A New Approach to Spin & Statistics

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Abstract

We give an algebraic proof of the spin-statistics connection for the parabosonic and parafermionic quantum topological charges of a theory of local observables with a modular P_1CT-symmetry. The argument avoids the use of the spinor calculus and also works in 1+2 dimensions. It is expected to be a progress towards a general spin-statistics theorem including also (1+2)-dimensional theories with braid group statistics.

1 Introduction

The spin-statistics theorem due to Fierz and Pauli [20, 28] is one of the great successes of general Quantum Field Theory. A general proof in the Wightman framework can be found in the monograph by Streater and Wightman [31]. However, the Bose-Fermi alternative enters the Wightman framework via the basic assumption of normal commutation relations.

In the algebraic approach due to Haag and Kastler [25, 24], the Bose-Fermi alternative is a result, not an axiom. The input of the theory is a net $\mathcal{A}$ of C*-algebras $\mathcal{A}(\mathcal{O})$ of bounded linear operators in a Hilbert space which are associated with every open, bounded space-time region $\mathcal{O} \subset \mathbf{R}^{1+s}$ in such a way that operators belonging to spacelike separated regions commute (*locality*). The basic structures of charged fields – including the possible particle statistics – can be recovered from the mere observable input [16, 17, 12]. In the most general case in lower dimensions, particles violating the familiar Bose-Fermi alternative can occur. The spin of these particles no longer needs to be integer or half-integer, it may be any real number. Such particles are expected to play a role in the theory of the fractional quantum Hall effect [30].

A field net consisiting of von Neumann algebras which generates in particular all massive parabosonic and parafermionic sectors from the vacuum and exhibits normal commutation relations has been constructed by Doplicher and Roberts [19] for any local net of observables satisfying the standard assumptions, and such a field is unique up to unitary equivalence.

For the algebraic framework, the spin-statistics theorem in (1+3)-dimensional spacetime has been proven in [17] for charges which are localizable in bounded open sets and in [11] for charges which are localizable in open convex cones extended to spacelike infinity (spacelike cones, see the definition below); such charges appear in purely massive theories [12]. All these proofs use the spinor calculus. This structure relies on the fact that in 1+3 dimensions the universal covering of the rotation group SO(3) is of order two. In 1+2 dimensions, however, the rotation group is the circle, and the universal covering of the circle is of infinite order. This is why the familiar spinor structure does not describe the irreducible representations of $\widetilde{\mathcal{P}_+^\uparrow}$ in 1+2 dimensions and why we have decided to look for an alternative argument.

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In this article, we present an algebraic proof of the spin-statistics connection given in [26] for parabosonic and parafermionic charges localizable in spacelike cones. Our proof works in any theory of local observables in at least 1+2 dimensions with the property that a certain antiunitary operator $J_{\mathcal{A}}$, a modular conjugation associated with the net of observables and the vacuum vector by means of the modular theory of Tomita and Takesaki, is a P_1CT-operator; here P_1 denotes the reflection $(x_0, x_1, x_2, \ldots, x_s) \mapsto (x_0, -x_1, x_2, \ldots, x_s)$, and C and T denote, as usual, charge conjugation and time reflection. This assumption, which we shall refer to as *modular P_1CT-symmetry*, has been shown by Bisognano and Wichmann to hold for any $\widetilde{\mathcal{P}_+^\uparrow}$-covariant Wightman field [1, 2]. Recently Borchers [5] has proved that in 1+1 dimensions, every local net of observables may be extended to a local net which exhibits the modular symmetries established by Bisognano and Wichmann. For higher dimensions, Borchers' result implies that the modular objects considered have commutation relations with the translations like P_1CT-reflections or Lorentz-boosts, respectively. Using this fact, it can be shown that P_1CT-symmetry is the only symmetry that *can* be implemented on the net of observables by the considered modular conjugation (as soon as, in a well-defined, very general sense, *any* symmetry on the net of observables is implemented) [26]. On the other hand, Yngvason [36] has given examples of theories which are not Lorentz-covariant and where the modular objects considered do not implement any symmetry.

Our line of argument is as follows: from modular P_1CT-symmetry and a compactness assumption discussed below, we derive that any rotation is represented by a product of two P_1CT-operators (i.e. the P_1CT-operators with respect to two, in general different, Lorentz frames). We then transfer this result and modular P_1CT-symmetry from the net of observables to the $\widetilde{\mathcal{P}_+^\uparrow}$-covariant Bose-Fermi field constructed by Doplicher and Roberts. The straightforward computation of any rotation by 2π in the corresponding representation, finally, yields the Bose-Fermi operator of the field; this implies the familiar spin-statistics connection.

The first proof of the spin-statistics theorem using the structures established by the Bisognano-Wichmann theorem has been given by Fröhlich and Marchetti [21]. Their argument relies on the assumption of the *full* Bisognano-Wichmann structure not only for the net of local observables, but for the whole reduced field bundle (which does not consist of algebras). We here only make an assumption about the net of observables, namely modular P_1CT-symmetry; we need not assume anything concerning the Bisognano-Wichmann modular operator or modular group.

A proof of the spin-statistics theorem similar to ours has been found independently by Guido and Longo [23]. They have derived modular P_1CT-symmetry from the assumption that a certain modular group implements a one-parameter group of Lorentz boosts ("modular covariance").

2 Notation, Preliminaries, and Assumptions

For some integer $s \geq 2$, denote by $\mathbf{R}^{1+s}$ the (1+s)-dimensional Minkowski space, and let V_+ be the forward light cone. The set $\mathcal{K}$ of all *double cones*, i.e. the set of all open sets $\mathcal{O}$ of the form

$$\mathcal{O} := (a + V_+) \cap (b - V_+), \qquad a, b \in \mathbf{R}^{1+s},$$

is a convenient topological base of $\mathbf{R}^{1+s}$. Each nonempty double cone is fixed by two points, its upper and its lower apex, and the set $\mathcal{K}$ is invariant under the action of the Poincaré group.

In the sequel, we denote by $(\mathcal{H}_0, \mathcal{A})$ a *concrete local net of observables*: $\mathcal{H}_0$ is a Hilbert space, and the net $\mathcal{A}$ associates with every double cone $\mathcal{O} \in \mathcal{K}$ a (concrete) C*-algebra $\mathcal{A}(\mathcal{O})$ consisting of bounded operators in $\mathcal{H}_0$ and containing the identity operator; this mapping is assumed to be *isotonous*, i.e. if $\mathcal{O}_1 \subset \mathcal{O}_2$, $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$, then $\mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2)$, and *local*, i.e. if $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated double cones and if $A \in \mathcal{A}(\mathcal{O}_1)$, $B \in \mathcal{A}(\mathcal{O}_2)$, then $AB = BA$. Since $\mathcal{K}$ is a topological base of $\mathbf{R}^{1+s}$, we may for any open set $M \subset \mathbf{R}^{1+s}$ consistently define $\mathcal{A}(M)$ to be the C*-algebra generated by the C*-algebras $\mathcal{A}(\mathcal{O})$, $\mathcal{O} \in \mathcal{K}$, $\mathcal{O} \subset M$. We shall denote by $\widetilde{\mathcal{A}} := \mathcal{A}(\mathbf{R}^{1+s})$ the C*-algebra of *quasilocal observables*. Note that every state of the normed, involutive algebra $\mathcal{A}_{\mathrm{loc}} = \bigcup_{\mathcal{O} \in \mathcal{K}} \mathcal{A}(\mathcal{O})$ of all *local observables* has a continuous extension to a state of $\widetilde{\mathcal{A}}$.

For any subset M of $\mathbf{R}^{1+s}$, we denote by M' the *spacelike complement* of M, i.e. the set of all points in $\mathbf{R}^{1+s}$ which are spacelike with respect to all points of M, and for every algebra $\mathcal{M}$ of bounded operators in some Hilbert space $\mathcal{H}$, we denote by $\mathcal{M}'$ the algebra of all bounded operators which commute with all elements of $\mathcal{M}$. Using this notation, the above locality assumption reads $\mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')' \quad \forall \mathcal{O} \in \mathcal{K}$.

Another kind of regions in Minkowski space that will be used are the *spacelike cones*: for any open, salient, convex circular cone $\vec{\mathcal{C}}$ in $\mathbf{R}^s$, i.e. for any cone in $\mathbf{R}^s$ which is generated by some open ε-ball around a vector $\vec{x} \in \mathbf{R}^s$ with euclidean length $\|\vec{x}\|_2 > \varepsilon$, the causal completion $\mathcal{C}$ of $\vec{\mathcal{C}}$ and its Poincaré transforms will be called spacelike cones; their set will be denoted by Σ. Note that this definition, which is based on remarks in [11], singles out the *causally complete* spacelike cones in the sense of [12], i.e. cones with $\mathcal{C}'' = \mathcal{C}$.

Assumptions, I:

We shall assume that there exists in $\mathcal{H}_0$ a strongly continuous representation U of the universal covering $\widetilde{\mathcal{P}_+^\uparrow}$ of the restricted Poincaré group $\mathcal{P}_+^\uparrow$ and that $\mathcal{A}$ is covariant *with respect to U, i.e.*

$$U(g)\mathcal{A}(\mathcal{O})U(g)^* = \mathcal{A}(\Lambda(g)\mathcal{O}) \qquad \forall g \in \widetilde{\mathcal{P}_+^\uparrow},$$

where $\Lambda : \widetilde{\mathcal{P}_+^\uparrow} \to \mathcal{P}_+^\uparrow$ denotes the covering map.

We assume that the translations in U satisfy the spectrum condition, *i.e. the joint spectrum of their generators is contained in $\overline{V}_+$.*

We assume the existence and uniqueness up to a phase of a unit vector Ω in $\mathcal{H}_0$ which is invariant under U and cyclic *with respect to the concrete algebra $(\mathcal{H}_0, \widetilde{\mathcal{A}})$, i.e. $\overline{\widetilde{\mathcal{A}}\Omega} = \mathcal{H}_0$; Ω will be called the* vacuum vector.

The (vacuum) representation $(\mathcal{H}_0, id_{\widetilde{\mathcal{A}}})$ is assumed to be irreducible, i.e.

$$\widetilde{\mathcal{A}}' = \mathbf{C}\,\mathrm{id}_{\mathcal{H}_0},$$

and we assume $\mathcal{H}_0$ to be separable.

Buchholz and Fredenhagen have associated a unique irreducible vacuum representation $(\mathcal{H}_{\mathrm{vac}}, \pi_{\mathrm{vac}})$ with any massive single-particle representation $(\mathcal{H}, \pi)$ of $\widetilde{\mathcal{A}}$ ([12], Definition on p. 13 and Theorem 3.4). If $(\mathcal{H}, \pi)$ is irreducible, it is unitarily equivalent to $(\mathcal{H}_{\mathrm{vac}}, \pi_{\mathrm{vac}})$ when restricted to $\mathcal{A}(\mathcal{C}')$ for any spacelike cone $\mathcal{C}$ ([12], Theorem 3.5), so any irreducible massive single-particle representation may be regarded as an excitation of a vacuum. We may choose such a vacuum fixed for our purposes; therefore, we set $(\mathcal{H}_{\mathrm{vac}}, \pi_{\mathrm{vac}}) = (\mathcal{H}_0, \mathrm{id}_{\widetilde{\mathcal{A}}})$. We shall denote the set of all parabosonic and parafermionic spacelike-cone excitations (in the above sense) of the vacuum by Π_Σ. We need not confine ourselves to massive particle representations, however, we do not know any examples of representations which are localizable in spacelike cones and have finite statistics but do not arise from massive single-particle representations.

We shall repeatedly use the group homomorphism $r : \mathbf{R} \to \widetilde{\mathcal{P}_+^\uparrow}$ which is constructed as follows: denote by $\exp(i\cdot)$ the covering map $\phi \mapsto \exp(i\phi)$ from $\mathbf{R}$ onto S^1, and let $\iota : S^1 \to \mathcal{P}_+^\uparrow$ be the group homomorphism embedding S^1 into $\mathcal{P}_+^\uparrow$ as the group of rotations in the 1-2-plane. $\exp(i\cdot)$ and ι are continuous, so $\iota \circ \exp(i\cdot)$ is a continuous curve in $\mathcal{P}_+^\uparrow$. There is a unique lift of this curve to a continuous curve r in $\widetilde{\mathcal{P}_+^\uparrow}$ with $r(0) = 1_{\widetilde{\mathcal{P}_+^\uparrow}}$ (see, e.g., Theorem III.3.3. in [7]); r is a group homomorphism of $\mathbf{R}$ into $\widetilde{\mathcal{P}_+^\uparrow}$.

Denote now by W the wedge

$$W := \{x \in \mathbf{R}^{1+s} : x_1 \geq |x_0|\}.$$

From Theorem 1 in [4] (cf. also [10], p. 279) it follows as a modification of the Reeh-Schlieder theorem that Ω is cyclic with respect to $(\mathcal{H}_0, \mathcal{A}(W))$, so a fortiori with respect to $(\mathcal{H}_0, \mathcal{A}(W)'')$, and using a standard argument (see, e.g., Prop. 2.5.3 in [6]), one obtains from locality that Ω is also *separating* with respect to $(\mathcal{H}_0, \mathcal{A}(W)'')$, i.e. if $A \in \mathcal{A}(W)''$ and $A\Omega = 0$, then $A = 0$.

A triple $(\mathcal{H}, \mathcal{M}, \xi)$ consisting of a von Neumann algebra $(\mathcal{H}, \mathcal{M})$ and a cyclic and separating vector ξ is called a *standard von Neumann algebra*; such a triple is the setting of Tomita-Takesaki theory ([33], see also [6, 24]): the antilinear operator

$$S_0 : \mathcal{M}\xi \to \mathcal{M}\xi;\ A\xi \mapsto A^*\xi, \quad A \in \mathcal{M},$$

is closable, and its closure S admits a (unique) polar decomposition into an antiunitary operator J and a positive operator $\Delta^{\frac{1}{2}}$ defined on the domain of S such that

$$S = J\Delta^{\frac{1}{2}}.$$

J is called the *modular conjugation*, $\Delta = (\Delta^{\frac{1}{2}})^2$ the *modular operator* of the standard von Neumann algebra $(\mathcal{H}, \mathcal{M}, \xi)$. The unitary group $(\Delta^{it})_{t\in\mathbf{R}}$ is called the *modular group* of $(\mathcal{H}, \mathcal{M}, \xi)$. Δ, Δ^{it}, $t \in \mathbf{R}$, and J, which we refer to as the *modular objects* of the standard von Neumann algebra $(\mathcal{H}, \mathcal{M}, \xi)$, leave ξ invariant. From the well-known elementary relations of the modular objects (Theorem 7.1. in [33]) we shall use $J^2 = \mathrm{id}_{\mathcal{H}}$. We recall the theorem of Tomita and Takesaki [33]:

$$\begin{aligned} \Delta^{it}\mathcal{M}\Delta^{-it} &= \mathcal{M}; \\ J\mathcal{M}J &= \mathcal{M}'. \end{aligned}$$

We shall make use of two further basic facts from Tomita-Takesaki theory; we recall them for the reader's convenience:

2.1 Lemma

Let $(\mathcal{H}_1, \mathcal{M}_1, \xi_1)$ and $(\mathcal{H}_2, \mathcal{M}_2, \xi_2)$ be two standard von Neumann algebras with modular objects Δ_1, J_1 and Δ_2, J_2, respectively, and let $V : \mathcal{H}_1 \to \mathcal{H}_2$ be a unitary operator with $V\mathcal{M}_1V^* = \mathcal{M}_2$ and $V\xi_1 = \xi_2$. Then we have $V\Delta_1^{\frac{1}{2}}V^* = \Delta_2^{\frac{1}{2}}$ and $VJ_1V^* = J_2$.

2.2 Theorem (Takesaki, Winnink)

For any standard von Neumann algebra $(\mathcal{H},\mathcal{M},\xi)$, the modular automorphism group $(\mathrm{Ad}(\Delta^{it}))_{t\in\mathbf{R}}$ is the unique one-parameter group $(\sigma_t)_{t\in\mathbf{R}}$ of automorphisms of the von Neumann algebra $(\mathcal{H},\mathcal{M})$ which satisfies the following conditions:

(i) for any $A\in\mathcal{M}$, the function $t\mapsto\sigma_t(A)$, $t\in\mathbf{R}$, is a continuous function from $\mathbf{R}$ into the von Neumann algebra $(\mathcal{H},\mathcal{M})$ endowed with the strong operator topology;

(ii) $(\sigma_t)_{t\in\mathbf{R}}$ satisfies the KMS-condition (at the inverse temperature $\beta=1$) with respect to $(\mathcal{H},\mathcal{M},\xi)$*: for any $A,B\in\mathcal{M}$, the function $t\mapsto\langle\xi,A\sigma_t(B)\xi\rangle$, $t\in\mathbf{R}$, may be extended to a continuous function f on the complex strip* $0\le\operatorname{Im}z\le 1$ *which is analytic on the interior of this strip and satisfies*

$$f(t+i)=\langle\xi,\sigma_t(B)A\xi\rangle\qquad\forall t\in\mathbf{R}.$$

The proof of Lemma 2.1 is straightforward; for a proof of Theorem 2.2, see [33], Theorems 13.1 and 13.2; note that (ii) implies that $\langle\xi,\sigma_t(A)\xi\rangle=\langle\xi,A\xi\rangle\,\forall A\in\mathcal{M}$, $t\in\mathbf{R}$ since $t\in\mathbf{R}\mapsto\langle\xi,\sigma_t(A)\xi\rangle$ is a bounded function for any $A\in\mathcal{M}$.

As mentioned in the Introduction, Bisognano and Wichmann have shown that the modular objects $\Delta_{\mathcal{A}}^{\frac{1}{2}}$, $\Delta_{\mathcal{A}}^{it}$, $t\in\mathbf{R}$, and $J_{\mathcal{A}}$ of the standard von Neumann algebra $(\mathcal{H}_0,\mathcal{A}(W)'',\Omega)$ implement symmetries in any Wightman theory; in particular, $J_{\mathcal{A}}$ implements a P_1CT-symmetry. There is, at the moment, no reason to believe that this form of P_1CT-symmetry should only hold for Wightman fields.

Assumption II:

Denote by j the P_1T-reflection given by

$$j(x_0,x_1,x_2,\dots x_s):=(-x_0,-x_1,x_2,\dots,x_s);$$

we shall assume modular P_1CT-symmetry*:*

$$J_{\mathcal{A}}\mathcal{A}(\mathcal{O})J_{\mathcal{A}}=\mathcal{A}(j\mathcal{O})\qquad\forall\mathcal{O}\in\mathcal{K}.$$

Note that due to $\widetilde{\mathcal{P}_+^\uparrow}$-covariance, this condition automatically holds in all Lorentz frames as soon as it holds in one. Given modular P_1CT-symmetry, $J_{\mathcal{A}}$ indeed yields the correct charge conjugation [22]. In 1+3 dimensions, a full PCT-operator may be constructed as a product of such modular conjugations; in 1+2 dimensions, no full PCT-symmetry has ever been proved to exist (cf. also [27] for a discussion in the Wightman framework).

It follows from the Tomita-Takesaki theorem that modular P_1CT-symmetry implies *wedge duality*:

$$\mathcal{A}(W)''=\mathcal{A}(W')'.$$

This, again, implies the following duality assumption for spacelike cones:

$$\mathcal{A}(\mathcal{C}')'=\mathcal{A}(\mathcal{C})''\qquad\forall\mathcal{C}\in\Sigma$$

since for any two spacelike separated spacelike cones $\mathcal{C}_1$ and $\mathcal{C}_2$, there is a Poincaré transform $\hat{W}$ of W such that $\mathcal{C}_1\subset\hat{W}$ and $\mathcal{C}_2\subset\hat{W}'$.

The adjoint action $\mathrm{Ad}(j)$ of j on $\mathcal{P}_+^\uparrow$ has a unique lift to a group homomorphism of $\widetilde{\mathcal{P}_+^\uparrow}$ (cf. Section III.4 in [7]) which we shall denote by $\widetilde{\mathrm{Ad}}(j)$.

Assumption III:

> *The group of* internal symmetries *of* $(\mathcal{H}_0, \mathcal{A}, \Omega)$*, i.e. the group of all unitaries* γ *in* $\mathcal{H}_0$ *such that* $\gamma\Omega = \Omega$ *and* $\gamma\mathcal{A}(\mathcal{O})\gamma^* = \mathcal{A}(\mathcal{O})$ *for all* $\mathcal{O} \in \mathcal{K}$*, is assumed to be compact in the strong operator topology.*

This property has been derived in [15] from assumptions concerning the scattering theory of the system. Another sufficient condition is the distal split property [18]. The distal split property, again, has been derived by Buchholz and Wichmann from their so-called *nuclearity condition*, for which they have given a thermodynamical justification [13].

On the other hand, the compactness of the internal symmetries implies that all internal symmetries commute with all $U(g)$, $g \in \widetilde{\mathcal{P}_+^\uparrow}$, and that U is the unique strongly continuous unitary representation of $\widetilde{\mathcal{P}_+^\uparrow}$ in $\mathcal{H}_0$ with respect to which $\mathcal{A}$ is covariant and Ω is invariant [18, 8]. There are examples of $\widetilde{\mathcal{P}_+^\uparrow}$-covariant theories [32] which violate the familiar spin-statistics connection. They admit several unitary representations of $\widetilde{\mathcal{P}_+^\uparrow}$ under which they are covariant, so, a fortiori, our compactness assumption is violated.

Finally, we recall the definitions and results of the Doplicher-Roberts field construction performed in [19] which are used in our argument.

2.3 Definition

Let $(\mathcal{H}_0, \mathcal{A}, U, \Omega)$ be as above, let $\mathcal{H}$ be a (not necessarily separable) Hilbert space, and let $(\mathcal{F}(\mathcal{C}))_{\mathcal{C}\in\Sigma}$ be a net[1] of von Neumann algebras. Let π be a faithful representation of $\widetilde{\mathcal{A}}$ in $\mathcal{H}$, and let G be a strongly compact group of unitaries in $\mathcal{H}$. The quadruple $(\mathcal{H}, \mathcal{F}, \pi, G)$ is called an *extended field system with gauge symmetry* — we shall simply say: a *field* — *over* $(\mathcal{H}_0, \mathcal{A}, U, \Omega)$ if the following conditions are satisfied:

(i) $(\mathcal{H}, \pi)$ contains $(\mathcal{H}_0, \mathrm{id}_{\widetilde{\mathcal{A}}})$ as a subrepresentation;

(ii) $\mathcal{H}_0$ is the subspace of all G-invariant vectors in $\mathcal{H}$;

(iii) for any $\mathcal{C} \in \Sigma$, the maps $\mathrm{Ad}(\gamma)$, $\gamma \in G$, act as automorphisms on $\mathcal{F}(\mathcal{C})$, and $\pi(\mathcal{A}(\mathcal{C}))''$ is the algebra of those elements of $\mathcal{F}(\mathcal{C})$ which are invariant under all $\mathrm{Ad}(\gamma)$, $\gamma \in G$, i.e.:

$$\pi(\mathcal{A}(\mathcal{C}))'' = \mathcal{F}(\mathcal{C}) \cap G' \qquad \forall \mathcal{C} \in \Sigma;$$

(iv) $\mathcal{F}$ is *irreducible* and *weakly additive*:

$$\left(\bigcup_{a\in\mathbf{R}^{1+s}} \mathcal{F}(\mathcal{C}+a) \right)'' = \mathcal{B}(\mathcal{H}) \qquad \forall \mathcal{C} \in \Sigma;$$

(v) $\mathcal{F}$ has the *Reeh-Schlieder property for spacelike cones*:

$$\overline{\mathcal{F}(\mathcal{C})\Omega} = \mathcal{H} \qquad \forall \mathcal{C} \in \Sigma;$$

[1] Here we make an abuse of language: the index set Σ is not directed, so $(\mathcal{F}(\mathcal{C}))_{\mathcal{C}\in\Sigma}$ is not a net in the usual sense. In this paper, we also call a net any family (of algebras) indexed by a partial-ordered, not necessarily directed set (of regions in Minkowski space).

(vi) $\mathcal{F}$ is *local with respect to the net* $(\pi(\mathcal{A}(\mathcal{O})))_{\mathcal{O}\in\mathcal{K}}$:

$$\mathcal{F}(\mathcal{C}) \subset \pi(\mathcal{A}(\mathcal{C}'))' \qquad \forall \mathcal{C} \in \Sigma.$$

In this case, $\mathcal{F}$ is called the *field net*, and G is called the *(global) gauge group*.

A field $(\mathcal{H}, \mathcal{F}, \pi, G)$ is called *normal* if it satisfies the *normal commutation relations*, i.e., if the gauge group contains an involution k such that with the notations

$$F^{\pm} := \tfrac{1}{2}(F \pm kFk^*), \qquad F \in \mathcal{F}(\mathcal{C}), \mathcal{C} \in \Sigma,$$

we have for any two spacelike separated cones $\mathcal{C}_1$ and $\mathcal{C}_2$:

$$F_1^+F_2^+ = F_2^+F_1^+, \qquad F_1^+F_2^- = F_2^-F_1^+, \qquad F_1^-F_1^- = -F_2^-F_1^- \qquad \forall F_{1,2} \in \mathcal{F}(\mathcal{C}_{1,2}).$$

k is called a *Bose-Fermi operator.*

Using the separability of $\mathcal{H}_0$, Doplicher and Roberts have shown that, given any field $(\mathcal{H}, \mathcal{F}, \pi, G)$ over $(\mathcal{H}_0, \mathcal{A}, U, \Omega)$, every irreducible subrepresentation of $(\mathcal{H}, \pi)$ is contained in Π_Σ (Theorem 3.6. in [19], cf. also the remarks on p. 19 in [12]), i.e., for some index set I, there is a family $(\pi_\iota)_{\iota\in I}$ of irreducible representations in Π_Σ such that $\pi = \bigoplus_{\iota\in I}\pi_\iota$. If the field is normal and k is a Bose-Fermi operator of the field, then for every $\iota \in I$, the restriction of the Bose-Fermi operator k to $\mathcal{H}_\iota$ coincides with the sign of the statistics parameter of π_ι (Theorem 3.6. in [19]). Hence, k is uniquely determined by π.

If $(\mathcal{H}, \mathcal{F}, \pi, G)$ is a normal field over $(\mathcal{H}_0, \mathcal{A}, U, \Omega)$, the unitary operator defined by

$$V := \frac{1}{1+i}(\mathrm{id}_{\mathcal{H}} + ik)$$

implements a *twist* of the field: the field over $(\mathcal{H}_0, \mathcal{A}, U, \Omega)$ given by

$$\mathcal{F}^t(\mathcal{C}) := V\mathcal{F}(\mathcal{C})V^*, \qquad \mathcal{C} \in \Sigma,$$

is local with respect to $\mathcal{F}$, i.e. $\mathcal{F}(\mathcal{C}) \subset \mathcal{F}^t(\mathcal{C}')'$ for all $\mathcal{C} \in \Sigma$. Doplicher and Roberts even established *twisted duality*, i.e.

$$\mathcal{F}(\mathcal{C}) = \mathcal{F}^t(\mathcal{C}')' \qquad \forall \mathcal{C} \in \Sigma.$$

See Theorem 5.4. in [19]. The same arguments allow to show that the wedge duality of the net of observables implies *twisted wedge duality* of the field:

$$\mathcal{F}(W) = \mathcal{F}^t(W')',$$

where

$$\mathcal{F}(W) := \Bigg(\bigcup_{\substack{\mathcal{C}\in\Sigma\\ \mathcal{C}\subset W}} \mathcal{F}(\mathcal{C})\Bigg)''.$$

Note that the phase $\frac{1}{1+i}$ of V has been chosen such that V leaves Ω invariant; with this choice, $V^2 = k$.

Given, conversely, $(\mathcal{H}_0, \mathcal{A}, U, \Omega)$ as assumed above, Doplicher and Roberts have shown that there is an up to unitary equivalence unique normal field $(\mathcal{H}, \mathcal{F}, \pi, G)$ over $(\mathcal{H}_0, \mathcal{A}, U, \Omega)$ such that each irreducible representation in Π_Σ is unitarily equivalent to a subrepresentation of $(\mathcal{H},\pi)$ (Theorem 5.3 in [19][2]. There also is, up to unitary equivalence, a unique normal field $(\mathcal{H},\mathcal{F},\pi,G)$ over $(\mathcal{H}_0,\mathcal{A},U,\Omega)$ such that $(\mathcal{H},\pi)$ contains all irreducible $\widetilde{\mathcal{P}_+^\uparrow}$*-covariant* representations contained in the set Π_Σ and is, conversely, a direct sum of such representations ([19], top of p. 98). There is a unique strongly continuous unitary representation U_π of $\widetilde{\mathcal{P}_+^\uparrow}$ in $\mathcal{H}$ with

$$U_\pi(g)\pi(A)U_\pi(g)^* = \pi(U(g)AU(g)^*) \qquad \forall g\in\widetilde{\mathcal{P}_+^\uparrow},\ A\in\widetilde{\mathcal{A}}$$

([19], pp. 98-101, cf. also Lemma 2.2. in [17]). The vacuum vector is invariant under U_π, and the field net $\mathcal{F}$ is covariant with respect to U_π. Note that U_π does not depend on the field net $\mathcal{F}$ itself[3]. Such a field will be called a $\widetilde{\mathcal{P}_+^\uparrow}$*-covariant* (normal) field over $(\mathcal{H}_0,\mathcal{A},U,\Omega)$.

It follows from property (v) in Definition 2.3 that Ω is cyclic and separating with respect to the von Neumann algebra $(\mathcal{H},\mathcal{F}(W))$. We shall denote by $J_\mathcal{F}$ and $\Delta_\mathcal{F}$ the modular conjugation and operator of the standard von Neumann algebra $(\mathcal{H},\mathcal{F}(W),\Omega)$.

3 Results

3.1 Proposition

For any $g\in\widetilde{\mathcal{P}_+^\uparrow}$, we have

$$J_\mathcal{A}U(g)J_\mathcal{A} = U(\widetilde{\mathrm{Ad}}(j)g).$$

[2]Note that the full $\widetilde{\mathcal{P}_+^\uparrow}$-covariance is not needed at this stage; it would suffice to assume translation covariance. Furthermore we remark that Doplicher and Roberts make an additional assumption they call "property B'". However, the following is sufficient:

Borchers property for spacelike cones:

> A concrete net $(\mathcal{H}_0,\mathcal{A})$ of observables is said to have the *Borchers property for spacelike cones* if, given any two spacelike cones $\mathcal{C}_1$ and $\mathcal{C}_2$ with $\overline{\mathcal{C}_1}\subset\mathcal{C}_2$ which are chosen in such a way that there is a third spacelike cone $\mathcal{C}^\times$ with $\mathcal{C}^\times\subset\mathcal{C}_1'\cap\mathcal{C}_2$, we can find for each nonzero projection $E\in\mathcal{A}(\mathcal{C}_1)''$ an isometry $W\in\mathcal{A}(\mathcal{C}_2)''$ such that $WW^*=E$ (and, trivially, $W^*W=\mathrm{id}_{\mathcal{H}_0}$, i.e., E and $\mathrm{id}_{\mathcal{H}_0}$ are equivalent in $\mathcal{A}(\mathcal{C}_2)''$).

Noting that for any spacelike cone $\mathcal{C}$, we have additivity:

$$\left(\bigcup_{a\in\mathbf{R}^{1+s}}\mathcal{A}(\mathcal{C}+a)\right)'' = \widetilde{\mathcal{A}}'' = \mathcal{B}(\mathcal{H}_0),$$

and using the spectrum condition and irreducibility, the Borchers property for spacelike cones can be proven applying the arguments from [3]. We emphasize that Borchers proves in [3] the corresponding result for double cones and therefore has to *assume* for double cones the above additivity property.

Doplicher's and Roberts' property B' is stronger: the same assumption as in the Borchers property for spacelike cones is made for any two spacelike cones $\mathcal{C}_1$ and $\mathcal{C}_2$ with $\overline{\mathcal{C}_1}\subset\mathcal{C}_2$ even if there is no spacelike cone $\mathcal{C}^\times\subset\mathcal{C}_1'\cap\mathcal{C}_2$ (this is, e.g., the case if $\mathcal{C}_1$ is a translate of $\mathcal{C}_2$). However, in order to prove this stronger form of the Borchers property for spacelike cones by means of the arguments taken from [3], one has to assume weak additivity for double cones.

[3]As an example, consider some $\widetilde{\mathcal{P}_+^\uparrow}$-covariant field $(\mathcal{H},\mathcal{F},\pi,G)$ and its twisted field $(\mathcal{H},\mathcal{F}^t,\pi,G)$. Both are covariant under U_π although their field nets are different.

Proof: One easily verifies that the representation U and the strongly continuous unitary representation U^J of $\widetilde{\mathcal{P}_+^\uparrow}$ defined by

$$U^J(g) := J_{\mathcal{A}} U(\widetilde{Ad}(j)g) J_{\mathcal{A}}, \qquad g \in \widetilde{\mathcal{P}_+^\uparrow},$$

implement the same spacetime transformations on the net $\mathcal{A}$ and leave Ω invariant. As stated in the previous section, it follows from the strong compactness of the group of internal symmetries that there can be at most one such representation; this implies $U = U^J$. □

3.2 Corollary (modular rotation symmetry)

For every angle $\phi \in [0, 2\pi]$, denote by W_ϕ the rotation by ϕ of W in the 1-2-plane and by J_ϕ the modular conjugation of $(\mathcal{H}_0, \mathcal{A}(W_\phi)'', \Omega)$. With r as defined in the previous section, define $R(\phi) := U(r(\phi))$, $\phi \in \mathbf{R}$. We then have

$$R(\phi) = J_{\frac{\phi}{2}} J_{\mathcal{A}} \qquad \forall \phi \in \mathbf{R}.$$

The representation U does not only realize $\widetilde{\mathcal{P}_+^\uparrow}$-covariance, but even (restricted) Poincaré covariance of the net:

$$R(2\pi) = id_{\mathcal{H}_0}.$$

Proof: From Proposition 3.1, we get

$$J_{\frac{\phi}{2}} J_{\mathcal{A}} = R(\tfrac{\phi}{2}) J_{\mathcal{A}} R(-\tfrac{\phi}{2}) J_{\mathcal{A}} = R(\tfrac{\phi}{2}) R(\tfrac{\phi}{2}) J_{\mathcal{A}}^2 = R(\phi).$$

In particular,

$$R(2\pi) = J_\pi J_{\mathcal{A}} = J_{\mathcal{A}}^2 = \mathrm{id}_{\mathcal{H}_0};$$

in the second step we use that the modular conjugations of a standard von Neumann algebra and its commutant coincide. □

For the special case of the rotation by π in (1+1)-dimensional chiral theories, the above formula has already appeared in [29, 35].

3.3 Lemma

For any field $(\mathcal{H}, \mathcal{F}, \pi, G)$ over $(\mathcal{H}_0, \mathcal{A}, U, \Omega)$, we have

$$\begin{array}{lrcl}
(i) & \Delta_{\mathcal{F}}^{it}|_{\mathcal{H}_0} A \Delta_{\mathcal{F}}^{-it}|_{\mathcal{H}_0} & = & \Delta_{\mathcal{A}}^{it} A \Delta_{\mathcal{A}}^{-it} \quad \forall A \in \mathcal{A}(W)''; \\
(ii) & \Delta_{\mathcal{F}}^{it}|_{\mathcal{H}_0} & = & \Delta_{\mathcal{A}}^{it}; \\
(iii) & \Delta_{\mathcal{F}}^{\frac{1}{2}}|_{\mathcal{H}_0 \cap D(\Delta_{\mathcal{F}}^{\frac{1}{2}})} & = & \Delta_{\mathcal{A}}^{\frac{1}{2}}; \\
(iv) & J_{\mathcal{F}}|_{\mathcal{H}_0} & = & J_{\mathcal{A}}; \\
(v) & J_{\mathcal{F}} \pi(A) J_{\mathcal{F}} & = & \pi(J_{\mathcal{A}} A J_{\mathcal{A}}) \quad \forall A \in \tilde{\mathcal{A}}.
\end{array}$$

Proof: It follows from Lemma 2.1 that $\Delta_{\mathcal{F}}^{it}$ commutes with the elements of the gauge group G for any $t \in \mathbf{R}$. This implies that any such $\Delta_{\mathcal{F}}^{it}$ maps the G-invariant vectors in $\mathcal{H}$ into G-invariant vectors, i.e., $\Delta_{\mathcal{F}}^{it}\mathcal{H}_0 = \mathcal{H}_0$ because of property (ii) in Definition 2.3, and that its adjoint action acts as an automorphism on the commutant of G. This – together with the Tomita-Takesaki theorem and the identity $\mathcal{F}(W) \cap G' = \pi(\mathcal{A}(W))''$ following from property (iii) in Definition 2.3 – gives that $\mathrm{Ad}(\Delta_{\mathcal{F}}^{it})$ acts as an automorphism on $\pi(\mathcal{A}(W))''$.

Consider now the direct sum decomposition $\pi = \bigoplus_{\iota \in I} \pi_\iota$ of π into irreducible representations π_ι in Π_Σ. The restriction of each π_ι to $\mathcal{A}(W)$ has a faithful extension π_ι^W to $\mathcal{A}(W)''$ which is continuous with respect to all familiar operator topologies (cf. Lemma 4.1. in [12]). Noting that

$$\bigoplus_{\iota \in I} \pi_\iota^W(\mathcal{A}(W)'') = \bigoplus_{\iota \in I} (\pi_\iota^W(\mathcal{A}(W))'') = \left(\bigoplus_{\iota \in I} \pi_\iota^W(\mathcal{A}(W)) \right)'' = (\pi(\mathcal{A}(W)))''$$

(cf. Cor. II.3.6. in [34] for the second step), we obtain a faithful extension $\pi_W \colon \mathcal{A}(W)'' \to \pi(\mathcal{A}(W))''$ of π by setting

$$\pi_W(A) := \bigoplus_{\iota \in I} \pi_\iota^W(A), \qquad A \in \mathcal{A}(W)''.$$

Because of property (i) in Definition 2.3, the inverse is given by

$$\pi_W^{-1}(B) = B|_{\mathcal{H}_0} \qquad \forall B \in \pi_W(A(W))''.$$

It is obviously continuous with respect to the corresponding strong operator topologies (the same holds for the other familiar operator topologies).

We may now define a one-parameter group $(\sigma_t)_{t \in \mathbf{R}}$ of automorphisms of $(\mathcal{H}_0, \mathcal{A}(W)'')$ by

$$\sigma_t(A) := \pi_W^{-1}\left(\Delta_{\mathcal{F}}^{it} \pi_W(A) \Delta_{\mathcal{F}}^{-it}\right), \qquad A \in \mathcal{A}(W)'',$$

and since we have shown above that the $\Delta_{\mathcal{F}}^{it}$, $t \in \mathbf{R}$, leave the subspace $\mathcal{H}_0$ invariant, we conclude

$$\sigma_t(A) = \Delta_{\mathcal{F}}^{it}|_{\mathcal{H}_0} A \Delta_{\mathcal{F}}^{-it}|_{\mathcal{H}_0} \qquad \forall A \in \mathcal{A}(W)'', t \in \mathbf{R}.$$

From this and from Theorem 3.3 it follows that σ satisfies the conditions of Theorem 3.3 and therefore coincides with the modular automorphism group of $(\mathcal{H}_0, \mathcal{A}(W)'', \Omega)$; this proves (i).

(ii) follows from (i): for any $A \in \mathcal{A}(W)''$ and any $t \in \mathbf{R}$, we have

$$\Delta_{\mathcal{F}}^{it}|_{\mathcal{H}_0} A\Omega = \Delta_{\mathcal{F}}^{it}|_{\mathcal{H}_0} A \Delta_{\mathcal{F}}^{-it}|_{\mathcal{H}_0} \Omega = \Delta_{\mathcal{A}}^{it} A \Delta_{\mathcal{A}}^{-it} \Omega = \Delta_{\mathcal{A}}^{it} A\Omega,$$

so $\Delta_{\mathcal{F}}^{it}|_{\mathcal{H}_0}$ and $\Delta_{\mathcal{A}}^{it}$ coincide on a dense subspace of $\mathcal{H}_0$ and hence – being bounded – on all of $\mathcal{H}_0$.

(iii) follows from (ii) since the KMS-condition implies, in the sense of quadratic forms:

$$\langle A\Omega, \Delta_{\mathcal{F}}|_{D(\Delta_{\mathcal{F}}) \cap \mathcal{H}_0} B\Omega \rangle = \langle B^*\Omega, A^*\Omega \rangle = \langle A\Omega, \Delta_{\mathcal{A}} B\Omega \rangle \qquad \forall A, B \in \mathcal{A}(W)''.$$

Since any positive operator is uniquely determined by its quadratic form, we conclude $\Delta_{\mathcal{F}}|_{\mathcal{D}(\Delta_{\mathcal{F}}) \cap \mathcal{H}_0} = \Delta_{\mathcal{A}}$, and using the spectral theorem, we get (iii).

(iv) follows from (iii) since the range of $\Delta_{\mathcal{A}}^{\frac{1}{2}}$ is dense in $\mathcal{H}_0$ and

$$\left(J_{\mathcal{F}} \Delta_{\mathcal{F}}^{\frac{1}{2}}\right)|_{D(\Delta_{\mathcal{F}}^{\frac{1}{2}}) \cap \mathcal{H}_0} = J_{\mathcal{A}} \Delta_{\mathcal{A}}^{\frac{1}{2}}.$$

(v) follows from (iv): because of modular $\mathrm{P_1CT}$-symmetry, we have $J_{\mathcal{A}} \tilde{\mathcal{A}} J_{\mathcal{A}} = \tilde{\mathcal{A}}$, hence, twice using property (i) in Definition 2.3, we get

$$\begin{aligned} J_{\mathcal{F}} \pi(A) J_{\mathcal{F}} \Omega &= J_{\mathcal{F}} \pi(A) \Omega = J_{\mathcal{F}}|_{\mathcal{H}_0} \pi(A)|_{\mathcal{H}_0} \Omega \\ &= J_{\mathcal{A}} A\Omega = J_{\mathcal{A}} A J_{\mathcal{A}} \Omega = \pi(J_{\mathcal{A}} A J_{\mathcal{A}})|_{\mathcal{H}_0} \Omega \\ &= \pi(J_{\mathcal{A}} A J_{\mathcal{A}}) \Omega, \end{aligned}$$

and since Ω is separating with respect to $(\mathcal{H}, \mathcal{F}(W)') = (\mathcal{H}, \mathcal{F}^t(W'))$, statement (v) follows from the Tomita-Takesaki theorem. $\square$

Remark: In the above argument, the modular groups considered possibly do not implement any symmetry on the net $\mathcal{A}$. We mention that under the additional assumption that $\Delta_{\mathcal{A}}^{it}\widetilde{\mathcal{A}}\Delta_{\mathcal{A}}^{-it} \subset \widetilde{\mathcal{A}}$, $t \in \mathbf{R}$, one can derive

$$\Delta_{\mathcal{F}}^{it}\pi(A)\Delta_{\mathcal{F}}^{-it} = \pi(\Delta_{\mathcal{A}}^{it}A\Delta_{\mathcal{A}}^{-it}) \qquad \forall A \in \widetilde{\mathcal{A}}, t \in \mathbf{R}$$

from (ii) in the same way as we have obtained (v) from (iv) in the preceding proof.

3.4 Theorem (P_1CT-symmetry of the field)

Let $(\mathcal{H}, \mathcal{F}, \pi, G)$ be a $\widetilde{\mathcal{P}_+^\uparrow}$-covariant, normal field over $(\mathcal{H}_0, \mathcal{A}, U, \Omega)$. Then we have:

(i) $J_{\mathcal{F}}\mathcal{F}(\mathcal{C})J_{\mathcal{F}} = \mathcal{F}^t(j\mathcal{C}) \qquad \forall \mathcal{C} \in \Sigma$;

(ii) $J_{\mathcal{F}}U_\pi(g)J_{\mathcal{F}} = U_\pi(\widetilde{\mathrm{Ad}}(j)(g)) \qquad \forall g \in \widetilde{\mathcal{P}_+^\uparrow}$.

The antiunitary involution $\Theta_W := VJ_{\mathcal{F}} = J_{\mathcal{F}}V^*$ is the P_1CT-operator, i.e.

$$\Theta_W\mathcal{F}(\mathcal{C})\Theta_W = \mathcal{F}(j\mathcal{C}).$$

Proof: Note first that $VJ_{\mathcal{F}} = J_{\mathcal{F}}V^*$ follows from the definition of V by a straightforward computation. From the modular P_1CT-symmetry of $\mathcal{A}$, the preceding lemma, and the fact that the modular objects commute with internal symmetries, it follows that by

$$\begin{aligned} \mathcal{H}^J &:= J_{\mathcal{F}}\mathcal{H} = \mathcal{H}; \\ \mathcal{F}^J(\mathcal{C}) &:= J_{\mathcal{F}}\mathcal{F}(j\mathcal{C})J_{\mathcal{F}}, \qquad \mathcal{C} \in \Sigma; \\ \pi^J(A) &:= J_{\mathcal{F}}\pi(J_{\mathcal{A}}AJ_{\mathcal{A}})J_{\mathcal{F}} = \pi(A), \qquad A \in \widetilde{\mathcal{A}}, \\ G^J &:= J_{\mathcal{F}}GJ_{\mathcal{F}} = G \end{aligned}$$

a second $\widetilde{\mathcal{P}_+^\uparrow}$-covariant normal field $(\mathcal{H}^J = \mathcal{H}, \mathcal{F}^J, \pi^J = \pi, G^J = G)$ over $(\mathcal{H}_0, \mathcal{A}, U, \Omega)$ is defined; note that it follows from $\pi^J = \pi$ that $\mathcal{F}^J$ has the same Bose-Fermi operator as $\mathcal{F}$ and that $\mathcal{F}^J$ is covariant under $U_\pi = U_{\pi^J}$. To show that $\mathcal{F}^J = \mathcal{F}^t$, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two spacelike cones with $\mathcal{C}_1 \subset W$ and $\mathcal{C}_2 \subset W'$. Using the Tomita-Takesaki theorem, we get

$$\begin{aligned} \mathcal{F}^J(\mathcal{C}_1) &= J_{\mathcal{F}}\mathcal{F}(j\mathcal{C}_1)J_{\mathcal{F}} \subset J_{\mathcal{F}}\mathcal{F}(W')J_{\mathcal{F}} \\ &= J_{\mathcal{F}}V^*V\mathcal{F}(W')V^*VJ_{\mathcal{F}} = VJ_{\mathcal{F}}\mathcal{F}^t(W')J_{\mathcal{F}}V^* = VJ_{\mathcal{F}}\mathcal{F}(W)'J_{\mathcal{F}}V^* \\ &= V\mathcal{F}(W)V^* = \mathcal{F}^t(W) = \mathcal{F}(W')' \\ &\subset \mathcal{F}(\mathcal{C}_2)'. \end{aligned}$$

Since for any spacelike separated cones $\mathcal{C}_1$ and $\mathcal{C}_2$, we can find a Poincaré transform $\hat{W}$ of W such that $\mathcal{C}_1 \subset \hat{W}$ and $\mathcal{C}_2 \subset \hat{W}'$, the net $\mathcal{F}^J$ is easily shown to be local with respect to the net $\mathcal{F}$. Twisted duality implies $\mathcal{F}^J \subset \mathcal{F}^t$, hence $\Theta_W\mathcal{F}(\mathcal{C})\Theta_W \subset \mathcal{F}(j\mathcal{C})\,\forall\mathcal{C} \in \Sigma$. Since Θ_W is an involution, we conclude $\Theta_W\mathcal{F}(\mathcal{C})\Theta_W = \mathcal{F}(j\mathcal{C})$ and $\mathcal{F}^J = \mathcal{F}^t$. From this, the Theorem follows immediately. $\square$

3.5 Corollary (spin-statistics theorem)

Let $(\mathcal{H}, \mathcal{F}, \pi, G)$ be a covariant, normal field over $(\mathcal{H}_0, \mathcal{A}, U, \Omega)$.

For every angle $\phi \in [0, 2\pi]$, denote by W_ϕ the rotation of W by ϕ in the 1-2-plane, and let J_ϕ and Θ_{W_ϕ} be the modular conjugation and the corresponding P_1CT-operator of $(\mathcal{H}, \mathcal{F}(W_\phi), \Omega)$. With r as defined in the previous section, define $R_\pi(\phi) := U_\pi(r(\phi))$, $\phi \in \mathbf{R}$.

Then we have:

$$R_\pi(\phi) = J_{\frac{\phi}{2}} J_{\mathcal{F}} = \Theta_{W_{\frac{\phi}{2}}} \Theta_W.$$

In particular, $R_\pi(2\pi) = k$, i.e. the spin-statistics connection familiar from 1+3 dimensions holds.

Proof: The first statement immediately follows from the preceding theorem in the same way as Corollary 3.2 follows from Proposition 3.1.

To obtain the spin-statistics connection, note that $J_\pi = V J_{\mathcal{F}} V^*$ follows from wedge duality by Lemma 2.1. Since $V J_{\mathcal{F}} = J_{\mathcal{F}} V^*$, we obtain

$$R_\pi(2\pi) = J_\pi J_{\mathcal{F}} = V J_{\mathcal{F}} V^* J_{\mathcal{F}} = V^2 J_{\mathcal{F}}^2 = V^2 = k.$$

□

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