On the Layering Transition of an SOS Surface Interacting with a Wall. II. The Glauber Dynamics

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Abstract. We continue our study of the statistical mechanics of a 2D surface above a fixed wall and attracted towards it by means of a very weak positive magnetic field h in the solid on solid (SOS) approximation, when the inverse temperature β is very large. In particular we consider a Glauber dynamics for the above model and study the rate of approach to equilibrium in a large cube with arbitrary boundary conditions. Using the results proved in the first paper of this series we show that for all $h \in (h^*_{k+1}, h^*_k)$ ($\{h^*_k\}$ being the critical values of the magnetic field found in the previous paper) the gap in the spectrum of the generator of the dynamics is bounded away from zero uniformly in the size of the box and in the boundary conditions. On the contrary, for $h = h^*_k$ and free boundary conditions, we show that the gap in a cube of side L is bounded from above and from below by a negative exponential of L. Our results provide a strong indication that, contrarily to what happens in two dimensions, for the three dimensional dynamical Ising model in a finite cube at low temperature and very small positive external field, with boundary conditions that are opposite to the field on one face of the cube and are absent (free) on the remaining faces, the rate of exponential convergence to equilibrium, which is positive in infinite volume, may go to zero exponentially fast in the side of the cube.

Key Words: SOS model, Layering transition, Pirogov-Sinai theory, Relaxation time

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0. Introduction

This paper is the second part of a work, begun in [CM], about the equilibrium and non equilibrium statistical mechanics of a SOS surface above a fixed wall at low temperature and attracted towards it by a very weak external field. The equilibrium distribution of the model in a finite volume $V \subset \mathbb{Z}^2$ with boundary conditions $\{\psi(y)\}_{y\in\mathbb{Z}^2\setminus V}$ is described by the following Gibbs measure:

$$\mu_V^\psi(\varphi) = \frac{1}{Z^\psi(V)} \exp\Big[-\frac{\beta}{2} \sum_{\substack{x,y\in V;\\ |x-y|=1}} |\varphi(x)-\varphi(y)| \; - \; \beta h \sum_{x\in V} \varphi(x) \; - \sum_{\substack{x\in V\ y\in V^c\\ |x-y|=1}} |\varphi(x)-\psi(y)|\Big] \tag{0.1}$$

where $Z^\psi(V)$ is the associated partition function and the random variable $\varphi(x) \in \mathbb{Z}_+$ represents the height of the surface at $x \in V$ above the wall.

A kinetic version of the model is readily obtained by considering a Glauber dynamics for it, namely a single spin Markov process on the configuration space, reversible with respect to $\mu_V^\psi(\varphi)$ and such that each move consists in replacing at some site x, $\varphi(x)$ with $\varphi(x) \pm 1$.

We refer the reader to the introduction of [CM] for a detailed discussion of the main motivation which led us to consider such a model and for a brief description of the main results both for the equilibrium and the non equilibrium case. Here we just recall for the reader's convenience the main result of [CM], since it plays an essential role in our analysis of the non equilibrium case.

Theorem [CM]. *There exists β_0 such that for all $\beta \geq \beta_0$ there are positive numbers $\{h_k^*(\beta)\}_{k=1}^{k_{max}}$, with $k_{max} = \lfloor e^{\frac{\beta}{20000}} \rfloor$, such that the following holds for $k = 1, \ldots, k_{max}$*

- *(i)* $\frac{1}{4} e^{-4\beta k} \leq \beta h_k^*(\beta) \leq 4e^{-4\beta k}$
- *(ii) if $h_k^*(\beta) < h < h_{k-1}^*(\beta)$ (define $h_0^*(\beta) = +\infty$) then*
  - *(a) there exists a unique Gibbs measure for the interaction (0.1)*
  - *(b) there exist $m(\beta,h) > 0$, $C(\beta,h) > 0$ such that for any $N \geq \lfloor 8/h + 1 \rfloor$*

$$\sup_{\psi,\psi'\in\Omega} |E_{Q_N}^{h,\psi}\varphi(0) - E_{Q_N}^{h,\psi'}\varphi(0)| \leq C(\beta,h) e^{-m(\beta,h)\,N}$$

  where $E_{Q_N}^{h,\psi}(\varphi(0))$ denotes the expected value of the height of the surface at $x = 0$ in a square Q_N of side N and center at the origin, with boundary conditions ψ.
- *(iii) if $h = h_k^*(\beta)$ then both partitions functions $Z^k(Q_N)$ and $Z^{k+1}(Q_N)$, with boundary conditions $\psi \equiv k$ and $\psi \equiv k+1$ respectively, admit a convergent cluster expansion. Hence there are at least two distinct extreme Gibbs measures.*

In the present paper we turn our attention to the Glauber dynamics of the SOS surface in a square Q_N of side N, with boundary conditions ψ and magnetic field h; more precisely we analyze the $\text{gap}^{h,\psi}(Q_N)$ in the spectrum of its generator which is related to the relaxation time to equilibrium.

Our main result is

Theorem. *In the same setting as in Theorem [CM] we have, for all $k = 1, \ldots, k_{max}$,*

(i) if $h_k^*(\beta) < h < h_{k-1}^*(\beta)$ then there exist $L_0(\beta,h)$, $\kappa(\beta,h) > 0$ such that

$$\inf_{L \geq L_0} \inf_{\psi \in \Omega} \text{gap}^{h,\psi}(Q_L) \geq \kappa(\beta,h)$$

(ii) if $h = h_k^*(\beta)$, then there exist positive constants $C_1(\beta,h)$, $C_2(\beta,h)$ such that for all $N > 10/h$

$$C_1(\beta,h)\, e^{-100\beta k N} \leq \text{gap}^{h,\emptyset}(Q_N) \leq C_2(\beta,h)\, e^{-\frac{1}{40}\beta N}$$

where $\emptyset$ means free boundary conditions.

The proof of part (i), discussed in Section 2, is organized as follows. Using the methods of [MOS] and some apriori bounds on the moments of the random variable $\varphi(x)$ proved in [CM], one easily improve part (i) of Theorem [CM], by showing that a stronger form of weak dependence on the boundary conditions (strong mixing in the language of [MO1], [MO2], [MOS]) holds. Once strong mixing is established, a lower bound of the gap in any large enough cube, uniformly in the volume and in the boundary conditions, follows from the "block dynamics approach" envisaged in [MO1]. This method requires, however, as an input, a lower bound on the gap in a given fixed block. Such a bound would be trivial were the random variables $\varphi(x)$ be bounded. In our case the problem requires some extra work and our solution is based upon the so called "Cheeger inequality" (see [LS]).

The proof of the upper bound of part (ii) is discussed in Section 3 and it uses the variationl characterization of the gap. More precisely, using a suitable test function f, we show the existence of a bottleneck: the system, in order to relax to equilibrium, has to make an excursion to a region of the configuration space of very small equilibrium measure. Finally Section 4 contains the proof of the lower bound; our method relies upon a novel recursive estimate of the gap that allows us to overcome the problem of the unboundedness of the variables $\varphi(x)$.

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1. Preliminaries

For the reader's convenience we recall most of the basic definitions that were given in [CM].

1.1. General definitions

We consider the two dimensional lattice $\mathbb{Z}^2$ whose elements are called *sites* and its dual $\mathbb{Z}^2_* = \mathbb{Z}^2 + (1/2, 1/2)$. For $x, y \in \mathbb{R}^2$ we define two distances

$$d(x,y) = |x-y| = \sum_{i=1}^{2} |x_i - y_i| \quad \text{and} \quad d_\infty(x,y) = |x-y|_\infty = \max_{i=1,2} |x_i - y_i| \,.$$

$[x,y]$ is the *closed segment* with x, y as its endpoints. The *edges* of $\mathbb{Z}^2$ ($\mathbb{Z}^2_*$) are those $e = [x,y]$ with x, y nearest neighbors in $\mathbb{Z}^2$ ($\mathbb{Z}^2_*$). Given e edge of $\mathbb{Z}^2$, e^* is the unique edge in $\mathbb{Z}^2_*$ that intersects e. The *boundary of an edge* $e = [x,y]$ is $\delta e = \{x, y\}$ The *boundary of a subset of edges* α is the set of sites $\delta\alpha$ that belong to an odd number of edges of α. A set of edges is called *closed* if its boundary is empty.

We will often consider our model on a square

$$Q_N = \begin{cases} \{\, (x_1, x_2) \in \mathbb{Z}^2 : -L \le x_i \le L,\ i = 1,2 \,\} & \text{if } N = 2L+1 \\ \{\, (x_1, x_2) \in \mathbb{Z}^2 : -L+1 \le x_i \le L,\ i = 1,2 \,\} & \text{if } N = 2L \end{cases}$$

Λ and V will denote arbitrary subsets of $\mathbb{Z}^2$. If Λ is finite we write $\Lambda \subset\subset \mathbb{Z}^2$. The cardinality of Λ is denoted by $|\Lambda|$. We define four kinds of *boundaries*

$$\begin{aligned} \partial\Lambda &= \{x \in \Lambda : d(x, \Lambda^c) = 1\} \\ \bar\partial\Lambda &= \{x \in \Lambda : d_\infty(x, \Lambda^c) = 1\} \\ \partial^+\Lambda &= \{x \in \Lambda^c : d(x, \Lambda) = 1\} \\ \delta\Lambda &= \{e^* = [x,y]^* : \{x,y\} \cap \Lambda \ne \emptyset,\ \{x,y\} \cap \Lambda^c \ne \emptyset\} \end{aligned}$$

where $\Lambda^c = \mathbb{Z}^2 \setminus \Lambda$.

$(x_1, \dots, x_n)$ is called a *path* from x_1 to x_n if $|x_{i+1} - x_i| = 1$ for $i = 1, \dots, n-1$. A $*$–path is the same as a path with $|x_{i+1} - x_i| = 1$ replaced by $d_\infty(x_i, x_{i+1}) = 1$. A ($*$–) path is called *self-avoiding* if $x_i \ne x_j$ for all $\{i,j\}$ such that $i \ne j$ and $\{i,j\} \ne \{1,n\}$. If $x_1 = x_n$ the ($*$–) path is called *closed.*

$\Lambda \subset \mathbb{Z}^2$ is said to be *connected* ($*$–connected) if for all x, y in Λ there exists a path ($*$–path) from x to y which is entirely contained in Λ. $\Lambda \subset\subset \mathbb{Z}^2$ is called *simply connected* if Λ^c is $*$–connected. A set of edges α is *connected* if the union of all its edges is connected in $\mathbb{R}^2$.

We denote by C_B the set of all finite closed connected set of edges of $\mathbb{Z}^2_*$. If $\alpha \in C_B$ then we define the *interior* of α (see Fig. 1) as the set of all sites $x = (x_1, x_2) \in \mathbb{Z}^2$ such that the half line

$$\{x_1\} \times [x_2, +\infty)$$

intersects α in an odd number of points. The interior of α is denoted by $\bar{\alpha}$ and is always a (possibly disconnected) simply connected subset of $\mathbb{Z}^2$ for each $\alpha \in C_B$. $C_B(V)$ is the set of all α in C_B such that $\bar{\alpha} \subset V$.

1.2 The SOS model

The *configuration space* of the model is $\Omega = \mathbb{Z}_+^{\mathbb{Z}^2}$, or $\Omega_V = \mathbb{Z}_+^V$ for some $V \subset \mathbb{Z}^2$. An element of Ω_V will usually be denoted by $\varphi = \{\varphi(x), x \in V\}$. If $U \subset V \subset \mathbb{Z}^2$, and $\varphi \in \Omega_V$ we denote by φ_U the restriction of φ to the set U.

Given $V \subset\subset \mathbb{Z}^2$ and some *boundary condition* (b.c.) $\psi \in \Omega$, one defines the hamiltonian as

$$H_V^{J,h,\psi}(\varphi) = \frac{1}{2} \sum_{\substack{x,y \in V \\ |x-y|=1}} J(x,y)\, |\varphi(x) - \varphi(y)| + \sum_{\substack{x \in V,\, y \in V^c \\ |x-y|=1}} J(x,y)\, |\varphi(x) - \psi(y)| + h \sum_{x \in V} \varphi(x) \tag{1.1}$$

and we always assume $0 \le J(x,y) \le 1$ for all x, y. We write $J \in \delta V$ if $J(x,y) < 1$ only for the boundary terms, *i.e.* if $J(x,y) = 1$ unless $[x,y]^* \in \delta V$. If we take $J(x,y) = 0$ for all boundary terms, then we have *free boundary conditions* that we also denote with

$$H_V^{h,\emptyset}(\varphi) = \frac{1}{2} \sum_{\substack{x,y \in V \\ |x-y|=1}} |\varphi(x) - \varphi(y)| + h \sum_{x \in V} \varphi(x)\,. \tag{1.2}$$

The partition function is given by

$$Z^{J,h,\psi}(V) = \sum_{\varphi \in \Omega_V} e^{-\beta H_V^{J,h,\psi}(\varphi)}\,. \tag{1.3}$$

Given any set α of dual edges (for instance $\alpha = \delta V$) we define

$$|\alpha|_J = \sum\nolimits_{e^* = [x,y]^* \in \alpha} J(x,y)\,. \tag{1.4}$$

(if $J = 1$ everywhere this is just the ordinary *length* of α).

As in [CM] we often assume that J satisfies the hypothesis $H(\Lambda, t)$ for some $\Lambda \subset\subset \mathbb{Z}^2$ and $t \geq 0$, namely that

(i) $J \in \delta\Lambda$

(ii) $|\alpha|_J \geq t|\alpha|$ for all $\alpha \in C_B(\Lambda)$

We also define

$$\Delta(J) = \{x \in \mathbb{Z}^2 : J(x,y) \neq 1 \text{ for some } y \in \mathbb{Z}^2\} \tag{1.5}$$

If $H(\Lambda, t)$ holds for J then, clearly, for each $V \subset \Lambda$,

$$|\Delta(J) \cap V| \leq |\partial V| \leq |\delta V|\,.$$

When $J(x,y) = 1$ for all x, y, we drop the superscript J.

For $U \subset \mathbb{Z}^2$, let $\mathbf{F}_U$ be the σ–algebra generated by the collection of sets

$$\{\varphi \in \Omega : \varphi(x) = n\}_{x \in U, n \in \mathbb{Z}_+}$$

and let $\mathbf{F} = \mathbf{F}_{\mathbb{Z}^2}$. The (finite volume) conditional Gibbs measure on $(\Omega, \mathbf{F})$ associated with the Hamiltonian (1.1) is defined as

$$\mu_V^{J,h,\psi}(\varphi) = \begin{cases} \left(Z^{J,h,\psi}(V)\right)^{-1} e^{-\beta H_V^{J,h,\psi}(\varphi)} & \text{if } \varphi(x) = \psi(x) \text{ for all } x \in V^c \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

We also regard $\mu_V^{J,h,\psi}$ as a measure on Ω_V by extending each configuration $\varphi \in \Omega_V$ to the whole space in such a way that it agrees with the boundary conditions outside V. The expectation with respect to the measure (1.6) is denoted by $\mathbf{E}_V^{J,h,\psi}(\cdot)$. The set of probability measures (1.6) satisfies the compatibility conditions

$$\mu_\Lambda^{J,h,\psi}(\varphi) = \sum_{\varphi' \in \Omega} \mu_\Lambda^{J,h,\psi}(\varphi')\, \mu_V^{J,h,\varphi'}(\varphi) \qquad \text{for all } V \subset \Lambda \subset\subset \mathbb{Z}^2 \tag{1.7}$$

We refer to Section 1.2 of [CM] for the definition of (infinite volume) Gibbs measure, the definition of increasing (decreasing) functions, and the statement of the FKG properties.

1.3. The dynamics and our results

In the rest of this section, for readability purposes, we assume to have chosen β, h and J, so we don't write them explicitly if no confusion arises.

The stochastic dynamics we want to study is defined by the Markov generator

$$(L_V^\psi f)(\varphi) = \sum_{x \in V,\, s = \pm 1} c(x, \varphi, s)\, [f(\varphi^{x,s}) - f(\varphi)] \tag{1.8}$$

acting on $L^2(\Omega, d\mu_V^\psi)$, where

$$\varphi^{x,s}(y) = \begin{cases} \varphi(y) & \text{if } y \neq x \\ \varphi(y) + s & \text{if } y = x. \end{cases}$$

In (1.8) φ denotes a configuration on the whole lattice $\mathbb{Z}^2$ which, in view of (1.6), agrees with the b.c. ψ on V^c. In general we identify $L^2(\Omega, d\mu_V^\psi)$ with $L^2(\Omega_V, d\mu_V^\psi)$.

The nonnegative real quantities

$$c(x, \varphi, s) \quad x \in \mathbb{Z}^2, \quad \varphi \in \Omega, \quad s = \pm 1$$

are the *transition rates* for the process. We set $c(x, \varphi, s) = 0$ if $\varphi^{x,s} \notin \Omega$, that is if $\varphi(x) = 1$ and $s = -1$. The assumptions on the transition rates are

(H_1) *Tranlation invariance.* If there exists y such that $\widetilde{\varphi}(x) = \varphi(x + y)$ for all $x \in \mathbb{Z}^2$, then $c(x + y, \varphi, s) = c(x, \widetilde{\varphi}, s)$ for all $x \in \mathbb{Z}^2$, $s = \pm 1$

(H_2) *Nearest neighbor interactions* If $\varphi(y) = \widetilde{\varphi}(y)$ for all y such that $d(x, y) \leq 1$, then $c(x, \varphi, s) = c(x, \widetilde{\varphi}, s)$

(H_3) *Attractivity.* If $\varphi \leq \widetilde{\varphi}$ and $\varphi(x) = \widetilde{\varphi}(x)$, then

$$\begin{aligned} c(x, \varphi, +) &\leq c(x, \widetilde{\varphi}, +) \\ c(x, \varphi, -) &\geq c(x, \widetilde{\varphi}, -) \end{aligned} \tag{1.9}$$

(H_4) *Detailed balance.* For all x, φ, s such that $\varphi^{x,s} \in \Omega$

$$\exp\left[-\beta H^{\varphi}_{\{x\}}(\varphi(x))\right] c(x, \varphi, s) = \exp\left[-\beta H^{\varphi}_{\{x\}}(\varphi(x) + s)\right] c(x, \varphi^{x,s}, -s), \tag{1.10}$$

(H_5) *Positivity and boundedness.* There exist $c_m(\beta, h) > 0$ and $c_M(\beta, h) < \infty$ such that,

$$\begin{aligned} &\inf_{\varphi \in \Omega,\, s = \pm 1 :\, \varphi^{x,s} \in \Omega} c(x, \varphi, s) \geq c_m(\beta, h) \\ &\sup_{\varphi \in \Omega} \sum_{s = \pm 1} c(x, \varphi, s) \leq c_M(\beta, h) < \infty \end{aligned} \tag{1.11}$$

Two cases one may want to keep in mind are (χ is the characteristic function)

$$c(x, \varphi, s) = \min\bigl\{e^{-\beta \Delta_{x,s} H(\varphi)}\,,\, 1\bigr\}\, \chi\{\varphi^{x,s} \in \Omega\}$$

and

$$c(x, \varphi, s) = \bigl[1 + \exp\bigl(\beta \Delta_{x,s} H(\varphi)\bigr)\bigr]^{-1} \chi\{\varphi^{x,s} \in \Omega\}$$

where

$$\Delta_{x,s} H(\varphi) = H^{\varphi}_{\{x\}}(\varphi(x)+s) - H^{\varphi}_{\{x\}}(\varphi(x)) \, .$$

$(H_1)-(H_5)$ garantee that there exists a unique Markov process with semigroup $T^{\psi}_{V}(t)$ and generator L^{ψ}_{V}. L^{ψ}_{V} is a bounded operator on $L^2(\Omega, d\mu^{\psi}_{V})$. The process has a unique invariant measure given by μ^{ψ}_{V}. Moreover μ^{ψ}_{V} is *reversible* with respect to the process, *i.e.* L^{ψ}_{V} is self–adjoint on $L^2(\Omega, d\mu^{\psi}_{V})$. Given $\varphi \in \Omega$ we denote by φ_t the random configuration at time t evolving according to the process, so that

$$\mathbb{E}^{\varphi}_{V} \, f(\varphi_t) = \int f(\varphi_t) \, d\,\mathbb{P}^{\varphi}_{V} = (T^{\psi}_{V}(t)) \, f(\varphi) \, , \quad \forall \varphi \in \Omega \text{ such that } \varphi_{V^c} = \psi_{V^c} \, .$$

$\mathbb{E}^{\varphi}$ and $\mathbb{P}^{\varphi}$ stand respectively for the expectation and the probability measure associated with the process starting from φ_V at time zero and subject to b.c. φ_{V^c}.

Attractivity assumption implies (see for instance [L])

(1) If f is an increasing function on Ω_V then $T^{\psi}_{V}(t) f$ is also increasing for all $t \geq 0$
(2) If ρ_1, ρ_2 are two probability measures on $(\Omega_V, \mathbf{F}_V)$ such that $\rho_1 \leq \rho_2$ then $\rho_1 T^{\psi}_{V}(t) \leq \rho_2 T^{\psi}_{V}(t)$ for all $t \geq 0$
(3) For any φ, φ' in Ω such that $\varphi \leq \varphi'$, the *standard coupling* [L] $\mathbb{P}^{\varphi,\varphi'}_{V}$ of φ_t, φ'_t is such that $\mathbb{P}^{\varphi,\varphi'}_{V}\{\varphi_t \leq \varphi'_t\} = 1$, for all $t \geq 0$.

This last property allows us to define a *standard* coupling of two Gibbs measures which preserve the order of the b.c. Take in fact $\nu^{\psi,\psi'}_{V}$ as the unique invariant measure of the (standard) coupled process (φ_t, φ'_t). Then we have

(1) $\nu^{\psi,\psi'}_{V}\{(\varphi,\varphi') : \varphi = \varphi_0\} = \mu^{\psi}_{V}(\varphi_0)$ for all $\varphi_0 \in \Omega_V$
(2) $\nu^{\psi,\psi'}_{V}\{(\varphi,\varphi') : \varphi' = \varphi_0\} = \mu^{\psi'}_{V}(\varphi_0)$ for all $\varphi_0 \in \Omega_V$
(3) If $\psi \leq \psi'$, then $\nu^{\psi,\psi'}_{V}\{(\varphi,\varphi') : \varphi \leq \varphi'\} = 1$

A fundamental quantity associated with the dynamics of a reversible system is the gap of the generator, *i.e.*

$$\mathrm{gap}^{\psi}(V) = \mathrm{gap}(L^{\psi}_{V}) = \inf \mathrm{spec} \, (-L^{\psi}_{V} \restriction \mathbb{1}^{\perp})$$

where $\mathbb{1}^{\perp}$ is the subspace of $L^2(\Omega, d\mu^{\psi}_{V})$ orthogonal to the constant functions. The gap can be also characterized as

$$\mathrm{gap}^{\psi}(V) = \inf_{f \in L^2(\Omega, d\mu^{\psi}_{V})} \frac{\mathcal{E}^{\psi}_{V}(f,f)}{\mathrm{Var}^{\psi}_{V}(f)} \tag{1.12}$$

where $\mathcal{E}$ is the Dirichlet form associated with the generator L,

$$\mathcal{E}^{\psi}_{V}(f,f) = \frac{1}{2} \sum_{\varphi \in \Omega} \sum_{x \in V, \, s = \pm 1} \mu^{\psi}_{V}(\varphi) \, c(x,\varphi,s) \, [f(\varphi^{x,s}) - f(\varphi)]^2 \tag{1.13}$$

and $\mathrm{Var}_V^{h,\psi}$ is the variance relative to the probability measure $\mu_V^{h,\psi}$.

Then main result in this paper is

Theorem 1.1. *In the same setting as in Theorem CM of the Introduction, if the transition rates satisfy* (H_1)*–*(H_5)*, then we have for all* $k = 1, \cdots, k_{max}$

(i) if $h_k^*(\beta) < h < h_{k-1}^*(\beta)$ *then there exist* $L_0(\beta, h)$*,* $\kappa(\beta, h) > 0$ *such that*

$$\inf_{L \geq L_0} \inf_{\psi \in \Omega} \mathrm{gap}^{h,\psi}(Q_L) \geq \kappa(\beta, h)$$

(ii) if $h = h_k^*(\beta)$ *there exists a positive constant* $C_1(\beta, h)$ *such that for all* $N > 10/h$

$$C_1(\beta, h)\, e^{-100\beta k N} \leq \mathrm{gap}^{h,\emptyset}(Q_N) \leq c_M(\beta, h) e^{-\frac{1}{40}\beta N}$$

where c_M *is given by (1.11).*

2. Lower bound on the gap for $h \neq h_k^*(\beta)$

In this section we will show that, if $h_k^*(\beta) < h < h_{k-1}^*(\beta)$, then the finite volume Gibbs measures show a weak dependence (in a strong sense) on the boundary conditions.

As a consequence, we obtain a lower bound for the gap of the dynamics in a finite, large enough square Q_N. This lower bound is independent of the boundary conditions and of N. Thus we get part (i) of Theorem 1.1.

We will adapt to our model the ideas and techniques developed in [MO1], [MOS], supplemented by results of Section 3 of [CM] and by Proposition 7.1 of [CM]. As in [CM] we often need a small constant, so we set

$$\zeta = 1000^{-1}\,.$$

2.1. Strong mixing

Following the basic result of [MOS], we strengthen the result of Proposition 7.1 of [CM].

Theorem 2.1 *(Strong Mixing). Let β be large enough and $h \in (h_k^*(\beta), h_{k-1}^*(\beta))$, with $1 \le k \le k_{max} = \lfloor e^{\frac{\beta\zeta}{20}} \rfloor$. Then there exists $L_0(\beta,h)$, $C(\beta,h)$ and $m(\beta,h) > 0$ such that, for any $L \ge L_0$, any $\Delta \subset Q_L$ and any $y \in \partial^+ Q_L$, we have*

$$\sup_{\substack{\psi,\psi' \in \Omega \\ \psi'(x)=\psi(x)\,,\,\forall x \neq y}} \ \sup_{A \in \mathbf{F}_\Delta} |\mu_{Q_L}^{h,\psi}(A) - \mu_{Q_L}^{h,\psi'}(A)| \le C \sum_{x\in\Delta} e^{-m\, d(x,y)}$$

Remark. Following [MO1], we will say that *SM(L,C,m)* holds if the above inequality with the prescribed constants is valid for a square Q_L.

Proof. If the random variables were bounded then the result would simply follow from Proposition 7.1 of [CM], thanks to the main theorem of [MOS] which states that, for discrete bounded spin systems with finite range interaction, weak mixing implies strong mixing in two dimensions.

Thus we have to take care of the unboundedness of the $\varphi(x)$'s . If one examines closely the arguments of [MOS] one finds out that the two basic ingredients (besides the dimensionality and the finite range condition) are the following

- (i) Weak mixing in the form stated in Proposition 7.1 of [CM]
- (ii) let C, m be the "weak mixing" constants appearing in Proposition 7.1 of [CM], and let $l_0(\beta,h)$ be such that

$$C\sum_{j=l_0}^{\infty} e^{-mj} \le \frac{1}{2}\,.$$

Then there exists $\bar N(\beta,h)\ge l_0$ such that

$$\inf_{\psi\in\Omega}\mu^{\psi}_{Q_{\bar N}}\{\,\varphi(x)=\bar\varphi(x)\ \forall x\in Q_{\bar N}\,\}>0\quad\text{for some }\bar\varphi\in\Omega \tag{2.1}$$

While for bounded spins with finite range interaction, (2.1) is a trivial consequence of the boundedness of the interaction, in our case (2.1) follows from part (ii) of Proposition 3.2 of [CM], if one chooses the reference configuration $\bar\varphi$ identically equal to $+1$, and $\bar N=l_0\vee\lfloor 8/h+1\rfloor$. In fact from FKG inequality one gets

$$\inf_{\psi\in\Omega}\mu^{\psi}_{Q_{\bar N}}\{\,\varphi(x)=+1\ \forall\, x\in Q_{\bar N}\,\}\ge\prod_{x\in Q_{\bar N}}\inf_{\psi\in\Omega}\mu^{\psi}_{Q_{\bar N}}\{\varphi(x)=+1\}\ge b_2(\beta,h)^{|Q_{\bar N}|}$$

Thus (i) and (ii) hold also in our case and the theorem follows. $\square$

2.2. Proof of part (i) of Theorem 1.1

We are finally in a position to prove a lower bound for the gap of the dynamics defined in Section 1.3 in any finite, large enough square Q_N, independent of the boundary condition and of N.

We follow very closely Section 4 of [MO1]. Given an even integer $L_0=2K_0$ and a cube Q_N, $N=nL_0$, we consider its covering by squares $Q^i_{L_0}$ such that if two different squares $Q^i_{L_0}$ and $Q^j_{L_0}$ overlap than necessarily each one of them is the translated by K_0 of the other along at least one of the two coordinate axes. Let

$$\mathcal{D}_n=\{Q^i_{L_0}\}$$

and consider the block–dynamics defined in the Appendix, determined by β,h, the b.c. ψ and the collection $\mathcal{D}_n$, through the Markov generator $L^{h,\psi}_{\mathcal{D}_n}$ given by (A1.1). The same proof of Theorem 4.1 of [MO1] shows that, if $SM(L_0,C,m)$ holds and moreover L_0 is so large that

$$CL_0^2\exp(-m\sqrt{L_0})\le\frac{1}{L_0^4} \tag{2.2}$$

then the gap of $L^{h,\psi}_{\mathcal{D}_n}$ is bounded away from zero uniformly in ψ and in $N=nL_0$. Thus if we choose $L_0=L_0(\beta,h)$ large enough so that both Theorem 2.1 and (2.2) are satisfied we get

$$\inf_{\psi\in\Omega}\inf_{n\ge1}\operatorname{gap}(L^{h,\psi}_{\mathcal{D}_n})\ge\kappa_1(\beta,h)>0\,. \tag{2.3}$$

In Proposition 2.2, on the other side, we will show that, considering the usual single site dynamics on each block $Q^i_{L_0}$, if $L_0>\lfloor 8/h+1\rfloor$, we have

$$\inf_{\psi\in\Omega}\operatorname{gap}(L^{h,\psi}_{Q_{L_0}})\ge\kappa_2(\beta,h)>0\,. \tag{2.4}$$

Finally, Proposition A1.1 gives an estimate the for gap of the single site dynamics in the full volume Q_N, in terms of the quantities appearing in (2.3), (2.4). Thus, assuming to have chosen L_0 such that $L_0 \geq \lfloor 8/h + 1 \rfloor$ also holds, we get

$$\inf_{N=nL_0,\, n\in\mathbb{Z}_+} \inf_{\psi\in\Omega} \mathrm{gap}(L^{h,\psi}_{Q_N}) \geq \frac{1}{4}\kappa_1\,\kappa_2 > 0\,.$$

The factor $\frac{1}{4}$ is the "overlapping" factor appearing as the last term in (A1.2).

The extension of the theorem to all $N \geq L_0$ is straightforward after one realizes that Theorem 4.1 of [MO1] is valid for any coverings $\{Q^i_{L_0}\}$ of Q_N such that the following holds: for each site $x \in Q_N$ there is a square $Q^i_{L_0}$

$$d(x, (Q^i_{L_0})^c \setminus (Q_N)^c) \geq \frac{1}{10} L_0\,.$$

In this way we can "cover" any square Q_N with N greater than say $10L_0$. $\square$

We are thus left with the proof of (2.4).

Proposition 2.2. *For each $\beta, h > 0$, $N \geq N_1(h) = \lfloor 8/h + 1 \rfloor$, we have (remember that we always assume $0 \leq J(x,y) \leq 1$)*

$$g_N(\beta,h) \equiv \inf_{J\in\delta Q_N} \inf_{\psi\in\Omega} \mathrm{gap}(L^{J,h,\psi}_{Q_N}) > 0\,.$$

Proof. We recall that $J \in \delta Q_N$ means (see Section 1.2) that $J = 1$ everywhere with the possible exception of the boundary terms. Let then $\Lambda = Q_N$. As we did in Section 3 of [CM] we define

$$\bar{H}^{h,\infty}_V(\varphi) = \frac{1}{2} \sum_{\substack{x,y\in V\\ |x-y|=1}} |\varphi(x) - \varphi(y)| - \sum_{\substack{x\in V,\, y\in V^c\\ |x-y|=1}} \varphi(x) + h \sum_{x\in V} \varphi(x) \tag{2.5}$$

which satisfies

$$(n-1)\,|\delta V|_J + H^{J,h,1}_V(\varphi) \geq H^{J,h,n}_V(\varphi) \geq (n-1)\,|\delta V|_J + \bar{H}^{h,\infty}_V(\varphi) \tag{2.6}$$

For $X \subset \Omega_\Lambda$ we define ∂X as the set of all $\varphi \in X$ such that there exists $\varphi' \in X^c$ with

$$\sum_{x\in\Lambda} |\varphi(x) - \varphi'(x)| = 1.$$

The proof is based on the so called Cheeger inequality given in [LS], which in our case says

$$\operatorname{gap}(L_\Lambda^{J,h,\psi}) \geq \frac{q^2}{8M}$$

where, letting $\mu = \mu_\Lambda^{J,h,\psi}$,

$$\begin{aligned} q = & \inf_{\substack{X \subset \Omega_\Lambda \\ 0<\mu(X)\leq \frac{1}{2}}} \frac{1}{\mu(X)\,\mu(X^c)} \sum_{\varphi\in X} \mu(\varphi) \sum_{(x,s):\,\varphi^{x,s}\in X^c} c(x,\varphi,s) \geq \\ \geq & \inf_{\substack{X \subset \Omega_\Lambda \\ 0<\mu(X)\leq \frac{1}{2}}} c_m(\beta,h)\frac{\mu(\partial X)}{\mu(X)}\,. \end{aligned}$$

and

$$M = \sup_{\varphi\in\Omega_\Lambda} \sum_{x\in\Lambda}\sum_{s=\pm 1} c(x,\varphi,s) \leq |\Lambda| c_M(\beta,h)$$

Thanks the boundedness of the transition rates (1.11), the proposition is thus proven if we can show that

$$\inf_{J\in\delta V}\inf_{\psi\in\Omega} \inf_{\substack{X\subset\Omega_\Lambda \\ 0<\mu_\Lambda^{J,h,\psi}(X)\leq\frac{1}{2}}} \frac{\mu_\Lambda^{J,h,\psi}(\partial X)}{\mu_\Lambda^{J,h,\psi}(X)} > 0\,. \tag{2.7}$$

We denote by φ_1 the configuration on Λ which is equal to 1 everywhere, and we consider two cases depending on whether φ_1 belongs to X or not.

- $\varphi_1 \notin X$

For each $\varphi \in X$ we set

$$\begin{aligned} & m(\varphi) = \inf\{i : \varphi\wedge i \in X\} \\ & \varphi' = \varphi\wedge m(\varphi) \qquad \text{and} \qquad \varphi'' = \varphi\wedge(m(\varphi)-1) \end{aligned}$$

So that $\varphi' \in X$ and $\varphi'' \in X^c$. If we define $T\varphi = \varphi'$, then we claim that

$$\frac{\mu_\Lambda^{J,h,\psi}(X)}{\mu_\Lambda^{J,h,\psi}(TX)} \leq \sum_{\varphi\in\widetilde{\Omega}_\Lambda} e^{-\beta\bar{H}_\Lambda^{h,\infty}(\varphi)} \quad \forall\psi\in\Omega, \tag{2.8}$$

where $\widetilde{\Omega}_\Lambda = \{0,1,2,\ldots\}^\Lambda$. To prove (2.8) one has to observe that, since $\varphi(x)\geq\varphi(y)$ implies $\varphi'(x)\geq\varphi'(y)$, then

$$\begin{aligned} & |\varphi(x)-\varphi(y)| = |\varphi'(x)-\varphi'(y)| + |(\varphi-\varphi')(x)-(\varphi-\varphi')(y)| \\ & J(x,y)|\varphi(x)-\psi(y)| \geq -(\varphi-\varphi')(x) + J(x,y)|\varphi'(x)-\psi(y)|\,. \end{aligned}$$

This implies

$$H_\Lambda^{J,h,\psi}(\varphi) \geq H_\Lambda^{J,h,\psi}(\varphi') + \bar{H}_\Lambda^{h,\infty}(\varphi - \varphi')\,. \tag{2.9}$$

So, if we let $Z = Z^{J,h,\psi}(\Lambda)$, we have

$$\begin{aligned}
\mu_\Lambda^{J,h,\psi}(X) &= Z^{-1} \sum_{\varphi \in \Omega_\Lambda} e^{-\beta H_\Lambda^{J,h,\psi}(\varphi)} = Z^{-1} \sum_{\widetilde{\varphi} \in TX} \sum_{\varphi \in \Omega_\Lambda :\, T\varphi = \widetilde{\varphi}} e^{-\beta H_\Lambda^{J,h,\psi}(\varphi)} \leq \\
&\leq Z^{-1} \sum_{\widetilde{\varphi} \in TX} e^{-\beta H_\Lambda^{J,h,\psi}(\widetilde{\varphi})} \sum_{\varphi \in \Omega_\Lambda :\, T\varphi = \widetilde{\varphi}} e^{-\beta \bar{H}_\Lambda^{h,\infty}(\varphi - \widetilde{\varphi})} \leq \\
&\leq \mu_\Lambda^{J,h,\psi}(TX) \sum_{\varphi \in \widetilde{\Omega}_\Lambda} e^{-\beta \bar{H}_\Lambda^{h,\infty}(\varphi)}.
\end{aligned} \tag{2.10}$$

To complete the proof we are going to use the following facts

(a) For each X, for each $\widetilde{\varphi} \in TX$, there exists $\bar{\varphi} \in \partial X$ such that $\widetilde{\varphi} - 1 \leq \bar{\varphi} \leq \widetilde{\varphi}$, and

(b) $|H_\Lambda^{J,h,\psi}(\widetilde{\varphi}) - H_\Lambda^{J,h,\psi}(\bar{\varphi})| \leq 5N^2$

Statement (a) is a trivial consequence of the definition of $T\varphi$, while (b) follows easily from (a). Given (a) and (b), we can proceed as follows

$$\begin{aligned}
\mu_\Lambda^{J,h,\psi}(TX) &\leq Z^{-1} e^{5\beta N^2} \sum_{\widetilde{\varphi} \in TX} e^{-\beta H_\Lambda^{J,h,\psi}(\bar{\varphi})} = \\
&= Z^{-1} e^{5\beta N^2} \sum_{\bar{\varphi} \in \partial X} \#\{\widetilde{\varphi} \in TX : \widetilde{\varphi} \to \bar{\varphi}\}\, e^{-\beta H_\Lambda^{J,h,\psi}(\bar{\varphi})} \leq e^{5\beta N^2} 2^{N^2} \mu_\Lambda^{J,h,\psi}(\partial X)
\end{aligned} \tag{2.11}$$

The proof of (2.7) when $\varphi_1 \notin X$, follows then from (2.10), (2.11) and Proposition 3.1 of [CM] (the fact that Ω_Λ should be replaced by $\widetilde{\Omega}_\Lambda$ is not essential).

- $\varphi_1 \in X$.

The key ingredient is the following observation

(c) there exists $s(\beta, h, N)$ such that for all $X \in \Omega_\Lambda$ with $\mu_\Lambda^{J,h,\psi}(X) \leq 1/2$ for some $\psi \in \Omega$, there exists $\varphi \in X^c$ with $\sup_x \varphi(x) \leq s(\beta, h, N)$.

Proof of (c). By Proposition 3.2 of [CM] and Chebyshev inequality, we have

$$\mu_\Lambda^{J,h,\psi}\{\sup_x \varphi(x) > r\} \leq \sum_{x \in \Lambda} \mu_\Lambda^{J,h,\psi}\{\varphi(x) > r\} \leq \frac{1}{r}\, |\Lambda|\, b_1(\beta, h, 1).$$

So, by taking $s(\beta, h, N) = [4\,|\Lambda|\, b_1(\beta, h, 1)]^{-1}$, we get

$$\mu_\Lambda^{J,h,\psi}\{\sup_x \varphi(x) \leq s(\beta, h, N)\} \geq \frac{3}{4}$$

which implies (c).

Choose now some $\widetilde{\varphi} \in X^c$ such that $\sup_x \widetilde{\varphi}(x) \le s$. Since $\varphi_1 \in X$, there exists $\bar{\varphi} \in \partial X$ such that $\varphi_1 \le \bar{\varphi} \le \widetilde{\varphi}$. Hence, using the trivial bounds

$$H_\Lambda^{J,h,1}(\varphi) \le 5|\Lambda| \sup_x \varphi(x)$$

we obtain (remember (2.6)),

$$\mu_\Lambda^{J,h,\psi}(\partial X) \ge \mu_\Lambda^{J,h,\psi}(\bar{\varphi}) = (Z^{J,h,\psi}(\Lambda))^{-1} e^{-\beta H_\Lambda^{J,h,\psi}(\bar{\varphi})} \ge$$
$$e^{-\beta H_\Lambda^{J,h,1}(\bar{\varphi})} \Big[\sum_{\varphi \in \Omega_\Lambda} e^{-\beta \bar{H}_\Lambda^{h,\infty}(\varphi)} \Big]^{-1} \ge \Big[\sum_{\varphi \in \Omega_\Lambda} e^{-\beta \bar{H}_\Lambda^{h,\infty}(\varphi)} \Big]^{-1} e^{-5\beta|\Lambda| s(\beta,h,N)} .$$

This complete the proof of (2.7) and Proposition 2.2. $\square$

3. Upper bound on the gap at $h^*_k(\beta)$

In this section we prove the upper bound in part (ii) of Theorem 1.1. More precisely, we have

Theorem 3.1. *Let β be large enough and let $k \in \mathbb{Z}_+$ such that $1 \le k \le k_{max} = \lfloor e^{\frac{\beta\zeta}{20}} \rfloor$. If $N > 10/h^*_k(\beta)$, then*

$$\mathrm{gap}\big(L_{Q_N}^{h^*_k(\beta),\emptyset}\big) \le c_M(\beta, h^*_k(\beta))\, e^{-\frac{1}{40}\beta N}$$

where $c_M(\beta,h)$ is defined in (1.11).

We set for simplicity $h^* = h^*_k(\beta)$ and $\Lambda = Q_N$. In order to find an upper bound on $\mathrm{gap}^{h^*,\emptyset}(\Lambda)$ we are going to use the "look for the bottleneck" approach, *i.e.* we take advantage of the variational characterization for the gap (1.12) and choose an appropriate test function f_0 which illustrates how the system, in order to relax to equilibrium, has to make an excursion to a very unlikely region.

Given $\varphi \in \Omega$ we set

$$\sigma(x) = \mathrm{sign}(\varphi(x) - k - 1/2)$$

and, for $U \subset\subset \mathbb{Z}^2$

$$M_U(\sigma) = M_U(\sigma(\varphi)) = \sum\nolimits_{x \in U} \sigma(x)$$
$$m_U(\sigma) = m_U(\sigma(\varphi)) = |U|^{-1} M_U(\sigma)\,.$$

In analogy with the solution of the similar problem for the 2D Ising model in the phase coexistence region, we take as a test function

$$f_0(\varphi) = \chi\{M_\Lambda(\varphi) > 0\} - \chi\{M_\Lambda(\varphi) < 0\}\,.$$

Using (1.11), we find

$$\mathcal{E}_\Lambda^{h^*,\emptyset}(f_0,f_0) \le \mu_\Lambda^{h^*,\emptyset}\{|M_\Lambda(\varphi)| \le 2\} \sup_{\varphi\in\Omega} \sum_{s=\pm} c_\Lambda^{h^*,\emptyset}(x,\varphi,s) \le c_M(\beta,h)\, \mu_\Lambda^{h^*,\emptyset}\{|M_\Lambda(\varphi)| \le 2\}, \tag{3.1}$$

while

$$\mathrm{Var}_\Lambda^{h^*,\emptyset}(f_0) = \mu_\Lambda^{h^*,\emptyset}\{M_\Lambda(\varphi) \ne 0\} - \big(\mu_\Lambda^{h^*,\emptyset}\{M_\Lambda(\varphi) > 0\} - \mu_\Lambda^{h^*,\emptyset}\{M_\Lambda(\varphi) < 0\}\big)^2 \tag{3.2}$$

The proof of Theorem 3.1 is then a consequence of (1.12), (3.1), (3.2) and Lemmas 3.3, 3.4 proven below.

3.1. Bound on the Dirichlet form

We want to show that, if β is large enough and $N \geq 10/h^*_k(\beta)$, then

$$\mu_\Lambda^{h^*,\emptyset}\{|M_\Lambda(\varphi)| \leq 2\} \leq e^{-cost\,\beta|\delta\Lambda|}.$$

The idea is the following (see also [S]). Take $\Lambda' \subset \Lambda$. Then either there is a "contour" separating the phase $\sigma = 1$ from the phase $\sigma = -1$ or there is a circuit surrounding Λ' where σ is either all plus or all minus. So we need to prove that

- the probabilty of having a large contour of length l with free b.c. goes like $\exp(-\mathrm{cost}\,\beta\, l)$
- the probability of having $|M_{\Lambda'}| \leq 2$ conditioned to the existence of a $(+)$ circuit or a $(-)$ circuit surrounding Λ' goes like $\exp(-\mathrm{cost}\,\beta|\delta\Lambda'|)$

Before stating the next Lemma we have introduce another definition. Given $\sigma \in \{-1,1\}^V$ let $\sigma_+ \in \{-1,1\}^{\mathbb{Z}^2}$ be defined by

$$\sigma_+(x) = \begin{cases} \sigma(x) & \text{if } x \in V \\ +1 & \text{if } x \in V^c \end{cases}$$

The set $\{e^* : e = [x,y],\ \sigma_+(x) \neq \sigma_+(y)\}$ can be written as a union of its connected components $\alpha_1, \ldots, \alpha_r$ where $\alpha_i \in C_B(V)$. A closed set of dual edges α is called a *contour for* σ if $\alpha = \alpha_i$, for some i.

Lemma 3.2. *Let β be large enough, $1 \leq k \leq k_{max}$, $\Lambda = Q_N$. For each $\alpha \in C_B(\Lambda)$, let $\alpha_0 = \alpha \setminus \delta\Lambda$. Then, if $\alpha \cap \delta\Lambda \neq \emptyset$ and $|\alpha_0| > N/100$, we have*

$$\mu_\Lambda^{h^*_k(\beta),\emptyset}\{\varphi \in \Omega_\Lambda : \alpha \text{ is a contour for } \sigma(\varphi)\} \leq e^{-0.99\beta|\alpha_0|}\,.$$

Proof. Let $h^* = h^*_k(\beta)$. Given $\alpha \in C_B(\Lambda)$ we define the event

$$X_\alpha = \{\varphi \in \Omega_\Lambda : \alpha \text{ is a contour for } \varphi\}.$$

We also let $\Lambda_1 = \bar\alpha$, $\Lambda_2 = \Lambda \setminus \bar\alpha$ and

$$\Delta_1 = \{x \in \Lambda_1 : d(x, \Lambda_2) = 1\}$$
$$\Delta_2 = \{x \in \Lambda_2 : d(x, \Lambda_1) = 1\}.$$

We observe that both Λ_1 and Λ_2 are simply connected. In fact Λ_1 is the interior of a closed connected set of dual edges, and Λ_2 is is the interior of $(\alpha \cup \delta\Lambda) \setminus (\alpha \cap \delta\Lambda)$ which is again a closed connected set of dual edges because α touches $\delta\Lambda$. So we have

$$X_\alpha \subset X^1_\alpha \cup X^2_\alpha,$$

where

$$X^1_\alpha = \{\varphi \in \Omega_\Lambda : \sigma(x) = -1\ \forall x \in \Delta_1,\ \text{and } \sigma(x) = +1\ \forall x \in \Delta_2\}$$
$$X^2_\alpha = \{\varphi \in \Omega_\Lambda : \sigma(x) = +1\ \forall x \in \Delta_1,\ \text{and } \sigma(x) = -1\ \forall x \in \Delta_2\}$$

In the following we are going to show how to estimate $\mu_\Lambda^{h^*,\emptyset}(X^1_\alpha)$. $\mu_\Lambda^{h^*,\emptyset}(X^2_\alpha)$ can be dealt with in the same way. Choose now the couplings $J(e)$ which correspond to free b.c. on the boundary of Λ, *i.e.*

$$J(x,y) = \begin{cases} 1 & \text{if } [x,y]^* \notin \delta\Lambda \\ 0 & \text{if } [x,y]^* \in \delta\Lambda \end{cases} \tag{3.3}$$

In this way, since, for $\varphi \in X^1_\alpha$,

$$H_\Lambda^{h^*,\emptyset}(\varphi) = H_{\Lambda_1}^{J,h^*,k}(\varphi) + H_{\Lambda_2}^{J,h^*,k+1}(\varphi) + |\alpha_0|,$$

we can write

$$\mu_\Lambda^{h^*,\emptyset}(X^1_\alpha) = \frac{e^{-\beta|\alpha_0|}\, Z^{J,h^*,k}(\Lambda_1)\, Z^{J,h^*,k+1}(\Lambda_2)}{Z^{h^*,\emptyset}(\Lambda)}\,. \tag{3.4}$$

At this point our strategy depends on whether $|\alpha \cap \delta\Lambda|$ is greater than or less than $\frac{1}{2}|\delta\Lambda|$. In the first case we estimate the denominator as

$$Z^{h^*,\emptyset}(\Lambda) \geq Z^{J,h^*,k}(\Lambda_1)\, Z^{J,h^*,k}(\Lambda_2)$$

while in the second case we write

$$Z^{h^*,\emptyset}(\Lambda) \geq Z^{J,h^*,k+1}(\Lambda_1)\, Z^{J,h^*,k+1}(\Lambda_2)\,.$$

We treat only the first case because the second is identical. So, assuming $|\alpha \cap \delta\Lambda| \geq \frac{1}{2}|\delta\Lambda|$, we get

$$\mu_\Lambda^{h^*,\emptyset}(X^1_\alpha) \leq e^{-\beta|\alpha_0|}\frac{Z^{J,h^*,k+1}(\Lambda_2)}{Z^{J,h^*,k}(\Lambda_2)}. \tag{3.5}$$

What is left to complete the proof is to convince ourselves that (rememeber the definition of $H(\Lambda, t)$ given before (1.5) and that $\zeta = 1000^{-1}$)

$(*)$ $\qquad H(\Lambda_2, 10\zeta)$ *holds for* J

In fact, assuming $(*)$, we apply Corollary 4.2 of [CM] to estimate the above quotient and obtain

$$\mu_\Lambda^{h^*,\emptyset}(X^1_\alpha) \leq \exp\left[-\beta|\alpha_0| + |\delta\Lambda_2|e^{-\frac{1}{4}\beta\zeta}\right] \leq e^{-\beta|\alpha_0| + |\alpha_0| + |\delta\Lambda|} \leq e^{|\alpha_0|(-\beta+401)}$$

where we have used $|\alpha_0| \geq N/100$.

Proof of $(*)$. Let $\eta \in C_B(\Lambda_2)$. We are going to show that $|\eta|_J \geq \frac{1}{3}|\eta|$, which is more than enough. We recall that with our choice of J, $|\eta|_J = |\eta \setminus \delta\Lambda|$ and that $\delta\Lambda_2 = \alpha_0 \cup (\delta\Lambda \setminus \alpha)$, so that

$$|\eta \cap \delta\Lambda| \leq |\delta\Lambda_2 \cap \delta\Lambda| = |\delta\Lambda \setminus \alpha| \leq \frac{1}{2}|\delta\Lambda| \tag{3.6}$$

because of what we assumed right before (3.5). We consider three cases

(1) If η does not touch any two opposite sides of Λ then $|\eta| \geq 2|\eta \cap \delta\Lambda|$, so $|\eta|_J = |\eta| - |\eta \cap \delta\Lambda| \geq \frac{1}{2}|\eta|$

(2) If η touches exactly 3 sides of Λ, then, since it is closed, $|\eta \setminus \delta\Lambda| \geq \frac{1}{4}|\delta\Lambda|$, so, by (3.6), $|\eta \cap \delta\Lambda| \leq 2|\eta \setminus \delta\Lambda|$ which implies $|\eta|_J \geq \frac{1}{3}|\eta|$

(3) If η touches all 4 sides of Λ, then $|\eta| \geq |\delta\Lambda|$, so $|\eta|_J \geq \frac{1}{2}|\eta|$.

This proves $(*)$.

Thus we have obtained

$$\mu_\Lambda^{h^*,\emptyset}(X_\alpha) \leq 2e^{|\alpha_0|(-\beta+401)} \leq e^{-0.99\beta|\alpha_0|} \quad \square$$

We can now state our first key estimate

Lemma 3.3. *Let β be large enough, $1 \leq k \leq k_{max}$ and $\Lambda = Q_N$. Then, if $N \geq 1000$,*

$$\mu_\Lambda^{h_k^*(\beta),\emptyset}\{\varphi \in \Omega_\Lambda : |M_\Lambda(\varphi)| \leq 2\} \leq e^{-\frac{1}{12}\beta N}$$

Proof. We borrow the basic idea from [S].

Let $h^* = h_k^*(\beta)$. Consider a square $\Lambda' = Q_{N'}$ of side $N' = N - 2\lfloor N/20 \rfloor$. In this way $d(\delta\Lambda, \delta\Lambda') = \lfloor N/20 \rfloor$. It is then easy to check that

$$\mu_\Lambda^{h^*,\emptyset}\{|M_\Lambda(\varphi)| \leq 2\} \leq \mu_\Lambda^{h^*,\emptyset}\{|m_{\Lambda'}(\varphi)| \leq \frac{3}{8}\}. \tag{3.7}$$

Following [S] we consider the events

$$Y^\pm = \left\{\varphi \in \Omega_\Lambda : \begin{array}{l} \text{there exists a connected set } R \subset \Lambda \setminus \Lambda', \text{ surrounding } \Lambda' \\ \text{such that } \sigma(x) = \pm 1 \text{ for all } x \in R \end{array}\right\}.$$

and, letting $Y = Y^+ \cup Y^-$, we write

$$\mu_\Lambda^{h^*,\emptyset}\{|m_{\Lambda'}(\varphi)| \leq \frac{3}{8}\} \leq \mu_\Lambda^{h^*,\emptyset}\{m_{\Lambda'}(\varphi) \leq \frac{3}{8} \,|\, Y^+\} + \mu_\Lambda^{h^*,\emptyset}\{m_{\Lambda'}(\varphi) \geq -\frac{3}{8} \,|\, Y^-\} + \mu_\Lambda^{h^*,\emptyset}(Y^c). \tag{3.8}$$

The idea is to show that the first two terms are small because of Corollary 4.3 of [CM], while the last term is small because of Lemma 3.2. The first two terms can be taken care of in the same way, so we will deal only with the first.

Consider Λ' as a disjoint union of N' horizontal strips S_j, each strip being a $N' \times 1$ rectangle. Accordingly write

$$\Lambda' = S_1 \cup S_2 \cup \cdots \cup S_{N'}.$$

By conditioning on the "most external" R which surrounds Λ', and using the FKG inequality and the Markov property as we did in Proposition A1.1 of [CM], we find

$$\mu_\Lambda^{h^*,\emptyset}\{m_{\Lambda'}(\varphi) \le \frac{3}{8} \,|\, Y^+\} \le \sup_{\substack{V:\,\Lambda' \subset V \subset \Lambda \\ V \text{ connected and} \\ \text{simply connected}}} \mu_V^{h^*,k+1}\{m_{\Lambda'}(\varphi) \le \frac{3}{8}\}\,.$$

Applying Corollary 4.3 of [CM] to each S_j we get

$$\mu_\Lambda^{h^*,\emptyset}\{m_{\Lambda'}(\varphi) \le \frac{3}{8} \,|\, Y^+\} \le \sup_{\substack{V:\,\Lambda' \subset V \subset \Lambda \\ V \text{ connected and} \\ \text{simply connected}}} N' \sup_{1 \le j \le N'} \mu_V^{h^*,k+1}\{m_{S_j}(\varphi) \le \frac{3}{8}\} \le$$

$$\le N' e^{-\frac{4}{9}\beta\frac{5}{8}N'} \le N e^{-\frac{1}{4}\beta N} \le e^{-\frac{1}{5}\beta N} \tag{3.9}$$

We now complete the proof with an estimate of the last term in (3.8). The basic observation here is that if $\varphi \in Y^c$, then there is a contour α for φ touching both $\delta\Lambda$ and $\delta\Lambda'$, which implies

$$|\alpha_0| = |\alpha \setminus \delta\Lambda| \ge 2d(\delta\Lambda, \delta\Lambda') \ge 2\lfloor N/20 \rfloor\,.$$

It is easy to check that for any dual edge e^*

$$\#\{\alpha \in C_B(\Lambda) : \alpha \ni e^*,\, |\alpha_0| = l\} \le K^{l+4N+1}$$

where K^l is an upper bound for the number of α of length l containing a fixed edge. Then, by Lemma 3.2, setting $l_0 = 2\lfloor N/20 \rfloor$,

$$\mu_\Lambda^{h^*,\emptyset}(Y^c) \le \sum_{\substack{\alpha \in C_B(\Lambda) \\ \alpha \cap \delta\Lambda \ne \emptyset,\, |\alpha_0| \ge l_0}} \mu_\Lambda^{h^*,\emptyset}\{\alpha \text{ is a contour for } \varphi\ \} \le$$

$$\sum_{e^* \in \delta\Lambda} \sum_{\substack{\alpha \in C_B(\Lambda) \\ \alpha \ni e^*,\, |\alpha_0| \ge l_0}} \mu_\Lambda^{h^*,\emptyset}\{\alpha \text{ is a contour for } \varphi\ \} \le$$

$$\le |\delta\Lambda| \sum_{l=l_0}^{\infty} K^{l+4N+1} e^{-0.99\beta l} \le 2K^{l_0+4N+1} e^{-0.99\beta l_0}\,.$$

Together with (3.7), (3.8) and (3.9) this proves the theorem (if $N \geq 1000$ then $l_0 \geq \frac{1}{11}N$). $\square$

3.2. Bound on the variance of f_0

In order to find a lower bound on the variance of f_0, we prove

Lemma 3.4. *Let β be large enough, $1 \leq k \leq k_{max}$. Take $\Lambda = Q_N$ with $N > 10/h_k^*(\beta)$. Then*

$$\mu_\Lambda^{h_k^*(\beta),\emptyset}\{\varphi \in \Omega_\Lambda : M_\Lambda(\sigma(\varphi)) < 0\} \geq e^{-\frac{1}{20}\beta N}$$
$$\mu_\Lambda^{h_k^*(\beta),\emptyset}\{\varphi \in \Omega_\Lambda : M_\Lambda(\sigma(\varphi)) > 0\} \geq e^{-\frac{1}{20}\beta N}$$

Proof. Let $h^* = h_k^*(\beta)$. We assume, for simplicity that $N = 2L+1$ is odd and subdivide Λ into two rectangles, so we consider the central horizontal row

$$R = \{[-L, L] \cap \mathbb{Z}\} \times \{0\}\,.$$

Then we write

$$\Lambda = \Lambda_1 \cup R \cup \Lambda_2$$

where Λ_1 is the $N \times L$ rectangle above R and Λ_2 is the rectangle below R. Consider now the events

$$X = \{\varphi \in \Omega_\Lambda : \varphi(x) = k,\ \forall x \in R\}$$
$$Y_i = \{\varphi \in \Omega_\Lambda : M_{\Lambda_i}(\sigma) < 0\} \qquad i = 1, 2.$$

Then

$$\mu_\Lambda^{h^*,\emptyset}\{M_\Lambda(\sigma) < 0\} \geq \mu_\Lambda^{h^*,\emptyset}(Y_1 \cap Y_2 \,|\, X)\, \mu_\Lambda^{h^*,\emptyset}(X) = \mu_{\Lambda_1}^{J,h^*,k}(Y_1)\, \mu_{\Lambda_2}^{J,h^*,k}(Y_2)\, \mu_\Lambda^{h^*,\emptyset}(X) \tag{3.10}$$

where J is taken as in (3.3). To take care of the first two terms, we observe that $H(\Lambda_1, \frac{1}{4})$ holds for J by an argument very similar to the proof of statement $(*)$ in Lemma 3.2. So, using the row decomposition as in (3.9), and Corollary 4.3 of [CM], we get

$$\mu_{\Lambda_1}^{J,h^*,k}(Y_1) = 1 - \mu_{\Lambda_1}^{J,h^*,k}(Y_1^c) \geq 1 - Le^{-\frac{1}{9}\beta N} \geq \frac{1}{2} \tag{3.11}$$

(same for Y_2).

Let's consider now the last term in (3.10). If $\varphi \in X$ then

$$H_\Lambda^{h^*,\emptyset}(\varphi) = H_{\Lambda_1}^{J,h^*,k}(\varphi) + H_{\Lambda_2}^{J,h^*,k}(\varphi) + h^* kN$$

so

$$\mu_\Lambda^{h^*,\emptyset}(X) = \frac{\sum_{\varphi\in X} e^{-\beta H_\Lambda^{h^*,\emptyset}(\varphi)}}{Z^{h^*,\emptyset}(\Lambda)} = \frac{Z^{J,h^*,k}(\Lambda_1)\, Z^{J,h^*,k}(\Lambda_2)}{Z^{h^*,\emptyset}(\Lambda)}\, e^{-\beta h^* k N}\,. \tag{3.12}$$

The idea is again to use cluster expansion to estimate the quotient of partition functions. For this purpose we introduce a small coupling on the boundary of Λ

$$\bar{J}(x,y) = \begin{cases} 1 & \text{if } [x,y]^* \notin \delta\Lambda \\ 10\zeta & \text{if } [x,y]^* \in \delta\Lambda \end{cases} \tag{3.13}$$

where we have set $\zeta = 1000^{-1}$. Jensen inequality (see Proposition 4.5 of [CM]) and Proposition 3.5 (which is given below) yield

$$\frac{Z^{\bar{J},h^*,k}(\Lambda)}{Z^{h^*,\emptyset}(\Lambda)} \geq \exp\Big[-10\,\zeta\beta\,|\delta\Lambda| \sup_{x\in\partial\Lambda} \mathbf{E}_\Lambda^{h^*,\emptyset}\,|\varphi(x)-k|\Big] \geq e^{-11\zeta\beta|\delta\Lambda|}\,. \tag{3.14}$$

Hence, since $H(\Lambda_1,\frac{1}{4})$, $H(\Lambda_2,\frac{1}{4})$ holds for J, and $H(\Lambda,10\zeta)$ holds for $\bar{J}$ for all $V \subset\subset \mathbb{Z}^2$, all three partition functions

$$Z^{J,h^*,k}(\Lambda_1)\ ,\ Z^{J,h^*,k}(\Lambda_2)\ ,\ Z^{\bar{J},h^*,k}(\Lambda)$$

have a cluster expansion, thanks to Theorem 4.1 of [CM]. With the help of Theorem 4.4 of [CM] is not difficult to show that

$$\frac{Z^{J,h^*,k}(\Lambda_1)\, Z^{J,h^*,k}(\Lambda_2)}{Z^{\bar{J},h^*,k}(\Lambda)} \geq e^{\beta h^* k N - \varepsilon(\beta) N} \tag{3.15}$$

where $\varepsilon(\beta) \to 0$ when $\beta \to \infty$. Together with (3.12), (3.14) this implies

$$\mu_\Lambda^{h^*,\emptyset}(X) \geq e^{-45\zeta\beta N},$$

and then, by (3.10), (3.11)

$$\mu_\Lambda^{h^*,\emptyset}\{M_\Lambda(\sigma) < 0\} \geq e^{-50\zeta\beta N}.$$

The proof of the second statement is identical. $\square$

In the next Proposition we complete the proof, by showing how to get the second inequality in (3.14). We will actually give a more general statement than the one needed.

Proposition 3.5. *Let β be large enough, $1 \le k \le k_{max}$, $\Lambda = Q_N$, $N > 10/h^*_k(\beta)$. Then*

$$\sup_{x\in\Lambda} \mathbf{E}_\Lambda^{h^*_k(\beta),\emptyset} |\varphi(x) - k| \le 1.1$$

Proof. Let $h^* = h^*_k(\beta)$ and for any positive integer k let, as in [CM],

$$I_k(\beta) = [h^-_k(\beta), h^+_k(\beta)]\,, \quad h^-_k(\beta) = \frac{4}{\beta} e^{-4\beta k}\,, \quad h^+_k(\beta) = \frac{1}{4\beta} e^{-4\beta(k-1)}\,, \quad h^+_1 = \frac{1}{\beta} e^{-\frac{\beta}{25} J}\,.$$

By FKG we can write

$$\mu_\Lambda^{h^*,\emptyset}\{\varphi(x) \ge n\} \le \mu_V^{h,\emptyset}\{\varphi(x) \ge n\} \qquad \forall h \in I_{k+1}(\beta) \tag{3.16}$$
$$\mu_\Lambda^{h^*,\emptyset}\{\varphi(x) \le n\} \le \mu_V^{h,\emptyset}\{\varphi(x) \le n\} \qquad \forall h \in I_k(\beta) \tag{3.17}$$

Choose now $h \in I_{k+1}(\beta)$, and a positive integer j. We want to show that the RHS of (3.16) is less than $e^{-\frac{\zeta}{3}\beta j}$ when $n = k+1+j$. As in [CM], for $V \subset \Lambda$, we define the events

$$S_+(\Lambda, j, V) = \left\{ \varphi \in \Omega_\Lambda : \begin{matrix} \text{there exists a path } (x_1, \dots, x_s) \text{ such that} \\ x_1 \in V,\ x_s \in \partial\Lambda \text{ and } \varphi(x_i) \ge j \text{ for each } i \end{matrix} \right\}.$$

If $x \in \Lambda$ and $j > 0$, trivially we have that $\varphi(x) \ge k+j+1$ if and only if $\varphi \in S_+(\Lambda, k+1+j, \{x\})$. Thus, by Theorem 5.2 of [CM] we get

$$\mu_V^{h,\emptyset}\{\varphi(x) \ge k+1+j\} \le e^{-\zeta\beta j}\,.$$

If, on the contrary, $x \in \Lambda \setminus \partial\Lambda$, then we define the event F as

$$F = S_+(\Lambda, k+1+\lfloor j/2 \rfloor + 1, \partial^+\{x\})$$

So

$$\mu_\Lambda^{h,\emptyset}\{\varphi(x) \ge k+1+j\} \le \mu_\Lambda^{h,\emptyset}(\varphi(x) \ge k+1+j \,|\, F^c) + \mu_\Lambda^{h,\emptyset}(F)\,. \tag{3.18}$$

By Proposition A1.1 of [CM] one finds

$$\mu_\Lambda^{h,\emptyset}(\varphi(x) \ge k+1+j \,|\, F^c) \le \sup_{\substack{V \ni x \\ V \text{ connected and} \\ \text{simply connected}}} \mu_V^{h,k+1+\lfloor j/2 \rfloor}\{\varphi(x) \ge k+1+j\}.$$

Using FKG again and Proposition 3.5 of [CM], and letting $m = \lfloor j/2 \rfloor$, we get

$$\begin{aligned} &\mu_V^{h,k+1+m}\{\varphi(x) \ge k+1+j\} \le \mu_V^{h,k+1+m}(\varphi(x) \ge k+1+j \,|\, \varphi(y) > m\ \forall y \in V) = \\ &= \mu_V^{h,k+1}\{\varphi(x) \ge k+1+j-m\} \le e^{-\frac{1}{5}\beta(j-m)} \le e^{-\frac{1}{10}\beta j} \end{aligned}$$

On the other side Theorem 5.2 of [CM] says that

$$\mu_\Lambda^{h,\emptyset}(F) \le e^{-\zeta\beta(1+m)} \le e^{-\frac{1}{2}\zeta\beta j}\,.$$

These last two inequalities, together with (3.16), (3.18) give

$$\mu_\Lambda^{h^*,\emptyset}\{\varphi(x) \ge k+1+j\} \le e^{-\frac{\zeta}{3}\beta j}\,.$$

In a similar way one finds, for $j > 0$

$$\mu_\Lambda^{h^*,\emptyset}\{\varphi(x) \le k-j\} \le e^{-\frac{\zeta}{3}\beta j}\,.$$

Therefore, for all $x \in \Lambda$,

$$\mathbf{E}_\Lambda^{h^*,\emptyset}\,|\varphi(x)-k| \le 1 + 2\sum_{j=1}^{\infty} e^{-\frac{\zeta}{3}\beta j} + \le 1.1 \quad \square$$

4. Lower bound on the gap at $h_k^*(\beta)$

In this section we complete the proof of Theorem 1.1 by proving a lower bound on the gap of the generator of the dynamics for $h = h_k^*(\beta)$ and arbitrary boundary conditions, which is of the same order of the *upper* bound discussed in the previous section, namely a negative exponential of the *boundary* of the square Q_L.

As is [CM] we define $N_0(h) = (\zeta^3 h)^{-1}$, where $\zeta = 1000^{-1}$. We prove that (remember that we always assume $0 \le J(x,y) \le 1$ and that $J \in \delta Q$ has been defined in Section 1.2)

Theorem 4.1. *If β is large enough, then, for each $1 \le k \le k_{max} = \lfloor e^{\frac{\beta\zeta}{20}} \rfloor$, there exists a positive constant $G(\beta, h_k^*(\beta))$ such that*

$$\inf_{J\in\delta Q}\inf_{\psi\in\Omega}\mathrm{gap}\big(L_Q^{J,h_k^*(\beta),\psi}\big) \ge G(\beta, h_k^*(\beta))^{\log L}\, e^{-80\beta k L}$$

for any square Q of side $L \ge 20 N_0(h_k^*(\beta))$.

Remarks. (1) Free boundary conditions are recovered by simply setting $J(x,y) = 1$ everywhere but $J(x,y) = 0$ for all x, y such that $[x,y]^* \in \delta Q$. Moreover, using the fact that the gap is positive uniformly in the boundary conditions for each $L \ge \lfloor 8/h + 1 \rfloor$ (Proposition 2.2), we can easily find a constant $C_1(\beta, h) > 0$ such that statement (ii) of Theorem 1.1 holds.

(2) By using a more sophisticated approach which requires the so called "surgery" technique one can replace the factor $80\beta k$ with 80β.

Proof. Let $h^* = h_k^*(\beta)$, $N_0 = N_0(h^*)$ and $\hat N_0 = 10 N_0$. The proof is obtained by recursion as follows. For $L \in \mathbb{Z}_+$, let

$$g_L(\beta,h) = \inf_{J\in\delta Q_L}\inf_{\psi\in\Omega}\mathrm{gap}(L_{Q_L}^{J,h,\psi})$$

Proposition 2.2 says that $g_L > 0$ if $L \ge \lfloor 8/h + 1 \rfloor$. Given now $L \ge 2\hat N_0$, it is possible to find positive integers s, $\{L_i\}_{i=1}^s$ such that

$$\begin{aligned} &L = L_0 > L_1 > \cdots > L_s = \hat N_0 \\ &\frac{5}{4} L_i \le L_{i-1} \le \frac{7}{4} L_i \qquad i = 1, \dots, s \end{aligned}$$

(a proof of this is given in the Appendix A2). We clearly have

$$s \le \frac{\log L - \log \hat N_0}{\log(5/4)}\,.$$

Then we prove the following recursive inequality

$$g_{L_{i-1}}(\beta,h^*) \geq g_{L_i}(\beta,h^*)\, G_0(\beta,h^*)\, e^{-20\beta k L_i} \qquad \forall 1\leq i\leq s \tag{4.1}$$

for a suitable $G_0(\beta,h^*)>0$. Iterating (4.1) and observing that

$$\sum_{i=1}^{s} L_i \leq L\sum_{i=1}^{s}(4/5)^i \leq 4L$$

we get

$$g_L(\beta,h^*) \geq g_{\hat N_0}(\beta,h^*)\prod_{i=1}^{s}\Big[G_0(\beta,h^*)\, e^{-20\beta k L_i}\Big] \geq G(\beta,h^*)^{\log L} e^{-80\beta k L}$$

for some $G(\beta,h^*)>0$, and so the theorem is proven.

In order to establish (4.1), we actually prove the slightly more general inequality

$$\inf_{\frac{5}{4}N\leq L\leq\frac{7}{4}N} g_L(\beta,h^*) \geq g_N(\beta,h^*)\, G_0(\beta,h^*)\, e^{-20\beta k N} \qquad \forall N\geq \hat N_0\,.$$

We proceed as follows. Let $N\geq \hat N_0$, $5N/4<L\leq 7N/4$ and $\Lambda=Q_L$. Λ can be obtained in a unique way (apart from permutations) as a union of four overlapping squares of side N that we denote by Λ_i, $i=1,\dots,4$ (see Fig. 1). Let Λ_1 be the northeast square in Λ, Λ_2 the northwest one and so on, proceeding counterclockwise. Let $\mathcal{D}=\{\Lambda_1,\Lambda_2,\Lambda_3,\Lambda_4\}$, and consider the block–dynamics defined in Appendix A1, determined by β,h^*,J, the b.c. ψ and the collection $\mathcal{D}$, through the Markov generator $L_{\mathcal{D}}^{J,h^*,\psi}$ given by (A1.1).

Thanks to Proposition A1.1 we know that

$$\mathrm{gap}(L_\Lambda^{J,h^*,\psi}) \geq \mathrm{gap}(L_{\mathcal{D}}^{J,h^*,\psi})\, g_N(\beta,h^*) \tag{4.2}$$

Thus, in order to prove the theorem, it remains to show that there exists $G_0(\beta,h^*)>0$ such that

$$\inf_{J\in\delta\Lambda}\inf_{\psi\in\Omega}\mathrm{gap}(L_{\mathcal{D}}^{J,h^*,\psi}) \geq G_0(\beta,h^*)\, e^{-20\beta k N} \tag{4.3}$$

By Proposition A1.2, this in turn follows, if we can prove that

$$\mathbb{P}_\Lambda^{J,h^*,\psi,(\varphi,\bar\varphi)}\{(\varphi_t,\bar\varphi_t):\varphi_1=\bar\varphi_1\} \geq G_0(\beta,h^*)\, e^{-20\beta k N} \quad \text{for all } \varphi,\bar\varphi \text{ such that } \varphi\leq\bar\varphi \tag{4.4}$$

where $(\varphi_t,\bar\varphi_t)$ is the coupled process defined in the Appendix A1. The idea at this point is that in order to have $\varphi_t=\bar\varphi_t$ on Λ it is "almost" sufficient to impose that the two configurations agree on a subset of Λ whose cardinality is of the order of the boundary. Let in fact (see Fig. 1)

$$A_i = \Lambda_i \cap \Big(\bigcup_{j\neq i} \partial^+ \Lambda_j \Big) \qquad i = 1, \dots, 4 \,.$$

Using the explicit construction and notation for this coupled process given in the Appendix A1, we have

$$\mathbb{P}_\Lambda^{J,h^*,\psi,(\varphi,\bar\varphi)} \{\varphi_1 = \bar\varphi_1\} \geq p_1\, p_2\, p_3 \tag{4.5}$$

where p_1 is the probability of having $\tau(6) \leq 1$, p_2 is the probability that the order in which the first six blocks are updated is given by

$$l(1), \dots, l(6) = 1, 3, 2, 4, 1, 3J \tag{4.6}$$

and p_3 is the probability of having $\varphi_{\tau(6)} = \bar\varphi_{\tau(6)}$, given (4.6). Define now the events (the random variables $(\eta, \bar\eta)$ have been introduced in the Appendix A1)

$$F_{i,j}^{(\varphi',\varphi'')} = \{(\eta(x), \bar\eta(x))_{i,j}^{(\varphi',\varphi'')} = (1,1) \ \forall x \in A_i\}$$

Then we observe, and this is the key remark, that, since the interaction is between nearest neighbors, if the event $F_{1,1}^{(\varphi_0,\bar\varphi_0)}$ and $F_{3,2}^{(\varphi_{\tau(1)},\bar\varphi_{\tau(1)})}$ happen, then necessarily (given (4.6)) $\varphi_{\tau(6)} = \bar\varphi_{\tau(6)}$. Thus, taking into account

(*i*) that $(\eta(x), \bar\eta(x)) = (1,1)$ if and only if $\bar\eta(x) = 1$

(*ii*) the properties of the standard coupling $\nu_{\Lambda_i}^{\varphi',\varphi''}(\eta, \bar\eta)$ given in Section 1.3

we get

$$\begin{aligned} p_3 &\geq \inf_{\varphi' \leq \varphi'',\, \tilde\varphi' \leq \tilde\varphi''} \mathrm{Prob}(F_{1,1}^{(\varphi',\varphi'')} \cap F_{3,2}^{(\tilde\varphi',\tilde\varphi'')}) \geq \\ &\geq \inf_{J \in \delta\Lambda_1} \inf_{\varphi' \in \Omega} \big(\mu_{\Lambda_1}^{J,h^*,\varphi'} \{\varphi \in \Omega_{\Lambda_1} : \varphi(x) = 1 \,,\ \forall x \in A_1\} \big)^2 \,. \end{aligned} \tag{4.7}$$

Clearly the event $\{\varphi(x) = 1 \ \forall x \in A_i\}$ is a decreasing event so that, using the FKG property we get

$$\mu_{\Lambda_1}^{J,h^*,\varphi'} \{\varphi(x) = 1 \ \forall x \in A_1\} \geq \prod_{x \in A_1} \mu_{\Lambda_1}^{J,h^*,\varphi'} \{\varphi(x) = 1\} \tag{4.8}$$

Now we subdivide the x's in A_1 into two groups, depending on whether they are close to the boundary or not. Let then

$$\begin{aligned} A_1' &= \{x \in A_1 : d(x, \partial\Lambda_1) \leq N_0\} \\ A_1'' &= A_1 \setminus A_1' \end{aligned}$$

We claim that

(a) $|A_1'| \le 4N_0$ and $|A_1''| \le 2N$

(b) if $x \in A_1'$ then for all $J \in \delta\Lambda_1$, $\varphi' \in \Omega$, $\mu_{\Lambda_1}^{J,h^*,\varphi'}\{\varphi(x)=1\} \ge b_2(b,h^*) > 0$

(c) if $x \in A_1''$ then for all $J \in \delta\Lambda_1$, $\varphi' \in \Omega$, $\mu_{\Lambda_1}^{J,h^*,\varphi'}\{\varphi(x)=1\} \ge \frac{1}{8}e^{-4\beta k}$

(a), (b) and (c) together with the obvious bound

$$p_1\, p_2 \ge e^{-\beta}$$

clearly imply (4.3) with

$$G_0(\beta,h^*) = e^{-\beta}\, b_2(\beta,h^*)^{8N_0}$$

and so the theorem would be proven.

• To prove (a) it is sufficient to observe that $A_1 = T_v \cup T_h$ (see Fig. 1), where T_v is a vertical line of length N whose distance from the vertical sides of Λ_1 is greater than

$$((L-N)\wedge(2N-L)) - 2 \ge \frac{1}{5}N \ge 2N_0$$

since $5N/4 < L \le 7N/4$. An analogous statement holds for T_h.

• (b) is proven in Proposition 3.2 of [CM].

• The proof of (c) is a little more involved and we have to use the fact the at distances greater than N_0 from the boundary is very likely to have $k \le \varphi \le k+1$. Let in fact $x \in A_1''$. Let V_0 be a square of side N_0 (or N_0+1 if N_0 is even) centered at x and let V_1 be the square still centered at x but with side $N_0 - 2\lfloor N_0/4\rfloor - 2$. By (1.7), taking into account that if $J \in \delta\Lambda_1$ then $J(x,y)=1$ for all $\{x,y\} \subset \Lambda_1$, we obtain

$$\inf_{J\in\delta\Lambda_1}\inf_{\varphi'\in\Omega}\mu_{\Lambda_1}^{J,h^*,\varphi'}\{\varphi(x)=1\} \ge \inf_{\varphi'\in\Omega}\mu_{V_0}^{h^*,\varphi'}\{\varphi(x)=1\}\,. \tag{4.9}$$

Take now $\bar h = \frac{1}{4\beta}e^{-4\beta k} \le h^*$. Because $\{\varphi(x)=1\}$ is a negative event, by the FKG property,

$$\mu_{V_0}^{h^*,\varphi'}\{\varphi(x)=1\} \ge \mu_{V_0}^{\bar h,\varphi'}\{\varphi(x)=1\}\,. \tag{4.10}$$

Furthermore, if we let

$$A = S_+(V_0, k+2, \partial^+V_1)$$

(the event $S_+(\Lambda,j,V)$ has been defined in Proposition 3.5), by 5.1 of [CM] and A1.1 of [CM] we have

$$\begin{aligned}\mu_{V_0}^{\bar h,\varphi'}\{\varphi(x)=1\} &\ge \mu_{V_0}^{\bar h,\varphi'}(\varphi(x)=1\,|\,A^c)\,\mu_{V_0}^{\bar h,\varphi'}(A^c) \ge \\ &\ge \frac{1}{2}\sup_{\substack{V:\,V_1\subset V\subset V_0\\ V\text{ connected and}\\ \text{simply connected}}}\mu_V^{\bar h,k+1}\{\varphi(x)=1\}\end{aligned} \tag{4.11}$$

On the other side, letting

$$F = \{\varphi \in \Omega_V : \varphi(y) = k+1 \ \forall y \in V, \text{ such that } |x-y| = 1\}$$

and using Proposition 3.5 of [CM] , we find, for each simply connected V containing V_1,

$$\begin{aligned}
&\mu_V^{\bar h,k+1}\{\varphi(x)=1\} \geq \mu_V^{\bar h,k+1}(\varphi(x)=1 \,|\, F)\, \mu_V^{\bar h,k+1}(F) \geq \\
&\geq \mu_V^{\bar h,k+1}(\varphi(x)=1 \,|\, F) \Big(1 - 4 \sup_{y\in V} \mu_V^{\bar h,k+1}\{\varphi(y) \neq k+1\}\Big) \geq \\
&\frac{1}{2}\, \mu_{\{x\}}^{\bar h,k+1}\{\varphi(x)=1\}
\end{aligned} \tag{4.12}$$

But an explicit computation, gives

$$\mu_{\{x\}}^{\bar h,k+1}\{\varphi(x)=1\} \geq \frac{1}{2} e^{-4\beta k}\,.$$

In this way we have found

$$\inf_{J\in\delta\Lambda_1} \inf_{\varphi'\in\Omega} \mu_{\Lambda_1}^{J,h^*,\varphi'}\{\varphi(x)=1\} \geq \frac{1}{8} e^{-4\beta k} \qquad \forall x \in A_1'' \quad \square$$

Appendix A1. The block dynamics

Given

(i) the hamiltonian (1.1), with parameters (β, h, J)

(ii) a boundary condition $\psi \in \Omega$

(iii) a collection $\mathcal{D} = \{\Lambda_1, \ldots, \Lambda_n\}$ of finite (possibly overlapping) subsets of $\mathbb{Z}^2$,

and letting

$$\Lambda = \cup_{i=1}^{n} \Lambda_i$$

we define as the standard block dynamics (with respect to $\beta, h, J, \psi, \mathcal{D}$) the Markov process on Ω_Λ whose generator is

$$(L_{\mathcal{D}}^{\psi} f)(\varphi) = \sum_{i=1}^{n} \sum_{\eta \in \Omega} \mu_{\Lambda_i}^{\varphi}(\eta)[f(\eta) - f(\varphi)] \qquad \text{for all } \varphi \text{ such that } \varphi_{\Lambda^c} = \psi_{\Lambda^c} \,. \tag{A1.1}$$

The sum over η is actually restricted to those configuration that agree with φ outside Λ_i because of (1.6). A straightforward computation shows that $L_{\mathcal{D}}^{\psi}$ is still a bounded self–adjoint operator on $L^2(\Omega_\Lambda, d\mu_\Lambda^{\psi})$. We assume to have chosen β, h, J and won't mention them explicitly in the following. Then

Proposition A1.1. *For all $\beta, h > 0$, all J, ψ, let L_Λ^{ψ} be given by (1.8) with transition rates satisfying $(H_1) - (H_5)$. Then*

$$\operatorname{gap}(L_\Lambda^{\psi}) \geq \operatorname{gap}(L_{\mathcal{D}}^{\psi}) \inf_i \inf_{\varphi \in \Omega} \operatorname{gap}(L_{\Lambda_i}^{\varphi}) \Big(\sup_{x \in \Lambda} \#\{i : \Lambda_i \ni x\} \Big)^{-1} \tag{A1.2}$$

Proof. Let

$$g = \inf_i \inf_{\varphi \in \Omega} \operatorname{gap}(L_{\Lambda_i}^{\varphi})$$

Thanks to (1.12), (A1.2) is proven if we can show that

$$\mathcal{E}_{\mathcal{D}}^{\psi}(f,f) \leq g^{-1} \sup_{x \in \Lambda} \#\{i : \Lambda_i \ni x\} \, \mathcal{E}_\Lambda^{\psi}(f,f) \qquad \text{for all } f \in L^2(\Omega_\Lambda, d\mu_\Lambda^{\psi}) \,. \tag{A1.3}$$

But, using (1.7), we find

$$\begin{aligned} \mathcal{E}_{\mathcal{D}}^{\psi}(f,f) &\leq \frac{1}{2} \sum_{\varphi \in \Omega} \mu_\Lambda^{\psi}(\varphi) \sum_i \sum_{\eta \in \Omega} \mu_{\Lambda_i}^{\varphi}(\eta) \, [f(\eta) - f(\varphi)]^2 = \\ &= \frac{1}{2} \sum_{\varphi \in \Omega} \sum_{\varphi' \in \Omega} \mu_\Lambda^{\psi}(\varphi') \sum_i \mu_{\Lambda_i}^{\varphi'}(\varphi) \sum_{\eta \in \Omega} \mu_{\Lambda_i}^{\varphi}(\eta) \, [f(\eta) - f(\varphi)]^2 = \\ &= \sum_{\varphi' \in \Omega} \mu_\Lambda^{\psi}(\varphi') \sum_i \operatorname{Var}_{\Lambda_i}^{\varphi'}(f) \leq g^{-1} \sum_{\varphi' \in \Omega} \mu_\Lambda^{\psi}(\varphi') \sum_i \mathcal{E}_{\Lambda_i}^{\varphi'}(f,f) \end{aligned} \tag{A1.4}$$

On the other side, by (1.7) again,

$$
\begin{aligned}
&\sum_{\varphi'\in\Omega}\mu^{\psi}_{\Lambda}(\varphi')\sum_i \mathcal{E}^{\varphi'}_{\Lambda_i}(f,f) = \\
&= \frac{1}{2}\sum_{\varphi'\in\Omega}\mu^{\psi}_{\Lambda}(\varphi')\sum_i\sum_{\varphi\in\Omega}\mu^{\varphi'}_{\Lambda_i}(\varphi)\sum_{x\in\Lambda_i,\, s=\pm1} c(x,\varphi,s)\,[f(\varphi^{x,s})-f(\varphi)]^2 \le \\
&\le \sup_{x\in\Lambda}\#\{i:\Lambda_i\ni x\}\,\mathcal{E}^{\psi}_{\Lambda}(f,f)
\end{aligned}
$$

(A1.3) and the proposition are thus proven. □

We want to introduce now a coupling between two copies of the block dynamic process in such a way to preserve the order given by the initial conditions. Given ψ, φ, $\bar\varphi$ such that $\psi_{\Lambda^c}=\varphi_{\Lambda^c}=\bar\varphi_{\Lambda^c}$, consider the the process $(\varphi_t,\bar\varphi_t)$ which starts from $(\varphi,\bar\varphi)$ at time zero and evolves according to the generator

$$
(\hat L^{\psi}_{\mathcal{D}}f)(\varphi,\bar\varphi)=\sum_i\sum_{\eta,\bar\eta\in\Omega}\nu^{\varphi,\bar\varphi}_{\Lambda_i}(\eta,\bar\eta)\,[f(\eta,\bar\eta)-f(\varphi,\bar\varphi)]
$$

where $\nu^{\varphi,\bar\varphi}_{\Lambda_i}$ is the standard coupling defined in Section 1.

Given the starting point $(\varphi,\bar\varphi)\in\Omega^2$, the process $(\varphi_t,\bar\varphi_t)$ can be constructed explicitly as follows. Let

- (a) $\tau(j)$, $j=1,2,\ldots$, be the jump times of a Poisson process with rate $n=$ number of blocks.
- (b) $(\eta,\bar\eta)^{(\varphi,\bar\varphi)}_{i,j}$, for $(\varphi,\bar\varphi)\in\Omega^2$, $i=1,\ldots,n$ and $j=1,2,\ldots$ be a collection of Ω^2 valued independent random variables (and independent of the τ_j's) distributed according to $\nu^{\varphi,\bar\varphi}_{\Lambda_i}(\eta,\bar\eta)$.
- (c) $l(j)$ for $j=1,2,\ldots$ be a collection of *i.i.d.* (and independent of what has been defined before) random variables with uniform distribution on $\{1,\ldots,n\}$

Define now

- (1) $(\varphi_0,\bar\varphi_0)=(\varphi,\bar\varphi)$
- (2) $(\varphi_t,\bar\varphi_t)=(\varphi_{\tau(j)},\bar\varphi_{\tau(j)})$, for all $\tau(j)\le t<\tau(j+1)$
- (2) $(\varphi_{\tau(j+1)},\bar\varphi_{\tau(j+1)})=(\eta,\bar\eta)^{(\varphi_{\tau(j)},\bar\varphi_{\tau(j)})}_{l(j+1),\,j+1}$

Then we have

Proposition A1.2. *With respect to the quantities previously defined*

- (i) *both φ_t and $\bar\varphi_t$ have the same distribution as the process associated with the generator (A1.1)* (*ii*) *if* $\varphi \le \bar\varphi$, *then*

$$\mathrm{Prob}\{\varphi_t \le \bar\varphi_t\} = 1 \qquad \text{for all } t \ge 0$$

(*iii*) *If* $\mathrm{Prob}\{\varphi_1 = \bar\varphi_1\} \ge p$ *for all starting points* $\varphi, \bar\varphi$ *such that* $\varphi \le \bar\varphi$ *(and such that they agree with* ψ *on* Λ^c*), then*

$$\mathrm{gap}(L_{\mathcal{D}}^{\psi}) \ge p$$

(i) is a standard computation. (ii) follows from the definition of $(\varphi_t, \bar\varphi_t)$ and from the properties of the coupling measure $\nu_\Lambda^{\varphi,\bar\varphi}$. To get (iii) we use the (time) Markov property which yieds

$$\mathrm{Prob}\{\varphi_t \ne \bar\varphi_t\} \le (1-p)^{\lfloor t \rfloor} \le e^{-(t-1)p} \quad \square$$

Appendix A2.

Proposition A2.1. *Let $N \geq 100$, $L \geq 2N$. Then there exist integers s, L_i, such that*

$$L = L_0 > L_1 > \cdots > L_s = N$$
$$\frac{5}{4}L_i \leq L_{i-1} \leq \frac{7}{4}L_i \qquad i = 1, \ldots, s$$

Proof. Let also X be the set of L's such that the proposition holds. We want to show that X contains all integers greater than or equal to $2N$. Let $N_0^+ = N_0^- = N$ and, for each $i \in \mathbb{Z}_+$ we set $K_i = [N_i^-, N_i^+]$, where

$$N_i^+ = \lfloor \tfrac{7}{4} N_{i-1}^+ \rfloor \quad \text{and} \quad N_i^- = \lceil \tfrac{5}{4} N_{i-1}^- \rceil$$

($\lceil x \rceil$ is the smallest integer not less than x). Obviously

$$K_i \supset \bigcup_{n \in K_{i-1}} \left[\, \lceil 5n/4 \rceil, \lfloor 7n/4 \rfloor \,\right]$$

and, since for $n \geq 100$,

$$\lfloor 7n/4 \rfloor \geq \lceil 5(n+1)/4 \rceil$$

we have in fact

$$K_i = \bigcup_{n \in K_{i-1}} \left[\, \lceil 5n/4 \rceil, \lfloor 7n/4 \rfloor \,\right].$$

So, for each $m \in K_i$ there exists $n \in K_{i-1}$ such that

$$\frac{5}{4} \leq \frac{m}{n} \leq \frac{7}{4}\,.$$

This implies

$$X \supset \bigcup_{i=1}^{\infty} K_i\,.$$

Furthermore, since $N \geq 100$, we have

$$N_2^- \leq (5/4)^2 N + 5/4 + 1 \leq (7/4)N - 1 \leq N_1^+$$

and it is easy to check, by induction, that

$$N_i^- \leq N_{i-1}^+ \qquad \forall i \geq 2\,.$$

Thus we get

$$\bigcup_{i=1}^{\infty} K_i \supset \bigcup_{i=2}^{\infty} [N_{i-1}^+, N_i^+] = [N_1^+, \infty) \supset [2N, \infty)\,. \quad \square$$

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Figure captions

(1) Proof of Proposition 4.1