On the Layering Transition of an SOS Surface Interacting with a Wall. I. Equilibrium Results

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Abstract. We consider the model of a 2D surface above a fixed wall and attracted towards it by means of a positive magnetic field h in the solid on solid (SOS) approximation, when the inverse temperature β is very large and the external field h is exponentially small in β. We improve considerably previous results by Dinaburg and Mazel on the competition between the external field and the entropic repulsion with the wall, leading, in this case, to the phenomenon of layering phase transitions. In particular we show, using the Pirogov Sinai scheme as given by Zahradník, that there exists a unique critical value $h_k^*(\beta)$ in the interval $(\frac{1}{4}e^{-4\beta k}, 4e^{-4\beta k})$ such that, for all $h \in (h_{k+1}^*, h_k^*)$ and β large enough, there exists a unique infinite volume Gibbs state. The typical configurations are small perturbations of the ground state represented by a surface at height $k+1$ above the wall. Moreover, for the same choice of the thermodynamic parameters, the influence of the boundary conditions of the Gibbs measure in a finite cube decays exponentially fast with the distance from the boundary. When $h = h_k^*(\beta)$ we prove instead the convergence of the cluster expansion for both k and $k+1$ boundary conditions. This fact signals the presence of a phase transition. In the second paper of this series we will consider a Glauber dynamics for the above model and we will study the rate of approach to equilibrium in a large finite cube with arbitrary boundary conditions as a function of the external field h. Using the results proven in this paper we will show that there is a dramatic slowing down in the approach to equilibrium when the magnetic field takes one of the critical values and the boundary conditions are free (absent).

Key Words: SOS model, Layering transition, Pirogov-Sinai theory, Relaxation time

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0. Introduction

This is the first of a series of two papers (the second part is [CM]) where we analyze a problem which originates from the study of the rate of approach to equilibrium of Glauber dynamics for discrete spins systems on the lattice when the thermodynamic parameters, e.g. the temperature and the magnetic field, are in the one phase region of the phase diagram. It is a widespread folklore that, in such a case, the relaxation to the equilibrium Gibbs measure should be exponentially fast with constants very weakly depending on the size of the volume, e.g. a large d-dimensional cube, and on the boundary conditions.

In the recent years there has been considerable progress in a deeper understanding of the above problem; we refer the reader in particular to the series of papers by Stroock and Zegarlinski (see [SZ] and references therein), Martinelli and Olivieri ([MO1] and [MO2]), Martinelli, Olivieri and Schonmann [MOS], Lu and Yau [LY].

In particular in [MO1] it was shown that, for ferromagnetic Ising type models in arbitrary dimension, if one considers the dynamics directly in the infinite volume, then the relaxation to equilibrium is indeed exponentially fast for any temperature above the critical one or for small temperature and external field different from zero.

The extension of the above result to the finite volume case with bounds on the relevant constants (e.g. gap in the spectrum of generator of the dynamics or the logarithmic Sobolev constant) uniform in the size of the system and in the boundary conditions, turned out to be quite hard and it required "mixing" hypotheses on the unique Gibbs state (e.g. complete analiticity in [SZ] or *strong mixing* in [MO1], [MO2] [LY]) that, in general, have been proved only deep inside the one phase region (see for more details [MO1]). There is one important exception represented by two dimensional systems. In such cases, in fact, it has been proved in [MOS] that the same condition on the unique Gibbs state that assures the exponential convergence to equilibrium for the Glauber dynamics directly in the infinite volume, also implies a uniform lower bound on the gap (and a uniform upper bound on the logarithmic Sobolev constant) for the same dynamics in any regular volume, e.g. a square or a rectangle.

Thus, in particular, the folklore mentioned at the beginning holds true for the two dimensional Ising model (see [MOS] for a proof when the temperature is just above the critical one or it is quite low and the external field is different from zero, and [SSh] for the remaining case).

For three dimensional systems the argument of [MOS] breaks down and no "finite volume results" are available near the boundary of the one phase region, particularly near a line of a first order phase transition. Already in [MO1] (see also [MOS]) it was discussed the possibility that, even if the thermodynamic parameters correspond to a unique phase, in finite volume one could have some sort of "boundary phase transition" which could slow down dramatically the approach to equilibrium. Such a case was refered to in [MO1] as the appearance of Bassuev states.

In order to clarify the discussion, let us consider the three dimensional Ising model in a cube Q of side L, at low temperature and small, positive external field h. As boundary conditions we take -1 on the bottom face $\mathcal{B}$ of the cube and free (*i.e.* absent) on the other faces. Since the magnetic field is positive, the typical configurations of the systems will be mostly pluses (plus phase) away from the bottom face. Thus there will be, with large probability, a unique dual contour Γ separating the plus bulk phase from the minus boundary conditions on the bottom face, and it is quite clear that the statistical properties of such a contour will play an important role in the mixing properties of the Gibbs measure of the system and on the rate of approach to equilibrium for an associated Glauber dynamics.

In order to attack the problem one can simplify the model by approximating the contour Γ with a two dimensional surface $\varphi = \{\varphi(x)\}_{x\in\mathcal{B}}$, where the random variable $\varphi(x)$ represents the height of the surface at x above the bottom face of the cube, and one may assume that the probability distribution of the surface is that of the solid on solid model (SOS)

$$\mu_{\mathcal{B}}(\varphi) \;=\; \frac{1}{Z_{\mathcal{B}}}\exp\Big[-\frac{\beta}{2}\sum_{\substack{x,y\in\mathcal{B};\\ |x-y|=1}} |\varphi(x)-\varphi(y)| \;-\; \beta h\sum_{x\in\mathcal{B}}\varphi(x)\Big] \tag{0.1}$$

where $Z_{\mathcal{B}}$ is the associated partition function; boundary conditions are missing here because of our particolar choice of free boundary conditions on the lateral faces of the cube Q, but they can be introduced without problems by simply adding a term

$$\sum_{\substack{x\in\mathcal{B},y\notin\mathcal{B};\\ |x-y|=1}} |\varphi(x)-\psi(y)|$$

to the Hamiltonian

It is important to remember that the random variables $\{\varphi(x)\}_{x\in\mathcal{B}}$ are constrained to be positive and therefore the above model is what is known in the literature as "a SOS surface interacting with a hard wall".

For such a model, and variants of it, it was realized already long ago (see e.g. the important papers by Bricmont, El Meloukki and Fröhlich [BEF], Frölich and Pfister [FP1], [FP2] and more recently the works by Lebowitz and Maes [LM], Bolthausen, Zeitouni and Deuschel [BZD], Dinaburg and Mazel [DM]) that the relevant statistical properties of the surface φ are determined by the competition between the attraction to the wall due to the external field and the entropic repulsion from the wall due to the wall itself. In particular in [DM] (the basic reference for our work) it has been shown for the first time the phenomenon of "layering phase transition":

if the external field h is chosen suitably small with β, more precisely

$$e^{-4\beta k+\frac{\beta}{100}} \leq \beta h \leq e^{-4\beta(k-1)-\frac{\beta}{100}}, \qquad k=1,2,\ldots$$

then

(*i*) the partition function $Z^k_{\mathcal{B}}$ with boundary conditions identically equal to k admits a convergent cluster expansion around the "ground state" surface

$$\varphi_{gs}(x) = k \quad \forall x$$

(*ii*) there is a unique infinite volume Gibbs state in the class of those states constructed using *uniformly bounded boundary conditions.*

The above result suggest that there exists a unique critical value of the magnetic field $h^*_k(\beta)$ in the interval

$$(e^{-4\beta k-\frac{\beta}{100}}, e^{-4\beta k+\frac{\beta}{100}})$$

such that, if $h = h^*_k(\beta)$, then, in absence of boundary conditions, the surface φ is undecided whether to stay at level k or at level $k+1$. Such a situation is reminiscent of the ordinary 2D Ising model at low temperature, zero external field and open boundary conditions, if we consider the new ±1 variables

$$\sigma(x) = \mathrm{sign}\Big(\varphi(x) - k - \frac{1}{2}\Big)$$

We introduce at this point a Glauber dynamics for the surface φ which is as close as possible to the original Glauber dynamics of the 3D Ising model in Q; namely at each updating we modify by only ±1 the heights $\{\varphi(x)\}_{x\in\mathcal{B}}$ at only one site x, with rates such that the resulting Markov process is reversible with respect to the SOS Gibbs measure $\mu_{\mathcal{B}}(\varphi)$. Then, keeping in mind the above analogy with the 2D Ising model, one may conjecture that the qualitative behaviour of the time evolution of the variables $\{\sigma(x)\}_{x\in\mathcal{B}}$ will be that of the Glauber dynamics for the 2D Ising model in the phase coexistence region. For the latter one of us (see [M]) has shown that its relaxation time t_{rel} is exponentially large in the side of the box: $t_{rel} \approx e^{\beta\tau_\beta L}$, where τ_β denotes the surface tension in the horizontal direction.

Thus we expect that, at the critical value $h^*_k(\beta)$ of the external field h, the relaxation time for both the SOS surface and the 3D Ising model in Q, will be exponentially large in the side L of Q. In analogy with the results of [M], we expect that the constant in the exponential is related to the so called step free energy (see e.g. [Mi]).

In this series of two papers we prove the above picture for the SOS model, but we do not compute the exact asymptotics of the relaxation time as $L \to \infty$ and we do not go beyond the SOS approximation, that is we do not establish any rigorous connection between our results and the three dimensional dynamical Ising model.

More precisely in this paper we consider only the equilibrium problem and we improve the results by Dinaburg and Mazel as follows

Theorem 1. *There exists β_0 such that for all $\beta \geq \beta_0$ there are positive numbers $\{h_k^*(\beta)\}_{k=1}^{k_{max}}$, with $k_{max} = \lfloor e^{\frac{\beta}{20000}} \rfloor$, such that the following holds for $k = 1, \ldots, k_{max}$*

(*i*) $\frac{1}{4} e^{-4\beta k} \leq \beta h_k^*(\beta) \leq 4 e^{-4\beta k}$

(*ii*) *if $h_k^*(\beta) < h < h_{k-1}^*(\beta)$ (define $h_0^*(\beta) = +\infty$) then*

  (*a*) *there exists a unique Gibbs measure for the interaction (0.1).*

  (*b*) *there exist $m(\beta, h) > 0$, $C(\beta, h) > 0$ such that for any $N \geq \lfloor 8/h + 1 \rfloor$*

$$\sup_{\psi, \psi' \in \Omega} |E_{Q_N}^{h,\psi} \varphi(0) - E_{Q_N}^{h,\psi'} \varphi(0)| \leq C(\beta, h) e^{-m(\beta,h)\, N}$$

  where $E_{Q_N}^{h,\psi}(\varphi(0))$ denotes the expected value of the height of the surface at $x = 0$ in a square Q_N of side N and center at the origin, with boundary conditions ψ.

(*iii*) *if $h = h_k^*(\beta)$ then both partitions functions $Z_{Q_N}^k$ and $Z_{Q_N}^{k+1}$, with boundary conditions $\psi \equiv k$ and $\psi \equiv k+1$ respectively, admit a convergent cluster expansion. Hence there are at least two distinct extreme Gibbs measures.*

In [CM], the second paper of this series, we turn our attention to the dynamics of the surface in a square Q_N of side N with boundary conditions ψ and magnetic field h; more precisely we focus on the analysis of the $\text{gap}_{Q_N}^{h,\psi}$ in the spectrum of its generator. Our main result in [CM] is

Theorem 2. *In the same setting as Theorem 1 we have for all $k = 1, \cdots, k_{max}$*

(*i*) *if $h_k^*(\beta) < h < h_{k-1}^*(\beta)$ then there exist $L_0(\beta, h)$, $\kappa(\beta, h) > 0$ such that*

$$\inf_{L \geq L_0} \inf_{\psi \in \Omega} \text{gap}^{h,\psi}(Q_L) \geq \kappa(\beta, h)$$

(*ii*) *if $h = h_k^*(\beta)$, then there exist positive constants $C_1(\beta, h)$, $C_2(\beta, h)$ such that for all $N > 10/h$*

$$C_1(\beta, h)\, e^{-100\beta k N} \leq \text{gap}^{h,\emptyset}(Q_N) \leq C_2(\beta, h)\, e^{-\frac{1}{40}\beta N}$$

where $\emptyset$ means free boundary conditions.

The paper, which unfortunately contains some very technical parts due to the fact that the random variables we are dealing with are unbounded, is organized as follows:

- In section 1 we define the model and give the main result.
- In section 2 we express the partition function as a gas of interacting cylinders and we recall several results, essentially due to Dinaburg and Mazel, on such models.
- In section 3 we prove some basic apriori bounds, uniformly in the boundary conditions, on the distribution of the variables $\{\phi(x)\}$ which will be used several times.
- In section 4, following the approach of Zahradník to the Pirogov and Sinai theory [Z], we prove the existence of the critical value $h^*_k(\beta)$ via cluster expansion methods and we prove part (i), $(ii-a)$ and (iii) of Theorem 1.
- In section 5, 6 we show that, if the external field h is different from the critical one $h^*_k(\beta)$, then the influence of boundary conditions is confined close to the boundary *independently* of the their strength. This proves part $(ii{-}b)$ of Theorem 1. A corollary (weak mixing in the language of [MO1]) of this result is discussed in Section 7.
- Section 8 contains a sketch of the proof that, in case (iii) all translation invariant Gibbs measures are convex combinations of the ones given by the boundary conditions k and $k+1$
- Finally in the appendix we have collected some minor technical results.

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1. Preliminaries and results

1.1. General definitions

We consider the two dimensional lattice $\mathbb{Z}^2$ whose elements are called *sites* and its dual $\mathbb{Z}^2_* = \mathbb{Z}^2 + (1/2, 1/2)$. For $x, y \in \mathbb{R}^2$ we define two distances

$$d(x,y) = |x-y| = \sum_{i=1}^{2} |x_i - y_i| \quad \text{and} \quad d_\infty(x,y) = |x-y|_\infty = \max_{i=1,2} |x_i - y_i| \,.$$

$[x,y]$ is the *closed segment* with x, y as its endpoints. The *edges* of $\mathbb{Z}^2$ ($\mathbb{Z}^2_*$) are those $e = [x,y]$ with x, y nearest neighbors in $\mathbb{Z}^2$ ($\mathbb{Z}^2_*$). Given e edge of $\mathbb{Z}^2$, e^* is the unique edge in $\mathbb{Z}^2_*$ that intersects e. The *boundary of an edge* $e = [x,y]$ is $\delta e = \{x, y\}$ The *boundary of a subset of edges* α is the set of sites $\delta\alpha$ that belong to an odd number of edges of α. A set of edges is called *closed* if its boundary is empty.

We will often consider our model on a square

$$Q_N = \begin{cases} \{\, (x_1,x_2) \in \mathbb{Z}^2 : -L \le x_i \le L,\ i = 1,2 \,\} & \text{if } N = 2L+1 \\ \{\, (x_1,x_2) \in \mathbb{Z}^2 : -L+1 \le x_i \le L,\ i = 1,2 \,\} & \text{if } N = 2L \end{cases}$$

Λ and V will denote arbitrary subsets of $\mathbb{Z}^2$. If Λ is finite we write $\Lambda \subset\subset \mathbb{Z}^2$. The cardinality of Λ is denoted by $|\Lambda|$. We define four kinds of *boundaries*

$$\begin{aligned} \partial\Lambda &= \{x \in \Lambda : d(x, \Lambda^c) = 1\} \\ \bar{\partial}\Lambda &= \{x \in \Lambda : d_\infty(x, \Lambda^c) = 1\} \\ \partial^+\Lambda &= \{x \in \Lambda^c : d(x, \Lambda) = 1\} \\ \delta\Lambda &= \{e^* = [x,y]^* : \{x,y\} \cap \Lambda \ne \emptyset,\ \{x,y\} \cap \Lambda^c \ne \emptyset\} \,, \end{aligned}$$

where $\Lambda^c = \mathbb{Z}^2 \setminus \Lambda$.

$(x_1, \ldots, x_n)$ is called a *path* from x_1 to x_n if $|x_{i+1} - x_i| = 1$ for $i = 1, \ldots, n-1$. A $*$–path is the same as a path with $|x_{i+1} - x_i| = 1$ replaced by $d_\infty(x_i, x_{i+1}) = 1$. A ($*$–) path is called *self-avoiding* if $x_i \ne x_j$ for all $\{i,j\}$ such that $i \ne j$ and $\{i,j\} \ne \{1,n\}$. If $x_1 = x_n$ the ($*$–) path is called *closed*.

$\Lambda \subset \mathbb{Z}^2$ is said to be *connected* ($*$–connected) if for all x, y in Λ there exists a path ($*$–path) from x to y which is entirely contained in Λ. $\Lambda \subset\subset \mathbb{Z}^2$ is called *simply connected* if Λ^c is $*$–connected. A set of edges α is *connected* if the union of all its edges is connected in $\mathbb{R}^2$.

We denote by C_B the set of all finite closed connected set of edges of $\mathbb{Z}^2_*$. If $\alpha \in C_B$ then we define the *interior* of α (see Fig. 1) as the set of all sites $x = (x_1, x_2) \in \mathbb{Z}^2$ such that the half line

$$\{x_1\} \times [x_2, +\infty)$$

intersects α in an odd number of points. The interior of α is denoted by $\bar{\alpha}$ and is always a (possibly disconnected) simply connected subset of $\mathbb{Z}^2$ for each $\alpha \in C_B$. $C_B(V)$ is the set of all α in C_B such that $\bar{\alpha} \subset V$.

1.2 The SOS model and results

The *configuration space* of the model is $\Omega = \mathbb{Z}_+^{\mathbb{Z}^2}$, or $\Omega_V = \mathbb{Z}_+^V$ for some $V \subset \mathbb{Z}^2$. An element of Ω_V will usually be denoted by $\varphi = \{\varphi(x),\, x \in V\}$. If $U \subset V \subset \mathbb{Z}^2$, and $\varphi \in \Omega_V$ we denote by φ_U the restriction of φ to the set U.

Given $V \subset\subset \mathbb{Z}^2$ and some *boundary condition* (b.c.) $\psi \in \Omega$, one defines the hamiltonian as

$$H_V^{h,\psi}(\varphi) = \frac{1}{2} \sum_{\substack{x,y \in V \\ |x-y|=1}} |\varphi(x) - \varphi(y)| + \sum_{\substack{x \in V,\, y \in V^c \\ |x-y|=1}} |\varphi(x) - \psi(y)| + h \sum_{x \in V} \varphi(x)\,, \tag{1.1}$$

for $\varphi \in \Omega_V$. For technical reasons we will need to consider a more general version of (1.1)

$$H_V^{J,h,\psi}(\varphi) = \frac{1}{2} \sum_{\substack{x,y \in V \\ |x-y|=1}} J(x,y)|\varphi(x) - \varphi(y)| + \sum_{\substack{x \in V,\, y \in V^c \\ |x-y|=1}} J(x,y)|\varphi(x) - \psi(y)| + h \sum_{x \in V} \varphi(x)\,, \tag{1.2}$$

and we always assume $0 \le J(x,y) \le 1$ for all x, y. We (improperly) write $J \in \delta V$ if $J(x,y) < 1$ only for the boundary terms, *i.e.* if $J(x,y) = 1$ unless $[x,y]^* \in \delta V$. If we take $J(x,y) = 0$ for all boundary terms, then we have *free boundary conditions* that we also denote with

$$H_V^{h,\emptyset}(\varphi) = \frac{1}{2} \sum_{\substack{x,y \in V \\ |x-y|=1}} |\varphi(x) - \varphi(y)| + h \sum_{x \in V} \varphi(x)\,. \tag{1.3}$$

The partition function is given by

$$Z^{J,h,\psi}(V) = \sum_{\varphi \in \Omega_V} e^{-\beta H_V^{J,h,\psi}(\varphi)}\,. \tag{1.4}$$

When $J(x,y) = 1$ for all x, y, we drop the superscript J. In what follows we assume to have chosen β large enough once and for all, so we do not usually write our quantities as β dependent. If $\psi(x) = n$ for each x then we say that the system has n–boundary condition.

Given any set α of dual edges (for instance $\alpha = \delta V$) we define

$$|\alpha|_J = \sum\nolimits_{e^*=[x,y]^*\in\alpha} J(x,y)\,. \tag{1.5}$$

(if $J=1$ everywhere this is just the ordinary *length* of α).
For $U \subset \mathbb{Z}^2$, let $\mathbf{F}_U$ be the σ–algebra generated by the collection of sets

$$\{\varphi \in \Omega : \varphi(x) = n\}_{x\in U, n\in\mathbb{Z}_+}$$

(if U is finite $\mathbf{F}_U$ has an obvious one–to–one correspondence with the set of all subsets of Ω_U), and let $\mathbf{F} = \mathbf{F}_{\mathbb{Z}^2}$. The (finite volume) conditional Gibbs measure on $(\Omega, \mathbf{F})$ associated with the Hamiltonian (1.2) is defined as

$$\mu_V^{J,h,\psi}(\varphi) = \begin{cases} \left(Z^{J,h,\psi}(V)\right)^{-1} e^{-\beta H_V^{J,h,\psi}(\varphi)} & \text{if } \varphi(x) = \psi(x) \text{ for all } x \in V^c \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

We also regard $\mu_V^{J,h,\psi}$ as a measure on Ω_V by extending each configuration $\varphi \in \Omega_V$ to the whole space in such a way that it agrees with the boundary conditions outside V. The expectation with respect to the measure (1.6) is denoted by $\mathbf{E}_V^{J,h,\psi}(\cdot)$. The set of measures (1.6) satisfies the compatibility conditions

$$\mu_\Lambda^{J,h,\psi}(\varphi) = \sum_{\varphi'\in\Omega} \mu_\Lambda^{J,h,\psi}(\varphi')\, \mu_V^{J,h,\varphi'}(\varphi) \qquad \text{for all } V \subset \Lambda \subset\subset \mathbb{Z}^2 \tag{1.7}$$

A probability measure μ on $(\Omega, \mathbf{F})$ is called a Gibbs measure (see for instance [G]) for the interaction (1.2) if, for each $V \subset\subset \mathbb{Z}^2$, $\varphi_0 \in \Omega_V$

$$\mu\{\varphi\in\Omega : \varphi = \varphi_0 \text{ on } V | \mathbf{F}_{V^c}\}(\psi) = \mu\{\varphi\in\Omega : \varphi = \varphi_0 \text{ on } V | \varphi = \psi \text{ on } \partial^+ V\} = \mu_V^{J,h,\psi}(\varphi)$$

for μ–almost every ψ. The first equality says that μ is a *Markov* random field.

One introduces a partial order on the Ω_V by saying that $\varphi \le \varphi'$ if $\varphi(x) \le \varphi'(x)$ for all $x \in V$. A function $f : \Omega_V \mapsto \mathbb{R}$ is called *monotone increasing (decreasing)* if $\varphi \le \varphi'$ implies $f(\varphi) \le f(\varphi')$ ($f(\varphi) \ge f(\varphi')$). An event is called *positive* (*negative*) if its characteristic function is increasing (decreasing). Given two probability measures μ, μ' on $(\Omega_V, \mathbf{F}_V)$, we write $\mu \le \mu'$ if $\mu(f) \le \mu'(f)$ for all increasing funcions f (with $\mu(f)$ we denote the expectation with respect μ).

In the following we will take advantage of the FKG inequalities [FKG] which state that

(1) if $\psi \le \psi'$, then $\mu_V^{J,h,\psi} \le \mu_V^{J,h,\psi'}$
(2) if f and g are increasing then $\mathbf{E}_V^{J,h,\psi}(fg) \ge \mathbf{E}_V^{J,h,\psi}(f)\, \mathbf{E}_V^{J,h,\psi}(g)$

Given two finite volume Gibbs measures $\mu_V^{J,h,\psi}$, $\mu_V^{J,h,\psi'}$ there exists a coupling between them which preserve the order of the b.c, *i.e.* a probability measure $\nu_V^{J,h,\psi,\psi'}$ on $\Omega \times \Omega$ such that (we drop J, h for simplicity)

(1) $\nu_V^{\psi,\psi'}\{(\varphi,\varphi') : \varphi = \varphi_0\} = \mu_V^{\psi}(\varphi_0)$ for all $\varphi_0 \in \Omega_V$
(2) $\nu_V^{\psi,\psi'}\{(\varphi,\varphi') : \varphi' = \varphi_0\} = \mu_V^{\psi'}(\varphi_0)$ for all $\varphi_0 \in \Omega_V$
(3) If $\psi \le \psi'$, then $\nu_V^{\psi,\psi'}\{(\varphi,\varphi') : \varphi \le \varphi'\} = 1$

Finally we recall for the reader's convenience the main result in [DM]

Theorem [DM]. *There exists β_0 such that for all $\beta \ge \beta_0$, if*

$$e^{-4\beta k + \frac{\beta}{100}} \le \beta h \le e^{-4\beta(k-1) - \frac{\beta}{100}}, \qquad k = 1, 2, \ldots$$

then the set of all Gibbs measures (for the interaction (1.1)) which can be obtained as an infinite volume limit of finite volume Gibbs measures with bounded boundary conditions contains exactly one element.

In this paper we prove the following results

Theorem 1.1. *There exists β_0 such that for all $\beta \ge \beta_0$ there are positive numbers $\{h_k^*(\beta)\}_{k=1}^{k_{max}}$, with $k_{max} = \lfloor e^{\frac{\beta}{20000}} \rfloor$, such that the following holds for $k = 1, \ldots, k_{max}$*

(i) $\frac{1}{4} e^{-4\beta k} \le \beta h_k^*(\beta) \le 4 e^{-4\beta k}$
(ii) if $h_k^*(\beta) < h < h_{k-1}^*(\beta)$ (define $h_0^*(\beta) = +\infty$) then
  (a) there exists a unique Gibbs measure for the interaction (1.1).
  (b) there exist $m(\beta,h) > 0$, $C(\beta,h) > 0$ such that for any $N \ge \lfloor 8/h + 1 \rfloor$

$$\sup_{\psi,\psi' \in \Omega} |E_{Q_N}^{h,\psi} \varphi(0) - E_{Q_N}^{h,\psi'} \varphi(0)| \le C(\beta,h) e^{-m(\beta,h)\, N} \tag{1.8}$$

(iii) if $h = h_k^*(\beta)$ then both partitions functions $Z^{h,k}(Q_N)$ and $Z^{h,k+1}(Q_N)$, with boundary conditions $\psi \equiv k$ and $\psi \equiv k+1$ respectively, admit a convergent cluster expansion. Hence there are at least two distinct extreme Gibbs measures.

Remarks. (1) The restriction $k \le e^{\frac{\beta}{20000}}$ can probably be eliminated at the expense of making the exposition more awkward. Without such restriction one has to take the size of elementary cylinders (see Section 2) increasing with their distance from the wall $\varphi = 0$ and decreasing with the magnetic field h. This is the point of view taken in [DM]. To get our results in this framework however requires additional work.

(2) In $(ii-b)$ N is restricted to be greater than $\lfloor 8/h+1\rfloor$, because otherwise the LHS of (1.8) may be infinite (think for instance of the case N=1). $m(\beta,h)$ can be taken equal to $\frac{\beta}{20000\,k^2}$.

(3) For what concerns (iii), one can prove, using the theory of Pirogov and Sinai as given in say [Z], that the Gibbs measures constructed with boundary conditions k and $k+1$ are in fact the only translation invariant extreme Gibbs measures. A detailed proof of this would require us to introduce quite a lot of extra definitions from [Z]. We will limit ourselves to sketch the proof in Section 8.

2. Cylinder models

General remark about this section. All results contained in this section are essentially due to Dinaburg and Mazel [DM]. Unfortunately our future needs are such that our results are given in a slightly different form for both hypotheses and theses. However *our* theses can be deduced from *our* hypotheses using *their* proofs (sometime a little extra care is required). For this reason we won't present the proofs again. At the end of the section some remarks will illustrate the main differences between our statements and those in [DM].

Following [DM] we are going to express the partition function of the SOS model as a sum over collections of *cylinders*. A cylinder is roughly speaking an elementary excitation of a certain ground state. The reason for introducing cylinders is that for large values of β, our model can be considered a gas of weakly interacting cylinders and this representation is suitable for the cluster expansion. Unfortunately the correct representation in terms of excitations with a small weight is not linked to the ordinary partition function in any obvious way, so one needs some preliminary work.

A *cylinder* is a triple $\gamma = (\widetilde{\gamma}, E(\gamma), I(\gamma))$ such that $\widetilde{\gamma} \in C_B$ (see Section 1.1), and $E(\gamma)$, $I(\gamma)$ are positive integers not equal to each other, called respectively the *external* and the *internal level* of γ. $\widetilde{\gamma}$ is the *base* of the cylinder. We also set $L(\gamma) = |I(\gamma) - E(\gamma)|$, and $S(\gamma) = sign(I(\gamma) - E(\gamma))$. $\bar{\gamma}$ is the *interior* of $\widetilde{\gamma}$ (see Section 1.1). The collection of all cylinders is denoted by C, while $C(V)$ stands for the set of all cylinders γ, such that $\bar{\gamma} \subset V$, and $C(V, n)$ is the set of all $\gamma \in C(V)$ such that $E(\gamma) = n$.

According to [DM] one defines the notion of compatibility of two cylinders in such a way to have a one–to–one correspondence between the set of configuration Ω_V and the set of all compatible collections of cylinders.

γ, γ' are *weakly compatible* (see Fig. 2) if either (1) or (2) hold

(1) $S(\gamma) = S(\gamma')$, $\bar{\gamma} \neq \bar{\gamma}'$ and
 either $\bar{\gamma} \cap \bar{\gamma}' = \emptyset$ and $\widetilde{\gamma} \cap \widetilde{\gamma}' = \emptyset$
 or $\bar{\gamma} \subset \bar{\gamma}'$
 or $\bar{\gamma}' \subset \bar{\gamma}$

(2) $S(\gamma) = -S(\gamma')$, $\bar{\gamma} \neq \bar{\gamma}'$ and
 either $\bar{\gamma} \cap \bar{\gamma}' = \emptyset$
 or $\bar{\gamma} \subset \bar{\gamma}'$ and $\widetilde{\gamma} \cap \widetilde{\gamma}' = \emptyset$
 or $\bar{\gamma}' \subset \bar{\gamma}$ and $\widetilde{\gamma} \cap \widetilde{\gamma}' = \emptyset$

γ, γ' are *compatible* if, in addition,

(3) $E(\gamma) = E(\gamma')$ if $\bar{\gamma} \cap \bar{\gamma}' = \emptyset$
 $E(\gamma) = I(\gamma')$ if $\bar{\gamma} \subset \bar{\gamma}'$
 $I(\gamma) = E(\gamma')$ if $\bar{\gamma}' \subset \bar{\gamma}$

(Hopefully) harmless notation ambiguity. Being $\widetilde{\gamma}$, $\widetilde{\gamma}'$ collections of dual edges, $\widetilde{\gamma}\cap\widetilde{\gamma}'$ would usually denotes the set of common edges. But when we write $\widetilde{\gamma}\cap\widetilde{\gamma}' = \emptyset$ $(\neq \emptyset)$ we *always* mean that $\widetilde{\gamma}\cup\widetilde{\gamma}'$ is connected (disconnected) as it was defined in Section 1.1. In other words if $\widetilde{\gamma}$, $\widetilde{\gamma}'$, considered as subsets of $\mathbb{R}^2$, have just one point in common, then we write $\widetilde{\gamma}\cap\widetilde{\gamma}' \neq \emptyset$.

Two cylinders γ', γ'' are said to be *separated* by a cylinder γ if $\bar{\gamma}' \neq \bar{\gamma} \neq \bar{\gamma}''$ and

either $\bar{\gamma}' \subset \bar{\gamma} \subset \bar{\gamma}''$	or $\bar{\gamma}'' \subset \bar{\gamma} \subset \bar{\gamma}'$
or $\bar{\gamma}' \subset \bar{\gamma}$ and $\bar{\gamma}'' \subset \bar{\gamma}^c$	or $\bar{\gamma}'' \subset \bar{\gamma}$ and $\bar{\gamma}' \subset \bar{\gamma}^c$

Given a collection of cylinders $\Gamma = \{\gamma\}$, we say that $\gamma', \gamma'' \in \Gamma$ are *neighbors* in Γ if there is no $\gamma \in \Gamma$ separating γ' and γ''. Γ is a *(weakly) compatible collection of cylinders* if all pairs of neighbors in Γ are (weakly) compatible. It is easy to see that if Γ is a weakly compatible collection of cylinders then each pair $\gamma, \gamma' \in \Gamma$ is weakly compatible (not just the pairs of neighbors). Γ_{ext} denotes the set of all external cylinders in Γ, *i.e.* the set of all $\gamma \in \Gamma$ such that $\bar{\gamma}$ is not contained in the interior of any other cylinder in Γ. We write $E(\Gamma) = n$ if $E(\gamma) = n$ for all $\gamma \in \Gamma_{ext}$.

We define

- $C_c^*(V,n)$ as the set of all finite compatible collections of cylinders $\Gamma \subset C(V)$ such that $E(\Gamma) = n$
- $C_w^*(V,n)$ as the set of all finite weakly compatible collections of cylinders $\Gamma \subset C(V)$ such that $E(\gamma) = n$ for all $\gamma \in \Gamma$.

The first result we present says that the partition function with constant boundary conditions can be expressed, apart from a trivial "ground state energy" term, as a sum over compatible collections of cylinders with suitable weights. So, given $V \subset\subset \mathbb{Z}^2$ and weights $z : C(V) \mapsto \mathbb{C}$ we define

$$\hat{Z}^n(V,z) = \sum_{\Gamma \in C_c^*(V,n)} \prod_{\gamma \in \Gamma} z(\gamma) \tag{2.1}$$

and we always assume

$$\sum_{\Gamma \in C_c^*(V,n)} \prod_{\gamma \in \Gamma} |z(\gamma)| < \infty \tag{2.2}$$

We then have ([DM])

Proposition 2.1. *Let $Z^{J,h,n}(V)$ be the partition function defined in (1.4). If $V \subset\subset \mathbb{Z}^2$ is simply connected, then there is a one–to–one correspondence between Ω_V and $C_c^*(V,n)$, and*

$$Z^{J,h,n}(V) = e^{-\beta h n |V|}\, \hat{Z}^n(V, w_{J,h})\,,$$

where

$$w_{J,h}(\gamma) = \exp[-\beta|\widetilde{\gamma}|_J L(\gamma) - \beta h S(\gamma)|\bar{\gamma}|L(\gamma)\,]$$
$$|\widetilde{\gamma}|_J = \sum_{[x,y]^*\in\widetilde{\gamma}} J(x,y)\,.$$

If $\Gamma \in C_c^*(V,n)$ is the collection of cylinders corresponding to the configuration φ, then we write $\varphi \sim \Gamma$. In Fig. 3 we give an example of how a certain configuration φ is obtained as a compatible collection of cylinders.

In the next three propositions we give a constructive characterization of the collection Γ of compatible cylinders corresponding to a certain configuration φ, and state without proof some more or less obvious properties of Γ.

Proposition 2.2. *Let $V \subset\subset \mathbb{Z}^2$ be simply connected. Choose $n \in \mathbb{Z}_+$, $\varphi \in \Omega_V$ and let $\Gamma \in C_c^*(V,n)$ such that $\Gamma \sim \varphi$. For each $m = 3/2, 5/2, 7/2, \ldots$, let*

$$U^m = \begin{cases} \{x \in V : \varphi(x) < m\} & \textit{if } 1 < m < n \\ \{x \in V : \varphi(x) > m\} & \textit{if } m > n\ . \end{cases}$$

Let's write now

$$\delta U^m = \alpha_1^m \cup \ldots \cup \alpha_{r(m)}^m,$$

where $\alpha_i^m \in C_B(V)$ are the connected components of δU^m. Then we have

(*a*) *for each m, i either $\varphi(x) > m$ for all $x \in \bar{\partial}\bar{\alpha}_i^m$ or $\varphi(x) < m$ for all $x \in \bar{\partial}\bar{\alpha}_i^m$. Accordingly we call α_i^m positive or negative.*

(*b*) *a cylinder $\gamma = (\widetilde{\gamma}, E(\gamma), I(\gamma))$ such that $I(\gamma) > E(\gamma)$ belongs to Γ if and only if*

$$\{s : \widetilde{\gamma} = \alpha_i^s \text{ for some } i,\ \alpha_i^s \text{ positive }\} = \{E(\gamma) + 1/2, \ldots, I(\gamma) - 1/2\}$$

(*c*) *a cylinder $\gamma = (\widetilde{\gamma}, E(\gamma), I(\gamma))$ such that $I(\gamma) < E(\gamma)$ belongs to Γ if and only if*

$$\{s : \widetilde{\gamma} = \alpha_i^s \text{ for some } i,\ \alpha_i^s \text{ negative }\} = \{I(\gamma) + 1/2, \ldots, E(\gamma) - 1/2\}$$

Proposition 2.3. *Under the same hypotheses as in the previous proposition, let x, y be nearest neighbors in V, and let e^* be the edge dual to $[x,y]$. Assume $\varphi(x) < \varphi(y)$ and let $\gamma \in \Gamma$ be such that $\widetilde{\gamma} \ni \varepsilon^*$. Then*

(*a*) *if $\bar{\gamma} \ni x$, then $S(\gamma) = -1$ and $\varphi(x) \le I(\gamma) < E(\gamma) \le \varphi(y)$*

(*b*) *if $\bar{\gamma} \ni y$, then $S(\gamma) = +1$ and $\varphi(x) \le E(\gamma) < I(\gamma) \le \varphi(y)$*

Proposition 2.4. *Under the same hypotheses as in the previous proposition, let $x \in \partial V$, $y \in V^c$ be nearest neighbors, and let e^* be the edge dual to $[x,y]$. Assume $\varphi(x) > n$ and let $\gamma \in \Gamma$ be such that $\widetilde{\gamma} \ni \varepsilon^*$. Then,*

$S(\gamma) = +1$ *and* $n \le E(\gamma) < I(\gamma) \le \varphi(x)$

The representation of the partition function given in Proposition 2.1 has the drawback of incorporating a very complicated constraint hidden inside the notion of compatibility between cylinders. The standard trick is then to define a new *renormalized weight* of cylinders in such a way that the partition function can be expressed as a sum over collections of *weakly* compatible cylinders (much easier to handle) with a renormalized weight. With this in mind we are going to swallow some more notation.

First of all we need to consider the cylinder partition function (2.1) with the additional constraint that all the external cylinders belong to some prescribed set. For this reason, if $\Pi \subset C$ is an arbitrary set of cylinders we define

$$\hat{Z}^n(V, z, \Pi) = \sum_{\substack{\Gamma \in C_c^*(V,n) \\ \Gamma_{ext} \subset \Pi}} \prod_{\gamma \in \Gamma} z(\gamma)$$

The constraints on the partition function naturally arise as compatibility conditions between cylinders. Thus, given an arbitrary collection of cylinders Γ, it is convenient to set

$$\Pi(\Gamma) = \{\gamma \in C : \gamma \text{ is weakly compatible with every cylinder in } \Gamma\}\,.$$

Since weak compatibility does not involve the $L(\gamma)$'s, $\Pi(\Gamma)$ *only depends on the bases and signs of the cylinders in* Γ. When Γ consists of just one cylinder we can use the following more compact notation: if $V \subset\subset \mathbb{Z}^2$ is the interior of some cylinder γ ($V = \bar{\gamma}$) then we define

$$\hat{Z}^n(V, z, \pm) = \hat{Z}^n(V, z, \Pi(\{\gamma'\}))$$

where $\gamma' = (\widetilde{\gamma}, n, n \pm 1)$. We can also write the *signed* partition function as a sum over particular configurations φ

$$Z^{J,h,n}(V, \pm) = e^{-\beta h n |V|} \hat{Z}^n(V, w_{J,h}, \pm) = \sum_{\varphi \in \Omega_V^{n,\pm}} e^{-\beta H_V^{J,hn}(\varphi)} \tag{2.3}$$

where

$$\Omega_V^{n,+} = \left\{ \varphi \in \Omega_V : \begin{array}{l} \varphi(y) \ge n \text{ for all } y \in \bar{\partial} V \text{ and} \\ \text{there exists } x \in \bar{\partial} V \text{ such that } \varphi(x) = n \end{array} \right\} \tag{2.4}$$

and $\Omega_V^{n,-}$ is defined in a similar way (replace $\ge$ with $\le$). The reason for introducing the signed partition function becomes clear if we write

$$\hat{Z}^n(V, z) = \sum_{\Gamma \in C_{c,ext}^*(V,n)} \prod_{\gamma \in \Gamma} z(\gamma)\, \hat{Z}^{I(\gamma)}(\bar{\gamma}, z, S(\gamma)) \tag{2.5}$$

where $C^*_{c,ext}(V,n)$ is the set of all $\Gamma \in C^*_c(V,n)$ such that Γ contains only external cylinders. Given a set of weights $z(\gamma)$ we define the *renormalized weight* of γ as

$$\widetilde{z}(\gamma) = z(\gamma)\ \frac{\hat{Z}^{I(\gamma)}(\bar{\gamma}, z, S(\gamma))}{\hat{Z}^{E(\gamma)}(\bar{\gamma}, z, S(\gamma))}. \tag{2.6}$$

Iterating (2.5) it's not difficult to show that (see [DM])

Proposition 2.5. *If $V \subset\subset \mathbb{Z}^2$ and z is a set of weights such that (2.2) holds, then*

$$\hat{Z}^n(V, z, \Pi) = \sum_{\substack{\Gamma \in C^*_w(V,n) \\ \Gamma_{ext} \subset \Pi}} \prod_{\gamma \in \Gamma} \widetilde{z}(\gamma) \tag{2.7}$$

Before stating the next results, we want to briefly explain what the goal of the rest of this section is. The importance of the previous proposition is the following: Γ is an element $C^*_w(V,n)$ if and only if

(1) for each $\gamma \in \Gamma$, $E(\gamma) = n$ and $\bar{\gamma} \subset V$
(2) each pair of cylinders in Γ is weakly compatible

This means that the partition function (2.7) is already in a form which is suitable for cluster expansion in the version given say in [KP]. The whole point is then to prove that the renormalized weights $\widetilde{w}_{J,h}$ are small, *i.e.* $\widetilde{w}_{J,h}(\gamma) \le \exp(-c\beta|\widetilde{\gamma}|_J L(\gamma))$ for some c. This will turn out to be true of course only for those cylinders whose external level $E(\gamma) = k$ is in the "right phase", that is only if there is a Gibbs state "close" to the constant configuration $\varphi = k$. On the other side, in order to estimate the renormalized weight of a cylinder starting from the right phase k, one has to estimate the quotient in (2.6) whose numerator is a partition function with boundary conditions $I(\gamma)$ which may very well be in the "wrong" or "unstable" phase. This partition function will then contain unstable cylinders, *i.e.* cylinders whose renormalized weight is large. For this reason one cannot hope to use directly cluster expansion in order to estimate the quotient in (2.6). The standard solution for this kind of problem is to study first a modified partition function which contains only *elementary* cylinders, that is cylinders which are *small enough to guarantee that they are all stable* (their renormalized weight is small), *independently of their external level.* The study of the elementary partition functions for all b.c. will tell us what the right phase is (Lemma 2.7) at least if the magnetic field is such that we are not too close to a phase transition ($h \in I_k(\beta)$ given in (2.15)). After that, one expresses the full partition function as a sum over collections of large cylinders (grouped into *contours*) on top of which one can have arbitrary collections of elementary cylinders (Proposition 2.6). The overall effect of the elementary cylinders will be a so called "entropic repulsion" which acts as a magnetic field pushing the surface away from the wall. The combination of the "entropic field" and the ordinary field h forces the surface to stay on the right phase (Corollary 2.8). In Section 4 we will finally complete our program and will prove a cluster expansion for (2.7) with the b.c. n equal to the right phase, even close to (and at) the phase transition.

Throughout this paper we will need many times some small constant independent of everything else. We call it ζ and its value is fixed to

$$\zeta = \frac{1}{1000}.$$

A cylinder γ is called *elementary* if $\operatorname{diam}\widetilde{\gamma} \leq \vartheta(\beta) = \exp(\frac{\beta\zeta}{10})$ (the diameter is taken in the $d(\cdot,\cdot)$ distance). The set of all elementary cylinders in $C(V)$ is denoted by $C_e(V)$. The partition function in which all external cylinders must be elementary is denoted with

$$\hat{Z}^n_e(V,z,\Pi) = \hat{Z}^n(V,z,\Pi\cap C_e(V)) = \sum_{\substack{\Gamma\in C^*_w(V,n)\\ \Gamma_{ext}\subset\Pi\cap C_e(V)}} \prod_{\gamma\in\Gamma} \widetilde{z}(\gamma)$$

Given $\Gamma \in C^*_c(V,n)$ we now define $\Gamma_l \subset \Gamma$ (l means *large*), as the collection of cylinders obtained by removing from Γ all those cylinders which are either elementary or contained (in the sense $\bar{\gamma} \subset \bar{\gamma}'$) in an elementary cylinder of Γ. The reason for this definition is that in the future we consider elementary cylinders whose maximum size depends on $E(\gamma)$, so that a cylinder can be non elementary, but contained in an elementary one.

Γ_l is still a (possibly empty) compatible collection of cylinders, because the operation of removing from Γ a cylinder together with all the other cylinders contained in it does not spoil the compatibility.

Given $\varphi \in \Omega_V$ (and a reference level n), φ_l stands for the unique configuration such that $\varphi \sim \Gamma$ and $\varphi_l \sim \Gamma_l$. We set

$$C^*_{c,l}(V,n) = \{\Gamma \in C^*_c(V,n) : \Gamma = \Gamma_l\}.$$

The map $\varphi \mapsto \Gamma_l$ induces a probability measure on $C^*_{c,l}(V,n)$, by setting

$$\bar{\mu}^{J,h,n}_V(\Gamma') = \mu^{J,h,n}\{\varphi \in V : \Gamma_l(\varphi) = \Gamma'\} \tag{2.8}$$

Let now $\Gamma \in C^*_{c,l}(V,n)$. Then it is possible to write Γ as the disjoint union

$$\Gamma = \Gamma_1 \cup \ldots \cup \Gamma_r \tag{2.9}$$

in such a way that, for each i,

(i) $\Gamma_i \in C^*_{c,l}(V,n)$ and there exists a unique cylinder which is external in Γ_i

(ii) if $\gamma \in \Gamma_i$ is not external then $E(\gamma) \neq E(\Gamma_i) = n$

(iii) if $\gamma \in \Gamma_i$ and $I(\gamma) = n$, then there is no other $\gamma' \in \Gamma_i$ such that $\bar{\gamma}' \subset \bar{\gamma}$.

The decomposition (2.9) is unique. Collections of cylinders $\Gamma \in C^*_{c,l}(V,n)$ that satisfy $(i - iii)$ are called *contours*. Fig. 3d shows a configuration which consists of 4 contours, (assuming all cylinders are non elementary) separated by the white regions where $\varphi = k$. We set

$$\mathrm{Con}(V,k) = \{\Gamma \in C^*_{c,l}(V,k) : \Gamma \text{ is a contour}\}.$$

For a contour Γ we define

$$\begin{aligned}
\Gamma_{int} &= \{\gamma \in \Gamma : I(\gamma) = E(\Gamma)\} \\
v(\Gamma,\gamma) &= \begin{cases} \bar{\gamma} \setminus \bigcup_{\gamma' \in \Gamma, \gamma' \neq \gamma} \bar{\gamma}' & \text{if } \gamma \in \Gamma \setminus \Gamma_{int} \\ \emptyset & \text{if } \gamma \in \Gamma_{int} \end{cases} \\
v(\Gamma) &= \bigcup\nolimits_{\gamma \in \Gamma} v(\Gamma,\gamma). \\
\Gamma_\partial &= \Gamma_{ext} \cup \Gamma_{int}
\end{aligned}$$

Obviously

$$\delta v(\Gamma) = \bigcup_{\gamma \in \Gamma_\partial} \widetilde{\gamma}\,.$$

We also let $E(\Gamma) = E(\Gamma_{ext})$ and $S(\Gamma) = S(\Gamma_{ext})$. Two contours Γ, Γ' are said to be *weakly compatible* if

(i) $v(\Gamma) \cap v(\Gamma') = \emptyset$

(ii) if $\gamma \in \Gamma_\partial$, $\gamma' \in \Gamma'_\partial$, then γ, γ' are weakly compatible (which, together with (i), implies that each cylinder in Γ is weakly compatible with every cylinder in Γ').

A contours Γ and an elementary cylinder ε are said to be *weakly compatible* if every $\gamma \in \Gamma_\partial$ is weakly compatible with ε.

Finally we denote by $W(V,n)$ the set of all finite weakly compatible collections

$$\mathcal{A} = (\{\Gamma\}, \{\varepsilon\})$$

of contours and elementary cylinders such that $\Gamma \in \mathrm{Con}(V,n)$ and $\varepsilon \in C_e(V,n)$ for all $\Gamma, \varepsilon \in \mathcal{A}$. $W_l(V,n)$ stands for the set of all $\mathcal{A} \in W(V,n)$ such that $\mathcal{A}$ contains only contours. $W_l(V,n)$ has a trivial one–to–one correspondence with $C^*_{c,l}(V,n)$ given by

$$\mathcal{A} \to \bigcup\nolimits_{\Gamma \in \mathcal{A}} \Gamma\,.$$

If we define the renormalized weight of a contour as

$$\widetilde{z}(\Gamma) = \frac{\prod_{\gamma \in \Gamma} z(\gamma)\, \hat{Z}_e^{I(\gamma)}(v(\Gamma,\gamma), z, \Pi(\Gamma))}{\hat{Z}_e^{E(\Gamma)}(v(\Gamma), z, \Pi(\Gamma_\partial))} \tag{2.10}$$

and sets $v(\mathcal{A}) = \cup_{\Gamma \in \mathcal{A}} v(\Gamma)$ and $\mathcal{A}_\partial = \cup_{\Gamma \in \mathcal{A}} \Gamma_\partial$, then one gets (see [DM])

Proposition 2.6. *If $V \subset\subset \mathbb{Z}^2$ and z is a set of weights such that (2.2) holds, then*

$$\hat{Z}^n(V,z) = \sum_{\mathcal{A} \in W_l(V,n)} \hat{Z}_e^n(V \setminus v(\mathcal{A}), z, \Pi(\mathcal{A})) \prod_{\Gamma \in \mathcal{A}} \prod_{\gamma \in \Gamma} z(\gamma)\, \hat{Z}_e^{I(\gamma)}(v(\Gamma,\gamma), z, \Pi(\mathcal{A})) =$$

$$= \sum_{\mathcal{A} \in W_l(V,n)} \hat{Z}_e^n(V, z, \Pi(\mathcal{A}_\partial)) \prod_{\Gamma \in \mathcal{A}} \widetilde{z}(\Gamma) = \sum_{\mathcal{A} \in W(V,n)} \prod_{\rho \in \mathcal{A}} \widetilde{z}(\rho)\,, \tag{2.11}$$

where $\widetilde{z}(\rho)$ is given by (2.6) if ρ is elementary, and by (2.10) if ρ is a contour.

Thanks to Proposition 2.6 we can use the one–to–one correspondence between $C_{c,l}(V,n)$ and $W_l(V,n)$ and write (see (2.8))

$$\bar{\mu}_V^{J,h,n}(\mathcal{A}) = \frac{\hat{Z}_e^n(V, w_{J,h}, \Pi(\mathcal{A}_\partial))}{\hat{Z}^n(V, w_{J,h})} \prod_{\Gamma \in \mathcal{A}} \widetilde{w}_{J,h}(\Gamma) \qquad \mathcal{A} \in W_l(V,n) \tag{2.12}$$

Now we want to introduce an hypothesis which allows us to keep the effects of J under control. So, given $\Lambda \subset\subset \mathbb{Z}^2$ and $0 \le t \le 1$, we say that J satisfies the hypothesis $H(\Lambda,t)$ if

- (i) $J \in \delta\Lambda$
- (ii) $|\alpha|_J \ge t|\alpha|$ for all $\alpha \in C_B(\Lambda)$

We also define

$$\Delta(J) = \{x \in \mathbb{Z}^2 : J(x,y) \neq 1 \text{ for some } y \in \mathbb{Z}^2\} \tag{2.13}$$

If $H(\Lambda,t)$ holds for J then, clearly, for each $V \subset \Lambda$,

$$|\Delta(J) \cap V| \le |\partial V| \le |\delta V| \tag{2.14}$$

We will start the analysis of the SOS model, by taking h in certain intervals $I_k(\beta)$, $(k = 1,2,3,\ldots)$ where the behavior of the system is easier to study, because the energy–entropy competition has a clear output. So we let, for $k \in \mathbb{Z}_+$,

$$I_k(\beta) = [h_k^-(\beta), h_k^+(\beta)]\,, \quad h_k^-(\beta) = \tfrac{4}{\beta} e^{-4\beta k}\,, \quad h_k^+(\beta) = \tfrac{1}{4\beta} e^{-4\beta(k-1)}\,, \quad h_1^+ = \tfrac{1}{\beta} e^{-\frac{\beta}{25} J}\,. \tag{2.15}$$

For simplicity we also let

$$l = l(b,h) = -\frac{1}{4\beta}\log(\beta h) \tag{2.16}$$

The following is a fundamental result in [DM]

Lemma 2.7. *Assume β is large enough and $h \le h_1^+(\beta)$. Let $V \subset\subset \mathbb{Z}^2$, and suppose $H(V,\zeta)$ holds for J ($\zeta = 1000^{-1}$). Let $V_0 = V \setminus \bar\partial V$. Let Π be a set of cylinders with the following properties*

(a) *if $\gamma \in \Pi$ and γ' is a vertical translate of γ (i.e. $\gamma' = (\widetilde\gamma, E(\gamma)+s, I(\gamma)+s)$ with $s \ge -(E(\gamma)\wedge I(\gamma))+1$), then $\gamma' \in \Pi$*

(b) *if $\bar\gamma \subset V_0$ then $\gamma \in \Pi$.*

Then (l is given by (2.16))

(i) *if ε is an elementary cilinder with $\bar\varepsilon \subset V$, then*

$$\widetilde w_{J,h}(\varepsilon) \le e^{-(\beta-1)|\widetilde\varepsilon|_J L(\varepsilon)}$$

(ii) *if $0 < m < n$, then*

$$\exp\left[|V_0|e^{-4\beta m} - |V|e^{-4\beta(m\wedge l)}e^{-\frac{\beta}{4}} - |V\cap\Delta(J)|e^{-4\beta\zeta(m\wedge l)-\frac{\beta\zeta}{4}}\right] \le$$
$$\frac{\hat Z_e^n(V,w_{J,h},\Pi)}{\hat Z_e^m(V,w_{J,h},\Pi)} \le \exp\left[|V|e^{-4\beta m} + |V|e^{-4\beta(m\wedge l)}e^{-\frac{\beta}{4}} + |V\cap\Delta(J)|e^{-\frac{\beta\zeta}{4}}\right]$$

(iii) *if $h \in I_k(\beta)$ and $n \ne k$, then*

$$e^{-(n-k)\beta h|V|}\frac{\hat Z_e^n(V,w_{J,h},\Pi)}{\hat Z_e^k(V,w_{J,h},\Pi)} \le \exp\left[-\frac{1}{2}\beta h|V||n-k| + |\delta V|e^{-\frac{\beta\zeta}{5}}\right]$$

(iv) *if $h \in [h_{k+1}^+(\beta), h_k^-(\beta)]$ and $n \notin \{k,k+1\}$, then*

$$e^{-(n-k-1)\beta h|V|}\frac{\hat Z_e^n(V,w_{J,h},\Pi)}{\hat Z_e^{k+1}(V,w_{J,h},\Pi)} \le \exp\left[-\frac{1}{2}\beta h|V||n-k-1| + |\delta V|e^{-\frac{\beta\zeta}{5}}\right]$$

Corollary 2.8. *Let β be large enough, $h \in I_k(\beta)$. Let $V \subset\subset \mathbb{Z}^2$, and suppose $H(V,\zeta)$ holds for J ($\zeta = 1000^{-1}$). Then*

$$\hat Z^k(V,w_{J,h}) = \sum_{\mathcal{A}\in W(V,k)}\prod_{\rho\in\mathcal{A}}\widetilde w_{J,h}(\rho)$$

and

(i) *if $\rho = \varepsilon$ is an elementary cylinder, then*

$$\widetilde w_{J,h}(\varepsilon) \le \exp\left[-(\beta-1)|\widetilde\varepsilon|_J L(\varepsilon)\right]$$

(*ii*) *if* $\rho = \Gamma$ *is a contour, then*

$$\widetilde{w}_{J,h}(\Gamma) \le \prod_{\gamma\in\Gamma} \exp\Big[-(\beta-1)|\widetilde{\gamma}|_J L(\gamma) - \frac{\beta h}{2}|v(\Gamma,\gamma)||I(\gamma)-k|\Big].$$

Proof of the Corollary. Part (*i*) was proved in Lemma 2.7. To prove part (*ii*) we recall the definition (2.10) and observe that

$$\hat{Z}_e^{E(\Gamma)}(v(\Gamma), z, \Pi(\Gamma_\partial)) \ge \hat{Z}_e^{E(\Gamma)}(v(\Gamma), z, \Pi(\Gamma)) \ge \prod_{\gamma\in\Gamma} \hat{Z}_e^{E(\Gamma)}(v(\Gamma,\gamma), z, \Pi(\Gamma)).$$

Then we apply part (*iii*) of Lemma 2.7 to each $V = v(\Gamma, g)$ (since $\Pi(\Gamma)$ satisfies (*a*) and (*b*)). Thus (*ii*) follows from the inequality

$$e^{-\frac{\beta\zeta}{5}} \sum_{\gamma\in\Gamma} |\delta v(\Gamma,\gamma)| \le e^{-\frac{\beta\zeta}{5}} 2\sum_{\gamma\in\Gamma}|\widetilde{\gamma}| \le \sum_{\gamma\in\Gamma}|\widetilde{\gamma}|_J \quad \square$$

An inportant consequence of Corollary 2.8 is that the sum of the weight all contours whose support contains a given site is "small". We state this results in a more general form in order to meet our future needs.

Proposition 2.9. *Let* β *be large enough,* $V \subset\subset \mathbb{Z}^2$ *and* $x \in V$. *Then*

(*i*) *for all* $c > \frac{\zeta}{10}$,

$$\sum_{\substack{\varepsilon\in C_e(V,k)\\ |\widetilde{\varepsilon}|\ge s,\ \bar{\varepsilon}\ni x}} e^{-c\beta|\widetilde{\varepsilon}|L(\varepsilon)} \le e^{-\frac{2}{3}cs\beta}$$

(*ii*) *for* $n \in \mathbb{Z}_+$, *let* $\bar{\vartheta}(\beta,h,n)$, $\bar{h}_n$ *be such that*

$$\beta\bar{h}_n \ge e^{-\frac{c}{25}\beta\bar{\vartheta}(\beta,h,n)}. \tag{2.17}$$

let Y *be the set of all* $\Gamma \in C_c^*(V,k)$ *such that*

(*a*) *there is a unique cylinder which is external in* Γ

(*b*) Γ *satisfies* (*ii*) *and* (*iii*) *in the definition of contours*

(*c*) *for each* $\gamma\in\Gamma$ *such that* $\widetilde{\gamma}\cap\delta V = \emptyset$, $\operatorname{diam}\widetilde{\gamma} \le \bar{\vartheta}(\beta,h,E(\gamma))$

Then, for all $c > \frac{\zeta}{10}$, $c' > \frac{\zeta}{10}$,

$$\sum_{\substack{\Gamma\in Y\\ |\tilde{\Gamma}_{ext}|\ge s,\ \bar{\Gamma}_{ext}\ni x}} \prod_{\gamma\in\Gamma} \exp\left[-c\beta|\widetilde{\gamma}|L(\gamma) - c'\beta\bar{h}_{I(\gamma)}\,|v(\Gamma,\gamma)|\,|I(\gamma)-k|\,\right] \le e^{-\frac{2}{3}cs\beta} \tag{2.18}$$

Remarks. (1) The first inequality is an easy consequence of the fact that the number of $\widetilde{\gamma}$ with lenght n is less than K^n for some fixed number K.

(2) The second inequality is trickier and relies on a construction in which one associates to each contour Γ a tree $T(\Gamma)$ (see [DM]).

(3) (ii) clearly holds if one takes $\bar{\vartheta}(\beta, h, n) = \vartheta(\beta) = e^{\frac{\beta\zeta}{10}}$ (and so $Y \supset \mathrm{Con}(V, k)$), and $\bar{h}(n) = h$ with

$$l = l(b, h) = -\frac{1}{4\beta}\log(\beta h) \leq 2k_{max} \equiv 2\lfloor e^{\frac{\beta\zeta}{20}}\rfloor\,. \tag{2.19}$$

Together with Corollary 2.8, this implies that the RHS of (2.18) is an upper bound for the sum of the weight of all contours whose interior contains x and such that $|\tilde{\Gamma}_{ext}| \geq s$.

Main differences with [DM].

(1) First of all in [DM] elementary cylinders are those with $\mathrm{diam}\,\widetilde{\gamma} \leq 100\,(k \wedge E(\gamma))$. In this way their results are valid for h arbitrarily small (see also the remark after Theorem 1.1), while we have sometimes to assume (2.19).

(2) They always have $J = 1$, but taking J arbitrary is not a real complication

(3) Statement (ii) of Lemma 2.7 is given with $|V|$ instead of $|V_0|$ in the LHS. With $|V|$ the statement is false because in the estimate of the *entropic repulsion* one cannot count for sure on those cyliders which touch the boundary of V. These, in fact, could be forbidden by the compatibility rules at the boundary. If one takes for instance $V = \{x\}$, then

$$\frac{\hat{Z}_e^n(V, w_{J,h}, +)}{\hat{Z}_e^m(V, w_{J,h}, +)} = 1$$

which is compatible with (ii) if one uses $|V_0|$.

Having $|V_0|$ instead of $|V|$ is the origin of the boundary term in the RHS of (iii). This term is harmless for the estimate of the weight of contours given in Corollary 2.8 (it just turns β into $\beta - 1$), but causes a good deal of trouble in Section 5 (Subsection 5.3 would be unnecessary in absence of this boundary term).

(4) Statement (iii) of Lemma 2.7 is given for

$$e^{-4\beta k + \frac{\beta}{100}} \leq \beta h \leq e^{-4\beta(k-1) - \frac{\beta}{100}}\,,$$

but one can easily check (given (ii)) that it is still valid for $h \in I_k(\beta)$

(5) Statement (ii) of Proposition 2.9 is given (with 2/3 replaced by 1/2) in the case of $c = \frac{1}{2}$, $c' = \frac{1}{4}$,

$$\bar{h}(n) = e^{-4\beta(n \wedge k)} \quad \text{and} \quad \bar{\vartheta}(\beta, h, n) = 100(n \wedge k)\,,$$

but the same proof proves our version.

3. A priori bounds

We collect in this section some basic estimates that will be useful in the future.

3.1 Bounds uniform in the boundary conditions

Often we are going to assume that our volume is large enough, so we set $N_1 = \lfloor 8/h + 1 \rfloor$. Let

$$\bar{H}_V^{h,\infty}(\varphi) = \frac{1}{2} \sum_{\substack{x,y \in V \\ |x-y|=1}} |\varphi(x) - \varphi(y)| - \sum_{\substack{x \in V,\, y \in V^c \\ |x-y|=1}} \varphi(x) + h \sum_{x \in V} \varphi(x). \tag{3.1}$$

Then we have, for all $J \in \delta V$,

$$(n-1)\, |\delta V|_J + H_V^{J,h,1}(\varphi) \geq H_V^{J,h,n}(\varphi) \geq (n-1)\, |\delta V|_J + \bar{H}_V^{h,\infty}(\varphi) \tag{3.2}$$

where $|\delta V|_J$ was defined in (1.5) and $J \in \delta V$ was defined in Section 1.2.

Proposition 3.1. *Let $\beta, h > 0$, $V = Q_N$ with $N \geq N_1 = \lfloor 8/h + 1 \rfloor$, and let $l(x) \geq 0$ for each $x \in V$. Then*

$$G_N(l) \equiv \sum_{\varphi \in \Omega_V} \prod_{x \in V} \varphi(x)^{l(x)} e^{-\beta \bar{H}_V^{h,\infty}(\varphi)} < \infty.$$

Proof. We can write

$$\bar{H}_V^{h,\infty}(\varphi) = H_V^{(h)}(\varphi) + H_V^{(v)}(\varphi)$$

where

$$H_V^{(h)}(\varphi) = \frac{1}{2} \sum_{\substack{x,y \in V \\ |x-y|=1}}^{(h)} |\varphi(x) - \varphi(y)| - \sum_{\substack{x \in V,\, y \in V^c \\ |x-y|=1}}^{(h)} \varphi(x) + \frac{h}{2} \sum_{x \in V} \varphi(x)$$

and the $\sum^{(h)}$ is the sum over all horizontal bonds ($H_V^{(v)}$ is defined in the same way with vertical bonds replacing horizontal ones). By the Schwarz inequality

$$G_N(l) \leq \Big[\sum_{\varphi \in \Omega_V} \prod_{x \in V} \varphi(x)^{l(x)} e^{-2\beta H_V^{(h)}(\varphi)} \Big]^{\frac{1}{2}} \Big[\sum_{\varphi \in \Omega_V} \prod_{x \in V} \varphi(x)^{l(x)} e^{-2\beta H_V^{(v)}(\varphi)} \Big]^{\frac{1}{2}} \equiv$$

$$\equiv \sqrt{G_N^{(h)}(l)\, G_N^{(v)}(l)}$$

Since both $G_N^{(h)}(l)$ and $G_N^{(v)}(l)$ factorize as product of 1–dimensional partition functions, all we have to do, in order to prove the Proposition, is to show that each of these factors is finite, *i.e.* that

$$\bar{G}_N(l) = \sum_{a \in \mathbb{Z}_+^N} \prod_{i=1}^{N} a_i^{l_i} e^{-2\beta H_N(a)} < \infty \tag{3.3}$$

where we have set

$$H_N(a) = -(a_1 + a_N) + \sum_{i=1}^{N-1} |a_{i+1} - a_i| + \frac{h}{2} \sum_{i=1}^{N} a_i \, . \tag{3.4}$$

Notice that (3.3) is false if N is small (just sum over all constant configurations). Let then

$$\bar{a} = \min_{1 \le i \le N} a_i \, .$$

Then

$$\sum_{i=1}^{N-1} |a_{i+1} - a_i| \ge (a_1 + a_N) - 2\bar{a} \, ,$$

and so, if $N \ge N_1$,

$$H_N(a) \ge -2\bar{a} + \frac{h}{4} N \bar{a} + + \frac{h}{4} \sum_{i=1}^{N} a_i \ge \frac{h}{4} \sum_{i=1}^{N} a_i \, .$$

In this way we get

$$\bar{G}_N(l) = \prod_{i=1}^{N} \sum_{a_i=1}^{\infty} a_i^{l_i} \, e^{-\frac{1}{2}\beta h a_i} < \infty \, .$$

Proposition 3.2. *For each $\beta, h > 0$, $s \ge 0$, there exist constants $b_1(\beta, h, s)$, $b_2(\beta, h) > 0$ such that for all $\Lambda \subset\subset \mathbb{Z}^2$, for all $J \in \delta\Lambda$, we have ($N_1(h) = \lfloor 8/h + 1 \rfloor$)*

(i) for each $x \in \Lambda$ such that there exists a square $\Lambda' = Q_{N_1} + y$, with $x \in \Lambda' \subset \Lambda$ and for all $s \ge 0$,

$$\sup_{\psi \in \Omega} \mathbf{E}_{\Lambda}^{J,h,\psi} \, \varphi(x)^s \le b_1(\beta, h, s)$$

(ii) for each $V \subset \Lambda$ such that there exists a square $\Lambda' = Q_{N_1} + y$ with $V \subset \Lambda' \subset \Lambda$

$$\inf_{\psi \in \Omega} \mu_{\Lambda}^{J,h,\psi} \{\varphi(x) = 1 \; \forall x \in V\} \ge b_2(\beta, h)$$

Proof. By (1.7) and the FKG and Markov properties,

$$\mathbf{E}_{\Lambda}^{J,h,\psi} \, \varphi(x)^s = \sum_{\psi' \in \Omega} \mu_{\Lambda}^{J,h,\psi}(\psi') \, \mathbf{E}_{\Lambda'}^{J,h,\psi'} \, \varphi(x)^s \le \sup_{\psi' \in \Omega} \mathbf{E}_{\Lambda'}^{J,h,\psi'} \, \varphi(x)^s =$$

$$= \sup_{n \in \mathbb{Z}_+} \mathbf{E}_{\Lambda'}^{J,h,n} \, \varphi(x)^s \le \Big[\sum_{\varphi \in \Omega_{\Lambda'}} e^{-\beta H_{\Lambda'}^{J,h,1}(\varphi)} \Big]^{-1} \sum_{\varphi \in \Omega_{\Lambda'}} \varphi(x)^s e^{-\beta \bar{H}_{\Lambda'}^{h;\infty}(\varphi)} \le b_1(\beta, h, n)$$

where we have used (3.2) and Proposition 3.1. The proof of the second statement is similar:

$$
\begin{aligned}
&\mu_\Lambda^{J,h,\psi}\{\varphi(x)=1\ \forall x\in V\}=\\
&=\sum_{\psi'\in\Omega}\mu_\Lambda^{J,h,\psi}(\psi')\mu_{\Lambda'}^{J,h,\psi'}\{\varphi(x)=j\ \forall x\in V\}\geq \inf_{\psi'\in\Omega}\mu_{\Lambda'}^{J,h,\psi'}\{\varphi(x)=1\ \forall x\in\Lambda'\}=\\
&=\inf_{n\in\mathbb{Z}_+}\mu_{\Lambda'}^{J,h,n}\{\varphi(x)=1\ \forall x\in\Lambda'\}\geq\Big[\sum_{\varphi\in\Omega_{\Lambda'}}e^{-\beta\bar H^{h,\infty}_{\Lambda'}(\varphi)}\Big]^{-1}e^{-\beta h|\Lambda'|}=b_2(\beta,h)
\end{aligned}
$$

3.2. The Peierls argument

We present here two inequalities which we will use several times, and which can be morally ascribed to Peierls.

Proposition 3.3. *Let Ω be any countable set and let $w:\Omega\mapsto[0,\infty)$ be such that $Z=\sum_{i\in\Omega}w(i)<\infty$. Define a probability measure μ on Ω by setting*

$$
\mu(X)=\frac{\sum_{i\in X}w(i)}{Z},\quad X\subset\Omega\,.
$$

Assume there exists an arbitrary set $\widetilde\Omega$, with weights $\bar w:\widetilde\Omega\mapsto[0,\infty)$ and two maps $f:\Omega\mapsto\widetilde\Omega$, $g:\Omega\mapsto\Omega$ such that

(1) $f(i)=f(j)$ and $g(i)=g(j)$ implies $i=j$

(2) $w(i)\leq\bar w(f(i))\,w(g(i))$ for each $i\in\Omega$.

Then, if $X\subset\Omega$,

(i)
$$
\mu(X)\leq\mu(g(X))\sum_{j\in f(X)}\bar w(j)
$$

(ii)
$$
\mu(X)\leq\sup_{i\in\Omega}\sum_{j\in f(X\cap g^{-1}(i))}\bar w(j)
$$

Remark. Notice that f and g need to be defined only on X and not on the whole space Ω.

Proof. In fact we have

$$
\begin{aligned}
\mu(X)&=\sum_{j\in f(X)}\sum_{i\in X:f(i)=j}\mu(i)\leq\sum_{j\in f(X)}\bar w(j)\sum_{i\in X:f(i)=j}\mu(g(i))\leq\\
&\leq\mu(g(X))\sum_{j\in f(X)}\bar w(j)
\end{aligned}
$$

where we have used (2) in the first inequality and (1) in the second one. Analogously

$$\mu(X) \le \sum_{i \in g(X)} \mu(i) \sum_{j \in X : g(j) = i} \bar{w}(f(j)) \le \sup_{i \in \Omega} \sum_{m \in f(X \cap g^{-1}(i))} \bar{w}(m) \quad \square$$

An immediate consequence of previous proposition is

Proposition 3.4. *Let $\beta, h > 0$, $V \subset\subset \mathbb{Z}^2$ simply connected, and $k \in \mathbb{Z}_+$. Then for any J and any $\{\Gamma_1, \dots, \Gamma_s\} \subset \mathrm{Con}(V,k)$ we have*

$$\mu_V^{J,h,k}\{\varphi \in \Omega_V : \{\Gamma_1, \dots, \Gamma_s\} \subset \mathcal{A}^{\varphi_l}\} \equiv \bar{\mu}_V^{J,h,k}\{\mathcal{A} : \{\Gamma_1, \dots, \Gamma_s\} \subset \mathcal{A}\} \le \prod_{i=1}^{s} \widetilde{w}_{J,h}(\Gamma_i)$$

where A^{φ_l} is the collection of contours $\mathcal{A}$ such that $\mathcal{A} \sim \varphi_l$ (φ_l was defined right before (2.8)).

Proof. Take as Peierls maps $f(\mathcal{A}) = \{\Gamma_1, \dots, \Gamma_s\}$, and $g(A) = \mathcal{A} \setminus \{\Gamma_1, \dots, \Gamma_s\}$. Choose

$$\bar{w}(f(\mathcal{A})) = \prod_{i=1}^{s} \widetilde{w}_{J,h}(\Gamma_i)\,.$$

Use the representation on the LHS of (2.11) for the partition function and notice that

$$\Pi(\mathcal{A}_\partial) \subset \Pi(g(\mathcal{A})_\partial).$$

Both (1) and (2) in Proposition 3.3 then follow. $\square$

3.3. The distribution of φ when $h \in I_k(\beta)$

We want to prove the following

Proposition 3.5. *Let β be large enough and $h \in I_k(\beta)$ with $1 \le k \le 2k_{max}$ ($k_{max} = \lfloor e^{\frac{\beta\zeta}{20}} \rfloor$). Let $V \subset\subset \mathbb{Z}^2$ be simply connected. Assume that $H(V,t)$ holds for J with $t \ge \zeta \equiv 1000^{-1}$. Let $U \subset V$ be such that*

$$|U \cap \bar{\gamma}| \le |\widetilde{\gamma}| \quad \forall \gamma \in C(V,k)\,. \tag{3.5}$$

Then, for all $c > \zeta$,

$$\mu_V^{J,h,k}\Big\{\varphi \in \Omega_V : \sum_{x \in U} |\varphi(x) - k| \ge c|U|\Big\} \le e^{-\frac{1}{5}\beta t c|U|}\,.$$

Proof. For each $\Gamma \in C_c^*(V,k)$ we define

- $\Gamma_{e,U}$ as the set of all $\gamma \in \Gamma$ such that $\bar\gamma \subset \bar\gamma'$ for some elementary γ' with $\bar\gamma' \cap U \neq \emptyset$ (γ' may coincide with γ. Notice that γ itself doesn't have to intersects U)
- $\Gamma_{l,U}$ as the set of all $\gamma \in \Gamma_l$ such that $\bar\gamma \cap U \neq \emptyset$ and $I(\gamma) \neq k$

If $\varphi \in \Omega_V$ we let $\Gamma^\varphi \in C_c^*(V,k)$ be such that $\varphi \sim \Gamma^\varphi$. Then we have, using also (3.5)

$$\sum_{x\in U} |\varphi(x) - k| \leq \sum_{x \in U} \sum_{\substack{\gamma\in\Gamma^\varphi \,:\, \bar\gamma \ni x \\ I(\gamma)\neq k}} L(\gamma) \leq$$

$$\leq \sum_{\gamma\in\Gamma^\varphi \,:\, I(\gamma)\neq k} |\bar\gamma \cap U|\, L(\gamma) \leq \sum_{\gamma\in\Gamma^\varphi_{l,U}} |\widetilde\gamma| L(\gamma) + \sum_{\gamma\in\Gamma^\varphi_{e,U}} |\widetilde\gamma| L(\gamma)\,.$$

Hence

$$\begin{aligned} &\mu_V^{J,h,k}\{\sum_{x\in U}|\varphi(x)-k| \geq c|U|\} \leq \\ &\leq \mu_V^{J,h,k}\{\sum_{\gamma\in\Gamma^\varphi_{l,U}} |\widetilde\gamma| L(\gamma) \geq \frac{c}{2}|U|\} + \mu_V^{J,h,k}\{\sum_{\gamma\in\Gamma^\varphi_{e,U}} |\widetilde\gamma| L(\gamma) \geq \frac{c}{2}|U|\} = \mu_1 + \mu_2 \end{aligned} \tag{3.6}$$

where μ_1 and μ_2 are defined in the obvious way. To evaluate the first term we observe that if

$$\sum_{\gamma\in\Gamma^\varphi_{l,U}} |\widetilde\gamma| L(\gamma) \geq \frac{c}{2}|U|$$

then, by regarding Γ^φ_l as a collection of contours $\mathcal{A}^{\varphi_l}$, there exists $s = 1,\ldots,|U|$, and a collection of contours $\{\Gamma_1,\ldots,\Gamma_s\} \subset \mathcal{A}^{\varphi_l}$ such that

(*i*) $v(\Gamma_i) \cap U \neq \emptyset$ for $i = 1,\ldots,s$

(*ii*) $\sum_{i=1}^s p(\Gamma_i) \geq \frac{c}{2}|U|$

where

$$p(\Gamma_i) = \sum_{\gamma\in\Gamma_i} |\widetilde\gamma| L(\gamma)\,.$$

By Proposition 3.4, we find

$$\mu_1 \leq \sum_{s=1}^{|U|} \sum_{\{\Gamma_1,\ldots,\Gamma_s\}\subset \mathrm{Con}(V,k)}{}' \prod_{i=1}^s \widetilde w_{J,h}(\Gamma_i)$$

where $\sum'$ means the sum must be taken over all collections of contours satisfying (i) and (ii). Now we can use Corollary 2.8, Proposition 2.9 (see the following Remark (3)) and the hypothesis on J, and get

$$\mu_1 \le e^{-\frac{c}{4}t\beta|U|}\sum_{s=1}^{|U|}\binom{|U|}{s}\Big[\sup_{x\in U}\sum_{\substack{\Gamma\in\mathrm{Con}(V,k)\\ \bar\Gamma\ni x}}\prod_{\gamma\in\Gamma}e^{-\frac{\beta}{3}|\tilde\gamma|_J L(\gamma)-\frac{1}{2}\beta h|v(\Gamma,\gamma)||I(\gamma)-k|}\Big]^s \le$$
$$\le e^{-\frac{c}{4}t\beta|U|}\sum_{s=1}^{|U|}\binom{|U|}{s}e^{-\frac{2}{9}\beta ts} \le e^{-\frac{c}{4}t\beta|U|}\ 2^{|U|}\,. \tag{3.7}$$

To evaluate μ_2, we write instead

$$\hat Z^k(V,w_{J,h}) = \sum_{\Gamma\in C^*_c(V,k)}\prod_{\gamma\in\Gamma}w_{J,h}(\gamma)\,.$$

Now we use the Peierls argument with $f(\Gamma)=\Gamma_{e,U}$, $g(\Gamma)=\Gamma\setminus\Gamma_{e,U}$ (these are well defined) and $\bar w(\Gamma_{e,U})=\prod_{\gamma\in\Gamma_{e,U}}w_{J,h}(\gamma)$. By Proposition 3.3 we get

$$\mu_2 \le \sup_{\Gamma'\in C^*_c(V,k)}\ \sideset{}{'}\sum_{\substack{\Gamma\in C^*_c(V,k)\\ \Gamma\setminus\Gamma_{e,U}=\Gamma'}}\ \prod_{\gamma\in\Gamma_{e,U}}w_{J,h}(\gamma)$$

where $\sum'$ means that the sum is restricted to those Γ such that

$$\sum_{\gamma\in\Gamma_{e,U}}|\tilde\gamma|L(\gamma)\ge\frac{c}{2}|U|\,.$$

For each collection of positive integers $n=\{n(x)\}_{x\in U}$ we define $Y(n)$ as the set of all Γ such that

(a) Γ is a compatible collection of elementary cylinders
(b) $\bar\gamma\cap U\ne\emptyset$ for each $\gamma\in\Gamma_{ext}$
(c) if $\gamma\in\Gamma_{ext}$ and $\bar\gamma\ni x$ for some $x\in U$, then $E(\gamma)=n(x)$
(d) $\sum_{\gamma\in\Gamma}|\tilde\gamma|L(\gamma)\ge\frac{c}{2}|U|$.

Then it is clear that

$$\mu_2\le\sup_n\sum_{\Gamma\in Y(n)}\prod_{\gamma\in\Gamma}w_{J,h}(\gamma)\,. \tag{3.8}$$

Since $\mathrm{diam}\,|\tilde\gamma|\le\vartheta=e^{\frac{\beta\zeta}{10}}$, we have $\beta h|\bar\gamma|\le\beta h\vartheta^2\le 1$, so

$$w_{J,h}(\gamma)\le e^{-(\beta-1)|\tilde\gamma|_J L(\gamma)}\,.$$

Because of this, the fact that $H(V,t)$ holds for J and property (d) of $Y(n)$,we get

$$\sum_{\Gamma\in Y(n)}\prod_{\gamma\in\Gamma} w_{J,h}(\gamma) \le e^{-\frac{c}{4}t(\beta-1)|U|} \sum_{\Gamma\in Y(n)}\prod_{\gamma\in\Gamma} e^{-\frac{1}{2}(\beta-1)|\widetilde{\gamma}|_J L(\gamma)}\,. \tag{3.9}$$

We perform the sum in the following way. Since the level of the external cylinders is fixed, an element $\Gamma\in Y(n)$ is completely determined by a set of bases $\{\widetilde{\gamma}\}$, signs $\{S(\gamma)\}$, and heights $\{L(\gamma)\}$ (some choices of the above quantities may not correspond to a compatible set of cylinders, but if we sum over all of them we get an upper bound). After we sum over the possible values of $\{S(\gamma)\}$ and $\{L(\gamma)\}$, we get

$$\sum_{\Gamma\in Y(n)}\prod_{\gamma\in\Gamma} w_{J,h}(\gamma) \le e^{-\frac{c}{4}t(\beta-1)|U|} \sum_{\{\widetilde{\gamma}\}}{}' \prod_{\widetilde{\gamma}\in\{\widetilde{\gamma}\}} e^{-\frac{1}{3}\beta|\widetilde{\gamma}|_J}\,. \tag{3.10}$$

where $\sum'$ means that $\{\widetilde{\gamma}\}$ is the set of bases of some $\Gamma\in Y(n)$. In order to sum over all $\{\widetilde{\gamma}\}$ we first fix the external ones and get

$$\sum_{\{\widetilde{\gamma}\}}{}' \prod_{\widetilde{\gamma}\in\{\widetilde{\gamma}\}} e^{-\frac{1}{3}\beta|\widetilde{\gamma}|_J} \le \sum_{\substack{\{\widetilde{\gamma}\}\\ \text{external}}}{}' \prod_{\widetilde{\gamma}\in\{\widetilde{\gamma}\}} e^{-\frac{1}{3}\beta|\widetilde{\gamma}|_J} B(\widetilde{\gamma}) \tag{3.11}$$

where, using Proposition A1.2 and the subsequent Remark,

$$B(\widetilde{\gamma}) = \sum_{\substack{\{\widetilde{\gamma}'\}\,:\,\bar{\gamma}'\subset\bar{\gamma}\\ \text{for each }\widetilde{\gamma}'\in\{\widetilde{\gamma}'\}}} \prod_{\widetilde{\gamma}'\in\{\widetilde{\gamma}'\}} e^{-\frac{1}{3}\beta|\widetilde{\gamma}'|_J} \le$$

$$\le \exp\left[|\bar{\gamma}|e^{-t\beta}\right] \le \exp\left[\vartheta^2 e^{-t\beta}\right] \le \exp\left[e^{-\frac{1}{2}t\beta}\right] \le 3\,.$$

Substituting back in the previous equation,

$$\text{LHS of (3.11)} \le \sum_{\substack{\{\widetilde{\gamma}\}\\ \text{external}}}{}' \prod_{\widetilde{\gamma}\in\{\widetilde{\gamma}\}} e^{-\frac{1}{4}\beta|\widetilde{\gamma}|_J}\,. \tag{3.12}$$

Because the $\{\widetilde{\gamma}\}$ are external, their interiors $\{\bar{\gamma}\}$ are pairwise disjoint. This, together with property (b) of $Y(n)$ allows us to use Proposition A1.2 (and subsequent Remark), which, since $|\widetilde{\gamma}|_J \ge t|\widetilde{\gamma}|$ and $|\widetilde{\gamma}|\ge 4$, implies

$$\sum_{\substack{\{\widetilde{\gamma}\}\\ \text{external}}}{}' \prod_{\widetilde{\gamma}\in\{\widetilde{\gamma}\}} e^{-\frac{1}{4}\beta|\widetilde{\gamma}|_J} \le \exp\left[|U|e^{-\frac{3}{4}\beta t}\right]\,.$$

Combining this with (3.8), (3.10), (3.11), (3.12), and using $c\ge\zeta$, we get

$$\mu_2 \le \exp\left[-\frac{c}{4}t(\beta-1)|U| + |U|e^{-\frac{3}{4}\beta t}\right] \le \exp\left[-\frac{c}{4}t(\beta-2)|U|\right]$$

which, together with (3.6), (3.7) gives the Proposition. $\square$

A straightforward consequence of the previous proposition is

Proposition 3.6. *Let β, h, V and J as in Proposition 3.5. Then*

$$\mathbf{E}_V^{J,h,k}\,|\varphi(x)-k| \le e^{-\frac{1}{4}\beta t} \qquad \forall x \in V\,.$$

4. Cluster expansion

4.1. Main results

Propositions 2.1, 2.6 say that if V is a simply connected finite volume, then the partition function can be written as

$$Z^{J,h,k}(V) = e^{-\beta hk|V|}\hat{Z}^k(V, w_{J,h}) = e^{-\beta hk|V|} \sum_{\Gamma\in C^*_w(V,k)} \prod_{\gamma\in\Gamma} \widetilde{w}_{J,h}(\gamma)$$

where the renormalized weights are given by (2.6). We want to show that for each integer $k \geq 1$ there exists $h^*_k(\beta) \in [h^+_{k+1}(\beta), h^-_k(\beta)]$ such that $\hat{Z}^k$ has a cluster expansion for $h \in [h^*_k(\beta), h^*_{k-1}(\beta)]$ (when $k = 1$ we prove cluster expansion for $h \in [h^*_1(\beta), h^+_1(\beta)]$). In particular if $h = h^*_k(\beta)$, then both $\hat{Z}^k$ and $\hat{Z}^{k+1}$ have a cluster expansion which implies the existence of at least two distinct Gibbs measures.

Given a cylinder γ, we define its *truncated weights* in the spirit of Zahradník's version [Z] of the the Pirogov–Sinai theory

$$w^{tr}_{J,h}(\gamma) = \widetilde{w}_{J,h}(\gamma) \wedge \exp\left[-\frac{\beta}{2}|\widetilde{\gamma}|_J L(\gamma)\right].$$

A cylinder γ is called *stable* if $w^{tr}_{J,h}(\gamma) = \widetilde{w}_{J,h}(\gamma)$. Accordingly we define the truncated partition function

$$Z^{J,h,n}_{tr}(V) = e^{-\beta hn|V|}\,\hat{Z}^n_{tr}(V, w_{J,h}) = e^{-\beta hn|V|} \sum_{\Gamma\in C^*_w(V,n)} \prod_{\gamma\in\Gamma} w^{tr}_{J,h}(\gamma)$$

and the truncated free energy

$$f^n_{tr}(\beta, h) = \lim_{N\to\infty} |Q_N|^{-1} \log Z^{h,n}_{tr}(Q_N) \tag{4.1}$$

(remember that absence of the superscript J means $J(x, y) = 1$ for all x, y). Let

$$a_k(\beta, h) = f^k_{tr}(\beta, h) - f^{k+1}_{tr}(\beta, h). \tag{4.2}$$

The main results in this section are the following

Theorem 4.1. *Let β be large enough, $V \subset\subset \mathbb{Z}^2$ simply connected, and let J be such that $H(V, t)$ holds with $t \geq 10\zeta$ ($\zeta = 1000^{-1}$). If $1 \leq k \leq k_{max} = \lfloor e^{\frac{\beta\zeta}{20}} \rfloor$, then*

- (*i*) *there exists a unique $h^*_k(\beta) \in [h^+_{k+1}(\beta), h^-_k(\beta)]$ such that $a_k(\beta, h^*_k(\beta)) = 0$*
- (*ii*) *if $h \in [h^*_k(\beta), h^+_k(\beta)]$, then all cylinders $\gamma \in C(V, k)$ are stable*
- (*iii*) *if $h \in [h^+_{k+1}(\beta), h^*_k(\beta)]$, then all cylinders $\gamma \in C(V, k + 1)$ are stable*
- (*iv*) *if $h \in [h^+_{k+1}(\beta), h^-_k(\beta)]$, then all cylinders $\gamma \in C(V, k) \cup C(V, k + 1)$ with $\operatorname{diam}\widetilde{\gamma} \leq |a_k(\beta, h)|^{-1}$ are stable*

Remark. The typical situation for what concerns J is $J = 1$ everywhere with the possible exception of the boundary of a square Λ, where $\sum_{e^*\in\delta\Lambda} J(e) \geq t|\delta\Lambda|$ holds.

Corollary 4.2. *Under the same hypotheses,*

$$|\log Z^{J,h_k^*(\beta),k}(V) - \log Z^{J,h_k^*(\beta),k+1}(V)| \leq |\delta V| e^{-\frac{1}{4}\beta\zeta} .$$

The third and last result needs some definitions. Given $\varphi \in \Omega$ we set

$$\sigma(x) = \mathrm{sign}(\varphi(x) - k - 1/2)$$

and, for $U \subset\subset \mathbb{Z}^2$

$$M_U(\sigma) = M_U(\sigma(\varphi)) = \sum\nolimits_{x\in U} \sigma(x)$$

$$m_U(\sigma) = m_U(\sigma(\varphi)) = |U|^{-1} M_U(\sigma) .$$

Then we have

Corollary 4.3. *Under the hypotheses of Theorem 4.1, if $U \subset V$ is such that*

$$|U \cap \bar{\alpha}| \leq \frac{1}{2}|\alpha| \qquad \text{for all } \alpha \in C_B(V) \tag{4.3}$$

then, for m such that $\zeta < m < 1 - \zeta$ we have

$$\mu_V^{h_k^*(\beta),k}\{m_U(\varphi) \geq -m\} \leq e^{-\frac{4}{9}\beta t(1-m)|U|}$$

$$\mu_V^{h_k^*(\beta),k+1}\{m_U(\varphi) \leq m\} \leq e^{-\frac{4}{9}\beta t(1-m)|U|}$$

4.2. General results in cluster expansion theory

Before proving Theorem 4.1 we present some standard results in cluster expansion theory adapted to our cylinder model. For this purpose, given a set of weights

$$z : C(\mathbb{Z}^2, k) \mapsto \mathbb{C}$$

and an arbitrary *finite* subset ξ of $C(\mathbb{Z}^2, k)$, we consider a generic partition function

$$\mathcal{Z}(\xi, z) = \sum_{\substack{\Gamma \in C_w^*(\mathbb{Z}^2,k) \\ \Gamma \subset \xi}} \prod_{\gamma\in\Gamma} z(\gamma)$$

(one should think of z as of the renormalized weight). A finite subset ξ of $C(\mathbb{Z}^2, k)$ is called a *cluster* if it cannot be decomposed into a union of two nonempty subsets $\xi = \xi_1 \cup \xi_2$, such that every cylinder in ξ_1 is weakly compatible with every cylinder in ξ_2.

For ξ finite subset of $C(\mathbb{Z}^2, k)$, we let

$$\|\xi\| = \sum_{\gamma\in\xi} |\widetilde{\gamma}| L(\gamma) \qquad \widetilde{\xi} = \cup_{\gamma\in\xi} \widetilde{\gamma} \qquad \bar{\xi} = \cup_{\gamma\in\xi} \bar{\gamma} \tag{4.4}$$

With a little abuse of notation we also set, for each $V \subset\subset \mathbb{Z}^2$,

$$\mathcal{Z}(V, z) = \mathcal{Z}(C(V, k), z)$$

($C(V, k)$ is an infinite set, but this won't cause any problem).

Theorem 4.4. *Let $c > 0$. There exists $\beta_0(c)$ such that if $\beta \geq \beta_0(c)$ and*

$$|z(\gamma)| \leq e^{-c\beta|\tilde{\gamma}|L(\gamma)}, \qquad \forall\gamma \in C(\mathbb{Z}^2, k)$$

then, if V is any finite subset of $\mathbb{Z}^2$, we have

(1) $\mathcal{Z}(\xi, z) \neq 0$ *for all* $\xi \subset\subset C(\mathbb{Z}^2, k)$ *and* $\mathcal{Z}(V, z) \neq 0$

(2)

$$\log \mathcal{Z}(\xi, z) = \sum_{\xi' \subset \xi} \Phi^T(\xi', z) \quad \forall\xi \subset\subset C(\mathbb{Z}^2, k)$$

$$\log \mathcal{Z}(V, z) = \sum_{\xi \subset\subset C(V,k)} \Phi^T(\xi, z)$$

where $\Phi^T(\xi, z) = 0$ unless ξ is a cluster, in which case

$$\Phi^T(\xi, z) = \sum_{\xi' \subset \xi} (-1)^{|\xi|-|\xi'|} \log \mathcal{Z}(\xi', z)$$

(3) denoting by $[\gamma]$ the set of all cylinders in $C(\mathbb{Z}^2, k)$ which are not weakly compatible with γ, we have

$$\sum_{\substack{\xi \subset\subset C(\mathbb{Z}^2,k) \\ \xi \cap [\gamma] \neq \emptyset}} |\Phi^T(\xi, z)| e^{\frac{3}{4}c\beta\|\xi\|} \leq \frac{1}{100}|\tilde{\gamma}|, \qquad \forall\gamma \in C(\mathbb{Z}^2, k)$$

(4) for each dual edge e^*,

$$\sum_{\substack{\xi \subset\subset C(\mathbb{Z}^2,k) \\ \tilde{\xi} \ni e^*}} |\Phi^T(\xi, z)| e^{\frac{3}{4}c\beta\|\xi\|} \leq 1$$

(5) if, for any finite volume Λ, we set

$$\Phi^k(\Lambda, z) = \sum_{\substack{\xi \subset\subset C(\mathbb{Z}^2,k) \\ \bar{\xi} = \Lambda}} \Phi^T(\xi, z)$$

then

$$\log \mathcal{Z}(V, z) = \sum_{\Lambda \subset V} \Phi^k(\Lambda, z)$$

and $|\Phi^k(\Lambda, z)| \leq \exp(-\frac{2}{3}c\beta K(\Lambda))$, where

$$K(\Lambda) \equiv \min\{\|\xi\| : \xi \text{ is a cluster and } \bar{\xi} = \Lambda\} \geq |\delta\Lambda|$$

(6)
$$\sum_{\substack{\Lambda\subset\subset\mathbb{Z}^2\\ \Lambda\ni x,\, K(\Lambda)\geq s}} |\Phi^k(\Lambda, z)| \leq e^{-\frac{c}{2}\beta s}$$

(7) if the weights $z(\gamma)$ are translation invariant, then

$$|\log \mathcal{Z}(V,z) - |V| f(z)| \leq |\delta V| e^{-2c\beta}$$

where

$$f(z) = \sum_{0\in\Lambda\subset\subset\mathbb{Z}^2} \frac{\Phi^k(\Lambda, z)}{|\Lambda|}$$

Remarks. Statements (1), (2) and (3) follow from the Kotecký and Preiss cluster expansion theory [KP], after having checked that their hypothesis (1) in the main theorem is verified with $a(\gamma) = |\widetilde{\gamma}|/(100)$ and $d(\gamma) = \frac{3}{4}\beta c|\widetilde{\gamma}|L(\gamma)$. The only problem here is that $C(V,k)$ is an infinite set, so one first defines

$$\mathcal{Z}_n(V,z) = \sum_{\substack{\Gamma\in C^*_w(V,k)\\ L(\gamma)\leq n\ \forall\gamma\in\Gamma}} \prod_{\gamma\in\Gamma} z(\gamma)$$

proves (1) and (2) for $\mathcal{Z}_n$ and then extends it to $\mathcal{Z}$ taking the $n\to\infty$ limit. Statements (4)–(7) follows from the first three (see, for instance [DKS]).

4.3. Preliminary results

We give now some results which will be needed in the proof of Theorem 4.1.

Sometime it will be useful to modify the strenght of the interaction J at the boundary of a certain volume V, so, given J and a real number b, we define

$$\bar{J}(b,\delta V)(x,y) = \begin{cases} bJ(x,y) & \text{if } [x,y]^* \in \delta V \\ J(x,y) & \text{otherwise} \end{cases}$$

Proposition 4.5. *Let $\beta, h > 0$, $V \subset\subset \mathbb{Z}^2$, $0<b<1$, and let $\bar{J} = \bar{J}(b,\delta V)$. Then for all $m, n \in \mathbb{Z}_+$,*

$$\frac{Z^{J,h,n}(V)}{Z^{J,h,m}(V)} \leq \exp\Big[\Big(b\beta|n-m| + (1-b)\beta \sup_{x\in\partial V} \mathbf{E}_V^{\bar{J},h,m} |\varphi(x) - m|\Big)|\delta V|_J\Big]$$

Proof. In fact

$$\frac{Z^{J,h,n}(V)}{Z^{J,h,m}(V)} \leq \frac{Z^{\bar{J},h,n}(V)}{Z^{J,h,m}(V)} \leq \exp\Big[\beta \sum_{[x,y]^*\in\delta V} \bar{J}(x,y)|n-m|\Big] \frac{Z^{\bar{J},h,m}(V)}{Z^{J,h,m}(V)}.$$

But
$$\sum_{[x,y]^*\in\delta V} \bar J(x,y) = b|\delta V|_J$$
and
$$H_V^{J,h,m}(\varphi) - H_V^{\bar J,h,m}(\varphi) = \sum_{[x,y]^*\in\delta V\,:\,x\in V} J(x,y)(1-b)|\varphi(x)-m|\,.$$
So, by the Jensen inequality,
$$\begin{aligned}\frac{Z^{\bar J,h,m}(V)}{Z^{J,h,m}(V)} = \left(\frac{Z^{J,h,m}(V)}{Z^{\bar J,h,m}(V)}\right)^{-1} &= \Big[\mathbf{E}_V^{\bar J,h,m} \exp[-\beta(H_V^{J,h,m}(\varphi) - H_V^{\bar J,h,m}(\varphi))\Big]^{-1} \le \\ &\le \exp\, \mathbf{E}_V^{\bar J,h,m}\big(\,\beta H_V^{J,h,m}(\varphi) - \beta H_V^{\bar J,h,m}(\varphi)\,\big) \le \\ &\le \exp\Big[\beta(1-b)|\delta V|_J \sup_{x\in\partial V} \mathbf{E}_V^{\bar J,h,m}\, |\varphi(x)-m|\Big] \quad \square\end{aligned}$$

Proposition 4.6. *Let β be large enough and $h\in I_k(\beta)$ with $1\le k\le 2k_{max}$. If V is a simply connected finite volume and $H(V,10\zeta)$ holds for J, then, for each $n>0$,*
$$\frac{Z^{J,h,n}(V)}{Z^{J,h,k}(V)} \le e^{0.11\beta|n-k||\delta V|_J}.$$

Proof. Let $\bar J = \bar J(b,\delta V)$ with $b=1/10$. Then $H(V,\zeta)$ holds for $\bar J$, so, by Proposition 3.6,
$$\mathbf{E}_V^{\bar J,h,k}\, |\varphi(x)-k| \le \frac{1}{100} \qquad \forall x\in V.$$
Apply now Proposition 4.5. $\square$

The previous result "almost" says that, if $h\in I_k(\beta)$, then cylinders starting from level k are stable. Unfortunately in the renormalized weight there is a quotient of "signed" partition functions. So we want to express "signed" partition functions in terms of unsigned ones

Proposition 4.7. *For any cylinder γ, if we let $\bar\gamma_x = \bar\gamma\setminus\{x\}$, then*

(*a*) $Z^{J,h,n}(\bar\gamma,\pm) \le \sum_{x\in\bar\partial\bar\gamma} Z^{J,h,n}(\bar\gamma_x)$

(*b*) $Z^{J,h,n}(\bar\gamma,+) \ge e^{-\beta hn} \sup_{x\in\bar\partial\bar\gamma} Z^{J,h,n}(\bar\gamma_x) \prod_{y\in\bar\partial\bar\gamma\setminus\{x\}} \mu_{\bar\gamma_x}^{J,h,n}\{\varphi(y)\ge n\}$

(*c*) $Z^{J,h,n}(\bar\gamma,-) \ge e^{-\beta hn} \sup_{x\in\bar\partial\bar\gamma} Z^{J,h,n}(\bar\gamma_x) \prod_{y\in\bar\partial\bar\gamma\setminus\{x\}} \mu_{\bar\gamma_x}^{J,h,n}\{\varphi(y)\le n\}$

Proof.

(a) By (2.3)

$$\frac{Z^{J,h,n}(\bar{\gamma},+)}{Z^{J,h,n}(\bar{\gamma})} = \mu^{J,h,n}_{\bar{\gamma}}(\Omega^{n,+}_{\bar{\gamma}}) \tag{4.5}$$

If we call X the RHS of (4.5), then

$$X \le \mu^{J,h,n}_{\bar{\gamma}}\left\{\varphi \in \Omega_V : \exists x \in \bar{\partial}\bar{\gamma},\, \varphi(x) = n\right\} \le \sum_{x \in \bar{\partial}\bar{\gamma}} \mu^{J,h,n}_{\bar{\gamma}}\{\varphi(x) = n\} =$$

$$= \left(Z^{J,h,n}(\bar{\gamma})\right)^{-1} \sum_{x \in \bar{\partial}\bar{\gamma}} \sum_{\substack{\varphi \in \Omega_{\bar{\gamma}_x} \\ \varphi(x)=n}} e^{-\beta H^{J,h,n}_{\bar{\gamma}}(\varphi)} \le \left(Z^{J,h,n}(\bar{\gamma})\right)^{-1} \sum_{x \in \bar{\partial}\bar{\gamma}} Z^{J,h,n}(\bar{\gamma}_x)$$

(b) On the other side, by (1.7) and the FKG property,

$$X \ge \sup_{x \in \bar{\partial}\bar{\gamma}} \mu^{J,h,n}_{\bar{\gamma}}\{\varphi(x) = n,\ \varphi(y) \ge n\ \forall y \in \bar{\partial}\bar{\gamma}\} =$$

$$= \sup_{x \in \bar{\partial}\bar{\gamma}} \mu^{J,h,n}_{\bar{\gamma}}\{\varphi(x) = n\}\, \mu^{J,h,n}_{\bar{\gamma}_x}\{\varphi(y) \ge n\ \forall y \in \bar{\partial}\bar{\gamma}\} \ge$$

$$\ge \sup_{x \in \bar{\partial}\bar{\gamma}} \left(Z^{J,h,n}(\bar{\gamma})\right)^{-1} Z^{J,h,n}(\bar{\gamma}_x) e^{-\beta h n} \prod_{y \in \bar{\partial}\bar{\gamma}\setminus\{x\}} \mu^{J,h,n}_{\bar{\gamma}_x}\{\varphi(y) \ge n\}$$

(c) can be obtained in the same way. $\square$

Proposition 4.8. *Let β be large enough and $h \in I_k(\beta)$ with $1 \le k \le 2k_{max}$. If V is a finite, simply connected volume, and $H(V, 10\zeta)$ holds for J, then, for each $\gamma \in C(V)$ with $E(\gamma) = k$ we have*

$$\widetilde{w}_{J,h}(\gamma) \le e^{-\frac{2}{3}\beta|\tilde{\gamma}|_J L(\gamma)} < w^{tr}_{J,h}(\gamma)$$

hence γ is stable.

Proof. If $\operatorname{diam}\widetilde{\gamma} \le \zeta^{-1}$, then γ is elementary and the statement follows from Lemma 2.7. Let, then $\operatorname{diam}\widetilde{\gamma} > \zeta^{-1}$ and assume $S(\gamma) = +1$ (the opposite case is analogous). Set also $n = I(\gamma)$. By Proposition 4.7

$$\frac{Z^{J,h,n}(\bar{\gamma},+)}{Z^{J,h,k}(\bar{\gamma},+)} \le e^{\beta h k} \sum_{x \in \bar{\partial}\bar{\gamma}} \frac{Z^{J,h,n}(\bar{\gamma}_x)}{Z^{J,h,k}(\bar{\gamma}_x)} \Big(\prod_{y \in \bar{\partial}\bar{\gamma}\setminus\{x\}} \mu^{J,h,k}_{\bar{\gamma}_x}\{\varphi(y) \ge k\}\Big)^{-1} \tag{4.6}$$

By Proposition 3.5, we know that

$$\Big(\prod_{y \in \bar{\partial}\bar{\gamma}\setminus\{x\}} \mu^{J,h,k}_{\bar{\gamma}_x}\{\varphi(y) = k\}\Big)^{-1} \le \Big(1 - e^{-\frac{1}{5}\beta\zeta}\Big)^{-|\bar{\partial}\bar{\gamma}|} \le \exp\Big[12|\widetilde{\gamma}|e^{-\frac{1}{5}\beta\zeta}\Big] \le e^{|\widetilde{\gamma}|}\,.$$

Moreover, if $x \in \bar{\partial}\bar{\gamma}$, $\bar{\gamma}_x$ is simply connected, so we apply Proposition 4.6 and obtain

$$\frac{Z^{J,h,n}(\bar{\gamma}_x)}{Z^{J,h,k}(\bar{\gamma}_x)} \le e^{0.11\beta|n-k||\delta(\bar{\gamma}_x)|_J} .$$

But, since $\operatorname{diam}\widetilde{\gamma} > \zeta^{-1}$ and $H(V,10\zeta)$ holds for J, we have

$$|\delta(\bar{\gamma}_x)|_J \le |\widetilde{\gamma}|_J + 4 \le |\widetilde{\gamma}|_J + 4\zeta|\widetilde{\gamma}| \le 1.4\,|\widetilde{\gamma}|_J . \tag{4.7}$$

Hence

$$\frac{Z^{J,h,n}(\bar{\gamma},+)}{Z^{J,h,k}(\bar{\gamma},+)} \le |\bar{\partial}\bar{\gamma}| e^{0.2\beta|n-k||\widetilde{\gamma}|_J+|\widetilde{\gamma}|+\beta hk} \le e^{0.3\beta|\widetilde{\gamma}|_J L(\gamma)}$$

and the results follows from the definition of renormalized weights (2.6). □

Proposition 4.9. *Let β, V and J satisfy the hypotheses of Theorem 4.1. With reference to the notation introduced in Theorem 4.4 we have, for each $h > 0$, $k \in \mathbb{Z}_+$*

$$f^k_{tr}(\beta,h) = -\beta hk + \sum_{0\in\Lambda\subset\subset\mathbb{Z}^2} \frac{\Phi^k(\Lambda, w^{tr}_h)}{|\Lambda|}$$

and

$$|\log Z^{J,h,k}_{tr}(V) - |V| f^k_{tr}(h)| \le |\delta V| e^{-\frac{1}{2}\beta\zeta}$$

Proof. By definition of truncated weights, we can apply the results of Theorem 4.4 to both $Z^{h,k}_{tr}(V)$ and $Z^{J,h,k}_{tr}(V)$. If $J = 1$ we are in the translation invariant case, so, by (7) of Theorem 4.4, we have

$$\left|\log Z^{h,k}_{tr}(V) + \beta hk - \sum_{0\in\Lambda\subset\subset\mathbb{Z}^2} \frac{\Phi^k(\Lambda, w^{tr}_h)}{|\Lambda|}\right| \le e^{-\beta} .$$

On the other side, since $\Phi^k(\Lambda, w^{tr}_h) = \Phi^k(\Lambda, w^{tr}_{J,h})$, unless Λ intersects $\Delta(J)$ (see (2.13)), by (5) and (6) of Theorem 4.4 we have

$$|\log Z^{h,k}_{tr}(V) - \log Z^{J,h,k}_{tr}(V)| \le \sum_{\Lambda\subset V} |\Phi^k(\Lambda, w^{tr}_h) - \Phi^k(\Lambda, w^{tr}_{J,h})| \le$$

$$\le \sum_{x\in\Delta(J)} \sum_{\Lambda\ni x} \left[|\Phi^k(\Lambda, w^{tr}_h)| + |\Phi^k(\Lambda, w^{tr}_{J,h})|\right] \le 2|\delta V| e^{-\beta\zeta} \quad \square$$

4.4. Proof of Theorem 4.1 and Corollary 4.2

We set $h^* = h_k^*(\beta)$ and $a(h) = a_k(\beta, h)$. The free energy is given

$$f(h) = \lim_{N\to\infty} |Q_N|^{-1} \log Z^{h,n}(Q_N)$$

and clearly does not depend on n. Proposition 4.8 implies, for each N

$$Z_{tr}^{h_k^-(\beta),k}(Q_N) = Z^{h_k^-(\beta),k}(Q_N)\,, \qquad Z_{tr}^{h_{k+1}^+(\beta),k+1}(Q_N) = Z^{h_{k+1}^+(\beta),k+1}(Q_N)$$

which gives

$$f_{tr}^k(h_k^-(\beta)) = f(h_k^-(\beta)) \geq f_{tr}^{k+1}(h_k^-(\beta))$$

and then

$$a(h_k^-(\beta)) \geq 0.$$

In the same way one obtains $a(h_{k+1}^+(\beta)) \leq 0$. Moreover a standard calculation (see for instance Proposition 1.9 in [Z]) gives

$$\frac{\partial f_{tr}^n(h)}{\partial h_\pm} = -\beta n + \alpha_n(\beta)$$

where $\lim_{\beta\to\infty} \alpha_n(\beta) = 0$. So $a(h)$ has a right and left derivative and both of them are greater than say $\beta/2$. This implies the strict monotonicity of $a(h)$ in the interval $[h_{k+1}^+(\beta), h_k^-(\beta)]$, and, by consequence, (i).

In the following we will make use of the fact that, for $0 < m < n$,

$$\frac{d}{dh}\frac{Z^{J,h,n}(V)}{Z^{J,h,m}(V)} = \sum_{x\in V}\Big[\mathbf{E}_V^{J,h,n}\,\varphi(x) - \mathbf{E}_V^{J,h,m}\,\varphi(x)\Big] \geq 0 \tag{4.8}$$

because of FKG.

We are now going to prove (ii), (iii) and (iv) by induction. Let

(A_s) : $\quad$ *(ii), (iii), (iv) hold for all cylinders γ such that $|\bar\gamma| \leq s$*

A_1 is true because of (i) of Lemma 2.7. Let's assume then A_s and we'll show that A_{s+1} follows. Actually we'll prove only statement (ii) and (iv), since (iii) can be obtained in the same way as (ii).

Choose a cylinder γ with $|\bar\gamma| = s+1$, $E(\gamma) = k$, $I(\gamma) = n$ and let γ' be another cylinder with the same basis $\widetilde{\gamma}' = \widetilde{\gamma}$ and $E(\gamma') = k+1$, $I(\gamma') = m$. Let $\bar\gamma_x = \bar\gamma \setminus \{x\}$. The proof can be conveniently broken into the following steps

(1) $\left|\log Z^{J,h^*,k}(\bar\gamma_x)-\log Z^{J,h^*,k+1}(\bar\gamma_x)\right|\le|\widetilde\gamma|e^{-\frac14\beta\zeta}\qquad\forall x\in\bar\partial\bar\gamma$

(2) If $h\in[h^*,h_k^-(\beta)]$, then $\left|\log Z^{J,h,k}(\bar\gamma,+)-\log Z^{J,h,k}(\bar\gamma,-)\right|\le|\widetilde\gamma|e^{-\frac14\beta\zeta}$

(3) If $h\in[h^*,h_k^-(\beta)]$, then for any n,

$$\log\frac{Z^{J,h,n}(\bar\gamma_x)}{Z^{J,h,k}(\bar\gamma_x)}\le 0.2\beta|n-k||\widetilde\gamma|_J\qquad\forall x\in\bar\partial\bar\gamma$$

(4) If $h\in[h^*,h_k^-(\beta)]$, then γ is stable

(5) if $h\in[h_{k+1}^+(\beta),h_k^-(\beta)]$ and $\operatorname{diam}\widetilde\gamma\le|a(\beta,h)|^{-1}$, then both γ and γ' are stable

Proof of (1). Since $|\bar\gamma_x|=s$ by induction all cylinders contributing to the two partition functions are stable, so

$$Z_{tr}^{J,h^*,i}(\bar\gamma_x)=Z^{J,h^*,i}(\bar\gamma_x)\qquad i=k,k+1\,.$$

Since $\bar\gamma_x$ is simply connected and since the property $H(V,t)$ is inherited by all subsets of V we can apply Proposition 4.9 to $\bar\gamma_x$ and use the fact that $a(h^*)=0$ to obtain (1).

Proof of (2). Both partition functions are sums over collection of cylinders γ' with $|\bar\gamma'|\le s$, which are stable by induction. So, setting $\bar\gamma_0=\bar\gamma\setminus\bar\partial\bar\gamma$, we obtain

$$Z_{tr}^{J,h,k}(\bar\gamma_0)e^{-\beta hk|\bar\partial\bar\gamma|}\le Z^{J,h,k}(\bar\gamma,\pm)=Z_{tr}^{J,h,k}(\bar\gamma,\pm)\le Z_{tr}^{J,h,k}(\bar\gamma)\,.$$

By Proposition 4.9

$$|\log Z_{tr}^{J,h,k}(\bar\gamma_0)-\log Z_{tr}^{J,h,k}(\bar\gamma)|\le\big(|\delta\bar\gamma_0|+|\widetilde\gamma|\big)e^{-\frac12\beta\zeta}+|\bar\partial\bar\gamma|f_{tr}^k(h)\le|\widetilde\gamma|e^{-\frac13\beta\zeta}$$

which implies

$$|\log Z^{J,h,k}(\bar\gamma,+)-\log Z^{J,h,k}(\bar\gamma,-)|\le|\widetilde\gamma|e^{-\frac13\beta\zeta}+\beta hk|\bar\partial\bar\gamma|\le|\widetilde\gamma|e^{-\frac14\beta\zeta}\,.$$

Proof of (3). We consider 3 cases

(a) If $\operatorname{diam}\widetilde\gamma\le e^{\frac{\beta\zeta}{10}}$ then

$$Z^{J,h,m}(\bar\gamma_x)=Z_e^{J,h,m}(\bar\gamma_x)=e^{-\beta hm|\bar\gamma_x|}\hat Z_e^m(\bar\gamma_x,w_{J,h})\,,\qquad\forall m$$

so we can use (4.8), (1) and Lemma 2.7–(iv), and get

$$\frac{Z^{J,h,n}(\bar\gamma_x)}{Z^{J,h,k}(\bar\gamma_x)}\le\frac{Z_e^{J,h,n}(\bar\gamma_x)}{Z_e^{J,h,k+1}(\bar\gamma_x)}\,\frac{Z^{J,h^*,k+1}(\bar\gamma_x)}{Z^{J,h^*,k}(\bar\gamma_x)}\le$$
$$\exp\Big[|\delta\bar\gamma_x|\big(e^{-\frac14\beta\zeta}+e^{-\frac15\beta\zeta}\big)\Big]\le e^{0.2\beta|n-k||\widetilde\gamma|_J}$$

(b) If $\operatorname{diam}\widetilde{\gamma} > e^{\frac{\beta\zeta}{10}}$ and $n < k$ then $|\delta\bar{\gamma}_x|_J \le 1.1|\widetilde{\gamma}|_J$, so, by (4.8) and Proposition 4.6

$$\frac{Z^{J,h,n}(\bar{\gamma}_x)}{Z^{J,h,k}(\bar{\gamma}_x)} \le \frac{Z^{J,h_k^-(\beta),n}(\bar{\gamma}_x)}{Z^{J,h_k^-(\beta),k}(\bar{\gamma}_x)} \le e^{0.11\beta|n-k||\delta(\bar{\gamma}_x)|_J} \le e^{0.2\beta|n-k||\widetilde{\gamma}|_J}.$$

(c) If $\operatorname{diam}\widetilde{\gamma} > e^{\frac{\beta\zeta}{10}}$ and $n \ge k+1$ we proceed in a similar way

$$\frac{Z^{J,h,n}(\bar{\gamma}_x)}{Z^{J,h,k}(\bar{\gamma}_x)} \le \frac{Z^{J,h_{k+1}^+(\beta),n}(\bar{\gamma}_x)}{Z^{J,h_{k+1}^+(\beta),k+1}(\bar{\gamma}_x)} \frac{Z^{J,h^*,k+1}(\bar{\gamma}_x)}{Z^{J,h^*,k}(\bar{\gamma}_x)} \le$$
$$\le \exp\left[0.2\beta|n-k-1||\widetilde{\gamma}|_J + e^{-\frac{1}{4}\beta\zeta}|\widetilde{\gamma}|\right] \le e^{0.2\beta|n-k||\widetilde{\gamma}|_J}.$$

Proof of (4).

(a) If $n > k$ then, by Proposition 4.7

$$\widetilde{w}_{J,h}(\gamma) = e^{-\beta|\widetilde{\gamma}|_J L(\gamma)} \frac{Z^{J,h,n}(\bar{\gamma},+)}{Z^{J,h,k}(\bar{\gamma},+)} \le e^{-\beta|\widetilde{\gamma}|_J L(\gamma)} R_1 R_2,$$

where

$$R_1 = \sup_{x\in\bar{\partial}\bar{\gamma}} \frac{Z^{J,h,n}(\bar{\gamma}_x)}{Z^{J,h,k}(\bar{\gamma}_x)} \le e^{0.2\beta|n-k||\widetilde{\gamma}|_J},$$

by (3), and

$$R_2 = |\bar{\gamma}| \sup_{x\in\bar{\partial}\bar{\gamma}} \Big(\prod_{y\in\bar{\partial}\bar{\gamma}\setminus\{x\}} \mu_{\bar{\gamma}_x}^{J,h,k}\{\varphi(y)\ge k\}\Big)^{-1} e^{\beta hk}.$$

To estimate R_2 we first use FKG to replace h with $h_k^-(\beta)$ in $\mu_{\bar{\gamma}_x}^{J,h,k}$ and then proceed as in the proof of Proposition 4.8, obtaining

$$R_2 \le |\bar{\gamma}| e^{|\widetilde{\gamma}|+\beta hk} \le e^{0.01\beta|\widetilde{\gamma}|_J}.$$

Thus we find

$$\widetilde{w}_{J,h}(\gamma) \le e^{0.79\beta|\widetilde{\gamma}|_J L(\gamma)} < w_{J,h}^{tr}(\gamma) \tag{4.9}$$

that is the cylinder is stable.

(b) If $n < k$ it is just a bit trickier,

$$\widetilde{w}_{J,h}(\gamma) = e^{-\beta|\widetilde{\gamma}|_J L(\gamma)} \frac{Z^{J,h,n}(\bar{\gamma},-)}{Z^{J,h,k}(\bar{\gamma},+)} \frac{Z^{J,h,k}(\bar{\gamma},+)}{Z^{J,h,k}(\bar{\gamma},-)} \le \exp\left[-\beta|\widetilde{\gamma}|_J L(\gamma) + e^{-\frac{1}{4}\beta\zeta}|\widetilde{\gamma}|\right] R_1 R_2$$

where we have used (2) and R_1, R_2 are the same as in (a). Hence

$$\widetilde{w}_{J,h}(\gamma) \le e^{0.78\beta|\tilde{\gamma}|_J L(\gamma)} < w^{tr}_{J,h}(\gamma)\,. \tag{4.10}$$

Proof of (5). We already know that γ is stable if $h \in [h^*_k(\beta), h^-_k(\beta)]$. Here we will show that γ' is stable for h in the same interval. The proof when $h \in [h^+_{k+1}(\beta), h^*_k(\beta)]$ is identical. If $m \neq k+1$, $S = \pm 1$, then, by (4.9) and (4.10),

$$\frac{Z^{J,h,m}(\bar{\gamma},S)}{Z^{J,h,k+1}(\bar{\gamma},S)} = \frac{Z^{J,h,m}(\bar{\gamma},S)}{Z^{J,h,k}(\bar{\gamma},S)}\,\frac{Z^{J,h,k}(\bar{\gamma},S)}{Z^{J,h,k+1}(\bar{\gamma},S)} \le e^{0.22\beta|\tilde{\gamma}|_J|m-k|}\frac{Z^{J,h,k}(\bar{\gamma},S)}{Z^{J,h,k+1}(\bar{\gamma},S)}\,. \tag{4.11}$$

Furthermore we know that

$$Z^{J,h,k}(\bar{\gamma},S) = Z^{J,h,k}_{tr}(\bar{\gamma},S) \le Z^{J,h,k}_{tr}(\bar{\gamma})$$

and that

$$Z^{J,h,k+1}(\bar{\gamma},S) \ge e^{-\beta h|\bar{\partial}\bar{\gamma}|(k+1)}\, Z^{J,h,k+1}(\bar{\gamma}_0) \ge e^{-\zeta|\tilde{\gamma}|} Z^{J,h,k+1}_{tr}(\bar{\gamma}_0)$$

where $\bar{\gamma}_0 = \bar{\gamma} \setminus \bar{\partial}\bar{\gamma}$. Now, using Proposition 4.9, we can write

$$\begin{aligned}
&\log Z^{J,h,k}(\bar{\gamma},S) - \log Z^{J,h,k+1}(\bar{\gamma},S) \le \log Z^{J,h,k}_{tr}(\bar{\gamma}) - \log Z^{J,h,k+1}_{tr}(\bar{\gamma}_0) + |\tilde{\gamma}| \le \\
&\le \left|\log Z^{J,h,k}_{tr}(\bar{\gamma}) - |\bar{\gamma}| f^k_{tr}\right| + \left|\log Z^{J,h,k+1}_{tr}(\bar{\gamma}_0) - |\bar{\gamma}_0| f^{k+1}_{tr}\right| + \left||\bar{\gamma}| f^k_{tr}| - |\bar{\gamma}_0| f^{k+1}_{tr}\right| + |\tilde{\gamma}| \le \\
&\le 2|\tilde{\gamma}| + a(h)|\bar{\gamma}|\,.
\end{aligned}$$

But, since $\operatorname{diam}\tilde{\gamma} \le |a(h)|^{-1}$, we get

$$a(h)|\bar{\gamma}| \le a(h)|\tilde{\gamma}|\ \operatorname{diam}\tilde{\gamma} \le |\tilde{\gamma}|\,.$$

So

$$\frac{Z^{J,h,m}(\bar{\gamma},S)}{Z^{J,h,k+1}(\bar{\gamma},S)} \le e^{0.22\beta|\tilde{\gamma}|_J|m-k|+3|\tilde{\gamma}|} \le e^{0.23\beta|\tilde{\gamma}|_J|m-k|} \le e^{0.46\beta|\tilde{\gamma}|_J|m-(k+1)|}$$

which means that γ' is stable. $\square$

Proof of Corollary 4.2. Same as the proof of (1). $\square$

4.5. Proof of Corollary 4.3

Given $\varphi \in \Omega_V$, let $\Gamma_\varphi \in C^*_c(V,k)$ be the collection of compatible cylinders corresponding to φ, *i.e.* $\Gamma_\varphi \sim \varphi$. Let

$$A(\varphi) = \{x \in V : x \in \bar{\gamma} \text{ for some } \gamma \in \Gamma_\varphi\ \}.$$

Then, by Chebyshev inequality,

$$\mu_V^{h^*,k}\{\varphi \in \Omega_V : m_U(\varphi) \geq -m\} \leq \mu_V^{h^*,k}\Big\{|A(\varphi) \cap U| \geq \frac{1}{2}(1-m)|U|\Big\} \leq$$
$$\leq e^{-c\beta\frac{(1-m)}{2}|U|}\,\mathbf{E}_V^{h^*,k}(e^{c\beta|A(\varphi)\cap U|}) \tag{4.12}$$

for all c such that the expectation is finite. The idea now is to introduce modified cylinder weights that allow us to express the expectation in (4.12) as a quotient of partition functions. This quotient will be estimated by cluster expansion.

So we let, for a cylinder γ,

$$w_h^U(\gamma) = w_h(\gamma)\, e^{c\beta|\bar{\gamma}\cap U|}.$$

Then

$$\mathbf{E}_V^{h^*,k}(e^{c\beta|A(\varphi)\cap U|}) = (\hat{Z}^k(V, w_{h^*}))^{-1}X$$

where

$$X = \sum_{\Gamma\in C_c^*(V,k)} \prod_{\gamma\in\Gamma_{ext}} w_{h^*}^U(\gamma) \prod_{\gamma\in\Gamma\setminus\Gamma_{ext}} w_{h^*}(\gamma) =$$
$$= \sum_{\Gamma\in C_w^*(V,k)} \prod_{\gamma\in\Gamma_{ext}} \widetilde{w}_{h^*}(\gamma)\, e^{c\beta|\bar{\gamma}\cap U|} \prod_{\gamma\in\Gamma\setminus\Gamma_{ext}} \widetilde{w}_{h^*}(\gamma) \leq$$
$$\leq \sum_{\Gamma\in C_w^*(V,k)} \prod_{\gamma\in\Gamma} \widetilde{w}_{h^*}(\gamma)\, e^{c\beta|\bar{\gamma}\cap U|} = \sum_{\Gamma\in C_w^*(V,k)} \prod_{\gamma\in\Gamma} \bar{w}_{h^*}^U(\gamma)$$

with

$$\bar{w}_{h^*}^U(\gamma) = \widetilde{w}_{h^*}(\gamma)\, e^{c\beta|\bar{\gamma}\cap U|}\,.$$

So we can write

$$\mathbf{E}_V^{h^*,k}(e^{c\beta|A(\varphi)\cap U|}) \leq \frac{\sum_{\Gamma\in C_w^*(V,k)} \prod_{\gamma\in\Gamma} \bar{w}_{h^*}^U(\gamma)}{\sum_{\Gamma\in C_w^*(V,k)} \prod_{\gamma\in\Gamma} \widetilde{w}_{h^*}(\gamma)}\,. \tag{4.13}$$

Now we can use cluster expansion for both numerator and denominator if c is small enough. In fact, by Theorem 4.1, we know that

$$\widetilde{w}_{h^*}(\gamma) \leq e^{-\frac{\beta}{2}t|\tilde{\gamma}|L(\gamma)}$$

and, on the other side, using (4.3),

$$\bar{w}_h^U(\gamma) \leq e^{-\frac{1}{2}\beta(t-c)|\tilde{\gamma}|L(\gamma)} \leq e^{-0.01t|\tilde{\gamma}|L(\gamma)}$$

if $c \leq 0.98t$. By Theorem 4.4 we get

$$\log \mathrm{RHS\ of\ } (4.13) \leq \sum_{\Lambda \subset V} \left| \Phi^k(\Lambda, \bar{w}^U_{h^*}) - \Phi^k(\Lambda, \widetilde{w}_{h^*}) \right| \leq$$

$$\leq \sum_{\substack{\Lambda \subset V \\ \Lambda \cap U \neq \emptyset}} \left| \Phi^k(\Lambda, \bar{w}^U_{h^*}) - \Phi^k(\Lambda, \widetilde{w}_{h^*}) \right| \leq |U| e^{-\frac{1}{200} t\beta}$$

Together with (4.12) this gives (take $c = 0.98t$)

$$\mu_V^{h^*,k}\{m_U(\varphi) \geq -m\} \leq e^{-\frac{4}{9}\beta t(1-m)|U|} \,.$$

In the same way one proves

$$\mu_V^{h^*,k+1}\{m_U(\varphi) \leq m\} \leq e^{-\frac{4}{9}\beta t(1-m)|U|} \quad \square$$

5. Arbitrary boundary condition I. Away from the phase transition

Throughout most of this section we assume

(K_1) β is large enough
(K_2) $h \in I_k(\beta) = [h_k^-(\beta), h_k^+(\beta)]$ (see (2.15)) with $k = 1, \ldots, k_{max} = \lfloor e^{\frac{\beta\zeta}{20}} \rfloor$ and $\zeta = 1000^{-1}$
(K_3) $\Lambda = Q_N$ with $N \geq N_0 = \lfloor (\zeta^3 h)^{-1} \rfloor$
(K_4) $\Lambda' = Q_{N'}$ where $N' = N - 2\lfloor N/4 \rfloor$

Our goal is to prove that *boundary conditions do not percolate inside the bulk.* So, for $\varphi \in \Omega_\Lambda$, we give a name to the set of clusters of the region where φ is at least j which are attached to the boundary

$$R_+(\Lambda, j, \varphi) = \left\{ x \in \Lambda : \begin{array}{c} \text{there exists a path } (x = x_1, \ldots, x_s) \text{ such that} \\ x_s \in \partial\Lambda \text{ and } \varphi(x_i) \geq j \text{ for each } i \end{array} \right\}.$$

$R_-(\Lambda, j, \varphi)$ is defined in the same way with $\geq j$ replaced by $\leq j$. We also define

$$\begin{aligned} S_\pm(\Lambda, j, \Lambda') &= \{\varphi \in \Omega_\Lambda : R_\pm(\Lambda, j, \varphi) \cap \Lambda' \neq \emptyset\} \\ S^0_\pm(\Lambda, j, \Lambda') &= S_\pm(\Lambda, j, \Lambda') \setminus S_\pm(\Lambda, j \pm 1, \Lambda') \end{aligned}$$

The goal of this section is to prove the following two results

Theorem 5.1. *Assume* (K_1)–(K_4). *Then for each* $j > 0$ *and for all* $\psi \in \Omega$

$$\mu_\Lambda^{h,\psi}\big(S_+(\Lambda, k+j, \Lambda') \cup S_-(\Lambda, k-j, \Lambda') \big) \leq e^{-\zeta\beta N j / k^2} . \tag{5.1}$$

Theorem 5.2. *Assume* (K_1)–(K_2) *and let* $\Lambda = Q_N$ *with* $N \geq 10h^{-1}$. *Then for each* $j > 0$ *and for all* $x \in \Lambda$,

$$\mu_\Lambda^{h,\emptyset}\big(S_+(\Lambda, k+j, \{x\}) \cup S_-(\Lambda, k-j, \{x\}) \big) \leq e^{-\zeta\beta j d(x, \Lambda^c)} .$$

Remark. Notice that with n b.c. we are saying that there is no percolation al level (say) at least $k+1$ reaching a distance greater than a certain constant divided by h. On the other side, if we have free b.c. one expects $\varphi(x) = k$ right *at* the boundary $\partial\Lambda$.

5.1. Percolation in terms of contours

We are going to

- define an event slightly larger than $S_+(\Lambda, k+j, \Lambda') \cup S_-(\Lambda, k-j, \Lambda')$, but with the advantage of being defined directly in terms of contours

• define a representation of the partition function $\hat{Z}^n$ analogous to (2.11), but with contours starting from the "right phase" k. In this representation, weights of contours, that must be modified to take into account b.c., are not small enough to guarantee cluster expansions, nevertheless they provide bound (5.1)

For reasons that will become more apparent shortly we don't want the boundary conditions to affect the weight of elementary cylinders, so we issue the following

Warning. In this section we modify the notion of *elementary cylinder* by considering as elementary only those cylinders γ such that

(i) $\operatorname{diam} \widetilde{\gamma} \leq \vartheta = e^{\frac{\varsigma\beta}{10}}$

(ii) $|\widetilde{\gamma} \cap \delta\Lambda| = 0$

Remark. Changing the definition of elementary cylinders is in principle a rather dangerous thing to do, since it is a very "low level" definition and may affect a whole lot of things. We observe here that in our case the main statement in Section 2, *i.e.* Corollary 2.8 is still valid with exactly the same proof. Also the hypothesis (c) in part (ii) of Proposition 2.9 is given in such a way to include the present definition of elementary cylinders.

The idea for proving Theorem 5.1 is to replace φ with φ_l (defined in Section 2), by eliminating all elementary cylinders first, and then use (2.12) in order to estimate the induced probability. We then start by defining

$$S_{\pm,l}(\Lambda, j, \Lambda') = \{\mathcal{A} \in W_l(\Lambda, k) : R_\pm(\Lambda, j, \varphi) \cap \Lambda' \neq \emptyset, \text{ where } \varphi \sim \mathcal{A}\}$$
$$S^0_{\pm,l}(\Lambda, j, \Lambda') = S_{\pm,l}(\Lambda, j, \Lambda') \setminus S_{\pm,l}(\Lambda, j \pm 1)$$

and showing that

Lemma 5.3. *If $\Lambda = Q_N$, $\Lambda' \subset \Lambda$, then for all β, h, j, n, N*

$$\mu_\Lambda^{h,n}(S_\pm(\Lambda, j, \Lambda')) \leq \mu_\Lambda^{h,n}\{\varphi : \varphi_l \in S_\pm(\Lambda, j, \Lambda')) \equiv \bar{\mu}_\Lambda^{h,n}(S_{\pm,l}(\Lambda, j, \Lambda')).$$

Proof. We will actually show that

$$R_+(\Lambda, j, \varphi) \subset R_+(\Lambda, j, \varphi_l) \tag{5.2}$$

which implies the thesis.

Let $x \in R_+(\Lambda, j, \varphi)$, and let $(x_1, \ldots, x_s)$ be a path connecting $x = x_1$ with $x_s \in \partial\Lambda$ such that $\varphi(x_i) \geq j$ for all i. Let Γ, Γ_l be the collections of cylinders in $C^*_c(\Lambda, k)$ corresponding respectively to φ and φ_l. Thus Γ_l is obtained by Γ by removing all elementary cylinders. Take any site x_i along the path. If there are no elementary cylinders γ such that $\bar{\gamma} \ni x_i$, then $\varphi_l(x_i) = \varphi(x_i) \geq j$. If, on the contrary, x_i is contained in the interior of some elementary cylinder, then

$$\varphi_l(x_i) = E(\gamma_*)$$

where γ_* is the most external elementary cylinder in Γ such that $\bar{\gamma}_* \ni x_i$.

But we know that $\bar{\gamma}_* \not\ni x_s$ because otherwise $\widetilde{\gamma}_*$ would have at least one edge in common with $\delta\Lambda$ and by consequence would not be elementary, by property (ii). So, there exists $j \in \{i, \dots, s-1\}$ such that $x_j \in \bar{\gamma}_*$ and $x_{j+1} \notin \bar{\gamma}_*$, so that $[x_j, x_{j+1}]^* \in \widetilde{\gamma}_*$. Hence, by Proposition 2.3,

$$E(\gamma_*) \geq \varphi(x_j) \wedge \varphi(x_{j+1}) \geq j.$$

So $\varphi_l(x_i) \geq j$ for all i, which implies (5.2). Analogously one proves that $R_-(\Lambda, j, \varphi) \subset R_-(\Lambda, j, \varphi_l)$ $\square$

In order to find a bound on $\bar{\mu}(S_{\pm,l}(\Lambda, j, \Lambda'))$, it is useful to express the partition function with n boundary condition as a sum over cylinders starting from k (the right phase) whose weight is modified if they touch the boundary. If one sets

$$w_h^n(\gamma) = w_h(\gamma)\, e^{2\beta q(\gamma) L_n(\gamma)} = \exp\big[-\beta|\widetilde{\gamma}|L(\gamma) + 2\beta q(\gamma) L_n(\gamma) + \beta h S(\gamma)|\bar{\gamma}|L(\gamma)\big] \tag{5.3}$$

where $q(\gamma) = |\widetilde{\gamma} \cap \delta\Lambda|$ and

$$L_n(\gamma) = |[k, n] \cap [E(\gamma), I(\gamma)]|\, \chi\{\mathrm{sign}(n-k)(I(\gamma) - E(\gamma)) = 1\} \tag{5.4}$$

then the partition function with n b.c. can be written as

$$Z^{h,n}(\Lambda) = e^{-\beta hk|\Lambda| - \beta|n-k||\delta\Lambda|} \hat{Z}^k(\Lambda, w_h^n) \tag{5.5}$$

Also, by Proposition 2.6

$$\hat{Z}^k(\Lambda, w_h^n) = \sum_{\mathcal{A} \in W_l(\Lambda, k)} w_h^n(\mathcal{A}) \tag{5.6}$$

where

$$w_h^n(\mathcal{A}) = \hat{Z}_e^k(\Lambda \setminus v(\mathcal{A}), w_h, \Pi(\mathcal{A})) \prod_{\Gamma \in \mathcal{A}} \prod_{\gamma \in \Gamma} \Big[w_h^n(\gamma)\, \hat{Z}_e^{I(\gamma)}(v(\Gamma, \gamma), w_h, \Pi(\mathcal{A}))\Big] \tag{5.7}$$

Notice that the elementary partition functions have w_h and not w_h^n because elementary cylinders don't touch $\delta\Lambda$, and so for them $q(\gamma) = 0$. On the other side, for the same reason if Γ is a contour, then it may contain arbitrarily small cylinders touching the boundary of Λ. As we will see they are not dangerous in our construction. By (5.5), (5.6) we get

$$\bar{\mu}_\Lambda^{h,n}(\mathcal{A}) \equiv \mu_\Lambda^{h,n}\{\varphi \in \Omega_\Lambda : \varphi_l \sim \mathcal{A}\} = \frac{w_h^n(\mathcal{A})}{\hat{Z}^k(\Lambda, w_h^n)}\,. \tag{5.8}$$

5.2. The cut and paste operation

In this subsection we want to find an upper bound for $\bar{\mu}_\Lambda^{h,n}(S_{+,l}(\Lambda, k+j, \Lambda'))$. The most natural approach leads one to try to estimate this quantity with Peierls-like sums like

$$\sum_{\substack{\Gamma\in\mathrm{Con}(\Lambda,k)\\ v(\Gamma)\cap\partial\Lambda\neq\emptyset,\, v(\Gamma)\cap\Lambda'\neq\emptyset}} \widetilde{w}_h^n(\Gamma).$$

Unfortunately the structure of contours is quite complicated and those techniques which yield for instance Proposition 2.9 are inadeguate in presence of an arbitrary b.c. because of the potentially dangerous factor $q(\gamma)L_n(\gamma)$ in the modified weight. In the following we'll see how the above sum can be replaced by another one containing *very simple* contours.

Define

$$\mathrm{Con}^{\pm}(\Lambda,k) = \{\Gamma\in\mathrm{Con}(\Lambda,k) : \forall\gamma\in\Gamma,\ \{E(\gamma), I(\gamma)\}\subset\{k, k\pm 1\}\}\,.$$

and

$$W_l^{\pm}(\Lambda,k) = \{\mathcal{A}\in W_l(\Lambda,k) : \forall\Gamma\in\mathcal{A},\ \Gamma\in\mathrm{Con}^{\pm}(\Lambda,k),\ v(\Gamma)\cap\partial\Lambda\neq\emptyset,\, v(\Gamma)\cap\Lambda'\neq\emptyset\} \tag{5.9}$$

The structure of these contours is fairly simple. In fact, if $\Gamma\in\mathrm{Con}^+(\Lambda,k)$, then

(i) if γ_0 is the external cylinder in Γ, then $E(\gamma_0)=k$, $I(\gamma_0)=k+1$

(ii) all the others $\gamma\in\Gamma$ have $E(\gamma)=k+1$, $I(\gamma)=k$ and there is no other γ' such that $\bar\gamma\subset\bar\gamma'\subset\bar\gamma_0$

(iii) if $\gamma\neq\gamma_0$, then $q(\gamma)=0$, because of the compatibility with γ_0.

The key result for proving the + half of Theorem 5.1 is

Lemma 5.4. *Assume (K_1)–(K_4). Then for each $n>k$, $j>k$*

$$\bar{\mu}_\Lambda^{h,n}(S^0_{+,l}(\Lambda,j,\Lambda')) \le \bar{\mu}_\Lambda^{h,n}(S^0_{+,l}(\Lambda,j-1,\Lambda'))\sum_{\mathcal{A}\in W_l^+(\Lambda,k)} \bar{w}(\mathcal{A})$$

where

$$\bar{w}(\mathcal{A}) = \prod_{\Gamma\in\mathcal{A}}\bar{w}(\Gamma) \quad\text{and}\quad \bar{w}(\Gamma) = e^{-\frac{\beta h}{3}|v(\Gamma)|}\prod_{\gamma\in\Gamma} e^{-\beta(|\tilde\gamma|-2q(\gamma))}$$

Remark. The above lemma is unfortunately false for $S^0_{-,l}(\Lambda,j,\Lambda')$ with $n<k$, $j<k$. The "negative" case is slightly trickier to handle and will be treated separately.

Proof. Thanks to Proposition 3.3 (i) and to (5.6), (5.7), (5.8), in order to prove Lemma 5.4 it is sufficient to find two maps f, g defined on $W_l(\Lambda,k)$, such that

(h_1) f maps $S^0_{+,l}(\Lambda, j, \Lambda')$ into $W^+_l(\Lambda, k)$ and g maps $S^0_{+,l}(\Lambda, j, \Lambda')$ into $S^0_{+,l}(\Lambda, j-1, \Lambda')$
(h_2) if $f(\mathcal{A}) = f(\mathcal{B})$, $g(\mathcal{A}) = g(\mathcal{B})$, then $\mathcal{A} = \mathcal{B}$
(h_3) $w^n_h(\mathcal{A}) \le w^n_h(g(\mathcal{A}))\, \bar{w}(f(\mathcal{A}))$

We set for simplicity $\mathcal{A}' = f(\mathcal{A})$ and $\mathcal{A}'' = g(\mathcal{A})$. Roughly speaking we want to define $\mathcal{A}'$ and $\mathcal{A}''$ as follows: let $\mathcal{A} \in S^0_{+,l}(\Lambda, j, \Lambda')$. This means the the corresponding configuration $\varphi \sim \mathcal{A}$ percolates from $\partial\Lambda$ to Λ' at level j but not at $j+1$. Then one could define a new configuration φ'' which is equal to $\varphi - 1$ on $R_+(\Lambda, k+1, \varphi)$ and equal to φ on the rest. In this way φ'' percolates from $\partial\Lambda$ to Λ' only at level $j-1$. Define now $\varphi' = k + \varphi - \varphi''$, $\mathcal{A}' \sim \varphi'$, $\mathcal{A}'' \sim \varphi''$ and more or less we are done. In other words $\mathcal{A}'$ is the portion of the slice $\varphi = k+1$ which is connected to $\partial\Lambda$, and $\mathcal{A}''$ is what we get by pasting together what remains after extracting that slice.

Things are a little more complicated because we want to define $\mathcal{A}'$ and $\mathcal{A}''$ directly in terms of cylinders, the reason being that it will be much easier to prove properties (h_1), (h_2) (h_3). So, we are going to mess around with cylinders a little bit, with the idea of reproducing a result similar to the one stated above.

We first define the *prime* and *double prime* operations on contours and then they will be extended to collections of contours. So, let $\Gamma \in \mathrm{Con}^+(\Lambda, k)$ with $S(\Gamma_{ext}) = +1$. Then set

$$\Gamma = \{\gamma_0\} \cup \Gamma_a \cup \Gamma_b \cup \Gamma_c$$

where γ_0 is the unique external cylinder in Γ and

$$\begin{aligned}
&\Gamma_c = \{\gamma \in \Gamma : I(\gamma) \le k, \text{ there is no other } \gamma' \in \Gamma \text{ s.t. } I(\gamma') \le k \text{ and } \bar{\gamma}' \supset \bar{\gamma}\} \\
&\Gamma_b = \{\gamma \in \Gamma \setminus \Gamma_c : \text{there is } \gamma' \in \Gamma_c \text{ such that } \bar{\gamma}' \supset \bar{\gamma}\} \\
&\Gamma_a = \{\gamma \in \Gamma \setminus (\Gamma_c \cup \Gamma_b \cup \{\gamma_0\}) : I(\gamma) \ge k+1\}
\end{aligned}$$

Define also, for any cylinder $\gamma = (\widetilde{\gamma}, E, I)$

$$\gamma_- = (\widetilde{\gamma}, E-1, I-1) \qquad \gamma_\downarrow = (\widetilde{\gamma}, E, I-1) \qquad \gamma_\vee = (\widetilde{\gamma}, E-1, I)$$

Now we let (see also Fig. 4)

$$\begin{aligned}
\Gamma' &= \{(\tilde{\gamma}_0, k, k+1)\} \cup \{(\widetilde{\gamma}, k+1, k) : \gamma \in \Gamma_c\} \\
\Gamma'' &= (\gamma_0)_\downarrow \cup \{\gamma_- : \gamma \in \Gamma_a\} \cup \{\gamma_\vee : \gamma \in \Gamma_c\} \cup \Gamma_b
\end{aligned}$$

Remarks.

- Some cylinders in $\{\gamma_\vee : \gamma \in \Gamma_c\} \cup \{(\gamma_0)_\downarrow\}$ may have $I(\gamma) = E(\gamma)$. It is understood that these zero height cylinders are not included in Γ''.

- In general Γ'' is not a contour but it splits into a collection of compatible contours as it appears in the example in Fig. 4, where Γ'' consists of two contours.

If $\mathcal{A} \in W_l(\Lambda, k)$, then let $\mathcal{A}_0$ be the subset of $\mathcal{A}$ containing all those Γ such that $S(\Gamma_{ext}) = +1$ and $v(\Gamma') \cap \partial\Lambda \neq \emptyset$, $v(\Gamma') \cap \Lambda' \neq \emptyset$. We then define

$$\begin{aligned} \mathcal{A}' &= \{\Gamma' : \Gamma \in \mathcal{A}_0\} \\ \mathcal{A}'' &= \{\Gamma'' : \Gamma \in \mathcal{A}_0\} \cup (\mathcal{A} \setminus \mathcal{A}_0) \end{aligned} \tag{5.10}$$

Proposition 5.5. *Let $\mathcal{A} \in S^0_{+,l}(\Lambda, j, \Lambda')$, $j > k$ and let $\varphi \sim \mathcal{A}$, $\varphi' \sim \mathcal{A}'$, $\varphi'' \sim \mathcal{A}''$. Then*

(i)

$$\varphi'(x) = \begin{cases} k+1 & \text{if } x \in v(\mathcal{A}') \\ k & \text{if } x \notin v(\mathcal{A}') \end{cases}$$

(ii)

$$\varphi''(x) = \begin{cases} \varphi(x) - 1 & \text{if } x \in v(\mathcal{A}') \\ \varphi(x) & \text{if } x \notin v(\mathcal{A}') \end{cases}$$

(iii) if B is a connected cluster of $R_+(\Lambda, k+1, \varphi)$ such that B intersects both $\partial\Lambda$ and Λ', there exists $\Gamma \in A$, such that $S(\Gamma) = +1$ and $B \subset v(\Gamma')$, i.e. $B \subset v(\mathcal{A}')$.

Proof. Parts (i) and (ii) are direct consequences of the definitions. ¿From Proposition 2.2, it is clear that there exists a contour Γ whose external cylinder γ_0 satisfies $\bar{\gamma}_0 \supset B$, $S(\gamma_0) = +1$.

In order to prove that $B \subset v(\Gamma')$ we must show that B does not intersect the interior of any cylinder in Γ_c. Let then $x \in B$, then we claim that there is no cylinder $\gamma \in \Gamma$ with $\bar{\gamma} \in x$ such that $S(\gamma) = -1$ and $I(\gamma) \le k$. In fact assume there is such a cylinder. Then, by Proposition 2.2, we have

$$\varphi(y) \le k, \quad \forall y \in \bar{\partial}\bar{\gamma}$$

and so there is a connected set surrounding x where $\varphi \le k$, which contradicts our hypotheses.

This means no $x \in B$ can be contained in the interior of a cylinder $\gamma \in \Gamma_c$, so, by consequence, $B \subset v(\Gamma')$. $\square$

Proposition 5.6. *Assume (K_1)–(K_4). Then for all $n, j > k$ properties (h_1), (h_2) and (h_3) hold.*

Proof.
(h_1) $\mathcal{A}' \in W_l^+(\Lambda, k)$ by definition. Moreover, it should be fairly easy to convince oneself that $\mathcal{A}''$ is a compatible collection of contours. Take now $\varphi, \varphi'' \in \Omega_\Lambda$ such that $\varphi \sim \mathcal{A}$ and

$\varphi'' \sim \mathcal{A}''$. Since φ percolates from $\partial\Lambda$ to Λ' at level j (but not $j+1$), and since $\varphi'' \geq \varphi - 1$, φ'' does percolate al level $j-1$. In order to prove that φ'' does not percolate at level j it is sufficient to observe that, by Proposition 5.5, $\varphi'' = \varphi - 1$ on a set which includes all clusters of $R_+(\Lambda, k+1, \varphi)$ touching Λ'. So, we have $\mathcal{A}'' \in S^0_{+,l}(\Lambda, j-1, \Lambda')$.

(h_2) follows from (i) and (ii) of Proposition 5.5.

We are left with the proof of property (h_3). ¿From (5.7) we get

$$\frac{w^n_h(\mathcal{A})}{w^n_h(\mathcal{A}'')} = R_1\, R_2$$

where

$$R_1 = \frac{\prod_{\Gamma\in\mathcal{A}}\prod_{\gamma\in\Gamma} w^n_h(\gamma)}{\prod_{\Gamma\in\mathcal{A}''}\prod_{\gamma\in\Gamma} w^n_h(\gamma)}$$

and

$$R_2 = \frac{\hat{Z}^k_e(\Lambda\setminus v(\mathcal{A}), w_h, \Pi(\mathcal{A}))\prod_{\Gamma\in\mathcal{A}}\prod_{\gamma\in\Gamma}\hat{Z}^{I(\gamma)}_e(v(\Gamma,\gamma), w_h, \Pi(\mathcal{A}))}{\hat{Z}^k_e(\Lambda\setminus v(\mathcal{A}''), w_h, \Pi(\mathcal{A}''))\prod_{\Gamma\in\mathcal{A}''}\prod_{\gamma\in\Gamma}\hat{Z}^{I(\gamma)}_e(v(\Gamma,\gamma), w_h, \Pi(\mathcal{A}''))}$$

¿From the definition of $\mathcal{A}''$, keeping in mind that

- (a) if $\gamma\in\Gamma_a$, and so both $E(\gamma), I(\gamma)$ are at least $k+1$, then $L_n(\gamma)\leq L_n(\gamma_-)$, so $w^n_h(\gamma)\leq w^n_h(\gamma_-)$
- (b) $w^n_h(\gamma_0)\leq w^n_h((\gamma_0)_\downarrow)\, w^n_h((\widetilde{\gamma}_0, k, k+1))$
- (c) if $\gamma\in\Gamma_c$, then $w_h(\gamma) = w_h(\gamma_\vee)\, w_h((\widetilde{\gamma}, k+1, k))$. (In this case since $S(\gamma) = -1$, w^n_h coincides with the unmodified weigth w_h).
- (d) for every contour Γ with $S(\Gamma) = +1$

$$v(\Gamma') = \bar{\gamma}_0 \setminus \cup_{\gamma\in\Gamma_c}\bar{\gamma} = \cup_{\gamma\in\{\gamma_0\}\cup\Gamma_a} v(\Gamma,\gamma)$$

one easily gets

$$R_1 \leq \prod_{\Gamma\in\mathcal{A}_0}\Big[w^n_h((\gamma_0, k, k+1))\prod_{\gamma\in\Gamma_c} w_h((\widetilde{\gamma}, k+1, k))\Big] = \prod_{\Gamma\in\mathcal{A}_0}\prod_{\gamma\in\Gamma'} e^{-\beta(|\gamma|-2q(\gamma))}e^{-\beta h v(\Gamma')}\,.$$

For what concerns R_2 we claim that

$$R_2 \leq \prod_{\Gamma\in\mathcal{A}_0}\prod_{\gamma\in\{\gamma_0\}\cup\Gamma_a}\frac{\hat{Z}^{I(\gamma)}_e(v(\Gamma,\gamma), w_h, \Pi(\mathcal{A}))}{\hat{Z}^{I(\gamma)-1}_e(v(\Gamma,\gamma), w_h, \Pi(\mathcal{A}))}\,. \tag{5.11}$$

Given (5.11) (we'll prove it in a moment), and using Lemma 2.7 (*ii*) to evaluate each quotient (remember that $I(\gamma) \geq k+1$ for all $\gamma \in \{\gamma_0\} \cup \Gamma_a$ and that $\Delta(J) = \emptyset$), we obtain

$$\frac{w_h^n(\mathcal{A})}{w_h^n(\mathcal{A}'')} \leq \prod_{\Gamma \in \mathcal{A}_0} \Big[\exp\big[\, |v(\Gamma')| \,\big(1.1 e^{-4\beta k} + \beta h e^{-\frac{\beta}{4}} \big) \big] \prod_{\gamma \in \Gamma'} e^{-\beta|\tilde{\gamma}| + 2\beta q(\gamma)} e^{-\beta h |v(\Gamma')|} \Big]$$

$$\leq \prod_{\Gamma \in \mathcal{A}_0} \Big[e^{-\frac{1}{3}\beta h |v(\Gamma')|} \prod_{\gamma \in \Gamma'} e^{-\beta|\tilde{\gamma}| + 2\beta q(\gamma)} \Big] = \bar{w}(\mathcal{A}') \tag{5.12}$$

which concludes the proof of Proposition 5.6 and Lemma 5.4.

Proof of claim (5.11). We observe that

(e) for each cylinder γ in $\mathcal{A}''$ (or more properly in some $\Gamma \in \mathcal{A}''$), there is a cylinder γ' in $\mathcal{A}$ with the same base and the same sign as γ. Thus if a cylinder is weakly compatible with every cylinder in $\mathcal{A}$, it is also weakly compatible with every cylinder in $\mathcal{A}''$. In other words $\Pi(\mathcal{A}) \subset \Pi(\mathcal{A}'')$, which implies

$$\hat{Z}_e^n(V, w_h, \Pi(\mathcal{A}'')) \geq \hat{Z}_e^n(V, w_h, \Pi(\mathcal{A})) \qquad \forall V \subset \Lambda$$

(f) for the same reason the partition of Λ

$$\Lambda = (\Lambda \setminus v(\mathcal{A})) \cup \bigcup_{\Gamma \in \mathcal{A}} \bigcup_{\gamma \in \Gamma} v(\Gamma, \gamma) \equiv \bigcup_i v_i$$

induced by the set of all cylinders in $\mathcal{A}$ is a refinement of the analogous partition of Λ induced by the cylinders in $\mathcal{A}''$

$$\Lambda = (\Lambda \setminus v(\mathcal{A}'')) \cup \bigcup_{\Gamma \in \mathcal{A}''} \bigcup_{\gamma \in \Gamma} v(\Gamma, \gamma) \equiv \bigcup_j \hat{v}_j \,.$$

Thus $\hat{v}_j = \cup_{i \in c(j)} v_i$ for some $c(j) = \{i_1, i_2, \ldots\}$.

In this way we have obtained for each factor appearing in the denominator of R_2

$$\hat{Z}_e^{m_j}(\hat{v}_j, w_h, \Pi(\mathcal{A}'')) \geq \hat{Z}_e^{m_j}(\hat{v}_j, w_h, \Pi(\mathcal{A})) \geq \prod_{i \in c(j)} \hat{Z}_e^{m_j}(v_i, w_h, \Pi(\mathcal{A})) \,.$$

By consequence,

$$R_2 \leq \prod_i \frac{\hat{Z}_e^{n_i}(v_i, w_h, \Pi(\mathcal{A}))}{\hat{Z}_e^{\bar{n}_i}(v_i, w_h, \Pi(\mathcal{A}))}$$

where $\bar{n}_i$ is equal to either n_i or to $n_i - 1$ and, because of the definition of $\mathcal{A}''$,

$$\{v_i : \bar{n}_i = n_i - 1\} = \{v(\Gamma, \gamma) : \Gamma \in \mathcal{A}_0 \text{ and } \gamma \in \{\gamma_0(\Gamma)\} \cup \Gamma_a\}$$

which proves the claim. $\square$

At this point we are almost done with the estimate of $\mu_\Lambda^{h,n}(\, S_+(\Lambda, k+j, \Lambda')\,)$ ("half" of Theorem 5.1). What is left is

Lemma 5.7. *Assume* (K_1)–(K_4) *and let*

$$\bar{w}(\mathcal{A}) = \prod_{\Gamma\in\mathcal{A}} \left[e^{-\frac{1}{3}\beta h|v(\Gamma)|} \prod_{\gamma\in\Gamma} e^{-\beta(|\tilde{\gamma}|-2q(\gamma))} \right].$$

Then

$$\sum_{\mathcal{A}\in W_l^+(\Lambda,k)} \bar{w}(\mathcal{A}) \le e^{-\frac{1}{30}\beta N} \tag{5.13}$$

Proof. We recall that ¿From the definition of $\mathrm{Con}^+(\Lambda,k)$ it follows that $q(\gamma)$ is nonzero only if γ is the external cylinder in Γ. Given $\mathcal{A}\in W_l^+(\Lambda,k)$, consider the set

$$U = \left(\bigcup\nolimits_{\Gamma\in\mathcal{A}} v(\Gamma)\right)^c = v(\mathcal{A})^c$$

and write its boundary as a union of connected components

$$\delta U = \eta_1 \cup \ldots \cup \eta_r = \{\eta\}\,.$$

In this way, to each $\mathcal{A}$ we can associate a collection $\{\eta\}$ of closed sets of dual edges $\eta\in C_B(\Lambda)$. It is easy to check that

- (i) if $\eta_1,\eta_2\in\{\eta\}$ with $\eta_1\neq\eta_2$, then $\bar{\eta}_1\cap\bar{\eta}_2=\emptyset$
- (ii) if $\eta\cap\delta\Lambda=\emptyset$ then $|\eta|\ge\vartheta$, where $\vartheta=e^{\frac{\beta\zeta}{10}}$ is the maximum diameter for an elementary cylinder
- (iii) for each $\mathcal{A}\in W_l^+(\Lambda,k)$

$$\sum_{\Gamma\in\mathcal{A}}\sum_{\gamma\in\Gamma}(|\tilde{\gamma}|-2q(\gamma)) = \sum_{\eta\in\{\eta\}}|\eta| - |\delta\Lambda| \tag{5.14}$$

$$\bar{w}(\mathcal{A}) = \exp\Big[\beta|\delta\Lambda| - \beta\sum_{\eta\in\{\eta\}}|\eta| - \frac{\beta h}{3}(|\Lambda| - \sum_{\eta\in\{\eta\}}|\bar{\eta}|)\Big]. \tag{5.15}$$

- (iv) $\{\eta\}$ determines $\mathcal{A}$ in a unique way, *i.e.* distincts $\mathcal{A}$'s produce distinct $\{\eta\}$'s.

We denote by Y the set of all collections $\{\eta\}$ which satisfy (i) and (ii).

Let's divide the $\mathcal{A}$'s in $W_l^+(\Lambda,k)$ into 2 groups, depending on whether $|v(\mathcal{A})|\ge\zeta^2|\Lambda|$ or $|v(\mathcal{A})|<\zeta^2|\Lambda|$. We call S_1 (S_2) the contribution of the first (second) group to the sum (5.13).

By definition of Y, Proposition A1.2 and subsequent remark

$$\sum_{\{\eta\}\in Y}\prod_{\eta\in\{\eta\}} e^{-\beta|\eta|} \le \exp\left[4Ne^{-3\beta} + N^2 e^{-\frac{3}{4}\beta\vartheta}\right].$$

Then, using (iii),

$$S_1 \le e^{-\beta \frac{h}{3}\zeta^2|\Lambda|+\beta|\delta\Lambda|} \sum_{\{\eta\}\in Y} \exp\Big[-\beta \sum_{\eta\in\{\eta\}} |\eta|\Big] \le$$

$$\exp\left[|\Lambda|\Big(-\frac{\beta h}{3}\zeta^2 + e^{-\frac{3}{4}\beta\vartheta}\Big) + |\delta\Lambda|\big(\beta + e^{-3\beta}\big)\right] \le \exp\left[-\frac{\beta h}{4}\zeta^2|\Lambda| + 2\beta|\delta\Lambda|\right] \le e^{-2\beta|\delta\Lambda|}\,.$$

We have used the inequalities $\beta h\zeta^2 > 100 e^{-\frac{1}{2}\beta\vartheta}$, $N \ge 64 h^{-1}\zeta^{-2}$, and the fact that β is large enough.

If $|v(\mathcal{A})| < \zeta^2|\Lambda|$ then, by Proposition A1.4 we know that $|\delta U| = \sum_{\eta\in\{\eta\}}|\eta| \ge 4.2N$. Furthermore, since $\sum_{\Gamma\in\mathcal{A}}\sum_{\gamma\in\Gamma} q(\gamma) \le 4N$, (5.14) implies

$$\sum_{\Gamma\in\mathcal{A}}\sum_{\gamma\in\Gamma}(|\widetilde{\gamma}| - 2q(\gamma)) \ge 0.2N \tag{5.16}$$

and

$$\sum_{\Gamma\in\mathcal{A}}\sum_{\gamma\in\Gamma}|\widetilde{\gamma}| \ge 2\sum_{\Gamma\in\mathcal{A}}\sum_{\gamma\in\Gamma} q(\gamma) + 0.2N \ge 2.05\sum_{\Gamma\in\mathcal{A}}\sum_{\gamma\in\Gamma} q(\gamma)\,. \tag{5.17}$$

¿From (5.16), (5.17) we get

$$\sum_{\Gamma\in\mathcal{A}}\sum_{\gamma\in\Gamma}(|\widetilde{\gamma}| - 2q(\gamma)) \ge \frac{N}{10} + \frac{1}{2}\sum_{\Gamma\in\mathcal{A}}\sum_{\gamma\in\Gamma}(|\widetilde{\gamma}| - 2q(\gamma)) \ge \frac{N}{10} + \frac{1}{100}\sum_{\Gamma\in\mathcal{A}}\sum_{\gamma\in\Gamma}|\widetilde{\gamma}|$$

and then

$$\bar{w}(\mathcal{A}) \le e^{-\frac{\beta}{10}N}\prod_{\Gamma\in\mathcal{A}}\Big[e^{-\frac{1}{3}\beta h|v(\Gamma)|}\prod_{\gamma\in\Gamma} e^{-\frac{\beta}{100}|\widetilde{\gamma}|}\Big]\,.$$

Therefore one finds

$$\begin{aligned} S_2 &\le e^{-\frac{\beta}{10}N}\sum_{\mathcal{A}\in W_l^+(\Lambda,k)}\prod_{\Gamma\in\mathcal{A}}\Big[e^{-\frac{1}{3}\beta h|v(\Gamma)|}\prod_{\gamma\in\Gamma} e^{-\frac{\beta}{100}|\widetilde{\gamma}|}\Big] = \\ &= e^{-\frac{\beta}{10}N}\sum_{\mathcal{A}\in W_l^+(\Lambda,k)}\prod_{\Gamma\in\mathcal{A}}\prod_{\gamma\in\Gamma} e^{-\frac{\beta}{100}|\widetilde{\gamma}|_J L(\gamma) - \frac{\beta h}{3}|v(\Gamma,\gamma)||I(\gamma)-k|} \le \\ &\le e^{-\frac{\beta}{10}N}\sum_{\mathcal{A}\in W_l(\Lambda,k)}{}' \prod_{\Gamma\in\mathcal{A}}\prod_{\gamma\in\Gamma} e^{-\frac{\beta}{100}|\widetilde{\gamma}|_J L(\gamma) - \frac{\beta h}{3}|v(\Gamma,\gamma)||I(\gamma)-k|} \end{aligned}$$

where $\sum'$ means that the sum is restricted to those $\mathcal{A}$'s such that

$$v(\Gamma)\cap\partial\Lambda \ne \emptyset \quad \text{and} \quad |\tilde{\Gamma}_{ext}| \ge N/4 \qquad \forall\Gamma\in\mathcal{A}$$

because, if $\mathcal{A} \in W_l^+(\Lambda, k)$, each $\Gamma \in \mathcal{A}$ must intersect both $\partial\Lambda$ and Λ'. We now proceed as in the proof of Proposition A1.2 and, using Proposition 2.9, we find

$$S_2 \le e^{-\frac{\beta}{10}N} \exp\left[|\partial\Lambda| e^{-\frac{\beta}{900}N}\right] \le e^{-\frac{\beta}{20}N}.$$

So we have obtained

$$\sum_{\mathcal{A}\in W_l^+(\Lambda,k)} \bar{w}(\mathcal{A}) \le S_1 + S_2 \le e^{-\frac{\beta}{30}N} \quad \square$$

5.3. Bounds on the probability of percolation below k and proof of Theorem 5.1

The arguments given above are not sufficient unfortunately to treat the negative case, *i.e.* to provide a good upper bound on the probability that a boundary condition $n < k$ percolates well inside the bulk. Our goal is then to add a few fixes and show that

Lemma 5.8. *Assume (K_1)–(K_4). Then for all $n < k$*

$$\bar{\mu}_\Lambda^{h,n}(S_{-,l}(\Lambda, k-1, \Lambda')) \le e^{-\frac{1}{800k}\beta N}.$$

The reason why our previous arguments need to be modified is the following: in (5.12) there is a term

$$\exp\left[-\beta h|v(\Gamma')| + |v(\Gamma')|\left(1.1e^{-4\beta k} + +\beta h e^{-\frac{\beta}{4}}\right)\right]$$

which, because $h \in I_k(\beta)$, is smaller than $e^{-\frac{\beta h}{3}|v(\Gamma')|}$, and this is enough to allow us to sum over contours. When we do the same with negative contours, the analogous term, because of Lemma 2.7, becomes

$$\exp\left[\beta h|v(\Gamma')| - |v_0(\Gamma')|e^{-4\beta(k-1)} + |v(\Gamma')|e^{-4\beta(k-1)-\frac{\beta}{4}}\right]$$

where we have set

$$v_0(\Gamma') = \bigcup_{\gamma\in\{\gamma_0\}\cup\Gamma_a} v_0(\Gamma,\gamma) \quad \text{and } v_0(\Gamma,\gamma) = v(\Gamma,\gamma) \setminus \bar{\partial} v(\Gamma,\gamma)\,. \tag{5.18}$$

The second term is usually dominant because of the factor $\exp[-4\beta(k-1)]$ which is larger than βh, but the trouble is that there are contours such that

$$|v_0(\Gamma')| \ll |v(\Gamma')|\,.$$

Think for instance of something like a negative pyramid obtained as a collection of centered cylinders, with square basis of sides $s, s-2, s-4, s-6, \dots, r$. In this case $|v(\Gamma')| = s^2$, while $|v_0(\Gamma')| \leq r^2$. For this kind of contours the operation of cut and paste is not useful, because one can have

$$\frac{w_h^n(\mathcal{A})}{w_h^n(\mathcal{A}'')} > 1\,.$$

So our idea is to "cut and paste" those contours for which, say $|v_0(\Gamma')| \geq \frac{1}{2}|v(\Gamma')|$. As for the other contours, they are supposedly very depressed because they have a lot of boundary, so it should not be too difficult to sum their weights directly.

Proposition 5.9. *Assume* (K_1)*–*(K_4)*. If* $\Gamma \in \mathrm{Con}(\Lambda, k)$ *with* $S(\Gamma) = -1$ *and* $v(\Gamma') \cap \partial\Lambda \neq \emptyset$*,* $v(\Gamma') \cap \Lambda' \neq \emptyset$*, then one of the following assertions holds*

(1) $\sum_{\gamma\in\Gamma} |\widetilde{\gamma}| \geq 4k|\delta\Lambda|$

(2) $|v_0(\Gamma')| \geq \frac{1}{2}|v(\Gamma')|$*, where* $|v_0(\Gamma')|$ *is given by (5.18)*

(3) $\sum_{\gamma\in\Gamma'}(|\widetilde{\gamma}| - 2q(\gamma)) \geq \frac{1}{160k}(\sum_{\gamma\in\Gamma'}|\widetilde{\gamma}| + \frac{1}{12}|v(\Gamma')|)$.

Proof. If $x \in v(\Gamma') \setminus v_0(\Gamma')$, then there exists $\gamma \in \Gamma$ such that $x \in \bar{\partial} v(\Gamma, \gamma)$, so there exists $\gamma' \in \Gamma$ such that $d_\infty(\widetilde{\gamma}', x) = 1/2$. On the other side, for each dual edge e^* there are 6 lattice sites whose ∞–distance from e^* is $1/2$. This implies

$$|v(\Gamma') \setminus v_0(\Gamma')| \leq 6 \sum_{\gamma\in\Gamma} |\widetilde{\gamma}|.$$

If one assumes now that both (1) and (2) fail, one gets

$$|v(\Gamma')| \leq 2|v(\Gamma') \setminus v_0(\Gamma')| \leq 12 \sum_{\gamma\in\Gamma} |\widetilde{\gamma}| \leq 48k|\delta\Lambda| \leq \zeta^2 |\Lambda|\,.$$

On the other side, by applying Proposition A1.4 to $U = \Lambda \setminus v(\Gamma')$ (see the proof of Lemma 5.7) and using the assumption that (1) fails, we obtain

$$\sum_{\gamma\in\Gamma'}(|\widetilde{\gamma}| - 2q(\gamma)) \geq \frac{1}{20}|\delta\Lambda| \geq \frac{1}{40\cdot 48k}|v(\Gamma')| + \frac{1}{40\cdot 4k}\sum_{\gamma\in\Gamma}|\widetilde{\gamma}| \quad \square$$

Proof of Lemma 5.8. The proof is quite similar to the one of Lemma 5.4 (plus the estimate of Lemma 5.7). So we are going to define two maps $\mathcal{A} \mapsto f(\mathcal{A}) = \mathcal{A}'$ and $\mathcal{A} \mapsto g(\mathcal{A}) = \mathcal{A}''$ in such a way that we can use Proposition 3.3 successfully. The main differences with Lemma 5.4 are (*a*) g maps $S_{-,l}(\Lambda, k-1, \Lambda')$ into the whole space $W_l(\Lambda, k)$, so we cannot iterate the inequality

(*b*) the definition of both f and g will depend on which condition in Proposition 5.9 is satisfied

(*c*) in Lemma 5.4 g is defined by modifying *all* contours Γ such that $v(\Gamma')$ intersects both $\partial\Lambda$ and Λ', because we wanted to make sure that g mapped $S^0_{+,l}(\Lambda, j, \Lambda')$ into $S^0_{+,l}(\Lambda, j-1, \Lambda')$. Here we cannot get that anyway, so we modify only the "first" contour which has a nonempty intersection with both $\partial\Lambda$ and Λ'.

Here is how it works. Let

$$Y = \{\Gamma \in \mathrm{Con}(\Lambda, k) : S(\Gamma) = -1 \text{ and } v(\Gamma') \cap \partial\Lambda \neq \emptyset\,, v(\Gamma') \cap \Lambda' \neq \emptyset\,\}$$

If $\mathcal{A} \in S_{-,l}(\Lambda, k-1, \Lambda')$ we choose $\Gamma_* \in \mathcal{A}$ as the "first" contour in $\mathcal{A} \cap Y$ (Proposition 5.5 (*iii*) shows that there exists at least one such contour, and one can choose the first by ordering the set of all contours in any arbitrary way). We then define (for any contour $\Gamma \in Y$) Γ_a, Γ_b, Γ_c, Γ', Γ'' exactly as in Lemma 5.4, but with *all signs reversed* (each $\leq$ is replaced by a $\geq$, each $+$ by a $-$ and viceversa).

Then we write

$$Y = Y_1 \cup Y_2 \cup Y_3$$

in such a way that the statement (j) of Proposition 5.9 holds for all $\Gamma \in Y_j$ for $j = 1, 2, 3$ (by taking differences one can assume that the Y_j's are pairwise disjoint). Accordingly, we define the maps f and g as

$$f(\mathcal{A}) = \begin{cases} \Gamma_* & \text{if } \Gamma_* \in Y_1 \\ \Gamma'_* & \text{if } \Gamma_* \in Y_2 \cup Y_3 \end{cases} \qquad \text{and} \qquad g(\mathcal{A}) = \begin{cases} \mathcal{A} \setminus \{\Gamma_*\} & \text{if } \Gamma \in Y_1 \\ \mathcal{A} \setminus \{\Gamma_*\} \cup \Gamma''_* & \text{if } \Gamma \in Y_2 \cup Y_3, \end{cases}$$

Now we want Y' to contain the image of $S_{-,l}(\Lambda, k-1, \Lambda')$ under f. So we set

$$\begin{aligned} &Y'_1 = Y_1, \quad Y'_i = \{\Gamma' : \Gamma \in Y_i\},\ i = 2, 3 \\ &Y' = Y'_1 \cup Y'_2 \cup Y'_3 \end{aligned}$$

(by taking differences again we assume that the Y'_i's are pairwise disjoint). If $\mathcal{A} \in S_{-,l}(\Lambda, k-1, \Lambda')$, then $f(\mathcal{A}) \in Y'$. We claim that

$$w^n_h(\mathcal{A}) \leq w^n_h(g(\mathcal{A}))\, \bar{w}(f(\mathcal{A})) \tag{5.19}$$

where, for any $\Gamma \in Y'$,

$$\bar{w}(\Gamma) = \begin{cases} e^{-\frac{1}{2}\beta k|\delta\Lambda|} \prod_{\gamma\in\Gamma} \exp\Big[-\frac{\beta}{5}|\widetilde{\gamma}| L(\gamma) - \frac{1}{3}\beta h |v(\Gamma,\gamma)||I(\gamma) - k|\Big] & \text{if } \Gamma \in Y'_1 \\[2ex] \exp\Big[-\frac{1}{5}\beta h|v(\Gamma)| - \beta \sum_{\gamma\in\Gamma}(|\widetilde{\gamma}| - 2q(\gamma))\Big] & \text{if } \Gamma \in Y'_2 \\[2ex] \exp\Big[-\frac{\beta}{160k}\Big(\frac{1}{15}|v(\Gamma)| + \sum_{\gamma\in\Gamma}|\widetilde{\gamma}|\Big)\Big] & \text{if } \Gamma \in Y'_3. \end{cases} \tag{5.20}$$

Once we believe in (5.19), (5.20), by Proposition 3.3 we get

$$\bar\mu_\Lambda^{h,n}(S_{-,l}(\Lambda,k-1,\Lambda')) \le \sum_{\Gamma\in Y'} \bar w(\Gamma) \tag{5.21}$$

We are going to estimate the RHS of (5.21) first, and then we'll prove (5.19), (5.20).

We write

$$\sum_{\Gamma\in Y'} \bar w(\Gamma) = X_1 + X_2 + X_3$$

where $X_i = \sum_{\Gamma\in Y_i'} \bar w(\Gamma)$. Then

$$X_1 \le e^{-\frac12\beta k|\delta\Lambda|} \sum_{\substack{\Gamma\in \mathrm{Con}(\Lambda,k)\\ v(\Gamma)\cap\partial\Lambda\neq\emptyset}} \prod_{\gamma\in\Gamma} \exp\left[-\frac{\beta}{5}|\widetilde\gamma|L(\gamma) - \frac13\beta h|v(\Gamma,\gamma)||I(\gamma)-k|\right] \le$$

$$\le e^{-\frac12\beta k|\delta\Lambda|}|\partial\Lambda| \sum_{\substack{\Gamma\in \mathrm{Con}(\Lambda,k)\\ v(\Gamma)\ni 0}} \prod_{\gamma\in\Gamma} \exp\left[-\frac{\beta}{5}|\widetilde\gamma|L(\gamma) - \frac13\beta h|v(\Gamma,\gamma)||I(\gamma)-k|\right] \le e^{-\frac13\beta k|\delta\Lambda|}$$

by Proposition 2.9. To estimate X_2 we use Lemma 5.7 (replacing $\frac{\beta h}{3}$ with $\frac{\beta h}{5}$ does not change the result nor the proof) and we get

$$X_2 \le \sum_{\mathcal{A}\in W_l^-(\Lambda,k)} e^{-\frac15\beta h|v(\Gamma)|} \prod_{\gamma\in\Gamma} e^{-\beta(|\widetilde\gamma|-2q(\gamma))} \le e^{-\frac{1}{30}\beta N}$$

As for X_3 we observe that $X_3\subset \mathrm{Con}^-(\Lambda,k)$, so, if $\Gamma\in X_3$

$$|v(\Gamma)| = \sum_{\gamma\in\Gamma} |v(\Gamma,g)|\,|I(\gamma)-k|\,.$$

Moreover each contour in X_3 intersects $\partial\Lambda$ and Λ' which implies $|\widetilde\Gamma_{ext}|\ge N/2$. Thus by Proposition 2.9

$$X_3 \le \sum_{\substack{\Gamma\in \mathrm{Con}(\Lambda,k)\\ v(\Gamma)\cap\partial\Lambda\neq\emptyset,\ |\widetilde\Gamma_{ext}|\ge N/2}} \prod_{\gamma\in\Gamma} \exp\left[-\frac{\beta}{160k}\Big(|\widetilde\gamma| + \frac{1}{15}|v(\Gamma,\gamma)|\,|I(\gamma)-k|\Big)\right] \le |\partial\Lambda| e^{-\frac{\beta N}{700k}}\,.$$

Adding up all terms, we obtain

$$\bar\mu_\Lambda^{h,n}(S_{-,l}(\Lambda,k-1,\Lambda')) \le \sum_{\Gamma\in Y'} \bar w(\Gamma) \le e^{-\frac{1}{800k}\beta N}.$$

Let's now go back and prove (5.19), (5.20).

- $\Gamma \in Y_1$.

Since $\mathcal{A}'' = \mathcal{A} \setminus \Gamma_*$, we have $\Pi(\mathcal{A}''_\partial) \supset \Pi(\mathcal{A}_\partial)$, so, by (5.7) and (2.10)

$$w_h^n(\mathcal{A}) = \hat{Z}_e^k(\Lambda, w_h, \Pi(\mathcal{A}_\partial)) \prod_{\Gamma \in \mathcal{A}} \widetilde{w}_h^n(\Gamma) \le w_h^n(\mathcal{A}'')\, \widetilde{w}_h^n(\Gamma_*)$$

and, by (ii) of Corollary 2.8,

$$\frac{w_h^n(\mathcal{A})}{w_h^n(\mathcal{A}'')} \le \prod_{\gamma \in \Gamma_*} \exp\Big[-(\beta-1)|\widetilde{\gamma}|L(\gamma) + 2\beta q(\gamma) L_n(\gamma) - \frac{1}{3}\beta h |v(\Gamma,\gamma)||I(\gamma)-k|\Big]$$

By observing that, since $S(\Gamma_{ext}) = -1$,

$$\sum_{\gamma \in \Gamma_*} 2q(\gamma)L_n(\gamma) \le 2k|\delta\Lambda| \quad \text{and} \quad \sum_{\gamma \in \Gamma_*} |\widetilde{\gamma}| \ge 4k|\delta\Lambda|$$

we get

$$\sum_{\gamma \in \Gamma_*} [|\widetilde{\gamma}|L(\gamma) - 2q(\gamma)L_n(\gamma)] \ge k|\delta\Lambda| + \frac{1}{4} \sum_{\gamma \in \Gamma_*} |\widetilde{\gamma}|L(\gamma)$$

hence

$$\frac{w_h^n(\mathcal{A})}{w_h^n(\mathcal{A}'')} \le e^{\frac{\beta}{2}k|\delta\Lambda|} \prod_{\gamma \in \Gamma_*} \exp\Big[-\frac{\beta}{5}|\widetilde{\gamma}|L(\gamma) - \frac{1}{3}\beta h|v(\Gamma,\gamma)||I(\gamma)-k|\Big].$$

- $\Gamma \in Y_2$.

In this case, by definition of Y_2, we have $|v_0(\Gamma')| \ge \frac{1}{2}|v(\Gamma')|$. Thus one can proceed as in the proof of Proposition 5.6 and get

$$\frac{w_h^n(\mathcal{A})}{w_h^n(\mathcal{A}'')} = R_1\, R_2$$

where, letting γ_0 be the external cylinder in Γ_*,

$$R_1 \le \Big[w_h^n((\gamma_0, k, k-1)) \prod_{\gamma \in (\Gamma_*)_c} w_h((\widetilde{\gamma}, k-1, k))\Big] = e^{-\beta h v(\Gamma'_*)} \prod_{\gamma \in \Gamma'_*} e^{-\beta(|\widetilde{\gamma}| - 2q(\gamma))}$$

and

$$R_2 \le \prod_{\gamma \in \{\gamma_0\} \cup (\Gamma_*)_a} \frac{\hat{Z}_e^{I(\gamma)}(v(\Gamma,\gamma), w_h, \Pi(\mathcal{A}))}{\hat{Z}_e^{I(\gamma)+1}(v(\Gamma,\gamma), w_h, \Pi(\mathcal{A}))}.$$

Using (ii) of Lemma 2.7, since since $\beta h \le \frac{1}{4} e^{-4\beta(k-1)}$, one gets

$$\frac{w_h^n(\mathcal{A})J}{w_h^n(\mathcal{A}'')} \le \exp\Big[\beta h|v(\Gamma'_*)| - |v_0(\Gamma'_*)|e^{-4\beta(k-1)} + |v(\Gamma'_*)|e^{-4\beta(k-1)-\frac{\beta}{4}}\Big] \prod_{\gamma\in\Gamma'_*} e^{-\beta|\tilde\gamma|+2\beta q(\gamma)} \le$$

$$\exp\big[\beta h|v(\Gamma'_*)| - 0.49|v(\Gamma'_*)|e^{-4\beta(k-1)}\big] \prod_{\gamma\in\Gamma'_*} e^{-\beta|\tilde\gamma|+2\beta q(\gamma)} \le \prod_{\gamma\in\Gamma'_*} e^{-\beta|\tilde\gamma|+2\beta q(\gamma)-\frac{1}{5}\beta h|v(\Gamma'_*)|}.$$

- $\Gamma \in Y_3$.

This case is similar to the previous one. Using (3) of Proposition 5.9 we obtain

$$\frac{w_h^n(\mathcal{A})}{w_h^n(\mathcal{A}'')} \le \exp\Big[\beta h|v(\Gamma'_*)| - |v_0(\Gamma'_*)|e^{-4\beta(k-1)} + |v(\Gamma'_*)|e^{-4\beta(k-1)-\frac{\beta}{4}}\Big] \prod_{\gamma\in\Gamma'_*} e^{-\beta|\tilde\gamma|+2\beta q(\gamma)} \le$$

$$\le \exp\Big[|v(\Gamma'_*)|(\beta h_1^+(\beta) + e^{-\frac{\beta}{4}})\Big] \prod_{\gamma\in\Gamma'_*} e^{-\beta|\tilde\gamma|+2\beta q(\gamma)} \le \prod_{\gamma\in\Gamma'_*} e^{-\frac{\beta}{160k}(|\tilde\gamma|+\frac{1}{15}|v(\Gamma'_*)|)} \qquad \square$$

We can finally give the

Proof of Theorem 5.1.

Let $n = \sup_{x\in\partial^+\Lambda}\psi(x)$. By FKG,

$$\mu_\Lambda^{h,\psi}\big(S_+(\Lambda,k+j,\Lambda')\big) \le \mu_\Lambda^{h,n}\big(S_+(\Lambda,k+j,\Lambda')\big) \le \lim_{n\to\infty}\mu_\Lambda^{h,n}\big(S_+(\Lambda,k+j,\Lambda')\big).$$

By Lemma 5.3, Lemma 5.4 and Lemma 5.7, we have, for all $n > k$,

$$\mu_\Lambda^{h,n}\big(S_+(\Lambda,k+j,\Lambda')\big) \le \bar\mu_\Lambda^{h,n}\big(S_{+,l}(\Lambda,k+j,\Lambda')\big) =$$

$$= \sum_{m=j}^{\infty} \bar\mu_\Lambda^{h,n}\big(S^0_{+,l}(\Lambda,k+m,\Lambda')\big) \le \sum_{m=j}^{\infty} \bar\mu_\Lambda^{h,n}\big(S^0_{+,l}(\Lambda,k,\Lambda')\big)\, e^{-\frac{1}{30}\beta N m} \le e^{-\frac{1}{50}\beta N j}.$$

On the other side, by FKG, Lemma 5.3 and Lemma 5.8

$$\mu_\Lambda^{h,\psi}\big(S_-(\Lambda,k-j,\Lambda')\big) \le \mu_\Lambda^{h,1}\big(S_-(\Lambda,k-j,\Lambda')\big) \le e^{-\frac{1}{800\,k}\beta N} \qquad \square$$

5.4. Free boundary conditions. Proof of Theorem 5.2

We prove the Theorem only for S_+, since the other case is identical.

As we did for n b.c. we express the partition function with free b.c. by means of cylinders starting from level k whose weight is modified at the boundary

$$w_h^{\emptyset}(\gamma) = \exp\big[-\beta|\widetilde{\gamma}|L(\gamma) + \beta q(\gamma)L(\gamma) - \beta h S(\gamma)|\bar{\gamma}|L(\gamma) \big]$$

where $q(\gamma) = |\widetilde{\gamma} \cap \delta\Lambda|$. Thanks to Lemma 5.3 we know that for any $x \in \Lambda$

$$\mu_\Lambda^{h,\emptyset}(\, S_+(\Lambda, j, \{x\})\,) \le \bar{\mu}_\Lambda^{h,\emptyset}(\, S_{+,l}(\Lambda, j, \{x\})\,)\,. \tag{5.22}$$

Furthermore, if $\mathcal{A} \in S_{+,l}(\Lambda, j, \{x\})$, then there exists a unique $\Gamma \in \mathcal{A}$ such that

$$R_+(\Lambda, j, \varphi_\Gamma) \ni x$$

where φ_Γ is the configuration corresponding to Γ. In other words there is a unique contour $\Gamma \in \mathcal{A}$ which is "responsible" for the percolation from the boundary to x at level (at least) j. Thus, using Proposition 3.4, we obtain

$$\bar{\mu}_\Lambda^{h,\emptyset}(\, S_{+,l}(\Lambda, j, \{x\})\,) \le \sum_{\Gamma:\, R_+(\Lambda, j, \varphi_\Gamma) \ni x} \widetilde{w}_h^{\emptyset}(\Gamma)\,.$$

By Corollary 2.8 we can write for all $\Gamma \in \mathrm{Con}(\Lambda, k)$

$$\widetilde{w}_h^{\emptyset}(\Gamma) \le \bar{w}(\Gamma) \equiv \prod_{\gamma\in\Gamma} \exp\left[-(\beta-1)|\widetilde{\gamma}|L(\gamma) + \beta q(\gamma)L(\gamma) - \frac{\beta h}{2}|v(\Gamma,\gamma)||I(\gamma) - k| \right]\,.$$

We now want to give an alternative expression for $\bar{w}(\Gamma)$ which corresponds to slicing the (positive and negative) skyscrapers into (possibly disconnected) slices of thickness one. Take then the sets U^m, $m = 3/2, 5/2, \ldots$ defined in Proposition 2.2. It is clear that

$$\bar{w}(\Gamma) = \prod_{m=3/2}^{\infty} \exp\left[-(\beta-1)|\delta U^m \setminus \delta\Lambda| - \frac{\beta h}{2}|U^m| \right]\,,$$

(only a finite number of terms are different from 1). Now we want to show that if $Nh \ge 10$, then

$$(\beta-1)\,|\delta U^m \setminus \delta\Lambda| + \frac{\beta h}{2}|U^m| \ge \frac{\beta}{11}|\delta U^m| + \frac{\beta h}{4}|U^m|\,. \tag{5.23}$$

Assume in fact that $|\delta U^m \setminus \delta\Lambda| \le \frac{1}{10}|\delta U^m|$. Then $|\delta U^m \cap \delta\Lambda| \ge \frac{9}{10}|\delta U^m|$, and, by Proposition A1.3,

$$|U^m| \ge \frac{1}{6}|\delta U^m| N\,.$$

Thus
$$\frac{\beta h}{2}|U^m| \geq \frac{\beta h}{4}|U^m| + \frac{1}{24}\beta h N|\delta U^m| \geq \frac{\beta h}{4}|U^m| + \frac{1}{11}\beta|\delta U^m|\,,$$

which implies (5.23). Moreover if $R_+(\Lambda, j, \varphi_\Gamma) \ni x$ then there is a $*$–path from $\partial\Lambda$ to x which is contained in each U^m for $m = k+1/2, \dots, k+j-1/2$. Hence

$$\sum_{m=3/2}^{\infty} \frac{\beta}{11}|\delta U^m| + \frac{\beta h}{4}|U^m| \geq \frac{\beta}{22} d(\Lambda^c, x) j + \sum_{m=3/2}^{\infty} \Big[\, \frac{\beta}{22}|\delta U^m| + \frac{\beta h}{4}|U^m| \,\Big]\,.$$

Going back to the standard representation in terms of cylinders we find that for all Γ such that $R_+(\Lambda, j, \varphi_\Gamma) \ni x$,

$$\bar{w}(\Gamma) \leq e^{-\frac{1}{22}\beta d(\Lambda^c, x) j} \prod_{\gamma \in \Gamma} \exp\left[-\frac{\beta}{22}|\widetilde{\gamma}| L(\gamma) - \frac{\beta h}{4}|v(\Gamma, \gamma)||I(\gamma) - k| \right]\,.$$

Collecting all the pieces together, and using Proposition 2.9 to sum over contours we finally obtain
$$\mu_\Lambda^{h,\emptyset}(\, S_+(\Lambda, j, \{x\})\,) \leq e^{-\frac{1}{22}\beta j d(\Lambda^c, x)} \qquad \square$$

6. Arbitrary boundary condition II. Close to the phase transition

Now we want to extend the validity of Theorem 5.1 to the intervals $h \in (h_k^*(\beta), h_{k-1}^*(\beta))$, provided that the size N of the box is large enough. In particular N has to go to infinity when h tends to one of the two endpoints $h_k^*(\beta), h_{k-1}^*(\beta)$. If we let (remember (4.2)) for each k

$$\bar{N}(\beta, h) = \begin{cases} N_0(h) = (\zeta^3 h)^{-1} & \text{if } h \in I_k(\beta) \\ \max\{N_0(h),\, \beta\, (\zeta^3 |a_k(\beta, h)|)^{-1}\} & \text{if } h \in (h_{k+1}^+(\beta), h_k^-(\beta)) \end{cases} \tag{6.1}$$

we have

Theorem 6.1. *Let β be large enough and $h \in (h_k^*(\beta), h_{k-1}^*(\beta))$, with $1 \le k \le k_{max} = \lfloor e^{\frac{\beta\zeta}{20}} \rfloor$. If $\Lambda = Q_N$ with $N > \bar{N}(\beta, h)$, and $\Lambda' = Q_{N'}$ with $N' = N - 2\lfloor 2N/5 \rfloor$, then (letting $\zeta = 1000^{-1}$)*

$$\sup_{\psi \in \Omega} \mu_\Lambda^{h,\psi}(\, S_+(\Lambda, k+j, \Lambda') \cup S_-(\Lambda, k-j, \Lambda')\,) \le e^{-\frac{\zeta\beta}{10k^2} N j}\,. \tag{6.2}$$

Corollary 6.2. *Let β, h, ζ be as in Theorem 6.1. Then there exists $C(\beta, h)$ such that for all $\Lambda = Q_N$ with $N \ge N_1 = \lfloor 8/h + 1 \rfloor$,*

$$\sup_{\psi, \psi' \in \Omega} |\, \mathbf{E}_\Lambda^{h,\psi}\, \varphi(0) - \mathbf{E}_\Lambda^{h,\psi'}\, \varphi(0)| \le C(\beta, h)\, e^{-\frac{\beta\zeta}{20k^2} N}$$

6.1. Proof of Theorem 6.1

In Theorem 5.1 we have already treated the case $h \in I_k(\beta)$ for $k = 1, \ldots, k_{max}$; thus we now assume

$$h \in \cup_{k=1}^{k_{max}} \left((h_k^*(\beta), h_k^-(\beta)] \cup [h_k^+(\beta), h_{k-1}^*(\beta)) \right).$$

The interval $[h_1^+(\beta), h_0^*(\beta))$ is somewhat special, because $h_0^*(\beta) = +\infty$, but it is also the easiest to deal with. In fact for h in this interval, we can choose any $\bar{h} \in I_1(\beta)$, so that, by FKG and Theorem 5.1, the LHS of (6.2) can be bounded by

$$\sup_{\psi \in \Omega} \mu_\Lambda^{h,\psi}(\, S_+(\Lambda, 1+j, \Lambda')\,) \le \sup_{\psi \in \Omega} \mu_\Lambda^{\bar{h},\psi}(\, S_+(\Lambda, 1+j, \Lambda')\,) \le e^{-\frac{\zeta\beta}{k^2} N j}\,.$$

Since the case $h \in [h_k^+(\beta), h_{k-1}^*(\beta))$ is similar to the case $h \in (h_k^*(\beta), h_k^-(\beta)]$, we will only consider this last possibility and choose then $h \in (h_k^*(\beta), h_k^-(\beta)]$. We also let $a(h) = a_k(\beta, h)$.

The difficulty we are going to face is the following: as long as $h \in I_k(\beta)$ we know, by Lemma 2.7–(iii) that

$$\frac{Z_e^{h,n}(V)}{Z_e^{h,k}(V)} \sim e^{-\frac{1}{2}\beta h|V|} \qquad \forall n \neq k$$

and this yields a bound on the weights on contours (Corollary 2.8) such that they can be summed (Proposition 2.9). On the other side we know from Theorem 4.1 that, if $h \in (h_k^*(\beta), h_k^-(\beta)]$ and we include in the partition function only cylinders whose diameter is less then $\frac{1}{|a(h)|}$, then all cylinders starting from k or $k+1$ are stable, so

$$\frac{Z_e^{h,k+1}(V)}{Z_e^{h,k}(V)} \sim e^{-a(h)|V|}\,.$$

This means that the bound for the weight of a contour given in Corollary 2.8 will be replaced by

$$\widetilde{w}_h(\Gamma) \leq \prod_{\gamma\in\Gamma} \exp\left[-(\beta-1)|\widetilde{\gamma}|_J L(\gamma) - \frac{\beta}{2} h'(I(\gamma))\,|v(\Gamma,\gamma)|\,|I(\gamma)-k|\right]$$

where $h'(n) = h$ except when $n = k+1$ in which case $h' \sim a(h)/\beta$. The problem is that, since $a(h) \searrow 0$ when $h \to h_k^*(\beta)$, then h' will be too small to "beat" the entropy given by the sum over cylinders whose minimun diameter is fixed, if h is too close to $h_k^*(\beta)$.

To fix this we *modify the definition of elementary cylinders*. A cylinder γ is here called elementary if does not touch the boundary $\delta\Lambda$ (as in Section 5) and

$$\operatorname{diam}\widetilde{\gamma} \leq \vartheta'(\beta, h, E(\gamma))$$

where

$$\vartheta'(\beta,h,n) = \begin{cases} \vartheta(\beta) = e^{\frac{\beta\zeta}{10}} & \text{if } n \notin \{k, k+1\} \\ (|a(h)|)^{-1} & \text{if } n = k, k+1\,. \end{cases}$$

This modification does of course affect the definition of many objects, like

$$C_e(V,k),\ \Gamma_l,\ \varphi_l,\ C^*_{c,l}(V,k),\ W(V,k),\ W_l(V,k),\ \mathrm{Con}(V,k)$$

which, when we use the new definition of elementary cylinder, will be respectively denoted by

$$C_{e'}(V,k),\ \Gamma_{l'},\ \varphi_{l'},\ C^*_{c,l'}(V,k),\ W'(V,k),\ W_{l'}(V,k),\ \mathrm{Con}'(V,k)\,.$$

Let also define

$$h'(n) = \begin{cases} h & \text{if } n \neq k+1 \\ |a(h)|/\beta & \text{if } n = k+1. \end{cases}$$

We observe that

$$\vartheta'(n) \geq \vartheta \quad \text{for all } n\,. \tag{6.3}$$

In fact, by (7) of Theorem 4.4 we have

$$a(h) = \beta h + \sum_{\Lambda \ni 0} \left[\frac{\Phi^k(\Lambda, w_h^{tr})}{|\Lambda|} - \frac{\Phi^{k+1}(\Lambda, w_h^{tr})}{|\Lambda|} \right]$$

and, by (6) of the same Theorem

$$|a(h)| \leq \beta h + \sum_{\Lambda \ni 0} \left[|\Phi^k(\Lambda, w_h^{tr})| + |\Phi^{k+1}(\Lambda, w_h^{tr})| \right] \leq \beta h_1^-(\beta) + 2e^{-\beta} \leq e^{-\frac{\beta}{2}}\,.$$

We now claim

Proposition 6.3. *Let β be large enough and $h \in (h_k^*(\beta), h_k^-(\beta)]$ with $1 \leq k \leq k_{max} = \lfloor e^{\frac{\beta\zeta}{20}} \rfloor$. If Π is as in Lemma 2.7, then for each simply connected finite V, $n \neq k$,*

$$e^{-(n-k)\beta h|V|} \, \frac{\hat{Z}^n_{e'}(V, w_h, \Pi)}{\hat{Z}^k_{e'}(V, w_h, \Pi)} \leq \exp\left[-\frac{1}{4}\beta h'(n)|V||n-k| + |\delta V| e^{-\frac{\beta\zeta}{6}} \right]$$

and for each $\Gamma \in \mathrm{Con}'(\Lambda, k)$

$$\widetilde{w}_h(\Gamma) \leq \prod_{\gamma \in \Gamma} \exp\Big[-(\beta-1)|\widetilde{\gamma}| L(\gamma) - \frac{1}{4}\beta h'(I(\gamma))\, |v(\Gamma,\gamma)|\, |I(\gamma) - k| \Big]\,.$$

Proof of Theorem 6.1 given Proposition 6.3. Let $h \in (h_k^*(\beta), h_k^-(\beta)]$. Then, by FKG, and Theorem 5.1

$$\mu_\Lambda^{h,\psi}(S_-(\Lambda, k-j, \Lambda')) \leq \mu_\Lambda^{h_k^-(\beta),\psi}(S_-(\Lambda, k-j, \Lambda')) \leq e^{-\frac{\zeta}{k^2}\beta N j} \qquad \forall j > 0\,.$$

Similarly, if $j \geq 2$,

$$\mu_\Lambda^{h,\psi}(S_+(\Lambda, k+j, \Lambda')) \leq \mu_\Lambda^{h_{k+1}^+(\beta),\psi}(S_+(\Lambda, k+j, \Lambda')) \leq e^{-\frac{\zeta}{(k+1)^2}\beta N(j-1)}\,. \tag{6.4}$$

So the only interesting part is proving that there is no percolation at level $k+1$. The idea is the following: we define a square Λ'' such that $\Lambda' \subset \Lambda'' \subset \Lambda$. Then we know that (with high probability) the b.c. don't percolate from $\partial\Lambda$ to Λ'' at level $k+2$. ¿From this it will follow that on the $\partial^+\Lambda''$ we have "almost" $k+1$ b.c. Then, all we have to do is to treat the case $\psi = k+1$.

Let then $\Lambda'' = Q_{N''}$, where $N'' = N - 2\lfloor N/4 \rfloor - 2$, and consider the event

$$F = S_+(\Lambda, k+2, \partial^+\Lambda'')\,.$$

By (6.4), Proposition A1.1,

$$\begin{aligned} \mu_\Lambda^{h,\psi}(S_+(\Lambda, k+1, \Lambda')) &\leq \mu_\Lambda^{h,\psi}(S_+(\Lambda, k+1, \Lambda') \,|\, F^c) + e^{-\frac{\zeta}{(k+1)^2}\beta N} \leq \\ &\leq e^{-\frac{\zeta}{(k+1)^2}\beta N} + \sup_{\substack{V:\Lambda''\subset V\subset\Lambda \\ V \text{ connected and} \\ \text{simply connected}}} \mu_V^{h,k+1}(S_+(\Lambda'', k+1, \Lambda')) \end{aligned} \tag{6.5}$$

so we have to estimate the second term. Let now

$$G = \Big\{ \varphi \in \Omega_\Lambda : \sum_{x\in\partial^+\Lambda''} \big(\, \varphi(x) - (k+1)\,\big)_+ \leq \zeta|\partial^+\Lambda''| \Big\}$$

where $(A)_+$ means $\max\{A, 0\}$, and, as usual $\zeta = 1000^{-1}$. Obviously G is a negative event Thanks to FKG and Proposition 3.5 (notice that $\partial^+\Lambda''$ does satisfy the hypotheses) we know that, for any simply connected V,

$$\mu_V^{h,k+1}(G^c) \leq \mu_V^{h^+_{k+1}(\beta),k+1}(G^c) \leq e^{-\frac{4}{5000}\beta N''} \leq e^{-\frac{1}{700}\beta N} \tag{6.6}$$

On the other side, using FKG again

$$\mu_V^{h,k+1}(S_+(\Lambda'', k+1, \Lambda') \,|\, G) \leq \sup_{\psi\in G} \mu_{\Lambda''}^{h,\widetilde{\psi}}(S_+(\Lambda'', k+1, \Lambda')) \tag{6.7}$$

where

$$\widetilde{\psi}(x) = \psi(x) \vee (k+1)\,.$$

Clearly

$$\sum_{x\in\partial^+\Lambda''} |\widetilde{\psi}(x) - (k+1)| \leq \zeta|\partial^+\Lambda''| \qquad \forall\psi\in G$$

so, regarding the probability as a quotient of two partition functions,

$$\sup_{\psi\in G} \mu_{\Lambda''}^{h,\widetilde{\psi}}(S_+(\Lambda'', k+1, \Lambda')) \leq e^{\frac{1}{50}\beta N''} \mu_{\Lambda''}^{h,k+1}(S_+(\Lambda'', k+1, \Lambda'))\,. \tag{6.8}$$

As we did in Section 5, we express the partition function with $k+1$ b.c. as a sum over collection of contours starting from level k and whose weight is modified at the boundary because of the b.c. So we write

$$Z^{h,k+1}(\Lambda'') = e^{-\beta hk|\Lambda''|-\beta|\delta\Lambda''|}\hat{Z}^k(\Lambda, w_h^{k+1})$$

where

$$\hat{Z}^k(\Lambda, w_h^{k+1}) = \sum_{\mathcal{A}\in W_{l'}(\Lambda,k)} \hat{Z}^k_{e'}(\Lambda, w_h, \Pi(\mathcal{A}_\partial)) \prod_{\Gamma\in\mathcal{A}} \widetilde{w}_h^{k+1}(\Gamma)$$

and the modified weight for a cylinder $w_h^{k+1}(\gamma)$ is defined in (5.3), (5.4). Given a contour Γ, we now define Γ' as in Section 5.2. Then, letting

$$Y = \{\Gamma \in \mathrm{Con}'(\Lambda'', k) : S(\Gamma) = +1,\, v(\Gamma')\cap\partial\Lambda'' \neq \emptyset,\, v(\Gamma')\cap\Lambda' \neq \emptyset\}$$

we claim that

$$\mu^{h,k+1}_{\Lambda''}(S_+(\Lambda'', k+1, \Lambda')) \leq \sum_{\Gamma\in Y} \widetilde{w}_h^{k+1}(\Gamma)\,. \tag{6.9}$$

In fact, Lemma 5.3 and statement (iii) of Proposition 5.5 imply

$$\begin{aligned}
&\mu^{h,k+1}_{\Lambda''}(S_+(\Lambda'', k+1, \Lambda')) \leq \bar{\mu}^{h,k+1}_{\Lambda''}(S_{+,l}(\Lambda'', k+1, \Lambda')) \leq \\
&\leq \bar{\mu}^{h,k+1}_{\Lambda''}\{\mathcal{A}\in W_{l'}(\Lambda'', k+1) : \exists\Gamma\in\mathcal{A},\, \Gamma\in Y\} \leq \\
&\leq \sum_{\Gamma\in Y} \bar{\mu}^{h,k+1}_{\Lambda''}\{\mathcal{A}\in W_{l'}(\Lambda'', k+1) : \mathcal{A}\ni\Gamma\}\,.
\end{aligned}$$

(6.9) now follows from Proposition 3.4. ¿From the definition of $w_h^{k+1}(\gamma)$, it is clear that, given a contour Γ with $S(\Gamma) = +1$, only the weight of its external cylinder is affected by the modification at the boundary. Thus, if γ_* is the external cylinder in Γ, we have

$$w_h^{k+1}(\gamma_*) = w_h(\gamma_*)\, e^{2\beta q(\gamma_*)}$$

while $w_h^{k+1}(\gamma) = w_h(\gamma)$ for all the other cylinders in Γ. So, by Proposition 6.3

$$\widetilde{w}_h^{k+1}(\Gamma) \leq e^{2\beta q(\gamma_*)}\, \bar{w}(\Gamma, \beta-1, \frac{\beta}{4}) \qquad \forall\Gamma\in\mathrm{Con}'(\Lambda'', k)$$

where we have set

$$\bar{w}(\Gamma, c, c') = \prod_{\gamma\in\Gamma} \exp\left[-c|\widetilde{\gamma}|L(\gamma) - c'h'(I(\gamma))|v(\Gamma,\gamma)||I(\gamma)-k|\right]\,.$$

We also observe that, because of Proposition 2.9 we get

$$\sum_{\substack{\Gamma\in \mathrm{Con}'(\Lambda'',k)\\ \bar\Gamma\ni x}} \bar w(\Gamma,\frac{\beta}{200},\frac{\beta}{8}) \le 1 \qquad \forall x\in\Lambda''\,. \tag{6.10}$$

At this point we proceed roughly as in the proof of Lemma 5.7, *i.e.* we divide the terms in the sum (6.9) into two groups: those with $|v(\Gamma)|\ge \zeta^2|\Lambda''|$ and the others. Let's call Y_1 the first group and Y_2 the second. Since for each $\Gamma\in Y_1$ we have

$$\bar w(\Gamma,\beta-1,\frac{\beta}{4}) \le e^{-\frac{1}{8}\beta h_m \zeta^2|\Lambda''|}\bar w(\Gamma,\beta-1,\frac{\beta}{8})$$

where $h_m=\min\{h,h'(k+1)\}$, thus, taking into account that $N''\ge 200(\zeta^2 h_m)^{-1}$ and $q(\gamma_*)\le|\delta\Lambda''|$, and using (6.10), we find

$$\sum_{\Gamma\in Y_1}\widetilde w_h^{k+1}(\Gamma)\le |\delta\Lambda''|\,e^{-\frac{1}{8}\beta h_m\zeta^2|\Lambda''|+2\beta|\delta\Lambda''|}\le e^{-\beta|\delta\Lambda''|}\le e^{-2\beta N}\,. \tag{6.11}$$

If $\Gamma\in Y_2$, then we claim

$$(\beta-1)\sum_{\gamma\in\Gamma}|\widetilde\gamma|L(\gamma)-2\beta q(\gamma_*)\ge \frac{\beta}{11}N''+\frac{\beta}{200}\sum_{\gamma\in\Gamma}|\widetilde\gamma|L(\gamma)\,. \tag{6.12}$$

If one believes it for a second, then one gets

$$\sum_{\Gamma\in Y_2}\widetilde w_h^{k+1}(\Gamma)\le e^{-\frac{1}{11}\beta N''}\sum_{\Gamma\in Y_2}\bar w(\Gamma,\frac{\beta}{200},\frac{\beta}{4})\le|\delta\Lambda''|e^{-\frac{1}{11}\beta N''}\le e^{-\frac{1}{22}\beta N}\,. \tag{6.13}$$

To prove (6.12) we write, for $\Gamma\in Y_2$,

$$\sum_{\gamma\in\Gamma}|\widetilde\gamma|L(\gamma)=\sum_{\gamma\in\Gamma'}|\widetilde\gamma|+R \tag{6.14}$$

where $R\ge 0$. But, since $|v(\Gamma')|\le|v(\Gamma)|\le\zeta^2|\Lambda''|$, $v(\Gamma')\cap\partial\Lambda''\ne\emptyset$ and $v(\Gamma')\cap\Lambda'\ne\emptyset$, we can use the argument in the proof of Lemma 5.7 and get $(q(\gamma_*)=q(\Gamma'_{ext}))$

$$\sum_{\gamma\in\Gamma'}|\widetilde\gamma|-2q(\gamma_*)\ge\frac{1}{10}N''+\frac{1}{100}\sum_{\gamma\in\Gamma'}|\widetilde\gamma|$$

which, together with (6.14), implies (6.12).

¿From (6.7), (6.8), (6.9), (6.11), (6.13) we see that, for each symply connected V such that $\Lambda'' \subset V$,

$$\mu_V^{h,k+1}(S_+(\Lambda'', k+1, \Lambda') \,|\, G) \le e^{-\frac{1}{50}\beta N}$$

which, together with (6.5), (6.6), gives

$$\mu_\Lambda^{h,\psi}(S_+(\Lambda, k+1, \Lambda')) \le e^{-\frac{\zeta}{2(k+1)^2}\beta N} \quad \square$$

Proof of Proposition 6.3. Let V be a simply connected finite volume. By Theorem 4.1 all cylinders with $E(\gamma) = k$ are stable, as well as those with $E(\gamma) = k+1$ and $\operatorname{diam}\gamma \le \vartheta'(k+1)$. So

$$Z^{h,k}(V) = Z_{tr}^{h,k}(V) \quad \text{and} \quad Z_{e'}^{h,k+1}(V) = Z_{e',tr}^{h,k+1}(V)$$

where we have set

$$Z_{e',tr}^{h,n}(V) = e^{-\beta h n |V|}\, \hat{Z}_{e'}^{n}(V, w_h^{tr})\,.$$

Then we have

$$\frac{Z_{e'}^{h,k+1}(V)}{Z_{e'}^{h,k}(V)} = \frac{Z_{e',tr}^{h,k+1}(V)}{Z_{e',tr}^{h,k}(V)} \le \frac{Z_{tr}^{h,k+1}(V)}{Z_{e',tr}^{h,k}(V)} = \frac{Z_{tr}^{h,k+1}(V)}{Z_{tr}^{h,k}(V)}\,\frac{Z_{tr}^{h,k}(V)}{Z_{e',tr}^{h,k}(V)}\,.$$

Using the cluster expansion for the truncated partition function we get, for $n > 0$,

$$\log Z_{tr}^{h,n}(V) - \log Z_{e',tr}^{h,n}(V) = \sum_{\substack{\Lambda\subset V\\ \operatorname{diam}\Lambda > \vartheta'(n)}} \Phi^n(\Lambda, w_h^{tr}) \le |V| e^{-\frac{1}{4}\beta\vartheta'(n)} \tag{6.15}$$

where we have used Theorem 4.4–(6). Also, by Proposition 4.9

$$|\log Z_{tr}^{h,k+1}(V) - \log Z_{tr}^{h,k}(V) + a(h)| \le |\delta V| e^{-\frac{1}{3}\beta\zeta}\,. \tag{6.16}$$

So we have obtained

$$\frac{Z_{e'}^{h,k+1}(V)}{Z_{e'}^{h,k}(V)} \le \exp\left[-|V|\big(a(h) - e^{-\frac{1}{4}\beta\vartheta'(k)}\big) + |\delta V| e^{-\frac{1}{3}\beta\zeta}\right] \le$$
$$\exp\left[-\frac{1}{2}a(h)|V| + |\delta V| e^{-\frac{1}{3}\beta\zeta}\right]\,.$$

Furthermore, if $n \neq k, k+1$, then, $\vartheta'(n) = \vartheta(n)$ while $\vartheta'(k+1) \geq \vartheta(k+1)$ by (6.3). Thus, by (iv) of Lemma 2.7,

$$\frac{Z^{h,n}_{e'}(V)}{Z^{h,k}_{e'}(V)} = \frac{Z^{h,n}_{e'}(V)}{Z^{h,k+1}_{e'}(V)}\,\frac{Z^{h,k+1}_{e'}(V)}{Z^{h,k}_{e'}(V)} \leq \frac{Z^{h,n}_{e}(V)}{Z^{h,k+1}_{e}(V)}\,\frac{Z^{h,k+1}_{e'}(V)}{Z^{h,k}_{e'}(V)} \leq$$
$$\leq \exp\left[-\frac{1}{2}\beta h|V||n-(k+1)| + |\delta V|\big(e^{-\frac{1}{5}\beta\zeta} + e^{-\frac{1}{3}\beta\zeta}\big)\right] \leq$$
$$\leq \exp\left[-\frac{1}{4}\beta h|V||n-k| + |\delta V| e^{-\frac{1}{6}\beta\zeta}\right].$$

The second statement follows easily from the first (see the proof of Corollary 2.8). $\square$

6.2. A cluster expansion for $\mathbf{E}\,\varphi(0)$.

Here we want to find a formula for $\mathbf{E}^{h,k}_V\,\varphi(0)$, based on cluster expansion that will be crucial in the proof of Corollary 6.2.

We introduce a modification of the magnetic field at $x=0$ and define for $\lambda \in \mathbb{R}$,

$$h_\lambda(x) = \begin{cases} h & \text{if } x \neq 0 \\ h - \lambda/\beta & \text{if } x = 0. \end{cases}$$

In this way, the weight of a cylinder γ such that $\bar\gamma \ni 0$ is given by

$$w_{h_\lambda}(\gamma) = w_h(\gamma)\, e^{\lambda S(\gamma) L(\gamma)}$$

while $w_{h_\lambda}(\gamma) = w_h(\gamma)$ for cylinders whose interior does not contain the origin. The partition function can be written as

$$Z^{h_\lambda,k}(V) = e^{-\beta h_\lambda k|V|}\hat{Z}^k(V, w_{h_\lambda})$$

where

$$\hat{Z}^k(V, w_{h_\lambda}) = \sum_{\Gamma \in C^*_w(V,k)} \prod_{\gamma\in\Gamma} \widetilde{w}_{h_\lambda}(\gamma)\,.$$

We want to prove the following

Proposition 6.4. *Let β be large enough, $h \in (h^*_k(\beta), h^*_{k-1}(\beta))$ with $k \leq k_{max} = \lfloor e^{\frac{\beta\zeta}{20}} \rfloor$. Then there exists $\lambda_0 = \lambda_0(\beta,h) > 0$ such that $\widetilde{w}_{h_\lambda}(\gamma)$ is differentiable with respect to λ in the interval $|\lambda| < \lambda_0$, and, for each cylinder γ with $E(\gamma) = k$*

$$\widetilde{w}_{h_\lambda}(\gamma) \leq e^{-\frac{1}{2}\beta|\tilde\gamma|L(\gamma)} \tag{6.17}$$

$$\left|\frac{d}{d\lambda}\widetilde{w}_{h_\lambda}(\gamma)\right| \leq e^{-\frac{1}{3}\beta|\tilde\gamma|L(\gamma)}, \tag{6.18}$$

for all $|\lambda| < \lambda_0$. Furthermore for each simply connected finite V containing the origin

$$\mathbf{E}_V^{h,k}\,\varphi(0) = k + \sum_{\substack{\gamma\in C(V,k)\\ \bar\gamma\ni 0}} \frac{d}{d\lambda}\widetilde{w}_{h_\lambda}(\gamma)\Big|_{\lambda=0} \exp\Big(-\sum_{\substack{\xi\subset\subset C(V,k)\\ \xi\cap[\gamma]\neq\emptyset}} \Phi^T(\xi,\widetilde{w}_h)\Big) \tag{6.19}$$

where $[\gamma]$ is the set of all cylinders which are not weakly compatible with γ and Φ^T is defined in Theorem 4.4.

An important consequence of this Proposition is the following

Corollary 6.5. *Let β, h be as above. Take $\Lambda = Q_N$ and $V \subset\subset \mathbb{Z}^2$ simply connected such that $V \supset \Lambda$. Then*

$$\Big|\mathbf{E}_V^{h,k}\,\varphi(0) - \mathbf{E}_\Lambda^{h,k}\,\varphi(0)\Big| \le e^{-\frac{1}{15}\beta N}\,.$$

Proof. Let

$$g_V(\gamma) = \exp\Big(-\sum_{\substack{\xi\subset\subset C(V,k)\\ \xi\cap[\gamma]\neq\emptyset}} \Phi^T(\xi,\widetilde{w}_h)\Big)\,.$$

Then, by Proposition 6.4

$$\begin{aligned}\Big|\mathbf{E}_V^{h,k}\,\varphi(0) - \mathbf{E}_\Lambda^{h,k}\,\varphi(0)\Big| \le& \sum_{\substack{\gamma\in C(V,k)\setminus C(\Lambda,k)\\ \bar\gamma\ni 0}} \Big|\frac{d}{d\lambda}\widetilde{w}_{h_\lambda}(\gamma)\Big|_{\lambda=0} |g_V(\gamma)| + \\ &+ \sum_{\substack{\gamma\in C(\Lambda,k)\\ \bar\gamma\ni 0}} \Big|\frac{d}{d\lambda}\widetilde{w}_{h_\lambda}(\gamma)\Big|_{\lambda=0} |g_V(\gamma)|\,|1-e^{X(\gamma)}| = S_1 + S_2\end{aligned} \tag{6.20}$$

where S_1, S_2 are the values of the two sums, and

$$X(\gamma) = \log\frac{g_\Lambda(\gamma)}{g_V(\gamma)} = \sum_{\substack{\xi\subset\subset C(V,k)\\ \xi\not\subset C(\Lambda,k),\ \xi\cap[\gamma]\neq\emptyset}} \Phi^T(\xi,\widetilde{w}_h)\,. \tag{6.21}$$

Since $|g_V(\gamma)| \le 1$, then first term in (6.20) can be (upper) bounded by

$$S_1 \le \sum_{\substack{\gamma\in C(V,k)\\ \bar\gamma\ni 0,\ |\widetilde\gamma|\ge N/2}} \Big|\frac{d}{d\lambda}\widetilde{w}_{h_\lambda}(\gamma)\Big|_{\lambda=0} \le e^{-\frac{1}{8}\beta N} \tag{6.22}$$

thanks to (6.18). As for the second term in (6.20), we observe that each ξ appearing in the sum (6.21) has $\bar{\xi} \cap (V \setminus \Lambda) \neq \emptyset$. Also, since it contains at least one cylinder γ' which is not weakly compatible with γ, we have

$$d(\bar{\xi}, \bar{\gamma}) \leq 2$$

and so (see (4.4))

$$\|\xi\| \geq d(\bar{\gamma}, \partial\Lambda)\,.$$

By consequence, using (3) of Theorem 4.4 (remember that the weights $\widetilde{w}_h(\gamma)$ satisfy Theorem's hypotheses with $c = 1/2$), we obtain

$$X(\gamma) \leq e^{-\frac{3}{8}\beta d(\bar{\gamma},\partial\Lambda)} \sum_{\substack{\xi \subset\subset C(V,k) \\ \xi \cap [\gamma] \neq \emptyset}} |\Phi^T(\xi, \widetilde{w}_h)| e^{\frac{3}{8}\beta \|\xi\|} \leq |\widetilde{\gamma}| e^{-\frac{3}{8}\beta d(\bar{\gamma},\partial\Lambda)}\,. \tag{6.23}$$

We now divide the terms contributing to S_2 into two groups: Y' is the set of all cylinders γ such that $d(\bar{\gamma}, \partial\Lambda) \geq N/4$ (and $\bar{\gamma} \ni 0$), while Y'' contains those γ with $d(\bar{\gamma}, \partial\Lambda) < N/4$ (and $\bar{\gamma} \ni 0$). Accordingly we write $S_2 = S_2' + S_2''$. If $\gamma \in Y'$, then, since $|\widetilde{\gamma}| \leq 4N^2$, we find $X(\gamma) \leq \frac{1}{10}$, and so

$$\left|1 - e^{X(\gamma)}\right| \leq 2|X(\gamma)| \leq 8N^2 e^{-\frac{3}{8}\beta d(\bar{\gamma},\partial\Lambda)} \leq e^{-\frac{1}{3}\beta d(\bar{\gamma},\partial\Lambda)}\,.$$

Hence, considering that $d(\bar{\gamma}, \partial\Lambda) + |\widetilde{\gamma}| \geq N/2$ and using (6.19), we find

$$S_2' \leq \sum_{\gamma \in Y'} e^{-\frac{1}{3}\beta d(\bar{\gamma},\partial\Lambda)} e^{-\frac{1}{3}\beta|\widetilde{\gamma}|L(\gamma)} \leq \sum_{\substack{\alpha \in C_B(\Lambda) \\ \bar{\alpha} \ni 0}} e^{-\frac{1}{3}\beta d(\bar{\alpha},\partial\Lambda)} e^{-\frac{1}{4}\beta|\alpha|} \leq$$

$$\leq e^{-\frac{1}{9}\beta N} \sum_{\substack{\alpha \in C_B(\Lambda) \\ \bar{\alpha} \ni 0}} e^{-\frac{1}{100}|\alpha|} \leq e^{-\frac{1}{9}\beta N}\,.$$

If, on the contrary, $\gamma \in Y''$, then $|\widetilde{\gamma}| \geq N/4$ and

$$\left|1 - e^{X(\gamma)}\right| \leq e^{X(\gamma)} \leq e^{|\widetilde{\gamma}|}\,.$$

Thus

$$S_2'' \leq \sum_{\substack{\gamma \in C(\Lambda,k) \\ \bar{\gamma} \ni 0,\, |\widetilde{\gamma}| \geq N/4}} e^{-\frac{1}{3}\beta|\widetilde{\gamma}|L(\gamma) + |\widetilde{\gamma}|} \leq e^{-\frac{1}{13}\beta N}\,.$$

At last

$$\left|\mathbf{E}_V^{h,k}\,\varphi(0) - \mathbf{E}_\Lambda^{h,k}\,\varphi(0)\right| \leq S_1 + S_2' + S_2'' \leq e^{-\frac{1}{15}\beta N} \quad \square$$

Before starting with the proof of Proposition 6.4 it is convenient to present some simple facts as Lemmas.

Lemma 6.6. *Let $\beta, h > 0$. Then there exists $\lambda_0(\beta, h) > 0$ such that if $|\lambda| < \lambda_0$ then,*

(*i*) *for all $V \subset\subset \mathbb{Z}^2$ such that $V \ni 0$ and for all $n > 0$*

$$\mathbf{E}_V^{h_\lambda, n}\, \varphi(0) = \frac{d}{d\lambda} \log Z^{h_\lambda, n}(V)\,.$$

(*ii*) *for each cylinder γ such that $\bar\gamma \ni 0$ and for all $n > 0$ (remember (2.4))*

$$\mathbf{E}_V^{h_\lambda, n}(\varphi(0) \,|\, \Omega_{\bar\gamma}^{n,\pm}) = \frac{d}{d\lambda} \log Z^{h_\lambda, n}(\bar\gamma, \pm)\,.$$

Proof. The only thing to check (remember (2.3)) is that the quantities

$$\sum_{\varphi \in \Omega_V} e^{-\beta H_V^{h_\lambda, n}(\varphi)} \qquad \sum_{\varphi \in \Omega_{\bar\gamma}^{n,\pm}} e^{-\beta H_V^{h_\lambda, n}(\varphi)}$$

can be differentiated term by term. For this to be true it is sufficient to check that for instance

$$\sum_{\varphi \in \Omega_V} \sup_{|\lambda| < \lambda_0} \left| \frac{d}{d\lambda} e^{-\beta H_V^{h_\lambda, n}(\varphi)} \right| \le \sum_{\varphi \in \Omega_V} \varphi(0)\, e^{-\beta H_V^{h(\lambda_0), n}(\varphi)} < \infty \quad \square$$

Lemma 6.7. *Let β be large enough, $h \in (h_k^*(\beta), h_{k-1}^*(\beta))$ with $1 \le k \le k_{max}$. Then there exists $\lambda_0 = \lambda_0(\beta, h) > 0$ such that if $|\lambda| < \lambda_0$, then for all cylinders γ such that $\bar\gamma \ni 0$, for all $n > 0$, $S = \pm 1$,*

$$\mathbf{E}_{\bar\gamma}^{h_\lambda, n}(\varphi(0) \,|\, \Omega_{\bar\gamma}^{n,S}) \le n + k + 1$$

Proof. Let

$$G = \{\varphi \in \Omega_{\bar\gamma} : \varphi(y) \ge n \text{ for all } y \in \bar\gamma\}\,.$$

Using an opportune coupling of two Glauber dynamics it is easy to show that

$$\mu_{\bar\gamma}^{h_\lambda, n}(\cdot \,|\, \Omega_{\bar\gamma}^{n,S}) \le \mu_{\bar\gamma}^{h_\lambda, n}(\cdot \,|\, G) \tag{6.24}$$

in the FKG sense. The proof is quite standard and one can see for instance Theorem 2.9 in Chapter II of [L]. That theorem deals with $\{0, 1\}$ random variables and with the probability measures which assign a positive probability to each configuration, but it is not too difficult to modify the proof in such a way that our situation can fit in. One can take for instance the *standard coupling* [L] of two "Metropolis" dynamics reversible with respect to the two probability measures appearing in (6.24). In this way one avoids any occurence of zero denominators in the transition rates.

Thus by just changing variables $\varphi(x) \to \varphi(x) + (n-1)$, we get

$$\mathbf{E}_{\bar\gamma}^{h_\lambda, n}(\varphi(0) \,|\, G) = \mathbf{E}_{\bar\gamma}^{h_\lambda, 1}(\varphi(0) + (n-1))\,.$$

If we now choose some $\bar h \in I_{k+1}(\beta)$, by FKG and Proposition 3.6, we have

$$\mathbf{E}_{\bar\gamma}^{h_\lambda, 1}(\varphi(0) + (n-1)) \le n - 1 + \mathbf{E}_{\bar\gamma}^{\bar h,\, k+1}\,\varphi(0) \le n - 1 + k + 2 = n + k + 1 \quad \square$$

We now go back to the

Proof of Proposition 6.4. Choose λ_0 such that $h \pm \lambda_0 \in (h_k^*(\beta), h_{k-1}^*(\beta))$ and let γ be a cylinder with $E(\gamma) = k$, $I(\gamma) = n$, $S(\gamma) = S$ and such that $\bar\gamma \ni 0$. To show that

$$\widetilde{w}_{h_\lambda}(\gamma) \le e^{-\frac{1}{2}\beta|\widetilde\gamma| L(\gamma)}$$

one can repeat the proof of Theorem 4.1 step by step, replacing in the steps (2), (3) and (4) the hypotesis $h \in [h_k^*(\beta), h_k^-(\beta)]$ with $\tilde h(x) \in [h_k^*(\beta), h_k^-(\beta)]$ for all x, where $\tilde h$ is now a variable magnetic field. There is only one thing to change: in the proof of (3) we consider apart the case of elementary cylinders in order to avoid the possibility that

$$|\delta\bar\gamma_x|_J >> |\widetilde\gamma|_J$$

(it could happen that $J(e) = \zeta$ for all $e \in \widetilde\gamma$, but $J = 1$ on $\delta\bar\gamma_x \setminus \widetilde\gamma$), and make an explicit appeal to (iv) of Lemma 2.7. In our case, however $J = 1$ everywhere, so we always have

$$|\delta\bar\gamma_x| \le |\widetilde\gamma| + 4 \le 2|\widetilde\gamma|\,.$$

Therefore we can treat small cylinders together with the large ones, suppress case (a) in the proof of (3), and avoid any reference to Lemma 2.7 (of course Lemma 2.7 *is still valid* in presence of a small enough λ, but we don't want to dig that deep into the proof).

In order to prove the second bound (6.18), we notice that $\widetilde{w}_{h_\lambda}(\gamma)$ is differentiable because both $w_{h_\lambda}(\gamma)$ and $Z^{h_\lambda, n}(\bar\gamma, \pm)$ (Lemma 6.6) are. Moreover

$$\begin{aligned} \frac{d}{d\lambda}\widetilde{w}_{h_\lambda}(\gamma) &= \widetilde{w}_{h_\lambda}(\gamma)\frac{d}{d\lambda}\log\frac{Z^{h_\lambda, n}(\bar\gamma, S)}{Z^{h_\lambda, k}(\bar\gamma, S)} = \\ &= \widetilde{w}_{h_\lambda}(\gamma)\left[\mathbf{E}_{\bar\gamma}^{h_\lambda, n}(\varphi(0) \,|\, \Omega_{\bar\gamma}^{n,S}) - \mathbf{E}_{\bar\gamma}^{h_\lambda, k}(\varphi(0) \,|\, \Omega_{\bar\gamma}^{n,S})\right]\,. \end{aligned}$$

By Lemma 6.7

$$\begin{aligned} &\left|\frac{d}{d\lambda}\widetilde{w}_{h_\lambda}(\gamma)\right| \le (n + k + 1)e^{-\frac{1}{2}\beta|\widetilde\gamma| L(\gamma)} \le \\ &\le (L(\gamma) + 2k_{max} + 1)e^{-\frac{1}{2}\beta|\widetilde\gamma| L(\gamma)} \le e^{-\frac{1}{3}\beta|\widetilde\gamma| L(\gamma)} \end{aligned}$$

Let now V be a simply connected finite volume containing the origin. We have to prove (6.19). Since

$$\sum_{\Gamma\in C_w^*(V,k)} \sup_{|\lambda|<\lambda_0} \left|\frac{d}{d\lambda}\prod_{\gamma\in\Gamma}\widetilde{w}_{h_\lambda}(\gamma)\right| \le \sum_{\Gamma\in C_w^*(V,k)}\sum_{\gamma\in\Gamma}\sup_{|\lambda|<\lambda_0}\left|\frac{d}{d\lambda}\widetilde{w}_{h_\lambda}(\gamma)\right| \prod_{\gamma'\in\Gamma\setminus\{\gamma\}}\widetilde{w}_{h_\lambda}(\gamma)\le$$

$$\le \sum_{\gamma\in C(V,k)}\sup_{|\lambda|<\lambda_0}\left|\frac{d}{d\lambda}\widetilde{w}_{h_\lambda}(\gamma)\right| \sum_{\substack{\Gamma\in C_w^*(V,k)\\ \Gamma\ni\gamma}}\prod_{\gamma'\in\Gamma\setminus\{\gamma\}}\widetilde{w}_{h_\lambda}(\gamma)\le$$

$$\le \hat{Z}^k(V,\widetilde{w}_{h_\lambda})\sum_{\gamma\in C(V,k)} e^{-\frac{1}{3}\beta|\tilde{\gamma}|L(\gamma)}<\infty$$

then the quantity

$$\sum_{\Gamma\in C_w^*(V,k)}\prod_{\gamma\in\Gamma}\widetilde{w}_{h_\lambda}(\gamma)$$

can be differentiated term by term, and a similar computation shows that

$$\frac{d}{d\lambda}\log Z^{h_\lambda,k}(V)=k+\frac{d}{d\lambda}\log\hat{Z}^k(V,w_{h_\lambda})=k+\sum_{\gamma\in C(V,k)} g_V(\gamma,\lambda)\frac{d}{d\lambda}\widetilde{w}_{h_\lambda}(\gamma)$$

where

$$g_V(\gamma,\lambda)=\left(Z^{h_\lambda,k}(V)\right)^{-1}\sum_{\substack{\Gamma\in C_w^*(V,k)\\ \Gamma\ni\gamma}}\prod_{\gamma'\in\Gamma\setminus\{\gamma\}}\widetilde{w}_{h_\lambda}(\gamma)=\left(Z^{h_\lambda,k}(V)\right)^{-1}\sum_{\substack{\Gamma\in C_w^*(V,k)\\ \Gamma\cap[\gamma]=\emptyset}}\prod_{\gamma'\in\Gamma}\widetilde{w}_{h_\lambda}(\gamma)\,.$$

Because of (6.17) we can use cluster expansion (Theorem 4.4 with $c=1/2$) and obtain

$$\log g_V(\gamma,\lambda)=\sum_{\substack{\xi\subset\subset C(V,k)\\ \xi\cap[\gamma]=\emptyset}}\Phi^T(\xi,\widetilde{w}_{h_\lambda})-\sum_{\xi\subset\subset C(V,k)}\Phi^T(\xi,\widetilde{w}_{h_\lambda})=$$

$$=-\sum_{\substack{\xi\subset\subset C(V,k)\\ \xi\cap[\gamma]\neq\emptyset}}\Phi^T(\xi,\widetilde{w}_{h_\lambda})\quad\Box$$

6.3. Proof of Corollary 6.2

If $\Lambda=Q_N$ with $N_1(h)\le N\le\bar{N}(\beta,h)$, then, by Proposition 3.2

$$|\,\mathbf{E}_\Lambda^{h,\psi}\,\varphi(0)-\mathbf{E}_\Lambda^{h,\psi'}\,\varphi(0)|\le b_1(\beta,h,2)\qquad\forall\psi,\psi'\in\Omega\,.$$

Assume now that $N > \bar{N}(\beta, h)$ and take another square Λ' inside Λ, given by $\Lambda' = Q_{N'}$, with $N' = N - 2\lfloor 2N/5 \rfloor - 2$. Let

$$n = \sup\{\psi(x) \vee \psi'(x) : x \in \partial^+\Lambda\}.$$

Then, by FKG and Markov property

$$|\, \mathbf{E}_\Lambda^{h,\psi}\, \varphi(0) - \mathbf{E}_\Lambda^{h,\psi'}\, \varphi(0)| \le \mathbf{E}_\Lambda^{h,n}\, \varphi(0) - \mathbf{E}_\Lambda^{h,1}\, \varphi(0).$$

If we let

$$S_\pm = S_\pm(\Lambda, k \pm 1, \partial^+\Lambda')$$

we have

$$\mathbf{E}_\Lambda^{h,n}\, \varphi(0) \le \mathbf{E}_\Lambda^{h,n}(\, \varphi(0)\, \chi\{\varphi \in S_+\}\,) + \mathbf{E}_\Lambda^{h,n}(\, \varphi(0)\,|\, S_+^c\,)$$

(χ is the characteristic function). By Proposition A1.1, we have

$$\mathbf{E}_\Lambda^{h,n}(\, \varphi(0)\,|\, S_+^c\,) \le \sup_{\substack{V:\Lambda' \subset V \subset \Lambda \\ V \text{ connected and} \\ \text{simply connected}}} \mathbf{E}_V^{h,k}\, \varphi(0)\,.$$

Moreover, by Schwarz inequality, Proposition 3.2 and Theorem 6.1,

$$\begin{aligned} &\mathbf{E}_\Lambda^{h,n}(\varphi(0)\, \chi\{\varphi \in S_+\}) \le \left(\mathbf{E}_\Lambda^{h,n}(\varphi(0)^2)\right)^{1/2} \left(\mu_\Lambda^{h,n}(S_+)\right)^{1/2} \le \\ &\le (b_1(\beta, h, 2))^{1/2}\, e^{-\frac{\zeta}{20k^2}\beta N}\,. \end{aligned}$$

On the other side we get

$$\mathbf{E}_\Lambda^{h,1}\, \varphi(0) \ge \mathbf{E}_\Lambda^{h,1}(\, \varphi(0)\,|\, S_-^c\,)\, \mu_\Lambda^{h,1}(\, S_-^c\,) \ge \left(1 - e^{-\frac{\zeta}{10k^2}\beta N}\right) \inf_{\substack{V:\Lambda' \subset V \subset \Lambda \\ V \text{ connected and} \\ \text{simply connected}}} \mathbf{E}_V^{h,k}\, \varphi(0)\,.$$

By FKG and Proposition 3.5,

$$\mathbf{E}_V^{h,k}\, \varphi(0) \le \mathbf{E}_V^{h_{k+1}^+(\beta),\, k+1}\, \varphi(0) \le k+2, \qquad \text{for all simply connected } V \subset\subset \mathbb{Z}^2.$$

Furthermore, by Corollary 6.5, if V is a simply connected finite volume such that $\Lambda' \subset V$,

$$|\, \mathbf{E}_V^{h,k}\, \varphi(0) - \mathbf{E}_{\Lambda'}^{h,k}\, \varphi(0)| \le e^{-\frac{1}{15}\beta N'} \le e^{-\frac{1}{100}\beta N}$$

In this way we have found

$$\begin{aligned} &|\, \mathbf{E}_\Lambda^{h,\psi}\, \varphi(0) - \mathbf{E}_\Lambda^{h,\psi'}\, \varphi(0)| \le \\ &\le e^{-\frac{1}{100}\beta N} + \left(k + 2 + (b_1(\beta, h, 2))^{1/2}\right) e^{-\frac{\zeta}{20k^2}\beta N} \le C(\beta, h)\, e^{-\frac{\zeta}{20k^2}\beta N} \quad \square \end{aligned}$$

7. Weak mixing and uniqueness of the Gibbs measure for $h \neq h_k^*(\beta)$

An important consequence of Corollary 6.2 is the weak mixing property discussed in [MO1], [MOS] which in turn implies the uniqueness of the Gibbs measure.

In order to state it we first need one additional definition. We will say that a set $\Lambda \subset\subset \mathbb{Z}^2$ is h-admissible if for each $x \in \Lambda$ there exists a square $Q_{N_1} + y$ of side $N_1(h) = \lfloor 8/h + 1 \rfloor$ contained in Λ such that $x \in Q_{N_1}$. Then we have:

Proposition 7.1 *(Weak Mixing). Let β be large enough and $h \in (h_k^*(\beta), h_{k-1}^*(\beta))$, with $1 \le k \le k_{max} = \lfloor e^{\frac{\beta\zeta}{20}} \rfloor$. Then there exists $C(\beta, h)$ such that for each h-admissible finite volume Λ and for each $\Delta \subset \Lambda$*

$$\sup_{\psi,\psi' \in \Omega} \sup_{A \in \mathbf{F}_\Delta} |\mu_\Lambda^{h,\psi}(A) - \mu_\Lambda^{h,\psi'}(A)| \le C(\beta, h) \sum_{x \in \Delta} e^{-m_0(\beta,h)\, d(x, \partial\Lambda)}$$

where $m_0(\beta, h) = \frac{\beta\zeta}{10k^2}$.

Proof. Using the coupling $\nu_\Lambda^{h,\psi,\psi'}$ defined in Section 1.2 between the two Gibbs measures $\mu_\Lambda^{h,\psi}$, $\mu_\Lambda^{h,\psi'}$ we write

$$\begin{aligned} |\mu_\Lambda^{h,\psi}(A) - \mu_\Lambda^{h,\psi'}(A)| &\le \max\Big\{ \sum_{\varphi \in A,\, \varphi' \notin A} \nu_\Lambda^{h,\psi,\psi'}(\varphi, \varphi'),\ \sum_{\varphi \notin A,\, \varphi' \in A} \nu_\Lambda^{h,\psi,\psi'}(\varphi, \varphi') \Big\} \le \\ &\le \sum_{x \in \Delta} \sup_n \nu_\Lambda^{h,n,1}(\varphi(x) \neq \varphi'(x)) \end{aligned} \tag{7.1}$$

Since the coupling measure $\nu_\Lambda^{h,n,1}$ is above the diagonal,

$$\begin{aligned} \nu_\Lambda^{h,n,1}(\varphi(x) > \varphi'(x)) &= \sum_k \nu_\Lambda^{h,n,1}(\varphi(x) = k,\ \varphi'(x) < k) \\ &\le \sum_k \nu_\Lambda^{h,n,1}(\varphi(x) \ge k,\ \varphi'(x) < k) = \sum_k \big[\nu_\Lambda^{h,n,1}(\varphi(x) \ge k) - \nu_\Lambda^{h,n,1}(\varphi'(x) \ge k)\big] = \\ &= \mathbf{E}_\Lambda^{h,n}\, \varphi(x) - \mathbf{E}_\Lambda^{h,1}\, \varphi(x) \end{aligned} \tag{7.2}$$

Let us now first suppose that $d(x, \partial\Lambda) \ge N_1(h)$ (see (6.1)) and let $Q(x)$ be the largest square centered at x and contained in Λ. Then we use (1.7) and Corollary 6.2 to get

$$\sup_n \mathbf{E}_\Lambda^{h,n}\, \varphi(x) - \mathbf{E}_\Lambda^{h,1}\, \varphi(x) \le \sup_{\psi,\psi' \in \Omega} |\mathbf{E}_{Q(x)}^{h,\psi}\, \varphi(x) - \mathbf{E}_{Q(x)}^{h,\psi'}\, \varphi(x)| \le C(\beta, h)\, e^{-m_0\, d(x, \partial\Lambda)} \tag{7.3}$$

If instead $d(x, \partial\Lambda) < N_1(h)$, we use the fact that the set Λ is h-admissible and Proposition 3.2

$$\sup_n \mathbf{E}_\Lambda^{h,n}\, \varphi(x) - \mathbf{E}_\Lambda^{h,1}\, \varphi(x) \le \sup_n \mathbf{E}_\Lambda^{h,n}\, \varphi(x) \le b_1(\beta, h, 1) \tag{7.4}$$

Clearly (7.1) $\ldots$(7.4) prove our statement provided we redefine $C(\beta,h)$ in a suitable way. $\square$

We can now prove that if β and h are as in the previous proposition, then there is a unique Gibbs measure for the interaction (1.1). Assume in fact that μ_1 and μ_2 are both Gibbs measures for such an interaction. Let $V \subset\subset \mathbb{Z}^2$ and $X \in \mathbf{F}_V$. Then, by definition of Gibbs measure, we have for all squares $\Lambda = Q_N$ with $N \geq N_1(h)$ such that $\Lambda \supset V$

$$|\mu_1(X) - \mu_2(X)| \leq \int_{\Omega\times\Omega} \mu_1(d\varphi)\, \mu_2(d\varphi')\, |\mu_\Lambda^\varphi(X) - \mu_\Lambda^{\varphi'}(X)| \leq C\,|V|\, e^{-m\, d(V,\partial\Lambda)}\,.$$

The argument is completed by letting N go to ∞.

8. A comment on the completeness of the phase diagram

As we remarked after the statement of Theorem 1.1, using the techniques developed in [Z] one can prove that if $h = h_k^*(\beta)$ with $1 \leq k \leq k_{max}$, then

$$\begin{array}{l}\textit{all translation invariant Gibbs measures are} \\ \textit{convex combinations of } \mu^k \textit{ and } \mu^{k+1}\end{array} \tag{8.1}$$

where

$$\mu^n = \operatorname*{weak\,lim}_{N\to\infty} \mu^n_{Q_N} \qquad n = k,\, k+1\,. \tag{8.2}$$

(β large enough and $h = h_k^*(\beta)$ have been chosen). Unfortunately [Z] deals with systems of random variables $\varphi(x)$ which can take only a finite number of values. Thus if one can show that

(1) the results of [Z] can be applied to our model

(2) k and $k+1$ are the only *stable* values in the sense of Zahradník

then (8.1) would follow from Corollary 3.2 in [Z].

We claim here that Zahradník's theory can be used in our case thanks to the following facts

- In [Z], with the exception of Section 3.2, one never takes into account the boundary term in the hamiltonian. For this reason the only problem caused by having an infinite number of values for $\varphi(x)$ is due to the following: the basic assumption in [Z] is that *contours* (we are talking of Zahradník's contours and in the rest of this discussion we use Zahradník's notation) have a weight which is bounded by an exponential of the volume of their support

$$\text{weight of } \Gamma^q = \exp(-\Phi(\Gamma^q)) \leq \exp(-\tau|\operatorname{supp}\Gamma^q|) \tag{8.3}$$

for some large enough τ. This bound is clearly ineffectual when one has to sum over an infinite number of contours with the same support. But it is not difficult to realize that all results in [Z] (except those in Section 3.2) are still valid if one drops the assumption of a finite single spin state space and replaces the above condition with the following

$$\sum_{\Gamma^q :\, \operatorname{supp}\Gamma^q = A} e^{-\Phi(\Gamma^q)} \leq e^{-\tau\,|A|} \qquad \forall A \subset\subset \mathbb{Z}^2\,. \tag{8.4}$$

(8.4) can be easily verified in our case.

- In Section 3.2 of [Z] the boundedness of the interaction (that we don't have) is used to prove Theorem 3.2. This theorem implies that the expectation of the number of *unstable points* in a volume Λ is of the order of the boundary of Λ. Showing that the number of unstable points grows like $|\Lambda|^\alpha$ with $\alpha \le 1$ is, in turn, the key ingredient for proving the main result, namely Corollary 3.2, which says that every translation invariant Gibbs measure is a convex combination of those Gibbs measures given by "stable" boundary conditions.

Let then $X_\Lambda(\varphi)$ be the number of $x \in \Lambda$ such that x is Λ–unstable for φ. We now sketch how to prove that, if $\Lambda = Q_N$ and $\Delta = Q_{N+4}$, then for all N large enough

$$\sup_{\psi \in \Omega} \mathbf{E}^\psi_\Delta(X_\Lambda) \le 2N^{\frac{5}{3}} \,. \tag{8.5}$$

(8.5) is enough to prove Corollary 3.2 of [Z]. To get (8.5) we are going to use the fact that, on $\partial\Delta$, thanks to Proposition 3.2, the configuration is likely to stay below say $\sqrt{N}$. Let $\Lambda' = Q_{N+2}$, and consider the event

$$F = \{\varphi \in \Omega : \varphi(x) \le \sqrt{N} \quad \forall x \in \partial\Delta\} \,.$$

Then

$$\mathbf{E}^\psi_\Delta(X_\Lambda) \le \mathbf{E}^\psi_\Delta(X_\Lambda \,|\, F) + |\Lambda| \, \mu^\psi_\Delta(F^c) \,.$$

Then, by Proposition 3.2 and Chebyshev inequality, we have

$$\mu^\psi_\Delta(F^c) \le |\partial\Delta| \, \frac{b_1(\beta, h, 4)}{N^2} \le \frac{c}{N} \,.$$

As for the other term, we get

$$\mathbf{E}^\psi_\Delta(X_\Lambda \,|\, F) \le N^{\frac{5}{3}} + N^2 \, \mu^\psi_\Delta(X_\Lambda \ge N^{\frac{5}{3}} \,|\, F)$$

and, using (1.7)

$$\mu^\psi_\Delta(X_\Lambda \ge N^{\frac{5}{3}} \,|\, F) \le \sup_{\varphi \in F} \, \mu^\varphi_{\Lambda'}(X_\Lambda \ge N^{\frac{5}{3}})$$

Moreover we notice that for each $\varphi, \eta \in \Omega$

$$H^1_{\Lambda'}(\eta) - \sum_{\substack{x \in \Lambda',\, y \in (\Lambda')^c \\ |x-y|=1}} |\varphi(y) - 1| \le H^\varphi_{\Lambda'}(\eta) \le H^1_{\Lambda'}(\eta) + \sum_{\substack{x \in \Lambda',\, y \in (\Lambda')^c \\ |x-y|=1}} |\varphi(y) - 1| \,.$$

Thus, if $\varphi \in F$, taking into account that X_Λ is $\mathbf{F}_{\Lambda'}$–measurable, we have

$$\mu^\varphi_{\Lambda'}(X_\Lambda \ge N^{\frac{5}{3}}) \le e^{32\beta(N+2)\sqrt{N}} \, \mu^1_{\Lambda'}(X_\Lambda \ge N^{\frac{5}{3}}) \,.$$

In this way we have gotten rid of the arbitrary boundary conditions. So we can use Proposition 3.1 of [Z] to estimate the last term, and we get

$$\mu_\Delta^\psi(X_\Lambda \geq N^{\frac{5}{3}} \mid F) \leq e^{c' N^{\frac{3}{2}}}\, e^{-\alpha N^{\frac{5}{3}} + c'' N^{\frac{3}{2}}}$$

for some $\alpha(\beta, h) > 0$, $c''(\beta, h) > 0$. We have thus obtained

$$\mathbf{E}_\Delta^\psi(X_\Lambda) \leq cN + N^{\frac{5}{3}} + N^2\, e^{c' N^{\frac{3}{2}}}\, e^{-\alpha N^{\frac{5}{3}} + c'' N^{\frac{3}{2}}} \leq 2N^{\frac{5}{3}}$$

if N is large enough. This proves (8.5) and by consequence Corollary 3.2 of [Z]. Hence every translation invariant Gibbs measure is a convex combination of the Gibbs measure produced by "stable" boundary conditions. So the last thing to observe in order to prove (8.1) is that k and $k+1$ are the only stable values in the sense of Zahradník. Assume in fact that there exists another $n \notin \{k, k+1\}$ such that n is stable. By Corollary 1.7 of [Z] there should be a Gibbs state which is a perturbation of the configuration $\varphi \equiv n$. But this is impossible because, by FKG, Corollary 6.2 and Proposition 3.6 one easily gets, for N large enough,

$$k - \tfrac{1}{100} \leq \mathbf{E}_{Q_N}^\psi\, \varphi(0) \leq k + 1 + \tfrac{1}{100} \qquad \forall \psi \in \Omega\,.$$

So the only values that can be stable are k and $k+1$. If only one of them, say k, is stable, then we should have $\mu^k = \mu^{k+1}$. But this again can't be, because of, for instance, Corollary 4.3.

Appendix A1

Proposition A1.1. *Let $\Lambda = Q_N$ and let Λ' be a connected subset of Λ such that $d(\Lambda^c, \Lambda') \geq 2$. Consider the event*

$$Y = \left\{ \varphi \in \Omega_\Lambda : \begin{array}{l} \textit{there exists a path } (x_1, \ldots, x_n) \textit{ from } \partial\Lambda \textit{ to } \partial^+\Lambda' \\ \textit{such that } \varphi(x_i) > k \textit{ for all } i \end{array} \right\}.$$

Then, for any positive event $X \in \mathbf{F}_{\Lambda'}$, for all $\psi \in \Omega$, we have

$$\mu_\Lambda^{h,\psi}(X \mid Y^c) \leq \sup_{\substack{V : \Lambda' \subset V \subset \Lambda \\ V \text{ connected and} \\ \text{simply connected}}} \mu_V^{h,k}(X).$$

Proof. This is a more or less straightforward consequence of FKG and Markov properties of our Gibbs measures, but its proof requires some care, so we give the details.

Given $A \subset\subset \mathbb{Z}^2$ we define $\text{ext}(A)$ as the unique infinite connected component of A^c and we let $\text{int}(A) = A^c \setminus \text{ext}(A)$. If $\varphi \in Y^c$, then there must exists V such that

(1) $\Lambda' \subset V \subset (\Lambda \setminus \partial\Lambda)$
(2) V is connected and simply connected
(3) $\varphi(y) \leq k$ for all $y \in \partial^+ V$

Furthermore there is a unique "largest" V among those satisfying (1), (2) and (3), meaning a unique V for which

(4) V is not a proper subset of any U which also satisfies (1), (2) and (3).

If we assume in fact that V_1 and V_2 both satisfy (1)–(4) with $V_1 \neq V_2$, then it will follow that

$$U = V_1 \cup V_2 \cup \text{int}(V_1 \cup V_2)$$

satisfies (1)–(3) which is a contradiction, because either V_1 or V_2 is a proper subset of U.

Clearly $\Lambda' \subset U \subset (\Lambda \setminus \partial\Lambda)$. Since V_1, V_2 have a nonempty intersection (they both contain Λ'), U is connected. Moreover $U^c = \text{ext}(V_1 \cup V_2)$, so U^c is connected which implies that U is simply connected. To prove (3) we observe that

$$\partial^+ \,\text{int}(V_1 \cup V_2) \subset V_1 \cup V_2$$

and, by consequence,

$$\partial^+ U = \partial^+ U \setminus (V_1 \cup V_2) \subset \partial^+(V_1 \cup V_2) \subset \partial^+ V_1 \cup \partial^+ V_2 \,.$$

In this way we have shown that, given $\varphi \in Y^c$, there is a unique set $\bar V(\varphi)$ such that (1)–(4) hold. So, if we let G be the set of all V's which satisfy (1) and (2), we can write Y^c as a union of disjoint events

$$Y^c = \bigcup_{V\in G} Y^c(V),$$

where

$$Y^c(V) = \{\varphi \in \Omega_\Lambda : \bar V(\varphi) = V\}.$$

The reason for using the "largest" V is that $Y^c(V) \in \mathbf{F}_{V^c}$ which implies for any event X

$$\mu_V^{h,\varphi}(X \cap Y^c(V)) = \mu_V^{h,\varphi}(X)\, \chi\{\varphi \in Y^c(V)\}\,. \tag{A1.1}$$

So, if X is a positive event in $\mathbf{F}_{\Lambda'}$, using (A1.1), (1.7), the Markov property and the FKG inequality, we get

$$\begin{aligned}
&\mu_\Lambda^{h,\psi}(X\cap Y^c) = \sum_{V\in G} \mu_\Lambda^{h,\psi}(X\cap Y^c(V)) = \\
&= \sum_{V\in G}\sum_{\varphi\in\Omega} \mu_\Lambda^{h,\psi}(\varphi)\, \mu_V^{h,\varphi}(X\cap Y^c(V)) \le \sum_{V\in G} \mu_\Lambda^{h,\psi}(Y^c(V)) \sup_{\varphi\in Y^c(V)} \mu_V^{h,\varphi}(X) \le \\
&\le \sum_{V\in G} \mu_\Lambda^{h,\psi}(Y^c(V))\, \mu_V^{h,k}(X) \le \mu_\Lambda^{h,\psi}(Y^c) \sup_{\substack{V:\Lambda'\subset V\subset\Lambda \\ V \text{ connected and} \\ \text{simply connected}}} \mu_V^{h,k}(X) \quad \square
\end{aligned}$$

Proposition A1.2. *Let β be large enough. Let $V \subset\subset \mathbb{Z}^2$ and let Y be a set of finite collections $\{\eta\} \subset C_B$. Assume that*

- (i) *for each $\{\eta_1,\dots,\eta_n\} \in Y$ there exists $\{x_1,\dots,x_n\} \subset V$ with $|\{x_1,\dots,x_n\}| = n$ and such that $\bar\eta_i \ni x_i$ for all i's*
- (ii) *for each $\{\eta\} \in Y$ and for each $\eta \in \{\eta\}$ we have $|\eta| \ge M$*

Then

$$\sum_{\{\eta\}\in Y}\prod_{\eta\in\{\eta\}} e^{-\beta|\eta|} \le \exp\left[|V|e^{-\frac{3}{4}\beta M}\right]$$

Remark. Hypothesis (i) holds in the following two particular cases

1. (1) the $\{\eta\}$ are the bases $\{\widetilde\gamma\}$ of a compatible or weakly compatible collection $\{\gamma\}$ of cylinders such that $\bar\gamma \subset V$ for each $\gamma \in \{\gamma\}$
2. (2) the $\{\eta\}$ have pairwise disjoint interiors and $\bar\eta \cap V \ne \emptyset$ for all $\eta \in \{\eta\}$.

Proof. Remembering that there exists a fixed constant K such that the number of η's of length l such that $\bar\eta \ni x$ is bounded by K^l, we get

$$\sum_{\{\eta\}\in Y}\prod_{\eta\in\{\eta\}} e^{-\beta|\eta|} = \sum_{s=0}^{|V|}\sum_{\substack{\{\eta\}\in Y\\ |\{\eta\}|=s}}\prod_{\eta\in\{\eta\}} e^{-\beta|\eta|} \le$$

$$\le \sum_{s=0}^{|V|}\sum_{\{x_1,\dots,x_s\}\subset V}\prod_{i=1}^{s}\Big[\sum_{\substack{\eta\in C_B:\\ \bar\eta\ni x_i\,,\,|\eta|\ge M}} e^{-\beta|\eta|}\Big] \le \sum_{s=0}^{|V|}\binom{|V|}{s}\left[\sum_{l=M}^{\infty}K^l e^{-\beta l}\right]^s \le$$

$$\le \sum_{s=0}^{|V|}\binom{|V|}{s}\left[e^{-\frac34\beta M}\right]^s = \left[1+e^{-\frac34\beta M}\right]^{|V|} \le \exp\left[|V|e^{-\frac34\beta M}\right] \quad \square$$

Proposition A1.3 Let $\Lambda = Q_N$ and $U \subset \Lambda$. *If*

$$|\delta U \cap \delta\Lambda| \ge \frac{9}{10}|\delta U|$$

then

$$|U| \ge \frac{24}{25}|\Lambda| \quad \text{and} \quad |U| \ge \frac{1}{6}|\delta U| N\,.$$

Proof. Let $V = \Lambda \setminus U$. Since

$$\delta U \setminus \delta\Lambda = \delta V \setminus \delta\Lambda$$

and

$$(\delta U \cap \delta\Lambda) \cup (\delta V \cap \delta\Lambda) = \delta\Lambda$$

we get

$$|\delta U| - 2|\delta U \cap \delta\Lambda| = |\delta V| - |\delta\Lambda|\,. \tag{A1.2}$$

If $|\delta U \cap \delta\Lambda| \ge \frac{9}{10}|\delta U|$, then δU must intersects all 4 sides of $\delta\Lambda$, so

$$|\delta U| \ge 4N\,.$$

By consequence, using (A1.2),

$$|\delta V| \le |\delta U| - \frac{18}{10}|\delta U| + |\delta\Lambda| \le \frac{1}{5}|\delta\Lambda|$$

which implies

$$|V| \le \left(\frac14|\delta V|\right)^2 \le \frac{1}{25}N^2\,.$$

The second statement follows from the first and the inequality

$$|\delta U| \le \frac{10}{9}|\delta U \cap \delta\Lambda| \le \frac{40}{9}N \quad \square$$

Proposition A1.4. *Let $\Lambda = Q_N$, $\Lambda' \subset \Lambda$. Let $U \subset \Lambda$ be such that*

(*i*) $|U| \ge (1-\varepsilon^2)|\Lambda|$ *for some* $\varepsilon > 0$

(*ii*) *there exists a $*$–path from Λ' to $\partial\Lambda$ which does not intersect U*

(*iii*) $2\varepsilon N \le d(\Lambda', \delta\Lambda)$.

Then

$$|\delta U| \ge |\delta\Lambda| + d(\Lambda', \delta\Lambda) - 11\varepsilon N - 11.$$

Proof. For further convenience we translate Q_N in such a way that it coincides with the square

$$\{x = (x_1, x_2) \in \mathbb{Z}^2 : 1 \le x_i \le N,\ i = 1, 2\}\,.$$

Let then $L = N - 2l$, with $l = \lfloor \varepsilon N + 1 \rfloor$ and let $\bar\Lambda = Q_L + (l, l)$, so that Λ and $\bar\Lambda$ have the same center.

For each $e \in \delta\bar\Lambda$, we denote with e' the unique dual edge in $\delta\Lambda$ such that $d(e, e') = l$. If now $(x_1, \dots, x_l)$ is the unique (straight) path of length l such that $d(x_1, e) = d(x_l, e') = \frac{1}{2}$, we set (see Fig. 5)

$$S(e) = \{x_1, \dots, x_l\}.$$

Because of hypothesis (*i*) we have

$$\#\{e \in \delta\bar\Lambda : S(e) \cap U = \emptyset\} \le l. \tag{A1.3}$$

On the other side, for each $S(e)$ which intersects U there is at least one edge $[x_i, x_{i+1}]^*$ which belongs to δU (by x_{l+1} we mean the unique $x \in \Lambda^c$ with $d(x_l, x_{l+1}) = 1$).

By (*ii*) there is a $*$–path $(y_1, \dots, y_q)$ such that $y_1 \in \Lambda'$, $y_q \in \partial\Lambda$ and

$$y_i \notin U, \quad i = 1, \dots, q.$$

Let

$$p = \min\{i = 1, \dots, q : y_i \in \partial\bar\Lambda\},$$

in such a way that $y_i \in \bar\Lambda$ for each $i = 1, \dots, p$. We can assume that y_p belongs to the bottom side $\partial_1\bar\Lambda$ of $\bar\Lambda$, *i.e.* that

$$y_p \in \partial_1\bar\Lambda = \{x = (x_1, x_2) \in \bar\Lambda : x_2 = l + 1\}.$$

Let $n = d(y_1, \partial_1 \bar{\Lambda}) + 1$, and, for $i = 1, \ldots, n$, let

$$T_i = \{x = (x_1, x_2) \in \bar{\Lambda} : x_2 = l + i\}.$$

Each T_i must intersects at least one y_j with $j \leq p$. Moreover $y_j \notin U$, so either $T_i \cap U = \emptyset$ or there is one edge $[x, y]^* \in \delta U$ such that $\{x, y\} \subset T_i$. But again, because of (i),

$$\#\{i = 1, \ldots, n : T_i \cap U = \emptyset\} \leq l.$$

Together with (A1.3) this implies

$$|\delta U| \geq |\delta \bar{\Lambda}| - l + n - l = |\delta \Lambda| + n - 10l \geq |\delta \Lambda| + d(\Lambda', \partial \Lambda) - 11\varepsilon N - 11 \quad \square$$

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Figure captions

(1) The interior $\bar{\alpha}$ of an element $\alpha \in C_B$

(2) Four pairs of compatible cylinders

(3) The configuration in (d) is constructed by successive addition of compatible cylinders. Each figure shows the internal level of the cylinders, while the external level can be deduced from the figure preceeding the first one where the cylinder appears. All cylinders in (a) have external level equal to k.

(4) An example of how the *prime* and *double prime* maps act on contours. Notice that Γ'' is not a contour, but it consists of two compatible contours.

(5) Proof of Proposition A1.4