_1.$$

We come to a contradiction, because the first inequality is strict. It remains tio prove that there is no cycle of odd order $m = 2l + 1$. There are also two possibility: the function f_ϵ^* is monotonically nondecreasing or monotonically nonincreasing. In the first case the orientations of the pairs (z_i, z_{i+1}) are the same for all $i = 1, 2, \ldots, m - 1$. Therefore such sequence of points cannot be a cycle. In the second case the orientations of the subsequent pairs are reverse, because $f_\epsilon^* z_i = z_{i+1}$ by the construction of the derivative map. Hence the pairs $(f_\epsilon^* z_{2l+1}, f_\epsilon^* z_{2l+2})$ and (z_1, z_2) must have a reverse orientation. We come to a contradiction once more, because $f_\epsilon^* z_{2l+1} = z_1$ and $z_1 < z_i$ for any $1 < i \leq m$.Proof. Let $\bar{x} = (x_1, \ldots, x_n)$ be a cycle of the map f and let $U = \cup U_i$ be its neighborhood, consisting of disjoint neighborhoods of points of the cycle, in which there is a cycle $\bar{y} = (y_1, \ldots, y_k)$ of the ϵ-discretized map.

The multidimensional case was discussed in Section 2. Q.E.D.

Proof of Theorem 7. Let us begin from the one-dimensional case ($d = 1$). Let us denote by $I(x)$ the nearest integer to $x \in R^1$. The uniformly $1/N$-discretized dynamical system $(f_{1/N}^{(\alpha)}, T_{1/N})$ is ergodic if and only if the natural numbers N and $I(N\alpha)$ have no common multipliers. Therefore the statistical probability of ergodicity (if it exists) is equal to the density of numbers N such that N and $I(N\alpha)$ have no common multipliers.

Let us number the set of prime numbers by their order: $r_1 = 2 < r_2 = 3 < r_3 = 5 < \ldots$ and denote $r_i? = r_1 * r_2 * \ldots * r_i$. To calculate the density above we consider a partition of the set of natural numbers to nonintersected subsets Q_i, such that for each i the set Q_i consists of all natural numbers dividing by r_i and not dividing by r_1, r_2, $\ldots, r_{i-1}$. The set Q_i may be represented in the following form:

$$Q_i = \{m \in Z^+ : m r_i(r_{i-1}?n + q),\, n \in Z^0,\, q \in \Pi_i\}$$

where $Z^0 = Z^+ \cap \{0\}$ and the set Π_i consists of 1 and a set of all prime numbers larger or equal than r_i and less than $r_{i-1}?$. Hence the density of this set is well defined and is equal to $p_i = \#(\Pi_i)/r_i?$.

Now let us answer a question: when for a fixed pair (i, j) a number $n \in Q_i$ is such that n is divided by r_j and $I(n\alpha) \in Q_j$. This means that the system $(f_{1/n}^{(\alpha)}, T_{1/n})$ is not ergodic. ¿From the definition of the sets Q_i it follows that the necessary and sufficient condition for the event above is the following:

$$\mid r_j(r_{j-1}?m + q_j) - r_j r_i(r_{i-1}?n + q_i)\alpha \mid< 1/2,$$

where $m, n \in Z^0$ are arbitrary nonnegative integers, $q_i \in \Pi_i$ and $q_j \in \Pi_j$. This inequality is equivalent to

$$\mid m + q_j/r_{j-1}? - r_i q_i \alpha/r_{j-1}? - r_i?n\alpha/r_{j-1}? \mid< 1/(2r_j?).$$

Denoting $R_{ij} = r_i?/r_{j-1}?$ and $L_{ij} = (q_j - r_i q_i \alpha)/r_{j-1}?$, we obtain

$$\mid m + L_{ij} - nR_{ij}\alpha \mid< 1/(2r_j?).$$

The value m is arbitrary, therefore the last inequality is equivalent to a condition that the fractional part of $nR_{ij}\alpha$ lies in the smaller segment Δ_{ij} of the unit circle between the points $(L_{ij} \pm 1/(2r_j?))\,(\mathrm{mod}1)$. The length of this segment is equal to $1/r_j?$.

Now we consider a dynamical system (F, T) on a unit circle T, where F is a rotation through the irrational angle $R_{ij}\alpha$. Analogously to the proof of Theorem 4 the fraction of time that the trajectory of this system starting from the zero point spends in the segment Δ_{ij} is equal to its length $1/r_j?$. On another hand for a fixed value $q_j \in \Pi_j$ this means that the relative density of natural numbers such that $n \in Q_i$ is such that n is divided by r_j and $I(n\alpha) \in Q_j$ and $n = mr_i(r_{i-1}?n + q_j)$ is equal to $1/r_j?$ and does not depend on q_j. Therefore the whole relative density p_{ij} of natural numbers such that $n \in Q_i$ is such that n is divided by r_j and $I(n\alpha) \in Q_j$ is equal to $\#(\Pi_j)/r_j? = p_j$.

Now using these relative densities we can calculate the the statistical probability of ergodicity:

$$p(\alpha) = 1 - \sum_i p_i \sum_j p_{ij} = 1 - \sum_i p_i \sum_j p_j = 1 - (\sum_i p_i)^2 = 1 - 1 = 0.$$

In the multidimensional case the system $(f^{(\alpha)}_{1/N}, T_{1/N})$ is ergodic if and only if

(a) for all i natural numbers N and $I(N\alpha_i)$ have no common multipliers;

(b) for all $i \neq j$ natural numbers $I(N\alpha_i)$ and $I(N\alpha_j)$ have no common multipliers.

By the first part of the proof the density of all natural N satisfying the first condition is equal to zero. It remains to prove that the density of all natural N satisfying the second condition is well defined, but one can do this in the same manner as the first part of the proof. Q.E.D.

Proof of Theorem 12. At first we consider the linear case and certain generalizations of the weak local Markov conditions (1) - (3) for a sequence of linear mappings $\{f_i\}$:

$$(1') \quad \pi^+_{i+1}(f_i A_i \cap A_{i+1}) = \pi^+_{i+1}(A_{i+1}),$$

$$(2') \quad \pi^-_i(f^{-1}_{i+1}(f_i A_i \cap A_{i+1})) = \pi^-_i(A_i),$$

$$(3') \quad f_i(\partial A_i) \cap \partial A_{i+1} = f_i(\partial^+ A_i) \cap \partial^- A_{i+1},$$

We shall show that if the sequence of linear mappings $\{f_i\}$ satisfies conditions $(1')$ - $(3')$, then there exists a point $x \in A_0$ such that for all natural numbers n we have

$$f_n f_{n-1} \cdots f_2 f_1(x) \in A_n.$$

For each i we construct a new sequence of sets $\{A_i^{(k)}\}_k$, where $A_i^{(0)} = A_i$, and consecutive sets are defined inductively:

$$A_i^{(k+1)} = f_{i+1}^{-1}(f_i A_i^{(k)} \cap A_{i+1}^{(k)}).$$

By construction, $A_i^{(k+1)} \subset A_i^{(k)} \subset A_i$. Moreover, the linearity of the mappings f_i and condition $(3')$ imply that the sets $f_i^{(k)} A_i \cap A_i^{(k)}$ and $A_i^{(k+1)}$ are parallelograms for all i and k. Now from conditions $(1')$ and $(2')$ it follows that for each i and k the intersection of parallelograms $A_{i+1}^{(k)}$ and $f_i A_i^{(k)}$ is not empty.

Since the parallelograms $A_i^{(k)}$ on k are nested, there exists a nonempty set $A_i^* = \cap_k A_i^{(k)}$. However according to the definition,

$$f_{i+1}^{-1}(f_i A_i^* \cap A_{i+1}^*) = A_i^*.$$

Therefore, $f_i A_i^* \supset A_{i+1}^*$ for all i, i.e. the family of sets $\{A_i^*\}$ is invariant with respect to the sequence of mappings $\{f_i\}$, and this implies the desired statement.

It remains to consider the nonlinear case. To reduce this case to linear one, in a small neighborhood of each point x_i we carry out a change of variables g_i, transforming the mapping f locally to its linear part f_i. Note that by the assumptions of the theorem the mapping g_i is close to the identity in this neighborhood for all i.

Now, since the weak local Markov conditions (1) - (3) are satisfied for the mapping f with excess (replacing A_i by A_i^σ and A_{i+1} by $A_{i+1}^{-\sigma}$) and the parallelograms $\{A_i\}$ are small, the sequence of parallelograms $\{A_i\}$ satisfies conditions $(1')$ - $(3')$ for the sequence of linear mappings $\{f_i\}$. Therefore there exists a point $y \in A_0$ such that for all natural numbers n we have

$$f_n f_{n-1} \cdots f_2 f_1(y) \in A_n.$$

Carring out the inverse change of variables and setting $x = g_0^{-1}(y)$, we obtain the statement of the theorem.

When condition (1) or condition (2) breaks down, there is no intersection of the image and the preimage of some parallelograms (see Fig. 5(a,b)), therefore the set $A_0^* = \cap_n f^{-n}(A_n)$ is empty. In the case when the condition (3) breaks down, we come to the same situation as in the previous cases, only on the next step of the construction of the sets A_i (see Fig. 5(c)).

To verify the absence of excess in conditions (1) - (3) we consider a counterexample for this case, which is shown on the Fig. 5(d). If the image and preimage are close to fine strips (strong expansion along the positive direction and strong compression along the negative one) and are close to one of diagonals of parallelograms, then the nonlinearity of the mapping f may bring to their repeated intersections. As a result, A_i^{k+1} on the next iteration consists of several connected components, among which lies an image of the set A_{i-1}^k, so that the intersection is empty. Q.E.D.

9. Conclusion

It was shown that the influence of small deterministic perturbations due to the phase space discretizations to the dynamics of chaotic dynamical systems may be very drastic even in the case of sufficiently "good" systems. Results of type of "period multiplication" show that numerical investigations of the Feigenbaum scenario of the transition to chaos are not quite evident. The same problem appears in the numerical investigation of histograms along trajectories of dynamical systems. Due to the finiteness of the phase space X_ϵ of the ϵ-discretized dynamical systems, discretization results only in the destruction of chaotic properties. However section 6 shows that the emergence of chaos under partial discretization seems to be typical. Some applied results are given in sections 5 and 7. The method of section 5 gives a possibility to investigate statistical properties of chaotic systems by means of adding small suitably chosen random perturbations. The results of section 7 show that even in the general case one can verify numerically the existence of the genuine trajectory of the system in the neighborhood of a numerical trajectory.

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Figure Captions

Fig. 1. Period doubling due to round-off errors. Case of a period 2 cycle. By solid squares we denote the points of the parent cycle, and by lines the trajectory of the cycle of the discretized system.

Fig. 2. Invariant set of the discretized rotation of the plane ($\alpha = 0.1234$, $\epsilon = 0.01$).

Fig. 3. Destabilization of the generalized discretized rotation of the plane. By solid lines we denote one of the invariant curves for the genuine system, and by lines with arrows a trajectory of the discretized system.

Fig. 4. The region, where the discretized mapping monotonically increases the distance from the origin.

Fig. 5. Counterexamples to Theorem 12.