IMSc-93/21

Generalized Fock spaces, new forms of quantum statistics and their algebras

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Abstract

We formulate a theory of generalized Fock spaces which underlies the different forms of quantum statistics such as "infinite", Bose-Einstein and Fermi-Dirac statistics. Single-indexed systems as well as multi-indexed systems that cannot be mapped into single-indexed systems are studied. Our theory is based on a three-tiered structure consisting of Fock space, statistics and algebra. This general formalism not only unifies the various forms of statistics and algebras, but also allows us to construct many new forms of quantum statistics as well as many algebras of creation and destruction operators. Some of these are : new algebras for infinite statistics, q-statistics and its many avatars, a consistent algebra for fractional statistics, null statistics or statistics of frozen order, "doubly-infinite" statistics, many representations of orthostatistics, Hubbard statistics and its variations.

Key Words : Fock space – multioscillator algebras – q-deformation – quantum groups – generalized statistics – infinite statistics – fractional statistics – Hubbard model.

PACS Nos: 03.70 ; 05.30 ; 02.20 ; 71.27

2. Generalized Fock Spaces

Given a set of oscillators with indices g,h,i ... m, we construct the state vector

$$|n_g, n_h \ldots n_m; 1\rangle = |\underbrace{1_g \cdots 1_g}_{n_g} \underbrace{1_h \cdots 1_h}_{n_h} \cdots \underbrace{1_m \cdots 1_m}_{n_m}\rangle \tag{2.1}$$

On the right-hand-side of this equation, $1_g, 1_h \ldots 1_m$ appear $n_g, n_h \ldots n_m$ times respectively and $n_g \ldots n_m$ denote the number of quanta with indices g ... m respectively. Together with the state (2.1) we consider the set of all states obtained through all distinct permutations of the entries on the right-hand-side of (2.1). Thus, we have

$$|n_g, n_h \ldots n_m; 2\rangle = |\underbrace{1_g \cdots 1_g}_{n_g - 1} \; 1_h 1_g \underbrace{1_h \cdots 1_h}_{n_h - 1} \cdots \underbrace{1_m \cdots 1_m}_{n_m}\rangle \tag{2.2}$$

$$\vdots$$

$$|n_g, n_h \cdots n_m; 35\rangle = |1_g 1_g 1_m 1_m \underbrace{1_h \cdots 1_h}_{n_h} \ldots \underbrace{1_m \cdots 1_m}_{n_m - 2}\rangle \tag{2.3}$$

$$\vdots$$

$$|n_g, n_h \cdots n_m; s\rangle = |\underbrace{1_m \cdots 1_m}_{n_m} \ldots \underbrace{1_h \cdots 1_h}_{n_h} \underbrace{1_g \cdots 1_g}_{n_g}\rangle \tag{2.4}$$

Here we have given examples of a few permutations. In (2.2), the positions of one g-quantum and one h-quantum has been interchanged and in (2.3), we have indicated a few more interchanges. Collectively, we shall denote the set of all these states as

$$|n_g, n_h \ldots n_m; \mu\rangle \quad , \quad \mu = 1, 2 \ldots s \tag{2.5}$$

where s is the total number of distinct permutations and μ labels each of these states. It is easy to see that

$$s = \frac{(n_g + n_h + \ldots n_m)!}{n_g! n_h! \ldots n_m!} \tag{2.6}$$

(In eq.(2.3), μ has been put 35 rather arbitrarily). We assume the existence of a unique vacuum state corresponding to zero occupation number for all the oscillators :

$$|0\rangle \equiv |0, 0, 0 \ldots 0\rangle \tag{2.7}$$

We shall first consider the set of all the s states given in (2.5) as linearly independent. Although they are linearly independent, they may not be orthogonal to each other in general, nor are they normalized. However, a state in one sector characterized by the occupation numbers $(n_g, n_h \ldots n_m)$ is orthogonal to any state in another sector characterized by another set of occupation numbers $(n_g^{'}, n_h^{'} \ldots n_m^{'})$. We can summarize these statements by the equation :

$$\langle n_g^{'}, n_h^{'} \ldots n_m^{'}; \alpha | n_g, n_h \ldots n_m; \beta \rangle \; = \; \delta_{n_g^{'} n_g} \, \delta_{n_h^{'} n_h} \ldots \delta_{n_m^{'} n_m} \, M_{\alpha\beta} \tag{2.8}$$

Note in particular that the inner product vanishes even if a single occupation number does not match. Within the same sector, the inner product is given by

$$\langle n_g \ldots n_m; \alpha | n_g \ldots n_m; \beta \rangle \; = \; M_{\alpha\beta} \tag{2.8a}$$

where M is a $s \times s$ hermitian matrix. In fact, there is an infinite set of matrices of varying dimensions, one corresponding to each sector $\{n_g \ldots n_m\}$.

We choose all these matrices to be positive definite. This set of inner-product matrices M plays an important role in the general formalism.

¿From the set of linearly independent vectors given in (2.5), it is possible to construct an orthonormal set of vectors which we shall denote by a double-barred ket:

$$\parallel n_g \ldots n_m; \mu \gg; \quad \mu = 1.....s \tag{2.9}$$

These satisfy the orthonormality relation

$$\ll n_g^{'} \ldots n_m^{'}; \alpha \parallel n_g \ldots n_m; \beta \gg = \delta_{n_g^{'} n_g} \ldots \delta_{n_m^{'} n_m} \delta_{\alpha\beta} \tag{2.10}$$

There is no unique way of doing this and the resulting orthonormal set is not unique. One may use Gram - Schmidt orthogonalization procedure or calculate the eigenstates of the inner-product matrix M or follow any other method. Whatever may be the method, one can write the relation connecting the two sets of kets :

$$\parallel n_g \ldots n_m; \mu \gg = \sum_\nu X_{\nu\mu} |n_g \ldots n_m; \nu\rangle \tag{2.11}$$

and the inverse relation :

$$|n_g \ldots n_m; \alpha\rangle \; = \; \sum_\beta (X^{-1})_{\beta\alpha} \; \parallel n_g \ldots n_m; \beta \gg \tag{2.12}$$

where X is a nonsingular matrix. Although X is not unique (since it depends on the particular orthogonalization procedure used), it is possible to show, using (2.8), (2.10), (2.11) and (2.12), that $XX^\dagger$ is the inverse of the innerproduct matrix[1]:

$$M^{-1} = XX^{\dagger} \tag{2.13}$$

Thus, one simple way of ensuring positivity of the inner-product matrix is to choose a non-singular matrix X and then determine M using (2.13). Again it must be kept in mind that we are dealing with an infinite set of matrices, X, one for each sector $\{n_g \dots n_m\}$. Also the orthonormality relation holds between vectors from two different sectors and we have already used this in writing (2.10).

The completeness relation for the orthonormal set of states $\| n_g \dots n_m; \mu \gg$ can be written in the form:

$$I = \sum_{n_g \dots n_m} \sum_{\mu} \| n_g \dots n_m; \mu \gg \ll n_g \dots n_m; \mu \| \tag{2.14}$$

where I is the identity operator. Substituting from (2.11) into (2.14) and using (2.13), we get the resolution of the identity operator in terms of the non-orthonormal set of states :

$$I = \sum_{n_g \dots n_m} \sum_{\lambda,\nu} |n_g \dots n_m; \nu\rangle (M^{-1})_{\nu\lambda} \langle n_g \dots n_m; \lambda| \tag{2.14a}$$

It is convenient to define the projection operator

$$P(n_g \dots n_m) = \sum_{\mu} \| n_g \dots n_m; \mu \gg \ll n_g \dots n_m; \mu \| \tag{2.15}$$

$$= \sum_{\lambda,\nu} |n_g \dots n_m; \nu\rangle (M^{-1})_{\nu\lambda} \langle n_g \dots n_m; \lambda| \tag{2.16}$$

[1] Eq(2.13) is analogous to the relation between the metric tensor and the vierbien in general relativity.

so that we have

$$I = \sum_{n_g \ldots n_m} P(n_g \ldots n_m) \tag{2.17}$$

One can easily verify the following properties of the projection operators :

$$P(n_g \ldots n_m) \parallel n_g^{'} \ldots n_m^{'}; \mu \gg = \delta_{n_g n_g'} \ldots \delta_{n_m n_m'} \parallel n_g \ldots n_m; \mu \gg \tag{2.18}$$

$$P(n_g \ldots n_m)|n_g^{'} \ldots n_m^{'}; \mu\rangle = \delta_{n_g n_g'} \ldots \delta_{n_m n_m'} |n_g \ldots n_m; \mu\rangle \tag{2.19}$$

It is worth noting that $P(n_g \ldots n_m)$ projects out any single state not only from the orthonormal set $\parallel n_g \ldots n_m; \mu \gg$ but also from the non-orthonormal set $|n_g \ldots n_m; \mu\rangle$.

In terms of the above projection operators it is very easy to construct the number operators :

$$N_k = \sum_{n_g \ldots n_k \ldots n_m} n_k P(n_g \ldots n_k \ldots n_m) \tag{2.20}$$

which satisfy the following properties

$$N_k \parallel n_g \ldots n_k \ldots n_m; \mu \gg = n_k \parallel n_g \ldots n_k \ldots n_m; \mu \gg \tag{2.21}$$

$$N_k|n_g \ldots n_k \ldots n_m; \mu\rangle = n_k|n_g \ldots n_k \ldots n_m; \mu\rangle \tag{2.22}$$

$$[N_k, N_j] = 0 \quad \text{for any } k \text{ and } j \tag{2.23}$$

We now introduce the transition operators which connect states, lying in different sectors. Obviously it is enough to define the so called annihilation and creation operators c_j and $c_j^\dagger$. We define

$$c_j^\dagger = \sum_{n_g \ldots n_j \ldots n_m} \sum_{\mu' \nu} A_{\mu' \nu} \left| n_g \ldots (n_j+1) \ldots n_m; \mu^{'} \right\rangle \left\langle n_g \ldots n_j \ldots n_m; \nu \right| \quad (2.24)$$

and c_j as the hermitian conjugate of $c_j^\dagger$, where $A_{\mu' \nu}$ are a set of arbitrary (complex) numbers. Note that the span of $\mu^{'}$ is larger than that of ν and this is the reason for the prime on μ. Specifically,

$$\mu^{'} = 1 \ldots s^{'}; \; s^{'} = \frac{(n_g + \ldots (n_j+1) + \ldots n_m)!}{n_g! \ldots (n_j+1)! \ldots n_m!} \quad (2.25a)$$

$$\nu = 1 \ldots s; \; s = \frac{(n_g + \ldots n_j + \ldots n_m)!}{n_g! \ldots n_j! \ldots n_m!} \quad (2.25b)$$

Hence A is a rectangular matrix. Since A is arbitrary in general, the relation(2.24) provides the most general definition of the creation operator. Even in this general case, it is possible to verify the following commutation relation between the number operator defined in (2.20) and the creation operator defined in (2.24)

$$[c_j^\dagger, N_k] = -c_j^\dagger \, \delta_{jk} \quad (2.26)$$

The projection property given in (2.19) plays a crucial role in the proof of (2.26).

So far we did not specify how the ordered state vectors $|n_g \ldots n_m; \mu\rangle$ are constructed. In fact, in general there is no need to specify any procedure for their explicit construction. The formalism given so far holds whatever may be the explicit form of their construction. However, once annihilation and creation operators c and $c^\dagger$ are introduced, there exists a natural procedure to construct the set of ordered states using $c^\dagger$ and the vacuum state $|0\rangle$. This procedure has the advantage that the arbitrariness of the matrix A introduced in (2.24) disappears and A in fact gets determined in terms of M. Hence let us do it.

In terms of creation operators, the ordered state (2.2) for instance is constructed as follows :

$$|\underbrace{1_g\cdots 1_g}_{n_g-1}\, 1_h 1_g \underbrace{1_h\cdots 1_h}_{n_h-1}\ldots\underbrace{1_m\cdots 1_m}_{n_m}\rangle \;=\; (c_g^\dagger)^{n_g-1}c_h^\dagger c_g^\dagger (c_h^\dagger)^{n_h-1}\cdots(c_m^\dagger)^{n_m}\,|0\rangle \tag{2.27}$$

and other states are constructed in a similar fashion. We may introduce the notation

$$|n_g\ldots n_m;\mu> \;=\; (c_g^{\dagger n_g}\ldots c_m^{\dagger n_m};\mu)|0> \tag{2.28}$$

$$< n_g\ldots n_m;\mu| \;=\; <0|(c_m^{n_m}\ldots c_g^{n_g};\mu) \tag{2.29}$$

where $(c_g^{\dagger n_g}\ldots c_m^{\dagger n_m};\mu)$ is a permutation of the creation operators similar to the permutations defined in eqn(2.1) - (2.5) and $(c_m^{n_m}\cdots c_g^{n_g};\mu)$ is defined as the hermitian conjugate of $(c_g^{\dagger n_g}\ldots c_m^{\dagger n_m};\mu)$. For states constructed in this manner, there exists a simple formula connecting states in "adjacent" sectors. For instance,

$$|1_j,\underbrace{1_g\cdots 1_g}_{n_g}\ldots\underbrace{1_m\cdots 1_m}_{n_m}\rangle \;=\; c_j^\dagger|\underbrace{1_g\cdots 1_g}_{n_g}\ldots\underbrace{1_m\cdots 1_m}_{n_m}\rangle \tag{2.30}$$

More generally, we may write

$$c_j^\dagger|n_g\ldots n_j\ldots;\lambda\rangle \;=\; |1_j,n_g\ldots n_j\ldots;\lambda\rangle \tag{2.31}$$

where $|1_j, n_g \ldots n_j \ldots; \lambda\rangle$ on the right is a subset of states in which one 'j' quantum appears on the extreme left. Although the total number of states in the set $|n_g \ldots (n_j + 1) \ldots; \lambda^{'}\rangle$ is $s^{'}$ given by (2.25a), the total number of states in the subset $|1_j, n_g \ldots n_j \ldots; \lambda\rangle$ is s given by (2.25b). Further the states in the subset $|1_j, n_g \ldots n_j ...; \lambda\rangle$ are given the same ordinal number λ as in the set $|n_g \ldots n_j \ldots; \lambda\rangle$. This is possible since the quanta $\{n_g \ldots n_j \ldots\}$ are permuted among themselves without disturbing the extra j-quantum sitting on the extreme left.

We now substitute the expression for $c_j^\dagger$ given by (2.24) into the left hand side of (2.31). We have

$$c_j^\dagger |n_g \ldots n_j \ldots; \lambda\rangle$$

$$= \sum_{n_g^{'} \ldots n_j^{'} \ldots} \sum_{\mu^{'} \nu} A_{\mu^{'} \nu} \, |n_g^{'} \ldots (n_j^{'} + 1) \ldots; \mu^{'}\rangle \, \langle n_g^{'} \ldots n_j^{'} \ldots; \nu | n_g \ldots n_j \ldots; \lambda\rangle$$

$$= \sum_{\mu^{'} \nu} A_{\mu^{'} \nu} M_{\nu\lambda} \, |n_g \ldots (n_j + 1) \ldots; \mu^{'}\rangle \tag{2.32}$$

where we have used (2.8). On comparing with (2.31), we get

$$\sum_{\nu} A_{\mu^{'} \nu} \, M_{\nu\lambda} \, = \, \delta_{\mu^{'} \lambda} \tag{2.33}$$

From (2.33) we see that $\mu^{'} = \lambda$. This means that in (2.24), only the subset $|1_j, n_g \ldots n_j \ldots; \lambda\rangle$ contributes and hence A is in fact a square matrix and is equal to the inverse of M^{-1}:

$$A \, = \, M^{-1} \tag{2.34}$$

So, we may rewrite (2.24):

$$c_j^\dagger = \sum_{n_g \ldots n_j \ldots} \sum_{\lambda,\nu} (M^{-1})_{\lambda\nu} \left|1_j, n_g \ldots n_j \ldots ; \lambda\right\rangle \left\langle n_g \ldots n_j \ldots ; \nu\right| \tag{2.35}$$

Thus, A and hence c and $c^\dagger$ are completely determined in terms of the set of M matrices.

Next, consider the expression for the number operator (eqs.(2.20),(2.15) and (2.16)) :

$$N_k = \sum_{n_g \ldots n_k \ldots n_m} n_k \sum_{\mu} \parallel n_g \ldots n_k \ldots n_m ; \mu \gg \ll n_g \ldots n_k \ldots n_m ; \mu \parallel \tag{2.36}$$

$$= \sum_{n_g \ldots n_k \ldots n_m} n_k \sum_{\lambda\nu} |n_g \ldots n_k \ldots n_m ; \lambda > (M^{-1})_{\lambda\nu} < n_g \ldots n_k \ldots n_m ; \nu| \tag{2.37}$$

One may try to express this in terms of c and $c^\dagger$. Using eqs.(2.28) and (2.29),

$$N_k = \sum_{n_g \ldots n_k \ldots n_m} n_k \sum_{\lambda\nu} (c_g^{\dagger n_g} \ldots c_k^{\dagger n_k} \ldots c_m^{\dagger n_m} ; \lambda)|0 >< 0|(c_m^{n_m} \ldots c_k^{n_k} \ldots c_g^{n_g} ; \nu)(M^{-1})_{\lambda\nu} \tag{2.38}$$

Apart from $c^\dagger$ and c, the above expression contains the vacuum projector $|0 >< 0|$. Later we shall give examples where $|0 >< 0|$ is determined as products of $c^\dagger$ and c so that N_k can be expressed entirely in terms of $c^\dagger$ and c. However, in general, this does not lead to simple results, whereas eq.(2.36) provides us with a universal representation of number operators which is valid in all cases.

So far, we regarded the set of state vectors $|n_g, n_h \ldots, \mu >; \mu = 1 \ldots s$ (where s is given by (2.6)) to be linearly independent and the resulting generalized Fock space is the complete Fock space, which we shall call the *super Fock space*.

We shall now show how to construct reduced Fock spaces. The motivation for this is that many Fock spaces of physical interest such as the bosonic Fock space or fermionic Fock space are reduced Fock spaces. There are various ways of doing this. One may postulate relationships between states connected by permutations, or one may disallow certain permutations by equating them to null vectors. Yet another way to achieve this is to use the permutation group S_n acting on the n-particle state. The super Fock space we have constructed consists of all the representations of the permutation group. If we allow only certain representations of S_n, we get a reduced Fock space.

All these possibilities are contained in the statement that in the space of vectors $|n_g, n_h \ldots ; \mu >$, there are r null vectors $(r < s)$

$$\sum_{\mu} B^p_{\mu} |n_g, n_h \ldots ; \mu >= 0 \ ; \ p = 1, 2, \ldots r ; r < s. \tag{2.39}$$

where B^p_{μ} are constants. So, the dimension of the vector space in the sector $\{n_g, n_h \ldots\}$ is reduced to d given by

$$d \ = \ s \ - \ r \, . \tag{2.40}$$

An important class of reduced Fock spaces are those for which $d = 1$. Here, all the states connected by permutation of indices will be taken to be related to each other through equations of the type(2.39). In otherwords,the number of relations r in eq. (2.39) is $s - 1$. We shall call this space as the *bosonic Fock space.* If we impose the additional restriction : $n_g = 0$ or 1 only, the resulting space will be called *fermionic Fock space.* This restriction can also be stated in the form of eq.(2.39) :

$$|n_g, n_h \dots ; \mu > \, = \; 0 \quad \text{for} \quad n_g, n_h \dots \geq 2\,. \tag{2.41}$$

We can define a new reduced Fock space, also of dimension $d = 1$ in each sector, by taking the set of all the permuted states as null states except a single state (of a chosen order) which is taken to be the allowed state. We shall call this as the Fock space of frozen order.

Another important class of reduced Fock spaces are those associated with parastatistics [35-37], which we shall call parabosonic and parafermionic Fock spaces. For these, the number of relations r is smaller than for bosonic or fermionic Fock spaces, so that the dimension d satisfies

$$1 < d < s\,. \tag{2.42}$$

The formalism constructed in the present section is valid for all these reduced Fock spaces also, with the modification that all the summations over μ, ν etc. will now go over the range $1 \dots d$ and correspondingly, X, M, and A become $d \times d$ matrices. There is an arbitrariness in the choice of the d states. Any choice of d states will do, as long as they are non-null states.

All these Fock spaces, the super Fock space as well as the reduced ones will be collectively called generalized Fock spaces. To sum up this discussion we may note that a generalized Fock space is completely defined by stating what are the allowed states of the system.

We shall define statistics by the precise relationship linking states obtained by permutation. In general, many relationships can be envisaged and hence many different forms of statistics can reside within a particular reduced Fock space. However, in super Fock space, all the states obtained by permutation are independent and so there is a unique statistics associated with this Fock space, namely "infinite statistics", which is the same as Boltzmann statistics, but implemented quantum mechanically. Similarly, in the Fock space of frozen order too, there exists only a single statistics, named "null statistics".

In this section, we started with the generalized Fock space consisting of the set of allowed states of the system and constructed the creation, annihilation and number oeprators in terms of the outer products of state vectors. Do these c and $c^\dagger$ form an operator algebra? In general, c and $c^\dagger$ constructed in this way may not form a simple algebra, or even a closed algebra. Historically, it is the reverse route that has been followed ; one postulates an algebra of c and $c^\dagger$ and then deduces the states allowed by the algebra. In this sense, a given relation involving c and $c^\dagger$ implicitly defines an inner product and through it specifies the allowed and the null states of the system. In practice, starting with an algebra is an easier procedure and we shall use it in the later sections. Actually, it is complementary to the approach described in this section.

Therefore we have two equivalent ways of dealing with the generalized

Fock spaces. In the first approach, which we have formulated in this section we construct c and $c^\dagger$ in terms of the allowed states of the system and the algebra of c and $c^\dagger$ is then a derived consequence. This is the more fundamental approach and is of universal validity. In the second approach, which also we shall use in the latter sections, we start with an algebra of c and $c^\dagger$ and then determine the states of the system allowed by the $cc^\dagger$ algebra. Since the restrictions on the allowed states of the system can generally be stated in the form of cc relations, the first approach can be characterized as $cc \to cc^\dagger$ while the second is $cc^\dagger \to cc$. Within the second approach, we shall describe an elegant method to derive cc relations from $cc^\dagger$ algebras.

For infinite statistics, there is no restriction on the allowed states and cc relations do not exist. Hence, for this statistics, the first approach should be interpreted as "no cc relation" $\to cc^\dagger$ algebra while in the second approach, starting with any particular $cc^\dagger$ algebra describing infinite statistics one shows that there does not exist any cc relation. In those cases where the $cc^\dagger$ algebra depends on a continuous parameter q, one can determine the values of q where cc relation exists. Generally, these values of q correspond to the boundary of the region in the parameter space where infinite statistics with positive definite M exists. On this boundary, one or more eigenvalues of the M matrices become zero, thus leading to the emergence of the same number of null vectors in the Fock space which can equivalently be interpreted as the emergence of cc relations. Thus the formalism unifies infinite statistics residing on the super Fock space with the various forms of statistics residing on reduced Fock spaces.

It must be noted that the inner product matrices M occuring in eq.(2.8a) are quite arbitrary. Consequently, more than one realization or representation of creation and destruction operators is possible. In fact, it is this freedom to select arbitrary M which enables one to construct different algebras involving c and $c^{\dagger}$, all operating over the same Fock space.

To sum up, we construct a three-tiered structure consisting of Fock space, statistics and algebra. Fock space is specified by the set of the allowed states of the system. Statistics is defined by the nature of the symmetry of the allowed states under permutation. Algebra of the creation and destruction operators is determined by the choice of the inner product matrices M.

We shall construct different representations of infinite statistics in Sec.3. In the bosonic and fermionic Fock spaces, many forms of quantum statistics which include Bose and Fermi statistics are possible. These and the null statistics in the Fock space of frozen order will be taken up in Sec.4.

3. Super Fock Space and Infinite Statistics

3.1 The standard representation $(M = 1)$

Having taken all the s states to be independent, the simplest choice of the matrix X is the unit matrix which implies M and A also to be unit matrices (of appropriate dimensions) for all the sectors $\{n_g, n_h \ldots\}$:

$$X = M = A = 1 \tag{3.1}$$

For this choice, which we shall call the standard representation of infinite statistics, the ordered states (2.5) themselves form an orthonormal set :

$$\langle n_g, n_h \ldots ; \mu | n_g, n_h \ldots ; \nu \rangle \ = \ \delta_{\mu\nu} \tag{3.2}$$

Further, the creation and annihilation operators are given by

$$c_j^\dagger \ = \ \sum_{n_g, n_h \ldots} \sum_{\mu} |1_j, n_g, n_h \ldots ; \mu \rangle \, \langle n_g, n_h \ldots ; \mu | \tag{3.3}$$

$$c_j \ = \ \sum_{n_g, n_h \ldots} \sum_{\mu} |n_g, n_h \ldots ; \mu \rangle \, \langle 1_j, n_g, n_h \ldots ; \mu | \tag{3.4}$$

From (3.3) and (3.4), using the orthogonality and completeness relations, one can derive the $cc^\dagger$ algebra :

$$c_i c_j^\dagger \ = \ \delta_{ij} \tag{3.5}$$

This algebra was first proposed by Greenberg [13], in an important paper on infinite statistics.

In the standard representation, we also have the useful identity :

$$\sum_i c_i^\dagger c_i \ = 1 - |0 >< 0| \tag{3.6}$$

Although this identity can be obtained from the eqs(3.3) and (3.4) that define $c^\dagger$ and c, it is important to note that within Fock space it follows from the $cc^\dagger$ algebra (eq.(3.5)). We shall prove it by showing that eq.(3.6) is valid, when applied on any state in the generalized Fock space. First applying on $|0>$, we find that both sides are zero. Next we apply on the other states $c_k^\dagger c_\ell^\dagger \ldots |0>$:

$$\left\{ \sum_i c_i^\dagger c_i - 1 + |0><0| \right\} c_j^\dagger c_k^\dagger \ldots |0>= 0 \tag{3.7}$$

Using eq.(3.5) and noting $<0|c_j^\dagger \;=\; 0$ we see that the left side of eq(3.7) infact vanishes, thus completing the proof of eq(3.6).

In a different context, Cuntz [38] had studied the algebra defined by eq(3.5) and by the relation :

$$\sum_i c_i^\dagger c_i \;=\; 1\,. \tag{3.8}$$

Cuntz algebra is inconsistent with Fock space, as can be seen by applying both sides of eq (3.8) on $|0>$. In contrast, our eq(3.6), because of the inclusion of the vacuum projector term in it, is consistent with Fock space and is infact a consequence of the algebra of c and $c^\dagger$ in the standard representation of infinite statistics.

Putting $M = 1$, the number operator N_k given in eq(2.38) becomes

$$N_k \;=\; \sum_{n_g..n_k..n_m} n_k \sum_\mu \left(c_g^{\dagger n_g} \ldots c_k^{\dagger n_k} \ldots c_m^{\dagger n_m}; \mu \right) |0><0| \left(c_m^{n_m} \ldots c_k^{n_k} \ldots c_g^{n_g}; \mu \right) \tag{3.9}$$

Substitution of $|0><0|$ from (3.6) into this equation and a straightforward but tedius calcualtion finally leads to

$$N_k \;=\; \sum_{n_g..n_k..n_m} \sum_{\mu} \left(c_g^{\dagger n_g} \ldots c_k^{\dagger n_k} \ldots c_m^{\dagger n_m}; \mu \right) c_k^\dagger c_k \left(c_m^{n_m} \ldots c_k^{n_k} \ldots c_g^{n_g}; \mu \right) \quad (3.9a)$$

This expression is identical to Greenberg's formula for N_k [13], though it is written in a different form. This illustrates the derivation of the representation of N_k in terms of c and $c^\dagger$. As already mentioned, in general neither the procedure nor the result is simple and in any case one does not need it. The universal representation of N_k given in eq(2.36) is sufficient.

3.2 q-mutators with real q

Many other choices of M are possible. A particularly interesting choice is the one in which M is given as a function of a real parameter q. Consider the inner product between the n-particle state vectors with all occupation numbers unity : $|1_g, 1_h \ldots; \mu\rangle$. The inner product matrix M in this sector has dimension n ! $\times$ n! and its matrix element is taken to be

$$\langle 1_g, 1_h \ldots; \mu |\, 1_g, 1_h \ldots; \nu \rangle \;= q^J \qquad (3.10)$$

where q is a real number lying in the range $-1 < q < +1$ and J is the number of inversions required to transform the state $|1_g, 1_h \ldots; \nu\rangle$ into the state $|1_g, 1_h \ldots; \mu\rangle$. Number of inversions is the minimum number of successive interchanges between adjacent quanta that will take the state $|1_g, 1_h \ldots; \nu\rangle$ to $|1_g, 1_h \ldots; \mu\rangle$. For example,

$$\langle 1_h 1_g 1_k | 1_g 1_h 1_k \rangle \;=\; q \qquad (3.11)$$

$$\langle 1_h 1_k 1_g | 1_g 1_h 1_k \rangle \; = \; q^2 \tag{3.12}$$

The positivity of the q-dependent M matrices defined above has been proved by Fivel [15] and Zagier [16]. In particular, Zagier has given the explicit form of the determinant of the (n ! $\times$ n!) dimensional M matrix for arbitrary n :

$$detM \; = \; \prod_{k=1}^{n-1} [1 - q^{k(k+1)}]^{(n-k)n!/k(k+1)} \tag{3.13}$$

This determinant is positive in the range $-1 <$ q $< +1$.

The inner product for states with occupation numbers larger than unity (which are the same as states with repeated indices) is obtained from the above inner product for states with distinct indices by symmetrizing with respect to the repeated indices. For example, consider

$$\langle 2_g 1_m | 1_g 1_m 1_g \rangle \; = \; \langle 1_g 1_g 1_m | 1_g 1_m 1_g \rangle \tag{3.14}$$

Replace one of the g's by h in both the initial and final states and thus get a matrix element with distinct indices which can be calculated using(3.10). This replacement can be done in $(2!)^2$ ways. The sum of these $(2!)^2$ matrix elements divided by 2! is the required answer. Thus,

$$\langle 1_g 1_g 1_m | 1_g 1_m 1_g \rangle \; = \; \frac{1}{2} [\langle 1_g 1_h 1_m | 1_g 1_m 1_h \rangle + \langle 1_g 1_h 1_m | 1_h 1_m 1_g \rangle +$$

$$\langle 1_h 1_g 1_m | 1_g 1_m 1_h \rangle + \langle 1_h 1_g 1_m | 1_h 1_m 1_g \rangle] \tag{3.15}$$

$$= q + q^2 \tag{3.16}$$

It is clear that the matrix M for repeated indices is obtained from the higher dimensional matrix for distinct indices by a process of collapse or reduction. It can be shown that this process retains the positivity of the matrix.

It is worth noting that the norm for n-particle state with all the indices repeated is

$$< n_g|n_g >= [n_g!]_q = [n_g]_q[n_g-1]_q \dots [2]_q[1]_q \tag{3.17}$$

where the "q-number" $[n]_q$ is defined by

$$[n]_q \equiv \frac{1-q^n}{1-q} = 1+q+q^2+\dots q^{n-1}. \tag{3.18}$$

Inverting these M matrices and defining the creation operator through (2.35), one can derive the algebra of c and $c^\dagger$. This is more difficult than the reverse procedure which is the way this subject really developed. Greenberg [14] proposed the q-mutator algebra :

$$c_i c_j^\dagger - q c_j^\dagger c_i = \delta_{ij} \tag{3.19}$$

The q-dependent M matrices defined through (3.10), (3.15) and (3.16) follow from this algebra. Fivel and Zagier then proved the positivity of these M matrices for $-1 < q < +1$. Inspite of the fact that we have not derived eq(3.19) from the M matrices defined here, we can assert its validity because the M matrices completely determine c and $c^\dagger$. It is in order to emphasize this point that we have presented the M matrices first and the algebra of c and $c^\dagger$ as a derived consequence.

3.3. New representations of infinite statistics

First we briefly consider the 2-parameter algebra

$$c_i c_j^\dagger - q_1 c_j^\dagger c_i - q_2 \delta_{ij} \sum_k c_k^\dagger c_k \ = \ \delta_{ij} \tag{3.20}$$

where q_1 and q_2 are real parameters. This may be regarded as another representation of infinite statistics. Although the determination of the full region in the $\{q_1, q_2\}$ parameter space for which M is positive definite is still an unsolved problem, one can show [19] positivity of M on the straight line defined by $q_1 = 0; -1 < q_2 < \infty$. Infact it is possible to map this whole line on to the point $q_1 = 0; q_2 = 0$ so that we just get back the standard representation defined by eq (3.5).

We next present two new algebras :

$$c_i c_j^\dagger - c_j^\dagger c_i \ = \ \delta_{ij} p^{2\sum_{k<i} N_k} p^{N_i} \tag{3.21}$$

and

$$\left.\begin{aligned} c_i c_j^\dagger - p^{-1} c_j^\dagger c_i \ &= 0 \quad \text{for} \quad i \neq j \\ c_i c_i^\dagger - c_i^\dagger c_i \ &= \ p^{N_i} \end{aligned}\right\} \tag{3.22}$$

where p is a real parameter and N_i are number operators already written down in Sec.2. Of these two algebras, the first one (eq.3.21) is based on ordered indices, that is, given any two indices i and j, one must be considered larger or smaller than the other. So, we may take the indices to be the natural numbers $1, 2, 3 \ldots$.

Eqs (3.21) and (3.22) are again representations of the same infinite statistics, for it is possible to map both these algebras on to Greenberg's q-

mutator algebra of eq (3.19) with the following identification of the parameters: $p = q^{-1}$. Temporarilly renaming the $(c, c^\dagger)$ of eqs (3.21) and (3.22) by $(b, b^\dagger)$ and $(d, d^\dagger)$ respectively, the mapping or transformation equations are

$$b_i \;=\; p^{\sum_{k<i} N_k} p^{\frac{1}{2}N_i} c_i \tag{3.23}$$

$$d_i \;=\; p^{\frac{1}{2}N_i} c_i \,. \tag{3.24}$$

Hence, it is clear that both the algebras of eqs (3.21) and (3.22) lead to positive definite M matrices for $p > +1$ and $p < -1$ (corresponding to $-1 < q < +1$).

An interesting feature of the algebra (3.21) is the validity of ordinary commutation relation between c_i and $c_j^\dagger$ for $i \neq j$ as in Bose statistics ; nevertheless the full algebra (3.21) describes infinite statistics!

In contrast to the situation for the q-mutator algebra of eq (3.19), where the number operators can be expressed in terms of c and $c^\dagger$ only after considerable algebraic manouvers [16,17] and the resulting expressions are quite complicated, the corresponding expressions for the algebras (3.21) and (3.22) are simple. For (3.21), we get (taking $p > 0$)

$$\left.\begin{aligned} N_1 \;&=\; \tfrac{1}{log\ p}\, log\ (c_1 c_1^\dagger - c_1^\dagger c_1) \\ &\ \ \vdots \\ N_i \;&=\; \tfrac{1}{log\ p}\, log\ (c_i c_i^\dagger - c_i^\dagger c_i) - 2\textstyle\sum_{k<i} N_k \end{aligned}\right\} \tag{3.25}$$

For (3.22), the number operator is even simpler :

$$N_i \;=\; \frac{1}{log\ p}\, log\ (c_i c_i^\dagger - c_i^\dagger c_i) \quad \text{for all} \quad i\,. \tag{3.26}$$

Inspite of our ability to write down such formal expressions for N_i in terms of c's and $c^\dagger$s, we must also point out that they are not of much use. All that one ever needs of the number operators are the properties contained in the eqs (2.21) - (2.23) and (2.26) and as for explicit representation, eqs (2.36) and (2.37) will do.

3.4. q-mutators with complex q

We now generalize Greenberg's q-mutator algebra to complex q. This generalized algebra is based on ordered indices and is defined by the following equations :

$$c_i c_j^\dagger - q c_j^\dagger c_i = 0 \quad \text{for} \quad i < j \tag{3.27}$$

$$c_i c_i^\dagger - p c_i^\dagger c_i = 1 \tag{3.28}$$

where q and p are complex and real parameters respectively and the indices i,j etc. refer to any of the natural numbers 1,2,3, ... The relation for the opposite order $i > j$ is derivable from (3.27) by hermitian conjugation :

$$c_i c_j^\dagger - q^* c_j^\dagger c_i = 0 \quad \text{for} \quad i > j \tag{3.27$'$}$$

and so it is not an independent relation.

Let us now calculate the inner product matrix for this algebra. For distinct indices, we find

$$\langle 1_g 1_h 1_k \ldots ; \mu | 1_g 1_h 1_k \ldots ; \nu \rangle \;=\; (q^*)^{J_+} q^{J_-} \tag{3.29}$$

where J_+ and J_- are the number of positive and negative inversions in the permutation $\nu \rightarrow \mu$. The total number of inversions (the sum of positive and negative inversions) is the same as the number of inversions already defined below eq.(3.10). We further define an inversion as positive if it is a transposition of indices from the ascending order to the descending order and as negative if it is the reverse transposition. For example, (1,2) $\to$ (2,1) is a positive inversion while (2,1) $\to$ (1,2) is a negative inversion. Thus we have

$$\langle 1_3 1_1 1_2 | 1_2 1_1 1_3 \rangle \ = \ (q^*)^2 q \tag{3.29$'$}$$

since the permutation (213) $\to$ (312) contains two positive and one negative inversions as shown in Fig.1.

The relationship between the algebra defined by eqs(3.27) and (3.28) with complex q (and $p = |q|$) and Greenberg's algebra defined by eq.(3.19) with real q can be given. Calling the creation operators for the former and latter algebras as $c^\dagger(q)$ and $c^\dagger(|q|)$ respectively, the relationship is

$$c_i^\dagger(q) \ = \ e^{i\theta \sum_{k<i} N_k} c_i^\dagger(|q|) \tag{3.30}$$

where θ is the phase of q :

$$q \ = \ |q| e^{i\theta} \tag{3.31}$$

and N_k are the number operators defined in eq.(2.37).

As a consequence, the inner product matrices for the two algebras are related by θ-dependent unitary matrices:

$$M(q) \ = \ T^\dagger(\theta) M(|q|) T(\theta) \tag{3.32}$$

where

$$T^\dagger T \;=\; T T^\dagger \;=\; 1\,. \tag{3.33}$$

We have already given in eq.(3.13), Zagier's result for the determinant of M for real q. For our algebra with complex q, because of eq.(3.32), the determinant of M for n particles with distinct indices is

$$\det M_n(q) \;=\; \prod_{k=1}^{n-1} (1- \mid q \mid^{k(k+1)})^{\frac{(n-k)n!}{k(k+1)}} \tag{3.34}$$

Hence, $M_n(q)$ for complex q is positive for $|q| < 1$, and we have thus extended the Fivel-Zagier result for positivity to complex q.

In contrast to the situation for Greenberg's algebra, the inner product matrix M for states with repeated indices cannot be derived from that for states with distinct indices, for the present algebra. For n-particle states with all the indices repeated, the norm is the same as in eqn(3.17) and (3.18), with q replaced by p :

$$\langle n_g \mid n_g \rangle \;=\; (1+p)(1+p+p^2)\ldots(1+p+p^2+\ldots p^{n_g-1}) \tag{3.35}$$

which is positive for p > -1. For states with only some indices repeated, M is a function of both q and p and the problem is more complicated ; but we have verified positivity of the M's upto 3-particle states for $-1 < p <\mid q \mid^{-2}$.

If we choose $p = -1$, all states with repeated indices are forbidden. This is just Pauli's exclusion principle. Thus, the algebra defined by eqs.(3.27) and (3.28) with $p = -1$ leads to *infinite or Boltzmann statistics with exclusion principle.*

Infact, we can restrict both eqs(3.5) and (3.19) to $i \neq j$ and for $i = j$ replace them by

$$c_i c_i^\dagger + c_i^\dagger c_i = 1 \tag{3.36}$$

In all these cases, we have Boltzmann statistics with exclusion principle.

We give a compilation of the various algebras all representing infinite statistics in Table I.

3.5. Unitary transformations

We must point out an important difference between the algebras described by eqs(3.5), (3.19) and (3.20) on the one hand and the algebras of eqs(3.21), (3.22), (3.27) and (3.28) on the other hand. It is easy to see that the former are covariant under the unitary transformations on the indices:

$$c_k \ \rightarrow \ \sum_m \ U_{km} \ c_m \ ; \ U^\dagger U \ = \ UU^\dagger \ = \ 1\,. \tag{3.37}$$

In fact, one can show [19] that under certain conditions eq.(3.20) is the most general bilinear algebra of c_i and $c_j^\dagger$ invariant under this unitary transformation. This property is violated for the complex q-algebra of eqs(3.27) and (3.28) as well as the algebras of eqs (3.21) and (3.22). Covariance under the unitary transformation is desirable in a general context since it is closely connected to the superposition principle in quantum mechanics [3]. Nevertheless, algebras violating this requirement have been proposed in the recent literature, either because of the possibility of applications to specific systems in condensed matter physics or because of mathematical motivation. Hence, we include such algebras in our investigation.

[12pt]article

4. Statistics and Algebras in Bosonic, Fermionic and Frozen Fock Spaces

We introduce two types of reduced Fock spaces in sub section 4.1. In one, we take any two states obtained by permutation to be related to each other and we allow all occupation numbers : $n_i = 0, 1, 2, \ldots$ for all i. In the other, any two states obtained by permutation are again related to each other but occupation numbers are restricted by exclusion principle : $n_i = 0$ and 1 only. We shall call the former as bosonic Fock space and the latter as fermionic Fock space.

In subsection 4.2, we introduce the Fock space of frozen order, in which permutations are forbidden and this also comes in two varieties, bosonic and fermionic.

4.1 Bosonic and Fermionic Fock Spaces

A consistent way of defining a general relationship among permuted states for both the bosonic and fermionic Fock spaces is to take

$$|n_1, n_2 \ldots ; \mu\rangle \; = \; q^J |n_1, n_2 \ldots ; 1\rangle \tag{4.1}$$

where q is a complex number and J is the number of inversions in the permutation : $1 \rightarrow \mu$. The number of inversions was already defined in Sec.3. Although J was defined there for the case in which the occupation numbers were restricted to unity, the same definition can be extended to larger occupation numbers. We assume $q \neq 0$. The case $q = 0$ will be considered separately. Many different forms of statistics as well as various algebras of c and $c^\dagger$ can be shown to be contained as particular cases of eq.(4.1).

We first make some general remarks amplifying the meaning of eq(4.1). This equation can be obtained by repeated application of the elementary relation :

$$|\ldots 1_i\, 1_j \ldots > = \, q|\ldots 1_j 1_i \ldots >, \quad \text{for} \quad i > j \tag{4.2}$$

where, except for the two adjacent quanta of indices i and j which are interchanged, all other quanta are left unchanged. In eq.(4.2), we have used the same notation for the state vectors as on the right hand side of eqs(2.1)-(2.4). It is clear that, for any q other than ± 1, eq.(4.2) makes sense only if the indices i and j, are ordered through an inequality (to be specific, $i > j$). Hence we have taken the indices to be the natural numbers $1, 2, 3 \ldots$in writing eq.(4.1).

The relationship among the state vectors given in eq(4.1) or (4.2) can be equivalently expressed as a quadratic relation between two creation operators or two annihilation operators. For any state $|\ldots>$, using eq.(2.30) or (2.31) we have

$$c_i^\dagger c_j^\dagger|\ldots> = \, |1_i 1_j \ldots>$$

$$c_j^\dagger c_i^\dagger|\ldots> = \, |1_j 1_i \ldots>$$

Comparing with eq.(4.2), we get

$$c_i^\dagger c_j^\dagger|\ldots> = \, q c_j^\dagger c_i^\dagger|\ldots>, \quad \text{for} \quad i > j$$

Since this is valid for any state $| \ldots >$, we can write

$$c_i^\dagger c_j^\dagger - q c_j^\dagger c_i^\dagger = 0 \quad \text{for} \quad i > j$$

or,

$$c_j c_i - q^\star c_i c_j \;=\; 0 \quad \text{for} \quad i > j. \tag{4.3}$$

The above cc relation (eq 4.3) is common to both bosonic and fermionic spaces. But for the fermionic space there exists the additonal restriction :

$$c_i^\dagger c_i^\dagger \;=\; 0\,, \quad \text{or} \quad c_i c_i \;=\; 0 \tag{4.4}$$

All the states $|n_1, n_2 \ldots ; \mu >$ for fixed occupation numbers $(n_1, n_2 \ldots)$ but different values of μ being related to each other through eq.(4.1), the dimension of the vector space in any sector $(n_1, n_2 \ldots)$ is reduced to unity. So, for each sector we choose one standard vector $|n_1, n_2 \ldots ; 1 >$ which we rewrite as $|n_1, n_2 \ldots >$, dispensing with μ completely.

The matrices X and M then become numbers related by

$$M^{-1} \;=\; X X^\star\,. \tag{4.5}$$

Eq.(2.11) becomes

$$\| \, n_1, n_2 \ldots \gg = \; X(n_1, n_2 \ldots) |n_1, n_2 \ldots \rangle \tag{4.6}$$

and we have

$$\ll n_1, n_2 \ldots \| \, n_1^{'}, n_2^{'} \ldots \gg = \; \delta_{n_1 n_1^{'}} \, \delta_{n_2 n_2^{'}} \cdots \tag{4.7}$$

From eq.(2.35), we get

$$
\begin{aligned}
c_j^\dagger &= \sum_{n_1 \ldots n_j \ldots} |1_j, n_1, n_2 \ldots n_j ..\rangle \langle n_1, n_2 \ldots n_j ..| \, M^{-1}(n_1, n_2 ..) \\
&= \sum_{n_1 \ldots n_j \ldots} q^{\sum_{i<j} n_i} |n_1, n_2 \ldots (n_j+1)..\rangle \langle n_1, n_2 \ldots n_j ..| \, M^{-1}(n_1, n_2 \ldots)
\end{aligned}
$$

where we have pushed the j quantum to the right of all i quanta for $i < j$ using eq.(4.2). Also using eqs.(4.5) and (4.6),

$$
c_j^\dagger = \sum_{n_1 \ldots n_j} q^{\sum_{i<j} n_i} \frac{X(n_1..n_j..)}{X(n_1..(n_j+1)..)} \parallel n_1, n_2 \ldots (n_j+1).. \gg \ll n_1, n_2 \ldots n_j .. \parallel \tag{4.8}
$$

Eq.(4.8) and its hermitian conjugate define the most general form of creation and destruction operators for the bosonic and fermionic Fock spaces.

If we assume factorization of the norm M as well as that of X :

$$
X(n_1, n_2 \ldots) = \phi(n_1)\phi(n_2) \ldots, \tag{4.9}
$$

then we get

$$
c_j^\dagger = \sum_{n_1 \ldots n_j \ldots} q^{\sum_{i<j} n_i} \frac{\phi(n_j)}{\phi(n_j+1)} \parallel n_1..n_j+1.. \gg \ll n_1..n_j.. \parallel \tag{4.10}
$$

$$
c_j = \sum_{n_1 \ldots n_j \ldots} (q^*)^{\sum_{i<j} n_i} \frac{\phi^*(n_j)}{\phi^*(n_j+1)} \parallel n_1..n_j.. \gg \ll n_1..(n_j+1).. \parallel \tag{4.11}
$$

We now construct the operator algebras for $c^\dagger$ and c given by eqs.(4.10) and (4.11). First of all, *for any form of* $\phi(n_i)$, we get

$$
\left.
\begin{aligned}
c_i c_j^\dagger - q c_j^\dagger c_i &= 0 \quad for \quad i < j \\
c_i c_j^\dagger - q^* c_j^\dagger c_i &= 0 \quad for \quad i > j
\end{aligned}
\right\} \tag{4.12}
$$

For i = j, the algebra depends on the choice of q and $\phi(n_i)$. We find

$$c_j c_j^\dagger = |q|^{2\sum_{i<j} N_i} \left| \frac{\phi(N_j)}{\phi(N_j+1)} \right|^2 \tag{4.13}$$

$$c_j^\dagger c_j = |q|^{2\sum_{i<j} N_i} \left| \frac{\phi(N_j-1)}{\phi(N_j)} \right|^2 \tag{4.14}$$

where $N_i, N_j \ldots$ are number operators and $\phi(N_i)$ is the function introduced in eq.(4.9). So, we can write the generalized commutation relation

$$\left.\begin{array}{rcl} c_j c_j^\dagger - p c_j^\dagger c_j &=& |q|^{2\sum_{i<j} N_i} f(N_j) \\ f(N_j) &=& \left|\frac{\phi(N_j)}{\phi(N_j+1)}\right|^2 - p \left|\frac{\phi(N_j-1)}{\phi(N_j)}\right|^2 \end{array}\right\} \tag{4.15}$$

where p is any (complex) number. Eqs.(4.12) and (4.15) constitute the $cc^\dagger$ algebra for q-statistics defined by the cc algebra of eq.(4.3).

What we have derived above can be regarded as the most general algebra of creation and annihilation operators for q-statistics, subject to the assumption of the factorization of the normalization factor implied by eq.(4.9). In particular, since the function $\phi(N_j)$ occuring in eq.(4.15) is arbitrary, we may regard our equations as constituting an infinite-parameter deformation of the algebra of c and $c^\dagger$. Exploiting the arbitrariness of $\phi(N_j)$ we can get many simpler forms of the algebra which we now describe.

(i) Choosing $\phi(n_j)$ to be a constant independent of n_j but noting that $\phi(-1)$ must vanish[2], eq.(4.15) becomes

$$c_j c_j^\dagger - p c_j^\dagger c_j = |q|^{2\sum_{i<j} N_i} \left\{1 - p(1 - \delta_{N_j,0})\right\} \tag{4.16}$$

[2]On applying both sides of eq.(4.14) on the vacuum state, the left hand side vanishes and so for consistency, $\phi(-1) = 0$

where $\delta_{N_j,0}$ is the Kronecker delta, which can also be represented as $sin\, \pi N_j/\pi N_j$.

(ii) Putting $p = 0$ in eq.(4.16) we get the simple algebra :

$$c_j c_j^\dagger \;=\; |q|^{2\sum_{i<j} N_i} \tag{4.17}$$

(iii) As a third possibility, we may choose $f(N_j)$ in eq.(4.15) to be unity and p real so that we have the algebra :

$$c_j c_j^\dagger - p c_j^\dagger c_j \;=\; |q|^{2\sum_{i<j} N_i} \tag{4.18}$$

This choice requires that the function $\phi(n)$ must satisfy the equation

$$\left|\frac{\phi(n_j)}{\phi(n_j+1)}\right|^2 \;-\; p\left|\frac{\phi(n_j-1)}{\phi(n_j)}\right|^2 \;=\; 1 \tag{4.19}$$

The solution of this functional equation (with the constraints that $\phi(0)$ is finite and $\phi(-1)$ vanishes) is

$$|\phi(n)|^{-2} \;=\; [(n)!]_p \equiv [n]_p[n-1]_p \ldots [1]_p \tag{4.20}$$

where

$$[n]_p \;=\; \frac{p^n - 1}{p-1} \;=\; 1 + p + p^2 + \ldots p^{n-1} \tag{4.21}$$

For $p = 1$, the “p-numbers” in eqs(4.20) and (4.21) become ordinary numbers.

(iv) As a final choice we put $p = |q|^2$ in eqs (4.18) - (4.21). As a consequence of this choice, the right hand side of eq (4.18) becomes a bilinear function of c and $c^\dagger$. For, from eq.(4.14), we can prove an identity :

$$(|q|^2 - 1)\sum_{k<j} c_k^\dagger c_k = \sum_{k<j} |q|^{2\sum_{i<k} N_i}\left|\frac{\phi(N_k-1)}{\phi(N_k)}\right|^2 (|q|^2 - 1)$$

$$= \sum_{k<j} |q|^{2\sum_{i<k} N_i} (|q|^{2N_k} - 1)$$
$$= |q|^{2\sum_{i<j} N_i} - 1 \tag{4.22}$$

where we have used eqs(4.20) and (4.21) with $p = |q|^2$. Hence eq.(4.18) can be replaced by

$$c_j c_j^\dagger - |q|^2 c_j^\dagger c_j \;=\; 1 + (|q|^2 - 1) \sum_{k<j} c_k^\dagger c_k \tag{4.23}$$

Any one of the eqs(4.16), (4.17), (4.18) or (4.23) together with eq.(4.12) constitute a $cc^\dagger$ algebra and one can construct many more examples of $cc^\dagger$ algebras, all of which correspond to the same q-statistics specified by eq.(4.3). We have so far assumed that the occupation numbers are unrestricted : $n_j \geq 0$ and so these algebras are valid for the bosonic q-statistics, namely q-statistics in the bosonic space.

The fermionic q-statistics is defined by eqs(4.3) and (4.4). In this case, n_j in eqs(4.10) and (4.11) can take the value 0 only, while all the other occupation numbers will range over 0 and 1. So, the only arbitrary parameter that enters the definition of c and $c^\dagger$ is $r = \phi(0)/\phi(1)$. The $cc^\dagger$ relation for $i \neq j$ is still the same as in eq(4.12) while the $c_j c_j^\dagger$ relation of eq(4.15) can be simplified to

$$c_j c_j^\dagger - p c_j^\dagger c_j \;=\; |q|^{2\sum_{i<j} N_i} |r|^2 \left\{ \delta_{N_j,0} - p\delta_{N_j,1} \right\} \tag{4.24}$$

Further we can generally prove the fermionic analogue of the identity which was earlier proved in eq.(4.22) for the bososnic case only for a special choice of $\phi(n)$:

$$\begin{aligned}
(|q|^2-1)\sum_{k<j} c_k^\dagger c_k &= (|q|^2-1)\sum_{k<j} |q|^{2\sum_{i<k} N_i}|r|^2\delta_{N_k,1} \\
&= \sum_{k<j} |q|^{2\sum_{i<k} N_i}(|q|^{2N_k}-1)|r|^2 \qquad (4.25)\\
&= (|q|^{2\sum_{i<j} N_i}-1)|r|^2
\end{aligned}$$

Hence, eq (4.24) can be rewritten as

$$c_j c_j^\dagger - p c_j^\dagger c_j = \{|r|^2 + (|q|^2-1)\sum_{k<j} c_k^\dagger c_k\}(\delta_{N_j,0} - p\delta_{N_j,1}) \qquad (4.26)$$

Eqs (4.12) and (4.26) constitute the general algebra for the fermionic q-statistics. In contrast to the eq(4.15) of the bosonic case, the fermionic case does not have the freedom of infinite - parameter deformation.

On the righthand side of eq.(4.26), apart from the curly bracket $\{\ \ \}$ which is bilinear in c and $c^\dagger$, there are kronecker deltas which depend on the operator N_j. A simpler relation is obtained for the choice $p=-1$, since $\delta_{N_j,0} + \delta_{N_j,1} = 1$. We then have

$$c_j c_j^\dagger + c_j^\dagger c_j = |r|^2 + (|q|^2-1)\sum_{k<j} c_k^\dagger c_k \qquad (4.27)$$

A further choice $|r|^2 = 1$ gives

$$c_j c_j^\dagger + c_j^\dagger c_j = 1 + (|q|^2-1)\sum_{k<j} c_k^\dagger c_k \qquad (4.28)$$

which is the fermionic analogue of eq.(4.23).

The algebra given by eqs(4.3), (4.12) and (4.23) (for real q) are covariant under *the quantum group* $SU_q(n)$ where n is the total number of indices.

This is true of the fermionic algebra of eqs(4.3), (4.12) and (4.28) also. See ref [20-31] for this relationship to the theory of quantum groups. However, our approach based on states related by q-statistics (eq(4.1)) does not involve any notion of quantum group *per se* and so we shall not describe it. The formalism of the generalized Fock space appears to be capable of incorporating many general algebraic structures. In particular, the $c_j c_j^\dagger$ algebra of eq(4.15) constitutes a general infinite-parameter deformation while the quantum-group related algebra of eq(4.23) is only a particular case of this.

We have constructed the general algebras for arbitrary complex q. The choice of q defines the symmetry of the state under permutation. Hence we call the different choices of q as different forms of statistics. Particular values of q such as $q = \pm 1$ or $q = e^{i\theta}$ are of special interest, although these cases are all contained in the general formulae already given. We shall call the statistics obtained for $q = +1, -1$ and $e^{i\theta}$ as Bose statistics, Fermi statistics or "fractional" statistics respectively. Either the bosonic or fermionic Fock space can be used to construct any one of the various forms of statistics. Thus, within our terminology, it is perfectly possible for instance to have Fermi statistics residing in bosonic Fock space or vice versa.

Within a particular statistics, different choices of M or ϕ correspond to different representations of c_j and $c_j^\dagger$, which in turn lead to different $cc^\dagger$ algebras. If we assume the factorization given in eq(4.9), for a given q, only the $i = j$ part of the $cc^\dagger$ algebra depends on the representation.

For the sake of clarity, all these forms of statistics and algebras are exhibited in Tables II and III for the bosonic and fermionic spaces respectively.

Several comments are in order, concerning the contents of these tables. The algebra

$$c_i c_j - c_j c_i = 0, \quad \text{for} \quad i \neq j \tag{4.29}$$

$$c_i c_j^\dagger - c_j^\dagger c_i = 0, \quad \text{for} \quad i \neq j \tag{4.30}$$

$$c_j c_j^\dagger \ = \ 1 \tag{4.31}$$

provide the simplest representation for Bose statistics based on the simple choice of M or ϕ as unity. This is analogous to the standard representation (eq(3.5)) for infinite statistics but very different from it. In fact they live in entirely different Fock spaces. In recent literature, eq(4.31) has been sometimes confused with infinite statistics. Eqs.(4.31) coupled with eqs(4.29) and (4.30) infact describes Bose statistics only, although in a noncanonical representation.

Attention may be drawn to the "anticommuting bosons" and the "commuting fermions" shown in Tables II and III respectively. The "commuting fermions" have been called hard-core bosons in condensed-matter-physics literature. Our terminology seems more appropriate since they are really fermions in disguise, living in the fermionic Fock space.

The precise connection of our "fractional" statistics to the exchange property of the wavefunctions proposed [39,40] for one and two-dimensional systems needs further study. In particular, a suitable mapping of the ordered indices to coordinates in one and two dimensions is necessary.

Among all the different algebras given in Tables II and III, only two of them, namely the canonical representations for the bosons and fermions have the distinction of being covariant under the unitary transformations that mix the indices (see eq.(3.37)). All the other algebras violate this requirement, although, as we have already mentioned, two of them, one in the bosonic and the other in the fermionic Fock spaces, are covariant under the quantum groups $SU_q(n)$.

The examples contained in these tables can be called deformed-oscillator algebras on which many papers [1-12,20-27,32,41-42] have been written recently. Inspite of the multiplicity of these deformed algebras, we must not ignore the basic fact that *they are all avatars of just two primary constructions which may be taken to be the canonical bosonic algebra for the bosonic Fock space and the canonical fermionic algebra for the fermionic Fock space.* All the different forms of statistics belonging to the bosonic Fock space as well as their various algebraic representations are related to each other and to the canonical bosonic algebra and similar is the situation for the fermionic Fock space. Given two different forms of statistics within the bosonic Fock space characterized by the statistics parameters q_1 and q_2 respectively, or/and two different representations characterized by the functions $\phi_1(n)$ and $\phi_2(n)$ respectively, it is easy to get from (4.10) the relationship

$$c_j^\dagger(2) \;=\; \left(\frac{q_2}{q_1}\right)^{\sum_{i<j} N_i} \frac{\phi_2(N_j-1)}{\phi_2(N_j)} \frac{\phi_1(N_j)}{\phi_1(N_j-1)} c_j^\dagger(1) \qquad (4.32)$$

The corresponding equation for the fermionic Fock space is

$$c_j^\dagger(2) \;=\; \left(\frac{q_2}{q_1}\right)^{\sum_{i<j} N_i} \left(\frac{r_2}{r_1}\right) c_j^\dagger(1) \tag{4.33}$$

Hence, from the point of view of Fock space, there is nothing new in all these deformed oscillators.

In the above equations, the number operator N_i are given by

$$N_i \;=\; \sum_{n_1\ldots n_i\ldots} n_i||n_1\ldots n_i\ldots\gg\ll n_1\ldots n_i\ldots|| \tag{4.34}$$

and, written in this fashion in terms of the normalized states $||\gg$, N_i *is the same in all the different forms of statistics (within the bosonic or fermonic Fock spaces) and in all the different representations.*

Using eq.(4.14) it is also possible to express the number operators in terms of c and $c^\dagger$ in any statistics or any representation within the bosonic or fermionic Fock space. Noting that for the canonical bosonic algebra

$$q = 1\ ;\ \phi(n) = \frac{1}{(n!)^{1/2}} \quad , \tag{4.35}$$

we get from eq.(4.14)

$$N_i = c_i^\dagger c_i. \tag{4.36}$$

For other statistics and representations also, the set of equations obtained from eq.(4.14) starting with j = 1 and successively increasing j, can be seen to be implicit recursion formulae for all N_j. Thus we have

$$\left.\begin{array}{rcl} c_1^\dagger c_1 &=& \left|\dfrac{\phi(N_1-1)}{\phi(N_1)}\right|^2 \\ c_2^\dagger c_2 &=& |q|^{2N_1}\left|\dfrac{\phi(N_2-1)}{\phi(N_2)}\right|^2 \\ \vdots && \end{array}\right\} \tag{4.37}$$

By choosing $\phi(N_j)$ one then gets the desired explicit expressions for N_j. However, as already emphasized, such expressions are not of much use; the universal representation of N_j given in eq.(4.34) is sufficient for all purposes.

The formalism of Fock space developed here has sufficient flexibility so as to allow many strightforward extensions or generalizations. For instance, the relationship between permuted states defined by eqs(4.1) or (4.2) can be generalized further. Instead of eq(4.3), we may take

$$c_j c_i - q_{ij}^* c_i c_j \;=\; 0 \quad \text{for} \quad i > j\,, \tag{4.38}$$

where q_{ij} are complex parameters. This may be called *multiparameter q-statistics*. The algebras corresponding to such extensions as well as further generalizations can all be constructed essentially by the same procedure as given in this section. Details will be presented elsewhere. Suffices it to say that, once the relationship among the permuted states is specified through the cc relation, the rest of the story follows.

Special cases of such algebras have been reported in the literature on quantum groups [20-33]. We have already referred to the generality of our approach as compared to quantum groups. As further points of comparison between the two approaches we must mention the following. From the point of view of quantum groups, the whole set of relations among c and $c^\dagger$ are taken for granted. In contrast, our analysis based on the underlying Fock space reveals the cc relation as the key to the whole algebraic structure, although the normalization function $\phi(n)$ also plays a role in determining the actual representation of the operators. Hence, depending on the mode of expressing the cc relation and the choice of $\phi(n)$, one can generate any number of algebras of c and $c^\dagger$. Thus, our approach helps to demystify the quantum - group related algebras by reducing them to their basic essentials which are identified to be simple properties of state vectors in the Fock space. Reversing the procedure, one can probably reconstruct the whole edifice of the quantum group itself starting from the more elementary notions relating to states in the Fock space. This of course lies outside the purview of the present paper.

4.2 Fock space of frozen order and null statistics

In subsection 4.1 we had taken any two states obtained by permutation to be related to each other. We now consider a limiting situation in which the particles are frozen in a particular order, with no permutation allowed. This is the *Fock space of frozen states* and the asosciated statistics is *the statistics of frozen states.* Whereas in infinite statistics each permutation leads to a new state, in the statistics of frozen states no permutation is allowed. To emphasize this contrast, the latter may be called *null statistics.* Although this may be obtained as the limit $q \to 0$ of the q-statistics, more properly, it must be regarded as an independent statistics. For, whereas the statistics for two (non-zero) values of q are related to each other through eqs(4.32) or (4.33), the statistics for $q = 0$ cannot be related to statistics for $q \neq 0$. Hence, we shall construct the algebra of c and $c^\dagger$ for this system of frozen states directly from the definition. Again, depending on whether the particles obey exclusion principle or not, we have two different versions of the system which we shall call the fermionic and the bosonic versions respectively.

Let us first consider the bosonic version. Referring to eq(4.1) the Fock space of frozen states is obtained by taking $|n_1 n_2 \ldots ; 1>$ as the allowed state and requiring

$$|n_1 n_2 \ldots ; \mu > \; = \; 0 \quad \text{for} \quad \mu \neq 1 \tag{4.39}$$

Or, equivalently

$$| \ldots 1_j 1_i \ldots > \; = \; 0 \quad \text{for} \quad i < j \tag{4.40}$$

and hence

$$c_i c_j \; = \; 0 \quad \text{for} \quad i < j \tag{4.41}$$

Assuming factorization of the norm as in eq(4.9), we can represent the creation operator in the form :

$$c_j^\dagger \; = \sum_{n_j, n_{j+1} \ldots} \frac{\phi(n_j)}{\phi(n_j + 1)} || 0_1, \ldots 0_{j-1}, (n_j{+}1), n_{j+1} \ldots \gg\ll 0_1, \ldots 0_{j-1}, n_j, n_{j+1} \ldots || \tag{4.42}$$

The zeroes in the state vectors arise from the fact that $c_j^\dagger$ can create a quantum with index j only if indices $i < j$ are unoccupied. From eq(4.41) and the orthogonality of the states, we get

$$c_k c_j^\dagger \;=\; 0 \quad \text{for} \quad k \neq j \tag{4.43}$$

The $c_j c_j^\dagger$ algebra will depend on the choice of $\phi(n_j)/\phi(n_j+1)$. We shall assume $\phi(n_j) = \phi(n_j+1)$ for simplicity. Then, by using the completeness relation for the states, one can verify

$$c_j c_j^\dagger \;=\; 1 - \sum_{k<j} c_k^\dagger c_k \tag{4.44}$$

Eqs (4.41), (4.43) and (4.44) together define the algebra for the bosonic version of the statistics of frozen order.

For the fermionic version, eq(4.41) is replaced by

$$c_i c_j \;=\; 0 \quad \text{for} \quad i \leq j \tag{4.45}$$

and one again gets

$$c_k c_j^\dagger \;=\; 0 \quad \text{for} \quad k \neq j \tag{4.46}$$

but, instead of eq (4.44), one finds

$$c_j c_j^\dagger \;=\; 1 - \sum_{k\leq j} c_k^\dagger c_k \tag{4.47}$$

Thus, the fermionic version of the statistics of frozen states is described by the algebra of eqs(4.45)-(4.47). It is interesting to note that the replacement of the sign $<$ in the bosonic algebra by the sign $\leq$ yields the fermionic algebra.

Finally we note that the expression for the number operators in terms of c and $c^\dagger$ for the above algebra of null statistics can be shown to be essentially the same as that for the standard representation of infinite statistics (eq 3.9a), but because of eqs (4.41) or (4.45), it can be simplified to read as

$$N_k = \sum_{n_1, n_2 \ldots n_k} c_1^{\dagger n_1} c_2^{\dagger n_2} \ldots c_k^{\dagger n_k} c_k^\dagger c_k c_k^{n_k} \ldots c_2^{n_2} c_1^{n_1} \tag{4.48}$$

where the sum over $n_1, n_2 \ldots n_k$ is unrestricted (≥ 0) for the bosonic version, but is restricted for the fermionic version : $n_1, n_2 \ldots n_{k-1}$ go over 0 and 1 while $n_k = 0$.

5. Derivation of cc relations from $cc^\dagger$ algebras

In the last section we showed how to construct the bilinear algebra of c and $c^\dagger$ starting from states related by eq.(4.1). In other words, $cc^\dagger$ algebra has been derived from the cc algebra given by eq.(4.3). The converse is also true ; the cc algebra can be derived from the $cc^\dagger$ algebra, as will be shown in this section. Thus, within the framework of Fock space it is unnecessary to give both cc and $cc^\dagger$ relations. Either the cc relation or the $cc^\dagger$ relation can be used to define the Fock space and the other can be derived. But there are some caveats:

(i) Although the cc relation does define the Fock space, the $cc^\dagger$ algebra does not follow uniquely ; as already pointed out, the operators $c, c^\dagger$ and their algebras depend on the choice of M.

(ii) In order to define the Fock space completely, the $cc^\dagger$ algebra must fulfill certain conditions. The $cc^\dagger$ algebra must be such that an arbitrary inner product $< 0|c_i c_j \ldots c_g^\dagger c_h^\dagger|0 >$ or infact the vacuum matrix element of any polynomial in the c's and $c^\dagger$'s arbitrarily ordered, can be calculated using the $cc^\dagger$ algebra and the definition of the vacuum state $|0>$:

$$c_k|0> \; = \; 0 \quad \text{for all} \quad k \,. \tag{5.1}$$

A general form of the $cc^\dagger$ algebra that satisfies this requirement is :

$$c_i c_j^\dagger \; = \; A_{ij} + \sum_{k,m} B_{ijkm} c_k^\dagger c_m \tag{5.2}$$

where A_{ij} and B_{ijkm} can be constants, but more generally they can be functions of the number operators. All the $cc^\dagger$ algebras considered in this paper are of this form.

There exists an elegant method [19] to derive cc relations from the $cc^\dagger$ algebras. Restricting ourselves to relations quadratic in c, we define

$$Q_{ij} \;=\; c_i c_j - q^{'} c_j c_i \tag{5.3}$$

where $q^{'}$ may be an arbitrary complex parameter. Suppose that, by using the given $cc^\dagger$ algebra, we are able to prove

$$Q_{ij} c_k^\dagger \;=\; \sum_{i'j'k'} F_{ijk;i'j'k'}\, c_{k'}^\dagger\, Q_{i'j'} \tag{5.4}$$

for all i,j and k where $F_{ijk;i'j'k'}$ may be a c-number or operator. Then, by applying this equation successively, we get

$$Q_{ij} c_k^\dagger c_m^\dagger \ldots \;=\; \sum_{i'j'k'} \sum_{i''j''m'} F_{ijk;i'j'k'}\, F_{i'j'm;i''j''m'}\, c_{k'}^\dagger\, c_{m'}^\dagger \ldots Q_{i''j''} \tag{5.5}$$

Allowing both sides of this equation to act on $|0>$, the right side vanishes and so we see that Q_{ij} acting on any Fock state $c_k^\dagger c_m^\dagger \ldots |0>$ gives zero. Hence we may write the operator identity :

$$Q_{ij} \;=\; 0 \tag{5.6}$$

which is the cc relation sought after. It may also be pointed out that often one finds

$$F_{ijk;i'j'k'} \;=\; f_{ijk} \delta_{ii'}\, \delta_{jj'}\, \delta_{kk'} \tag{5.7}$$

so that eq.(5.4) is simplified to

$$Q_{ij}\, c_k^\dagger \;=\; f_{ijk}\, c_k^\dagger Q_{ij} \tag{5.8}$$

Thus, the form-invariance of Q_{ij} on being pushed to the right of $c_k^\dagger$, as explicitly given in eqs.(5.4) or (5.8) is the necessary and sufficient condition for the existence of the cc relation. We may now apply this method on various $cc^\dagger$ algebras discussed in the previous sections.

5.1 q-mutator algebra with real q:

The $cc^\dagger$ algebra is

$$c_i c_j^\dagger - q c_j^\dagger c_i \;=\; \delta_{ij} \quad , \quad \forall i, j. \tag{5.9}$$

We define

$$Q_{ij} \;=\; c_i c_j - q' c_j c_i \quad , \quad \forall i, j \tag{5.10}$$

From eq.(5.9),

$$Q_{ij} c_k^\dagger \;=\; (1 - qq')c_i \delta_{kj} + (q - q')c_j \delta_{ki} + q^2 c_k^\dagger Q_{ij} \tag{5.11}$$

The form-invariance of Q_{ij} requires

$$q' = q = \pm 1. \tag{5.12}$$

So, we get the cc relations

$$c_i c_j \pm c_j c_i = 0 \tag{5.13}$$

corresponding to Bose and Fermi statistics at $q = \pm 1$. For $-1 < q < 1$, there are no cc relations and we have infinite statistics. The inner product matrix M which remains positive definite for $-1 < q < 1$, develops zero eigenvalues at $q = \pm 1$. (see eq.(3.13)) corresponding to the null states such as $(c_i^\dagger c_j^\dagger \pm c_j^\dagger c_i^\dagger)|0>$ arising from eq.(5.13).

5.2 q-mutator algebra with complex q

The $cc^\dagger$ relations are

$$c_i c_j^\dagger - q c_j^\dagger c_i = 0 \quad \text{for}\, i < j \tag{5.14}$$

$$c_i c_i^\dagger - p c_i^\dagger c_i = 1 \tag{5.15}$$

We define

$$Q_{ij} \;=\; c_i c_j - q' c_j c_i \quad \text{for} \quad i < j \tag{5.16}$$

$$= \; c_i c_i \quad \text{for} \quad i = j \tag{5.17}$$

Using eqs.(5.14) and (5.15), we find

$$\left.\begin{aligned} Q_{ij} c_k^\dagger &= q^2 c_k^\dagger Q_{ij} \quad for\, i \le j < k \\ &= {q^*}^2 c_k^\dagger Q_{ij} \quad for\, k < i \le j \\ &= |q|^2 c_k^\dagger Q_{ij} \quad for\, i < k < j \\ &= (q^* - q') c_j + p q^* c_i^\dagger Q_{ij} \quad for\, k = i < j \\ &= (1 - q q') c_i + p q c_j^\dagger Q_{ij} \quad for\, i < j = k \\ &= (1 + p) c_i + p^2 c_i^\dagger Q_{ii} \quad for\, i = j = k. \end{aligned}\right\} \tag{5.18}$$

The form-invariance of Q_{ij} for $i < j$ requires

$$q' \;=\; q^* = e^{-i\theta} \tag{5.19}$$

where θ is real. We thus get the cc relation corresponding to fractional statistics at the boundary $|q| = 1$ of the disc $|q| < 1$ in the complex plane.

We already know that the disc $|q| < 1$ corresponds to the region of positive-definite inner product matrix M for the complex q-mutator algebra of infinite statistics. At the boundary of this positivity region, we have fractional statistics living in a reduced Fock space (see Fig.2). It is the states of vanishing norm of the form

$$(\ldots c_i^\dagger c_j^\dagger \ldots |0>) \quad - e^{i\theta}(\ldots c_j^\dagger c_i^\dagger \ldots |0>) \quad \text{for} \quad i < j \tag{5.20}$$

which contribute to the zeroes of det M_n in eq.(3.34) at $|q| = 1$. Once we remove these states, we get the reduced Fock space of positive-definite norm.

The form-invariance of Q_{ij} for i = j requires

$$p = -1 \tag{5.21}$$

with the corresponding cc relation

$$c_i c_i \;=\; 0 \tag{5.22}$$

This leads to the exclusion principle namely $n_i = 0$ or 1 only ; equivalently, the norms of states with repeated indices are zero as seen by putting $p = -1$ in eq.(3.35).

To sum up, we may note four possibilities all contained in the algebra of eqs (5.14) and (5.15) :

(a) Infinite statistics with multiple occupation

$$|q| \neq 1 \quad ; \quad p \neq -1. \tag{5.23}$$

(b) Infinite statistics with exclusion principle

$$|q| \neq 1 \quad ; \quad p = -1 \tag{5.24}$$

(c) Fractional statistics with multiple occupation

$$q = e^{i\theta} \quad ; \quad p \neq -1 \tag{5.25}$$

(d) Fractional statistics with exclusion principle

$$q = e^{i\theta}; p = -1. \tag{5.26}$$

5.3 q-statistics

The $cc^\dagger$ relations are

$$c_i c_j^\dagger - q c_j^\dagger c_i = 0 \quad \text{for} \quad i < j \tag{5.27}$$

$$c_j c_j^\dagger - p c_j^\dagger c_j = |q|^{2\sum_{i<j} N_i} f(N_j) \tag{5.28}$$

We define

$$Q_{ij} = c_i c_j - q' c_j c_i \quad \text{for} \quad i > j \tag{5.29}$$

Using eqs.(5.27), (5.28) we get

$$
\left.\begin{aligned}
Q_{ij}c_k^\dagger &= q^2 c_k^\dagger Q_{ij} \quad \text{for} \quad i<j<k \\
&= q^{*^2} c_k^\dagger Q_{ij} \quad \text{for} \quad k<i<j \\
&= |q|^2 c_k^\dagger Q_{ij} \quad \text{for} \quad i<k<j \\
&= pq^* c_j^\dagger Q_{ij} + c_i |q|^{2\sum_{m<j} N_m} f(N_j)(1-q'q^*) \quad \text{for} \quad i>j=k \\
&= pq c_i^\dagger Q_{ij} + |q|^{2\sum_{m<i} N_m} f(N_i) c_j (q - q'|q|^2) \quad \text{for} \quad k=i>j
\end{aligned}\right\} \tag{5.30}
$$

Repeating the argument of form-invariance of Q_{ij}, we conclude

$$
c_i c_j - q^{\star^{-1}} c_j c_i = 0 \quad \text{for} \quad i>j \tag{5.31}
$$

In this proof we have not used any particular form of $f(N_j)$, but have used only the standard commutation relations among N_k and c_i (eqs(2.23) and (2.26).

This derivation of the $c_i c_j$ relation for $i \neq j$ is valid for the bosonic as well as the fermionic fock spaces. However, for the fermionic space, there exists the additional relation :

$$
c_i c_i \;=\; 0 \tag{5.32}
$$

To derive this, we now commute $c_k^\dagger$ through $c_i c_i$ using the same eqs(5.27 - 5.28), we get

$$
\left.\begin{aligned}
c_i c_i c_k^\dagger &= q^{*2} c_k^\dagger c_i c_i \quad \text{for} \quad i>k \\
&= q^2 c_k^\dagger c_i c_i \quad \text{for} \quad k>i \\
&= p^2 c_i^\dagger c_i c_i + |q|^{2\sum_{m<i} N_m} c_i \{ p f(N_i - 1) + f(N_i) \} \text{ for } i=k
\end{aligned}\right\} \tag{5.33}
$$

So, for the validity of eq(5.32), we require

$$c_i\{pf(N_i-1)+f(N_i)\} \;=\; 0 \tag{5.34}$$

Substituting the form of $f(N_i)$ for the fermionic Fock space (from eq.(4.24))

$$f(N_i) \;=\; |r|^2(\delta_{N_i,0}-p\delta_{N_i,1})\,, \tag{5.35}$$

we see that eq.(5.34) is satisfied, since in the fermionic Fock space, $\delta_{N_i,2}=0$ and $c_i\delta_{N_i,0}=0$.

One may also note that eq.(5.34) is also satisfied for $p=-1$ and $f(N_i)$ = constant (which may be chosen to be unity). So, for bosonic Fock space, i.e. $c_ic_i\neq 0$, we must avoid $p=-1$ in eq.(5.28) and (4.15).

The above derivations of cc relations have used the general $cc^\dagger$ algebra and hence includes the cases of quantum-group-covariant algebras, independent deformed oscillators, commuting fermions, anticommuting bosons etc. Further, it must be noted that the presence of the term with the factor $|q|^{2\sum_{i<j}N_i}$ in eq.(5.28) is crucial for the validity of the cc relations ; without this factor, there will be no cc relation and we will get infinite statistics.

6. Two-indexed Systems

We have so far considered generalized Fock spaces consisting of states $|n_g, n_h, \ldots; \mu>$ where the indices $g, h, \ldots$ may refer either to a single quantum number, or to a collection of quantum numbers, specifying the space, spin and other internal degrees of freedom. In the latter case, it may be supposed that one has mapped a collection of indices to a single index. What we are envisaging in this section are situations where such mapping is not possible. This can happen in various ways. To be specific, let us consider oscillators with a pair of indices, a latin index $(g, h, \ldots)$ and a greek index $(\alpha, \beta \ldots)$. There exists a class of systems in which the occupation numbers with a single index n_g or n_α are defined, but occupation numbers with both the indices $n_{g\alpha}$ are not defined. Such systems cannot be mapped into single-indexed systems. In another class of systems, $n_{g\alpha}$ do exist, but the subsidiary conditions that define the reduced Fock space depend on the two indices g and α in such a way that prevents mapping of (g, α) into a single index. We consider these two classes of systems in subsections 6.1 and 6.2 respectively.

It may also be mentioned that we first encountered such systems in the study of the Hubbard model in the limit of infinite Coulomb repulsion [43-46]. Although this was our original motivation, this has now opened the door to a more general framework encompassing novel forms of statistics and algebras.

6.1 Systems in which $n_{g\alpha}$ do not exist

We specify the states as $|n_g, n_h \ldots ; n_\alpha, n_\beta \ldots ; \mu >$, where $n_g, n_h \, \ldots$ are the numbers of quanta with indices $g, h \, \ldots$ respectively while $n_\alpha, n_\beta, \ldots$ are the numbers of quanta with indices $\alpha, \beta \ldots$ respectively and we have the constraint :

$$n_g + n_h + \ldots \;=\; n_\alpha + n_\beta + \ldots \tag{6.1}$$

We may regard n_g as the total number of quanta with index g whatever may be their greek index and similarly for n_α. In such a state, the occupancy number with both indices such as $n_{k\alpha}$ is not defined. The latin indices and the greek indices are independently permutated and this leads to a much enlarged space in each sector. Now μ goes over $1 \ldots s'$ where

$$s' \;=\; \frac{(n_g + n_h + \ldots)!}{n_g! n_h! \ldots} \times \frac{(n_\alpha + n_\beta + \ldots)!}{n_\alpha! n_\beta! \ldots} \tag{6.2}$$

This is to be compared to the size of the space of the states $|n_{g\alpha}, n_{h\beta} \ldots ; \mu >$ for which the range of μ is

$$s \;=\; \frac{(n_{g\alpha} + n_{h\beta} + \ldots)!}{n_{g\alpha}! n_{h\beta}! \ldots} \tag{6.3}$$

In general, s' is larger than s.

Let us consider the two-particle sector as an example. If we specify occupation numbers with both indices, we have the two states in the sector $(1_{k\alpha}, 1_{m\beta})$: $|1_{k\alpha}, 1_{m\beta}; \mu >$ with $\mu = 1$ or 2 which correspond to $|1_{k\alpha}, 1_{m\beta} >$ and $|1_{m\beta}, 1_{k\alpha} >$ and another two states in the sector $(1_{m\alpha}, 1_{k\beta})$: $|1_{m\alpha}, 1_{k\beta}; \mu >$ with $\mu = 1$ or 2 corresponding to $|1_{m\alpha}, 1_{k\beta} >$ and $|1_{k\beta}, 1_{m\alpha} >$. (Here we are not using $k\alpha, m\beta$ etc in their generic sense ; they are used to denote specific values of the indices.) On the other hand, with the new type of states with decoupled indices for which occupation numbers with both indices do not exist, we have the two-particle states $|1_k, 1_m; 1_\alpha, 1_\beta; \mu>$ and now μ goes over 1 to 4 corresponding to the four state vectors $|1_{k\alpha}, 1_{m\beta}>$, $|1_{m\beta}, 1_{k\alpha}>$, $|1_{m\alpha}, 1_{k\beta}>$, $|1_{k\beta}, 1_{m\alpha}>$. Thus, both the sectors considered earlier are combined to form a single enlarged sector. Such a regrouping of sectors with consequent enlargement occurs throughout the Fock space, in the case of the decoupled indices.

The construction of the orthonormal set as well as the other properties of the generalized Fock spaces given in Sec.2 goes through for the present case of decoupled indices also except that the matrices X, M etc will be of higher dimensions. The creation and destruction operators $c^\dagger_{k\alpha}, c_{k\alpha}$ also can be constructed in the same way as in Sec.2. The relevant equations and formulae with the appropriate changes incorporated are given in Appendix A. They are self-explanatory.

Just as in the case of single-indexed systems (cf.Sec.2), we can again have a super Fock space in which all the states connected by independent permutation of the latin and greek indices are taken to be independent. We shall call the associated statistics as "doubly-infinite" statistics since it is infinite statistics in latin and greek indices separately. By imposing relations among the permutated states, one can get many kinds of reduced

Fock spaces. Because of the larger number of available states in each sector, many new types of statistics become possible. These can be discussed and the associated algebras can be constructed by the same procedure as in Sec.3, 4 and 5. However, we shall be brief and restrict ourselves to presenting some of the important results only. Some of these algebras and the new kinds of statistics implied by them have been discussed by us in greater detail in the earlier papers [18,19,46].

Doubly-infinite statistics

Consider the $cc^\dagger$ algebra described by

$$c_{k\alpha}c^\dagger_{m\beta} - q\delta_{\alpha\beta}\sum_\gamma c^\dagger_{m\gamma}c_{k\gamma} \;=\; \delta_{km}\delta_{\alpha\beta} \tag{6.4}$$

where q is a real parameter lying in the range

$$-1 < q < 1\,. \tag{6.5}$$

It is the second term on the left of eq(6.4) in which α and β have been dissociated from k and m respectively that leads to the decoupling of the latin and greek indices and prevents the mapping of lattin and greek indices to a single index. One can show [19] that the inner-product matrices following from this algebra are all positive definite for $-1 < q < 1$. Further, one can show using arguments similar to those in Sec.5 that there is no cc relation for this algebra in the same parameter range and so all the states connected by *independent* permutation of the latin and greek indices are independent. The underlying Fock space is the full super Fock space of dimension given by eq.(6.2) and we have infinite statistics in latin and greek indices separately, or doubly-infinite statistics.

For $q = 0$ in eq(6.4), the two indices can be mapped into a single index and the algebra reduces to the standard representation of single-indexed infinite statistics described in Sec.3.1.

The algebra of eq.(6.4) is covariant under the unitary transformations on the latin indices :

$$d_{k\alpha} = \sum_m U_{km} c_{m\alpha} \; ; \; U^\dagger U \;=\; UU^\dagger \;=\; 1 \tag{6.6}$$

as well as under separate unitary transformations on the greek indices :

$$e_{k\alpha} \;=\; \sum_\lambda V_{\alpha\lambda} c_{k\lambda} \; ; \; V^\dagger V \;=\; VV^\dagger = 1. \tag{6.7}$$

But the algebra is not covariant under the enlarged unitary transformations involving both the latin and greek indices (for $q \neq 0$). A special case of the unitary transformations of eqs(6.6) and (6.7) is the phase transformation. Eq.(6.4) is covariant under either of the following phase transformations :

$$d_{k\alpha} \;=\; e^{i\phi_k} c_{k\alpha} \tag{6.8}$$

$$e_{k\alpha} \;=\; e^{i\phi_\alpha} c_{k\alpha} \tag{6.9}$$

As a consequence, the number operators N_k and N_α exist. However, eq(6.4) is not covariant under the transformation :

$$f_{k\alpha} \;=\; e^{i\phi_{k\alpha}} c_{k\alpha} \tag{6.10}$$

and correspondingly, $N_{k\alpha}$ does not exist.

We can make q in (6.4) complex provided we order the latin indices and thus we get an algebra which is the analogue of the q-mutator algebra with complex q for the single-indexed systems (Sec.3) :

$$c_{i\alpha}c^\dagger_{j\beta} - q\delta_{\alpha\beta}\sum_\gamma c^\dagger_{j\gamma}c_{i\gamma} \ = 0 \quad \text{for} \quad i<j \tag{6.11}$$

$$c_{j\alpha}c^\dagger_{j\beta} - p\delta_{\alpha\beta}\sum_\gamma c^\dagger_{j\gamma}c_{j\gamma} \ = \ \delta_{\alpha\beta} \tag{6.12}$$

where q is complex, but p is real. This again describes the same doubly-infinite statistics ; only the representation and algebra are different. This algebra is no longer covariant under the unitary transformations of eqs (6.6) and (6.7), but is still covariant under the phase transformations of eqs (6.8) and (6.9). Positivity of the inner-product matrices M requires $|q| < 1$. As for p, similar statements as in Sec.3 can be made.

Coming back to eq(6.4), at the boundary of the range of q given in eq(6.5), namely at $q = \pm 1$, we get two new forms of statistics called orthobose and orthofermi statistics [18,19,46] which reside in reduced Fock spaces defined by cc relations. The algebras for these are given below :

Orthobose Statistics

$$c_{k\alpha}c^\dagger_{m\beta} - \delta_{\alpha\beta}\sum_\gamma c^\dagger_{m\gamma}c_{k\gamma} \ = \ \delta_{km}\delta_{\alpha\beta} \tag{6.13}$$

$$c_{k\alpha}c_{m\beta} - c_{m\alpha}c_{k\beta} \ = \ 0\,. \tag{6.14}$$

Orthofermi Statistics

$$c_{k\alpha}c^{\dagger}_{m\beta} + \delta_{\alpha\beta}\sum_{\gamma} c^{\dagger}_{m\gamma}c_{k\gamma} \;=\; \delta_{km}\delta_{\alpha\beta} \tag{6.15}$$

$$c_{k\alpha}c_{m\beta} + c_{m\alpha}c_{k\beta} \;=\; 0\,. \tag{6.16}$$

The inner product matrices M for the reduced Fock spaces generated by these algebras (eqs(6.13) - (6.16)) can be shown to be positive. In both these statistics, any two states obtained by permuting the greek indices are independent. On the other hand, states obtained by permuting the latin indices are either equal to each other, or related by $(-1)^J$ where J is the number of inversions in the permutation of the latin indices, for the orthobose or orthofermi statistics respectively. Thus, we have infinite statistics in the greek indices and bose or fermi statistics in the latin indices. For orthofermi statistics, we have the further condition :

$$n_k \;=\; 0 \text{ or } 1 \text{ only}\,. \tag{6.17}$$

It must be noted that the exclusion implied by eq(6.17) is stronger than the usual Pauli exclusion principle ; eq(6.17) requires that there cannot be more than one particle with index k whatever may be its greek index.

Orthostatistics can be generalised to q-orthostatistics. We keep the infinite statistics in the greek indices but have q-bose or q-fermi statistics in the latin indices. More precisely, we must regard these as q-orthostatistics lying in orthobosonic and orthofermionic Fock spaces. We can call them q-orthobose or q-orthofermi statistics.

q - Orthobose Statistics

$$c_{i\alpha}c^{\dagger}_{j\beta} + q\delta_{\alpha\beta}\sum_{\gamma} c^{\dagger}_{j\gamma}c_{i\gamma} \;=\; 0 \quad \text{for} \quad i < j \tag{6.18}$$

$$c_{j\alpha}c^{\dagger}_{j\beta} - |q|^2\delta_{\alpha\beta}\sum_{\gamma} c^{\dagger}_{j\gamma}c_{j\gamma} \;=\; \delta_{\alpha\beta} + \delta_{\alpha\beta}(|q|^2 - 1)\sum_{k<j}\sum_{\gamma} c^{\dagger}_{k\gamma}c_{k\gamma} \tag{6.19}$$

$$c_{i\alpha}c_{j\beta} + q^{\star}c_{j\alpha}c_{i\beta} \;=\; 0 \quad \text{for} \quad i < j \tag{6.20}$$

q-Orthofermi Statistics

$$c_{i\alpha}c^{\dagger}_{j\beta} + q\delta_{\alpha\beta}\sum_{\gamma} c^{\dagger}_{j\gamma}c_{i\gamma} \;=\; 0 \quad \text{for} \quad i < j \tag{6.21}$$

$$c_{j\alpha}c^{\dagger}_{j\beta} + \delta_{\alpha\beta}\sum_{\gamma} c^{\dagger}_{j\gamma}c_{j\gamma} \;=\; \delta_{\alpha\beta} + \delta_{\alpha\beta}(|q|^2 - 1)\sum_{k<j}\sum_{\gamma} c^{\dagger}_{k\gamma}c_{k\gamma} \tag{6.22}$$

$$c_{i\alpha}c_{j\beta} + q^{\star}c_{j\alpha}c_{i\beta} \;=\; 0 \quad \text{for} \quad i < j \tag{6.23}$$

$$c_{j\alpha}c_{j\beta} \;=\; 0 \tag{6.24}$$

In eqs(6.18) - (6.24), q is an arbitrary complex parameter. For $q = e^{i\theta}$, and $q = 0$ we shall have fractional statistics and statistics of frozen order respectively, but in the latin indices only. Of course one can construct many other algebras corresponding to the same statistics, just as in the case of the single-indexed systems.

Further, these equations are analogous to eqs(4.3), (4.12),(4.23) and (4.28) which are covariant under quantum groups. So, another direction is indicated here for the further generalization of quantum-group-theoretic structures to two-indexed systems.

6.2 Systems in which $n_{g\alpha}$ exist

We shall now consider a different kind of double-indexed systems. Here, the occupation numbers $n_{g\alpha}$ exist, nevertheless mapping to single-indexed systems is not possible because of the subsidiary conditions that define the reduced Fock space. We give below three examples of such double indexed systems, described by the algebras :

(a)

$$c_{k\alpha}c^{\dagger}_{m\beta} + (1-\delta_{km})c^{\dagger}_{m\beta}c_{k\alpha} \;=\; \delta_{km}\delta_{\alpha\beta}\left(1-\sum_{\gamma} c^{\dagger}_{k\gamma}c_{k\gamma}\right) \tag{6.25}$$

$$c_{k\alpha}c_{m\beta} + (1-\delta_{km})c_{m\beta}c_{k\alpha} \;=\; 0\,. \tag{6.26}$$

(b)

$$c_{k\alpha}c^{\dagger}_{m\beta} - (1-\delta_{km})c^{\dagger}_{m\beta}c_{k\alpha} \;=\; \delta_{km}\delta_{\alpha\beta}\left(1+\sum_{\gamma} c^{\dagger}_{k\gamma}c_{k\gamma}\right) \tag{6.27}$$

$$(1-\delta_{km})(c_{k\alpha}c_{m\beta} - c_{m\beta}c_{k\alpha}) \;=\; 0\,. \tag{6.28}$$

(c)

$$c_{k\alpha}c^{\dagger}_{m\beta} - (1-\delta_{km})c^{\dagger}_{m\beta}c_{k\alpha} \;=\; \delta_{km}\delta_{\alpha\beta}\left(1-\sum_{\gamma} c^{\dagger}_{k\gamma}c_{k\gamma}\right) \tag{6.29}$$

$$c_{k\alpha}c_{m\beta} - (1-\delta_{km})c_{m\beta}c_{k\alpha} \;=\; 0\,. \tag{6.30}$$

Each of these algebras is covariant under the unitary transformation on the greek indices defined by eq.(6.7), but not covariant under the unitary transformation on the latin indices defined by eq.(6.6). However, all of them are covariant under not only the phase transformations in eqs(6.8) and (6.9) but also the $k\alpha$-dependent phase transformation of eq.(6.10). hence all the occupation numbers n_k, n_α as well as $n_{k\alpha}$ exist.

Algebra (a) leads to states which are antisymmetric for simultaneous interchange of latin and greek indices as in Fermi-Dirac statistics, but the usual Pauli exclusion principle is replaced by the stronger or more exclusive exclusion principle, as in orthofermi statistics (see eq.6.17) :

$$n_k \;=\; 0 \quad \text{or} \quad 1 \quad \text{only}\,, \tag{6.31}$$

where

$$n_k \;\equiv\; \sum_\alpha n_{k\alpha} \tag{6.32}$$

These are precisely the states that are allowed in the Hubbard model of strongly correlated electrons in the limit of infinite intrasite Coulomb repulsion if we interpret the latin index as the site and the greek index as the spin. Hence, this algebra and the statistics can be called Hubbard algebra and Hubbard statistics respectively. For more details, the reader is referred to [46]. Since $n_{k\alpha}$ exists, the states of the system can be characterized as $|n_{k\alpha}, n_{m\beta}\ldots; \mu >$. However, because of the above constraint (eq(6.31)) which can be rewritten in the form :

$$\sum_{\alpha} n_{k\alpha} \;=\; 0 \quad \text{or} \quad 1 \quad \text{only} \tag{6.33}$$

this sytem cannot be mapped into a single-indexed system.

Algebra (b) can be regarded as the bosonic "counterpoint" of algebra (a). States are symmetric under simultanueous exchange of latin and greek indices for quanta with different latin indices ($k \neq m$). But for quanta with identical latin indices ($k = m$), there is no restriction on the symmetry with respect to the greek indices. In other words, there is infinite statistics in greek indices if the corresponding latin indices are identical. Whereas algebra (a) leads to more exclusive states than allowed by Pauli, algebra (b) leads to "more inclusive" states than allowed by Bose. For this reason we may call algebra (b) as the "inclusive counterpoint" to algebra (a). Such a restriction on the allowed states of a two-indexed system which distinguishes $k = m$ from $k \neq m$ cannot be mapped into a condition for a single-indexed system.

Finally, algebra (c) leads to states that are symmetric for simultaneous exchange of latin and greek indices, but the stronger exclusion principle of eq (6.33) is also valid as in algebra (a). Hence, algebras (a) and (c) represent two forms of statistics which may be called antisymmetric and symmetric Hubbard statistics respectively both residing within the same reduced Fock space. On the other hand, algebra (b) and its statistics lie in a different reduced Fock space which is a Fock space with the new "inclusion" principle.

A compilation of the algebras and statistics for two-indexed systems is given in Table IV. The cc relations are not included since they can be shown to follow from the $cc^{\dagger}$ algebra, whenever they exist.

7. Summary and Discussion

We have formulated a theory of generalized Fock spaces which is sufficiently general so as to encompass the well known Fock spaces and many newer ones. We have shown that such a theory can be constructed without introducing creation and annihilation operators. The only requirements for constructing a generalized Fock space are to specify the set of allowed states, and to make it an inner product space. By freeing the notion of the underlying state space from c and $c^\dagger$, we are able to define different forms of quantum statistics in a representation independent manner. Subsequently, one can construct c and $c^\dagger$ and their algebras in any desired representation.

Our general formalism not only unifies the various forms of statistics and algebras proposed so far but also allows one to construct many new forms of quantum statistics as well as algebras of c and $c^\dagger$ in a systematic manner. Some of these are the following :

(a) Many new algebras for infinite statistics

(b) Complex q-statistics and a number of $cc^\dagger$ algebras representing them

(c) A consistent algebra of c and $c^\dagger$ for "fractional" statistics

(d) Null statistics or statistics of frozen order

(e) "Doubly-infinite" statistics and its representations

(f) q-orthobose and q-orthofermi statistics

(g) A statistics for two-indexed systems with a new "inclusion principle".

(h) A symmetric version of Hubbard statistics.

Our primary concept is that of generalized Fock space, of which many categories have been introduced in this paper. Next comes the notion of statistics which is defined by the type of symmetry or relationship among the state vectors residing in the particular type of Fock space. In a given Fock space, more than one type of symmetry can be postulated, the prime example of this being the symmetry, antisymmetry or q-symmetry in the bosonic and fermionic Fock spaces. For a given statistics, there can exist different representations of c and $c^\dagger$, leading to different $cc^\dagger$ relations. To summarize, a particular Fock space can admit different statistics, and a particular statistics can be represented by more than one $cc^\dagger$ algebra. But *the important point is that various statistics and algebras residing in a given Fock space are all inter-related.* These interconnections are given by generalized versions of the well-known Jordan-Wigner-Klein transformations. No such interconnections exist among statistics and algebras belonging to distinct Fock spaces.

For the sake of clarity, the above-described logical order of concepts as well as their logical interconnections are presented in the form of flow charts or block diagrams in Figs.3 and 4. The single-indexed systems are considered in Fig.3. The Fock spaces of higher dimension are shown to the right of those of lower dimension . The Fock space of frozen order as well as the bosonic and fermionic Fock spaces have the lowest dimension $d = 1$ in any sector $\{n_g, n_h \ldots\}$. Next come the parafermionic and parabosonic Fock spaces which have d > 1. At the extreme right, we have the super Fock space which has the largest dimension $d = s$ in each sector with s given by eq.(2.6). Null statistics and infinite statistics can be regarded as the opposite limiting cases of generalized statistics and hence these two forms of statistics along with their Fock spaces occupy the opposite ends of the diagram. Although not shown separately in Fig.3 because of lack of space, the bosonic and fermionic Fock spaces are distinct and each must be separately associated with the complete set of statistics and algebras shown. Same is true of the parabosonic and parafermionic Fock spaces. Further, there are two Fock spaces of frozen order, the bosonic and fermionic type. And finally, there exists another super Fock space with exclusion principle, which is not shown separately.

Within the parafermionic and parabosonic Fock spaces many "deformations" of parastatistics and many other representations and algebras apart from Green's trilinear algebra [35] are possible. These are indicated by the hanging arrows in Fig.3. Further, as shown by the dotted lines, there is enough room for many new varieties of Fock spaces and associated statistics and algebras. These possibilities may be pursued in the future.

Coming to Fig.4 depicting the systems with two indices, here again Fock spaces of higher dimension generally lie to the right. Although shown together, the orthobosonic and orthofermionic Fock spaces must be regarded distinct. Here, one can envisage a richer harvest of new Fock spaces, statistics and algebras because of the two indices and this again is for the future.

We now conclude with some general comments :

1. We must once again repeat and emphasize the point that most of the q-deformations on oscillators discussed in the literature amount to only a change of variable and hence must be regarded as different avatars of bosonic or fermionic systems. However, one must clearly distinguish those deformations such as the q-mutator algebra of Greenberg that require the construction of new types of Fock spaces. Obviously, Greenberg-type of deformations can never be reduced to change of variables living within the bosonic or fermionic Fock space. Some degree of confusion prevails in recent literature since this distinction is not kept in mind. (See for instance [7,32,41,42,47]).

2. In Sec.4, we have shown that algebras that are covariant under quantum groups are only a particular case of the more general class of algebras that can be derived from the formalism of generalized Fock spaces. This formalism is based on linear vector space and linear operators acting on this space; mathematically, no more sophistication is required. And yet it is capable of handling quantum-group related structures in a self-contained manner. It would seem that the basic concepts of quantum groups are contained in the theory of generalized Fock spaces and it must be possible to construct quantum group itself starting from this theory.

3. We have already referred to the desirability of covariance under unitary transformations that mix the indices as a requirement for the algebras of creation and annihilation operators. We shall call the algebras that satisfy this

requirement as covariant algebras. This property stems from the superposition principle in quantum mechanics. Since the indices describe quantum states of a single particle, if we demand that, for any orthonormal set of quantum states obtained by superposition of the original set of quantum states, the algebra should retain the same form, then covariance under unitary transformations follows. Many of the algebras presented in this paper violate this requirement. Such noncovariant algebras probably cannot be used in a general context, as for instance, in constructing a quantum field theory that respects many of the known invariance principles such as translational or rotational invariance. Nevertheless, these algebras may be useful to describe specific systems in specific states such as those encountered in condensed matter physics.

Some of these noncovariant algebras do have other nice properties, although these are motivated mainly from a mathematical point of view. This is the case of those algebras that are covariant under quantum groups.

Among the algebras for single-indexed systems that have been discussed, Greenberg's q-mutator algebra is the only q-deformation that is covariant under unitary transformations, but then one has to pay the price of the enlarged Fock space. Every other known q-deformation leads to a noncovariant algebra.

Greenberg's q-mutator algebra (including the case $q = 0$ which is the standard representation), the canonical bosonic and fermionic algebras and

Green's trilinear algebras for parabosons and parafermions are the covariant representatives living respectively in the super Fock space, bosonic and fermionic Fock spaces and the parabosonic and parafermionic Fock spaces. All the other algebras living in these three catagories of Fock spaces, although noncovariant, can be transformed to these covariant algebras through equations such as eq(4.32). This is not the case for the algebra of null statisitcs or the algebra of Boltzmann statistics with Pauli principle living respectively in the Fock space of frozen order and the super Fock space with Pauli principle. In these Fock spaces, covariant algebras do not exist.

4. Quantum mechanics is sometimes viewed as a deformation of classical mechanics since the commutator bracket of quantum mechanics can be related to the deformation of the classical Poisson bracket, the Planck's constant playing the role of the deformation parameter. Relying on similar reasoning it has been proposed that a deformation of canonical commutation relations will lead to fundamentally new mathematical or physical structures [48,49,50]. The analysis presented in this paper shows that nothing of this sort happens, if viewed within the framework of Fock space. The transition from classical to quantum mechanics requires the replacement of the notion of the phase space by that of the Hilbert space or Fock space. In contrast, we have seen that all the deformations of commutation relations can be formulated within the framework of Fock space. In fact most of the deformed structures proposed in the literature exist within the time-honoured bosonic

and fermionic Fock spaces only. Even Greenberg's infinite statistics lives within a Fock space, although an enlarged one.

5. While remaining within the framework of quantum mechanics, the general theory of Fock spaces presented here throws light on the enlarged framework within which the familiar quantum field theory and statistical mechanics reside and hence may lead to newer forms of quantum field theory and statistical mechanics. This is infact the main motivation behind our work. Apart from earlier work on parastatistics [37], we may mention as examples of new forms of quantum field theories, Greenberg's construction [13,14,51] of a nonrelativistic quantum field theory based on infinite statistics and our construction [52] of a local relativistic quantum field theory based on orthostatistics. Many other forms of quantum field theories based on the generalized Fock spaces may be possible. Their formulation and study is an agenda for the future.

6. Although one may not be able to construct local relativistic quantum field theories corresponding to many of the newer forms of statistics and algebras, nonrelativistic quantum field theories based on these are still possible. Condensed matter physics is a rich field where applications of such theories may be relevant. In fact there is no reason why the quasiparticles encountered in condensed matter systems should be bosons or fermions only. We have shown that any of the generalized Fock spaces provides a perfectly valid

quantum-mechanical framework for many-particle systems. Hence, quasiparticles living in a generalized Fock space offer an important field of study.

Appendix A : Generalised Fock Spaces for Two-indexed Systems

Here we consider only those two-indexed systems in which $n_{g\alpha}$ do not exist. See Sec.6.1.

The state vectors, inner products and orthonormal sets

$$\langle n_g^{'}, n_h^{'} \ldots ; n_\alpha^{'}, n_\beta^{'} \ldots ; \mu | n_g, n_h \ldots ; n_\alpha, n_\beta \ldots ; \nu \rangle$$

$$= \quad \delta_{n_g^{'} n_g} \delta_{n_h^{'} n_h} \ldots \delta_{n_\alpha^{'} n_\alpha} \delta_{n_\beta^{'} n_\beta} \ldots M_{\mu\nu} \tag{A.1}$$

$$\| n_g, n_h \ldots ; n_\alpha, n_\beta \ldots ; \mu \gg = \sum_\nu X_{\nu\mu} \, |n_g, n_h \ldots ; n_\alpha, n_\beta \ldots ; \nu \rangle \tag{A.2}$$

$$\ll n_g^{'}, n_h^{'} \ldots ; n_\alpha^{'}, n_\beta^{'} \ldots ; \mu \, \| \, n_g, n_h \ldots ; n_\alpha, n_\beta \ldots ; \nu \gg$$

$$= \quad \delta_{n_g^{'} n_g} \delta_{n_h^{'} n_h} \ldots \delta_{n_\alpha^{'} n_\alpha} \delta_{n_\beta^{'} n_\beta} \ldots \delta_{\mu\nu} \tag{A.3}$$

$$M^{-1} \, = \, X X^\dagger \tag{A.4}$$

$$I \, = \sum_{\substack{n_g, n_h .. \\ n_\alpha, n_\beta ..}} \sum_\mu \| \, n_g, n_h .. ; n_\alpha, n_\beta .. ; \mu \gg \ll n_g, n_h .. ; n_\alpha, n_\beta \ldots ; \mu \, \| \tag{A.5}$$

$$= \sum_{\substack{n_g, n_h \ldots \\ n_\alpha, n_\beta \ldots}} \sum_{\lambda,\nu} |n_g, n_h \ldots ; n_\alpha, n_\beta \ldots ; \nu \rangle \, (M^{-1})_{\nu\lambda} \, \langle n_g \ldots ; n_\alpha \ldots ; \lambda | \tag{A.6}$$

Projection Operators :

$$P(n_g, n_h \ldots ; n_\alpha, n_\beta \ldots)$$

$$= \sum_\mu \| \, n_g, n_h \ldots ; n_\alpha, n_\beta \ldots ; \mu \gg \ll n_g, n_h \ldots ; n_\alpha, n_\beta \ldots ; \mu \, \| \tag{A.7}$$

$$= \sum_{\lambda,\nu} |n_g, n_h \ldots ; n_\alpha, n_\beta \ldots ; \nu \rangle \, (M^{-1})_{\nu\lambda} \, \langle n_g \ldots ; n_\alpha \ldots ; \lambda | \tag{A.8}$$

$$I = \sum_{\substack{n_g, n_h \ldots \\ n_\alpha, n_\beta \ldots}} P(n_g, n_h \ldots; n_\alpha, n_\beta \ldots) \tag{A.9}$$

$$\begin{aligned} P(n_g, n_h \ldots; n_\alpha, n_\beta \ldots) \ \| \ n_g^{'}, n_h^{'} \ldots; n_\alpha^{'}, n_\beta^{'} \ldots; \mu \gg \\ = \quad \delta_{n_g n_g'} \ldots \delta_{n_\alpha n_\alpha'} \ldots \| \ n_g \ldots; n_\alpha \ldots; \mu \gg \end{aligned} \tag{A.10}$$

$$\begin{aligned} P(n_g \ldots; n_\alpha \ldots) \, |n_g^{'} \ldots; n_\alpha^{'} \ldots; \mu\rangle \\ = \quad \delta_{n_g n_g'} \ldots \delta_{n_\alpha n_\alpha'} \ldots |n_g \ldots; n_\alpha \ldots; \mu\rangle \end{aligned} \tag{A.11}$$

Number operators

$$N_k = \sum_{\substack{n_g \ldots n_k \ldots \\ n_\alpha \ldots}} n_k \, P(n_g \ldots n_k \ldots; n_\alpha \ldots) \tag{A.12}$$

$$N_\beta = \sum_{\substack{n_g \ldots \\ n_\alpha, n_\beta \ldots}} n_\beta \, P(n_g \ldots; n_\alpha, n_\beta \ldots) \tag{A.13}$$

$$\begin{aligned} N_k \ \| \ n_g \ldots n_k \ldots; n_\alpha, n_\beta \ldots; \mu \gg \\ = \quad n_k \ \| \ n_g \ldots n_k \ldots; n_\alpha, n_\beta \ldots; \mu \gg \end{aligned} \tag{A.14}$$

$$N_k |n_g \ldots n_k \ldots; n_\alpha, n_\beta \ldots; \mu >= n_k |n_g \ldots n_k \ldots; n_\alpha, n_\beta \ldots; \mu\rangle \tag{A.15}$$

$$\begin{aligned} N_\beta \ \| \ n_g \ldots n_k \ldots; n_\alpha, n_\beta \ldots; \mu \gg \\ = \quad n_\beta \ \| \ n_g \ldots n_k \ldots; n_\alpha, n_\beta \ldots; \mu \gg \end{aligned} \tag{A.16}$$

$$N_\beta |n_g \ldots n_k \ldots; n_\alpha, n_\beta \ldots; \mu\rangle = n_\beta |n_g \ldots n_k \ldots; n_\alpha, n_\beta \ldots; \mu\rangle \tag{A.17}$$

$$\begin{aligned} [N_k, N_j] \ = \ [N_\alpha, N_\beta] \ = \ [N_k, N_\alpha] \\ = \quad 0 \quad \text{for any } \ k, j \ \text{ and any } \ \alpha, \beta \end{aligned} \tag{A.18}$$

Total number operator is

$$N = \sum_k N_k = \sum_\alpha N_\alpha \qquad (A.19)$$

<u>Creation and destruction operators</u> :

$$c_{j\beta}^\dagger = \sum_{\substack{n_g, n_j \cdots \\ n_\alpha, n_\beta \cdots}} \sum_{\mu' \nu} A_{\mu' \nu} \, |n_g \ldots (n_j + 1) \ldots ; n_\alpha, n_\beta + 1 \ldots ; \mu' \rangle$$

$$\otimes \langle n_g \ldots n_j \ldots ; n_\alpha, n_\beta \ldots \nu | \qquad (A.20)$$

$$[c_{j\beta}^\dagger, N_k] = -c_{j\beta}^\dagger \delta_{jk} \qquad (A.21)$$

$$[c_{j\beta}^\dagger, N_\alpha] = -c_{j\beta}^\dagger \delta_{\alpha\beta} \qquad (A.22)$$

For some particular μ,

$$|3_g, 2_h; 4_\alpha, 1_\beta; \mu\rangle = |1_{g\alpha} 1_{g\alpha} 1_{g\beta} 1_{h\alpha} 1_{h\alpha}\rangle = (c_{g\alpha}^\dagger)^2 \, c_{g\beta}^\dagger \, (c_{h\alpha}^\dagger)^2 \, |0\rangle \qquad (A.23)$$

$$c_{j\beta}^\dagger \, |n_g \ldots \; n_{j \ldots}; n_\alpha, n_\beta \ldots ; \lambda\rangle = |\, \underbrace{1_{j\beta,}; n_g \cdots n_j \cdots ; n_\alpha, n_\beta \cdots ;}_{n_g \cdots n_j + 1 \cdots ; n_\alpha, n_\beta + 1} \lambda \rangle \qquad (A.24)$$

$$c_{j\sigma}^\dagger \, |n_g \ldots n_j \ldots ; n_\alpha \ldots n_\sigma \ldots ; \lambda\rangle$$

$$= \sum_{\substack{n_g' \cdots n_j' \cdots \\ n_\alpha' \cdots n_\sigma' \cdots}} \sum_{\mu' \nu} A_{\mu' \nu} \, |n_g' \ldots (n_j' + 1) \ldots ; n_\alpha' \ldots n_\sigma' + 1 \ldots ; \mu' \rangle \qquad (A.25)$$

$$\otimes \langle n_g' \ldots n_j' \ldots ; n_\alpha' \ldots n_\sigma' \ldots ; \nu | n_g \ldots n_j \ldots ; n_\alpha \ldots n_\sigma \ldots ; \lambda \rangle$$

$$= \sum_{\mu' \nu} A_{\mu' \nu} M_{\nu\lambda} \, |n_g \ldots (n_j + 1) \ldots ; n_\alpha \ldots n_\sigma + 1 \ldots ; \mu' \rangle \qquad (A.26)$$

$$\sum_\nu A_{\mu' \nu} M_{\nu\lambda} = \delta_{\mu' \lambda} \qquad (A.27)$$

$$A = M^{-1} \qquad (A.28)$$

$$c_{j\sigma}^{\dagger} = \sum_{\substack{n_g\ldots n_j\ldots \\ n_\alpha\ldots n_\sigma\ldots}} \sum_{\lambda\nu} (M^{-1})_{\lambda\nu} |1_{j\sigma}; n_g \ldots n_j \ldots; n_\alpha \ldots n_\sigma \ldots; \lambda\rangle$$
$$\otimes \langle n_g \ldots n_j \ldots; n_\alpha \ldots n_\sigma \ldots; \nu| \qquad (A.29)$$

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