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# On the lack of coercivity of the reduced Action-functional for zero total angular momentum in the planar Newtonian three-body problem

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**Abstract**

Combining hamiltonian reduction, variational methods and local analysis of the flow, we study for the planar three-body problem, the existence of non trivial periodic orbit without collisions, with zero total angular momentum. We prove that collision solutions are not minima, but the lack of compactness for the sub-levels of the Action-functional does not allow to prove the existence of $T$-periodic orbits without collisions. We restrict therefore to study periodic solutions which are odd under reflection (thorough some axis which may depend on the orbit). In this setting we prove that the loss of compactness is due to trajectories which are asymptotically collinear and we identify a class of sets which are compact and which are good candidates for the search of a minimum.

# 1 Introduction

In this paper we deal with the problem of the existence of periodic solutions in the planar Newtonian three-body problem. In particular we consider the problem under that the constraint of total angular momentum $J$ be zero.

For the Newtonian $N$-body problem in three dimensions, K.Sundman in a very old paper [24] proved that $J = 0$ is a necessary condition for total collision solutions. This fact implies that collision orbits must live in the submanifold of the phase space identified by the condition $J = 0$. Recently in [23] it is proved that in the $N$-body problem, under proper assumptions, collision orbits are not minima for the Action functional; (the assumptions are known to hold for $N \leq 4$) so that non-collision periodic solutions exist.

These facts led us to study the problem of the existence of periodic solutions and collision orbits by means of an Action functional reduced on the manifold of zero total angular momentum. Also in this case collision solutions are not minima, but now there is a lack of compactness and we cannot conclude immediately about the existence of minima. To go further, we study the geometry of the reduced configuration manifold; we show that the map which describes the reduction is a double covering of a set in $\mathbb{R}^3$, whose branching point is the set of collinear configurations. A weak Poincaré inequality shows that non-compactness is concentrated in a neighborhood of collinear orbits, i.e. when sides of the triangle (formed by the three bodies) become large while the relative angles become small.

The plan of the paper is the following: In sections 2, 3 we describe the construction of the Hamiltonian reduction and some of its technical properties. In section 4 we describe the geometry of the reduced configuration space. Collecting these geometrical facts, in section 5 we construct the reduced Action functional. In section 6 we prove that collision solutions cannot be minima of the Action. Finally in section 7 we construct a weak Poincaré inequality, we show the lack of compactness of sub-levels of the Action and moreover we define a class of subsets of the sub-levels which are $L^2$-compact. These sets could be good candidate to contain minima.

## 2 Hamiltonian reduction

In this section we present in a new form the hamiltonian reduction found by Van Kampen and Wintner in a very old paper [25].

The planar 3-body problem with newtonian potential can be described in hamiltonian formulation as:

configuration space:

$$Q \doteq \{\vec{q}_i \in \mathbb{R}^2; i = 1, 2, 3\} \simeq \mathbb{R}^6$$

In $Q$ there are configurations on which the dynamics of our system is not defined. Those are the *collision* configurations:

Double collisions

$$\{\vec{q}_1 = \vec{q}_2\} \quad \{\vec{q}_3 = \vec{q}_2\} \quad \{\vec{q}_1 = \vec{q}_3\}$$

Triple (total) collision

$$\{\vec{q}_1 = \vec{q}_2 = \vec{q}_3\}$$

we leave these points in the configuration space since we will study collisions later on.

The phase space is:

$$T^*Q \simeq \mathbb{R}^6 \times \mathbb{R}^6$$

equipped with the symplectic form:

$$\omega = \sum_{i=1}^{3} d\vec{q}_i \wedge d\vec{p}_i$$

where $\wedge$ denotes the wedge-product in $\mathbb{R}^2$. The hamiltonian function is:

$$H = \sum_{i=1}^{3} \frac{\|\vec{p}_i\|^2}{2m_i} - \sum_{i,j,i\neq j}^{3} \frac{m_i m_j}{\|\vec{q}_i - \vec{q}_j\|} \tag{1}$$

and the dynamics is given by the vector field $X_H \in T(T^*Q)$ defined as:

$$i_{X_H}\omega = dH \tag{2}$$

The flow generated by (2) preserves linear momentum $\vec{P}$ and angular momentum $J$

$$\vec{P} = \sum_{i=1}^{3} \vec{p}_i \tag{3}$$

$$J = \sum_{i=1}^{3} \vec{q}_i \wedge \vec{p}_i \tag{4}$$

We want to find a symplectic transformation which makes explicitly the dependence of $H$ on linear and angular momentum.

Let $U \subset T^*Q$ be an open set, let $\{U, \psi_1\}$ and $\{U, \psi_2\}$ be two different systems of coordinate:

$$\psi_i : U \longrightarrow \mathbb{R}^{12}$$

$$i = 1, 2$$

Consider the map:

$$\phi_i : U_1 \times U_2 \longrightarrow U_i$$

$$i = 1, 2$$

where:

$$U_i = \psi_i(U)$$

$$i = 1, 2$$

Denoting by $\omega_i$ the symplectic form $\omega$ expressed in the i-system of coordinate, then one constructs on $U_1 \times U_2$ the 2-form

$$\Omega = \phi_1^* \omega_1 - \phi_2^* \omega_2$$

It is easy to see that:

(i) $\Omega$ is symplectic

(ii) the map $f : U_1 \longrightarrow U_2$ is symplectic iff:

$$i_f^* \Omega = 0$$

where:

$$i_f : \Gamma_f \longrightarrow U_1 \times U_2 \quad , \quad \Gamma_f = \{(m, f(m)) \in U_1 \times U_2, m \in U_1\}$$

and $*$ denotes the pull-back.

In our case $\omega = -d\theta$ where $\theta$ is the canonical 1-form on $T^*Q$. Therefore there exists a 1-form $\Theta$ on $U_1 \times U_2$ such that $\Omega = -d\Theta$ and $\Theta = \phi_1^* \theta_1 - \phi_2^* \theta_2$. Thus:

$$i_f^* \Omega = 0 \Rightarrow d i_f^* \Theta = 0$$

and, if $U$ is contractible, by Poincaré lemma, there exists a map $S$

$$S : \Gamma_f \longrightarrow \mathbb{R}$$

defined locally and such that:

$$i_f^* \Theta = dS \tag{5}$$

$S$ is the generating function of the symplectic transformation $f$.

If we denote the coordinates on $U_1 \times U_2$ by $(\vec{q}_i, \vec{p}_i, \vec{\kappa}_i, \vec{\pi}_i)$ $i = 1,2,3$ where $(\vec{q}_i, \vec{p}_i)$ are coordinates of $U_1$ while $(\vec{\kappa}_i, \vec{\pi}_i)$ are coordinates in $U_2$, then from (5) we have:

$$dS = (\phi_1 \circ i_f)^* \theta_1 - (\phi_2 \circ i_f)^* \theta_2$$

but $(\phi_1 \circ i_f) = id_{U_1}$ and $(\phi_2 \circ i_f)$ are the coordinates of $U_2$ as function of coordinates of $U_1$ by means of $f$. Thus locally we can write:

$$dS = \sum_i \vec{p}_i \cdot d\vec{q}_i - \vec{\pi}_i \cdot d\vec{\kappa}_i$$

if we put

$$W = S - \sum_i \vec{\pi}_i \cdot \vec{\kappa}_i$$

we get

$$dW = \sum_i \vec{p}_i \cdot d\vec{q}_i + \vec{\kappa}_i \cdot d\vec{\pi}_i \tag{6}$$

$W$ is now a function on $\Gamma_f$ in the coordinates $\vec{q}_i, \vec{\pi}_i$ $i = 1,2,3$. Now, given W, the canonical transformation will be obtained by:

$$\begin{cases} \vec{p}_i = \frac{\partial W}{\partial \vec{q}_i} \\ \vec{\kappa}_i = \frac{\partial W}{\partial \vec{\pi}_i} \end{cases} \tag{7}$$

We define a particular canonical transformation, which allow us to write the hamiltonian in a form in which the integrals of motion appear explicitly. This transformation was found by Wintner and Van Kampen [25].

Let

$$\vec{q}_i = (q_i^1 \ q_i^2) \quad , \quad \vec{p}_i = (p_i^1 \ p_i^2)$$

$$i = 1,2,3$$

and define (cyclic permutations of the three indices $i, j, k = 1, 2, 3$)

$$\rho_i \doteq \sqrt{(q_j^1 - q_k^1)^2 + (q_j^2 - q_k^2)^2} \tag{8}$$

$$\cos\phi_i \doteq \frac{q_j^1 - q_k^1}{\rho_i} \tag{9}$$

Here

$$\theta_i \doteq \phi_j - \phi_k \tag{10}$$

are the exterior oriented angles at the vertices of the triangle formed by the vectors $\vec{q_i}$s.

¿From (10) we have:

$$\sum_{i=1}^{3} \theta_i = 0 \tag{11}$$

Let us define:

$$\xi \doteq 1/3 \sum_{i=1}^{3} \phi_i \tag{12}$$

so that:

$$\phi_i \to \phi_i + \pi \Rightarrow \xi \to \xi + \pi$$

Denote by $X_i$ the coordinates of the center of the mass:

$$\begin{cases} X_1 \doteq \frac{\sum_{i=1}^{3} m_i q_i^1}{M} \\ X_2 \doteq \frac{\sum_{i=1}^{3} m_i q_i^2}{M} \end{cases} \tag{13}$$

Using (8), (9), (12), (13), we can define the map:

$$\begin{aligned} F : Q \quad &\longrightarrow \quad C \times S^1 \times \mathbb{R}^2 \\ (\vec{q_i} \quad i = 1, 2, 3) \quad &\longrightarrow \quad (\rho_1, \rho_2, \rho_3, \xi, X_1, X_2) \end{aligned} \tag{14}$$

where:

$$C \doteq \left\{ \vec{\rho} = (\rho_1 \ \ \rho_2 \ \ \rho_3) \in \mathbb{R}_+^3 \ \ / \ \ \rho_i + \rho_j \geq \rho_k \ \ \textit{and cyclic permutations} \right\}$$

Observe that $C$ is a manifold with boundary given by the three relations:

$$\rho_i + \rho_j = \rho_k \ \ \textit{and cyclic permutations}$$

These are the *collinear* configurations. One can verify that $rank(dF)$ is less than 6 on collinear configurations. $F^{-1}$ is double valued on $C - \partial C$: let us call $\Gamma_+$ and $\Gamma_-$ the two sheets, so we have:

$$F^{-1}(C - \partial C) = \Gamma_+ \cup \Gamma_-$$

$\Gamma_+$ and $\Gamma_-$ are two open connected submanifold of $Q$ glued along the common boundary $F^{-1}(\partial C)$. By Existence-Uniqueness Theorem, for every regular trajectory $\{\rho_i(t)\}_{i=1}^3$, either $\rho_i(t) \in F^{-1}(\partial C)$ for all $t$ or the trajectory is transversal to $F^{-1}(\partial C)$ at the intersection points. This fact will allow us to define a global Action functional in the reduced coordinates. In $\Gamma_+ \cup \Gamma_-$ from (9) and (13) one has:

$$\begin{cases} q_k^1 = 1/M\{m_j \rho_i \cos\phi_i - m_i \rho_j \cos\phi_j\} + X_1 \\ q_k^2 = 1/M\{m_j \rho_i \sin\phi_i - m_i \rho_j \sin\phi_j\} + X_2 \end{cases} \tag{15}$$

where $\phi_i$'s are expressed in terms of $\xi$ and $\rho$'s by means of (9) and (12).

The canonical transformation can be obtained using the following generating function:

$$W(\pi_i, \vec{q}_i, \Xi, \vec{P}) = P_1 X_1(\vec{q}_i) + P_2 X_2(\vec{q}_i) + \Xi \xi(\vec{q}_i) + \sum_{l=1}^{3} \pi_l \rho_l(\vec{q}_i) \tag{16}$$

where the vector $(P_1 \; P_2)$ is the conjugate momentum to $(X_1 \; X_2)$, $\pi_l$s are conjugate momenta to $\rho_l$s, $\Xi$ is the momentum conjugate to $\xi$, and it is possible to prove that $\Xi = J$.

The full canonical transformation is given using (7):

$$\begin{aligned} \pi_i &= \frac{\partial W}{\partial \vec{q}_i} \\ \rho_l &= \frac{\partial W}{\partial \pi_l} \\ \xi &= \frac{\partial W}{\partial \Xi} \\ \vec{X} &= \frac{\partial W}{\partial \vec{P}} \end{aligned} \tag{17}$$

explicitly:

$$\rho_i = \sqrt{(q_j^1 - q_k^1)^2 + (q_j^2 - q_k^2)^2}$$

$$
\begin{aligned}
\xi &= 1/3 {\sum_{i,j,k}}' \arccos \frac{q_j^1 - q_k^1}{\rho_i} \\
X_1 &= \frac{\sum_{i=1}^3 m_i q_i^1}{M} \\
X_2 &= \frac{\sum_{i=1}^3 m_i q_i^2}{M} \\
p_i^1 &= \pi_j \cos\phi_j - \pi_k \cos\phi_k - 1/3\Xi \left( \frac{\sin\phi_j}{\rho_j} - \frac{\sin\phi_k}{\rho_k} \right) + (m_i/M) P_1 \\
p_i^2 &= \pi_j \sin\phi_j - \pi_k \sin\phi_k + 1/3\Xi \left( \frac{\cos\phi_j}{\rho_j} - \frac{\cos\phi_k}{\rho_k} \right) + (m_i/M) P_2
\end{aligned}
\tag{18}
$$

Computing $J$ in the new coordinate one finds:

$$
J \equiv \Xi
$$

and the hamiltonian takes the following form: (the sum is meant in cyclic sense ('))

$$
\begin{aligned}
H &= 1/2 {\sum_{i,j,k}}' 1/m_i \left[ \pi_j^2 + \pi_k^2 - 2\pi_j \pi_k \cos(\phi_j - \phi_k) \right] + \\
&+ \Xi^2/9 \left[ 1/\rho_j^2 + 1/\rho_k^2 - \frac{2\cos(\phi_j - \phi_k)}{\rho_j \rho_k} \right] + \\
&+ 2\Xi/3 \left[ \pi_k/\rho_j - \pi_j/\rho_k \right] \sin(\phi_j - \phi_k) + (m_i/M)^2 (P_1^2 + P_2^2) + \\
&+ 2m_i/M [P_1 (m_j \rho_i \cos\phi_i - m_i \rho_j \cos\phi_j) + + P_2 (m_j \rho_i \sin\phi_i - m_i \rho_j \sin\phi_j)] + \\
&+ 2m_i \Xi/3M \left[ P_2 \left( \frac{\cos\phi_j}{\rho_j} - \frac{\cos\phi_k}{\rho_k} \right) - P_1 \left( \frac{\sin\phi_j}{\rho_j} - \frac{\sin\phi_k}{\rho_k} \right) \right] - {\sum_{ijk}}' \frac{m_i m_j}{\rho_k}
\end{aligned}
\tag{19}
$$

Using the definitions of $\phi_i$ and $\theta_i$ and putting

$$
\vec{\rho} = \{\rho_i\}_{i=1}^3 \qquad \vec{\pi} = \{\pi_i\}_{i=1}^3
$$

it is easy to see that the hamiltonian (19) has the form:

$$
\begin{aligned}
H(\vec{\rho}, \vec{\pi}, \xi, \Xi, \vec{X}, \vec{P}) &= H(\vec{\rho}, \vec{\pi}, \Xi, \vec{P}) = \\
h_0(\vec{\rho}, \vec{\pi}) + h_1(\vec{\rho}) + \Xi^2 h_2(\vec{\rho}) &+ \Xi h_3(\vec{\rho}, \vec{\pi}) + \\
+\vec{P} \cdot \vec{h}_4(\vec{\rho}) &+ \Xi \vec{P} \cdot \vec{h}_5(\vec{\rho}, \vec{\pi}) + \left( \sum_{i=1}^3 (m_i/M)^2 \right) \|\vec{P}\|^2
\end{aligned}
\tag{20}
$$

where $h_k$ are suitable functions. This implies the following equation of motion:

$$\begin{aligned}
\dot{\vec{\rho}} = \frac{\partial H}{\partial \vec{\pi}} &= \frac{\partial h_0}{\partial \vec{\pi}} + \Xi \frac{\partial h_3}{\partial \vec{\pi}} \\
\dot{\xi} = \frac{\partial H}{\partial \Xi} &= 2\Xi h_2 + h_3 + \vec{h}_5 \cdot \vec{P} \\
\dot{\vec{X}} = \frac{\partial H}{\partial \vec{P}} &= \vec{h}_4 + \Xi \vec{h}_5 + 2a\vec{P} \\
\dot{\vec{\pi}} = -\frac{\partial H}{\partial \vec{\rho}} &= -\frac{\partial h_0}{\partial \vec{\rho}} - \frac{\partial h_1}{\partial \vec{\rho}} - \Xi \frac{\partial h_3}{\partial \vec{\rho}} - \vec{P} \cdot \frac{\partial \vec{h}_4}{\partial \vec{\rho}} - \Xi \vec{P} \cdot \frac{\partial \vec{h}_5}{\partial \vec{\rho}} \\
\dot{\Xi} = -\frac{\partial H}{\partial \xi} &= 0 \\
\dot{\vec{P}} = -\frac{\partial H}{\partial \vec{X}} &= 0
\end{aligned} \tag{21}$$

with $a = \sum_{i=1}^{3} (m_i/M)^2$. We can define the lagrangian function by the Legendre transformation:

$$L(\vec{\rho}, \dot{\vec{\rho}}, \xi, \dot{\xi}, \vec{X}, \dot{\vec{X}}) = \dot{\vec{\rho}} \cdot \vec{\pi} + \dot{\xi}\Xi + \dot{\vec{X}} \cdot \vec{P} - H(\vec{\rho}, \vec{\pi}, \Xi, \vec{P}) \tag{22}$$

here $\vec{\pi}$, $\Xi$, $\vec{P}$ are functions of $\vec{\rho}$, $\dot{\vec{\rho}}$, $\dot{\xi}$ and $\dot{\vec{X}}$. Since we are interested in the study of the motion of the system on the manifold where

$$\vec{P} = 0 \qquad J = \Xi = 0$$

the Lagrangian becomes

$$L(\vec{\rho}, \dot{\vec{\rho}}) = \dot{\vec{\rho}} \cdot \vec{\pi}(\vec{\rho}, \dot{\vec{\rho}}) - H(\vec{\rho}, \vec{\pi}, 0, 0) \tag{23}$$

and then the equations of motion turn out to be:

$$\begin{aligned}
\dot{\vec{\rho}} = \frac{\partial H}{\partial \vec{\pi}} &= \frac{\partial h_0}{\partial \vec{\pi}} \\
\dot{\vec{\pi}} = -\frac{\partial H}{\partial \vec{\rho}} &= -\frac{\partial h_0}{\partial \vec{\rho}} - \frac{\partial h_1}{\partial \vec{\rho}}
\end{aligned} \tag{24}$$

The term $h_0$ of (20) can be written as:

$$h_0(\vec{\rho}, \vec{\pi}) = \sum_{j,k}^{3} A_{jk}(\vec{\rho}) \pi_j \pi_k \tag{25}$$

where $A_{jk}$ is the following matrix, depending on $\vec{\rho}$:

$$A \doteq \begin{pmatrix} 1/2(1/m_2 + 1/m_3) & -\cos\theta_3/2m_3 & -\cos\theta_2/2m_2 \\ -\cos\theta_3/2m_3 & 1/2(1/m_1 + 1/m_3) & -\cos\theta_1/2m_1 \\ -\cos\theta_2/2m_2 & -\cos\theta_1/2m_1 & 1/2(1/m_1 + 1/m_2) \end{pmatrix}$$

where:

$$\cos\theta_i = \frac{\rho_i^2 - \rho_j^2 - \rho_k^2}{2\rho_j\rho_k}$$

with

$$\rho_i + \rho_j \geq \rho_k \qquad \textit{and cyclic permutations}$$

We will show that $A$ is invertible. Denoting with $B(\vec{\rho})$ the inverse of $A$, the Lagragian takes both on $\Gamma_+$ and on $\Gamma_-$ the form:

$$L(\vec{\rho}, \dot{\vec{\rho}}) = \sum_{ij}^{3} B_{ij}(\vec{\rho})\dot{\rho}_i\dot{\rho}_j + \sum_{i,j,k}{}' \frac{m_i m_j}{\rho_k} \tag{26}$$

We have reduced the number of degrees of freedom from 6 to 3. In section 4 we will describe the global structure of the reduced configuration space, and after that we construct, with this Lagrangian the Action functional.

# 3 Properties of the matrix $B$

**Proposition 3.1**

*The matrix $B(\vec{\rho})$ is symmetric and such that there exist $a_1, a_2 \in \mathbb{R}^+$ such that for all vectors $\vec{V} \in \mathbb{R}^3$:*

$$a_1\|\vec{V}\|^2 \leq (\vec{V}, B(\vec{\rho})\vec{V}) \leq a_2\|\vec{V}\|^2 \tag{27}$$

*where with $(.,.)$ we denote the natural scalar product in $\mathbb{R}^3$.*

**Proof**

The matrix $B(\vec{\rho})$ is the inverse of matrix $A(\vec{\rho})$.It is therefore sufficient to prove (27) for $A(\vec{\rho})$. The Sylvester Criterion says that the matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

is positive definite if:

(i) $a_{11} > 0$

(ii)

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} > 0$$

(iii) det $(A) > 0$.

In our case (i) trivially holds, and (ii) reads:

$$\frac{1}{4m_1}\left(\frac{1}{m_2}+\frac{1}{m_3}\right)+\frac{1}{4m_2m_3}+\frac{1}{4m_3^2}\left(1-\cos^2\theta_3\right)>0$$

$$\forall \;\; \theta_i \in S^1$$

condition (iii) is

$$\begin{aligned} \det A(\vec{\rho}) &= \frac{\sin^2\theta_2}{8m_2^2}\left(\frac{1}{m_1}+\frac{1}{m_3}\right)+\frac{\sin^2\theta_3}{8m_3^2}\left(\frac{1}{m_1}+\frac{1}{m_2}\right)+ \\ &\quad +\frac{\sin^2\theta_1}{8m_1^2}\left(\frac{1}{m_2}+\frac{1}{m_3}\right)+\frac{1}{4m_1m_2m_3}(1-\cos\theta_1\cos\theta_2\cos\theta_3) \end{aligned} \tag{28}$$

this is strictly positive for all $\theta_i \in S^1$.

Let $\lambda_+(\vec{\rho})$ respectively $\lambda_-(\vec{\rho})$ be the smallest (resp. largest) eigenvalue of $A$. For any $\vec{\rho}$ one has:

$$\frac{1}{\lambda_+(\vec{\rho})}>0 \quad , \quad \lambda_-(\vec{\rho})>0$$

both are continuous functions, and $A$ depends only on the angles $\theta_i$; i.e. is a function on $S^1 \times S^1 \times S^1$. By compactness:

$$\lambda_- = \min\left\{\lambda_-(\vec{\rho})\right\} > 0$$

$$\lambda_+{}^{-1} = \min\left\{\lambda_+^{-1}(\vec{\rho})\right\} > 0$$

Therefore $a_1\|\vec{V}\|^2 \leq (\vec{V}, B(\vec{\rho})\vec{V}) \leq a_2\|\vec{V}\|^2$ for any $\vec{V} \in \mathbb{R}^3$.

This proves the Proposition.

# 4 Geometry of the reduced configuration space

The group $O(2)$ acts on the configuration space $Q$ and its action is free if one excludes the origin of $Q$ ($\vec{q}=0$). We denote reflections with $S_2$. $O(2)$ is a symmetry for the system since it leaves the Lagrangian invariant.

In the reduction of the angular momentum, we considered the reduction of the $SO(2)$ action only. We will show that the reduced configuration space has therefore a non-trivial symmetry, which now we denote with $\sigma$, which takes an orbit of $SO(2)$ to the one obtained by reflection about an axis through the origin. Since reflections thorough two different axis are conjugate by a rotation, $\sigma$ does not depend on the axis chosen. The coordinates $\rho$'s are invariant under $\sigma$, the reduced configuration space is then a double covering of $C$ with boundary the collinear configurations (i.e. the fixed point of $\sigma$). In this section we describe the geometry of the reduced configuration space; then in the next section we will use the reduction to construct the space of the trajectories on which we define the Action functional.

The transformation $F$ is:

$$\begin{aligned} F : Q &\longrightarrow C \times S^1 \times \mathbb{R}^2 \\ (\vec{q}_i \quad i = 1, 2, 3) &\longrightarrow (\rho_1, \rho_2, \rho_3, \xi, X_1, X_2) \end{aligned} \tag{29}$$

where we recall:

$$C \doteq \left\{ \vec{\rho} = (\rho_1 \ \rho_2 \ \rho_3) \in \mathbb{R}^3_+ \quad / \quad \rho_i + \rho_j \geq \rho_k \ \ \textit{and cyclic permutations} \right\}$$

and

$$\partial C \doteq \left\{ \vec{\rho} = (\rho_1 \ \rho_2 \ \rho_3) \in \mathbb{R}^3_+ \quad / \quad \rho_i + \rho_j = \rho_k \ \ \textit{and cyclic permutations} \right\}$$

$C$ is locally defined by (8), (9), (12), (13). If we fix $\vec{X} = 0$ we reduce:

$$\tilde{Q} \doteq \left\{ \vec{q} = (\vec{q}_1 \ \vec{q}_2 \ \vec{q}_3) \in \ Q \ \ / \ \ \sum_{i=0}^{3} m_i \vec{q}_i = 0 \right\}$$

$\tilde{Q}$ is a four-dimensional plane in $\mathbb{R}^6$, so $\tilde{Q} \simeq \mathbb{R}^4$.

Hence we can define $F$ on $\tilde{C}$ and we denote it with $\tilde{F}$:

$$\begin{aligned} \tilde{F} : \tilde{Q} &\longrightarrow C \times S^1 \\ \vec{q} &\longrightarrow (\rho_1, \rho_2, \rho_3, \xi) \end{aligned} \tag{30}$$

$\tilde{F}$ is given by

$$\begin{aligned} \rho_i &= \sqrt{(q_j^1 - q_k^1)^2 + (q_j^2 - q_k^2)^2} \\ \xi &= 1/3 \sum_{i,j,k} \arccos \frac{q_j^1 - q_k^1}{\rho_i} \end{aligned} \tag{31}$$

with

$$
\begin{aligned}
\frac{\sum_{i=1}^{3} m_i q_i^1}{M} &= 0 \\
\frac{\sum_{i=1}^{3} m_i q_i^2}{M} &= 0
\end{aligned}
\tag{32}
$$

It is useful to recall (15) with the condition $\vec{X} = 0$:

$$
\begin{cases} q_k^1 = 1/M\{m_j \rho_i \cos\phi_i - m_i \rho_j \cos\phi_j\} \\ q_k^2 = 1/M\{m_j \rho_i \sin\phi_i - m_i \rho_j \sin\phi_j\} \end{cases}
\tag{33}
$$

Now we see how to transfer the action of $O(2)$ from $\tilde{Q}$ to $C \times S^1$. Let us begin with $SO(2) \simeq S^1$. If we denote with :

$$
R : SO(2) \times \tilde{Q} \longrightarrow \tilde{Q}
$$

the action of $SO(2)$ on $\tilde{Q}$, we want to find an action of $SO(2)$ on $C \times S^1$, denoted by $\bar{R}$:

$$
\bar{R} : SO(2) \times C \times S^1 \longrightarrow C \times S^1
$$

such that the following relation holds:

$$
\tilde{F} \circ R = \bar{R} \circ \tilde{F}
$$

We represent $R_\phi \in SO(2)$ with $\phi \in S^1$ as:

$$
R_\phi = \frac{1}{\|\vec{v}\|^2} \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}
$$

on $\tilde{Q}$ we consider the following action:

$$
R_\phi(\vec{q}) \doteq (R_\phi(\vec{q}_1)\ R_\phi(\vec{q}_2)\ R_\phi(\vec{q}_3))
$$

with

$$
\vec{q} = (\vec{q}_1\ \vec{q}_2\ \vec{q}_3) \in \tilde{Q}
$$

since $R_\phi$ acts linearly the condition $\sum_i \vec{q}_i = 0$ is fulfilled.

We define the action of $SO(2)$ on $C \times S^1$ as the map:

$$
\bar{R}_\phi : C \times S^1 \longrightarrow C \times S^1
$$

from the definition of $\vec{\rho}$ we have:

$$\bar{R}_\phi(\rho_i) = \|R_\phi(\vec{q}_j) - R_\phi(\vec{q}_k)\| = \|\vec{q}_j - \vec{q}_k\| = \rho_i$$

for all cyclic combination of $i, j, k$. Moreover since

$$R_\phi \begin{pmatrix} \cos\phi_i \\ \sin\phi_i \end{pmatrix} = \begin{pmatrix} \cos(\phi_i - \phi) \\ \sin(\phi_i - \phi) \end{pmatrix}$$

using (33) we obtain:

$$\bar{R}_\phi(\phi_i) = \phi_i - \phi$$

and remembering the definition of $\xi$:

$$\xi = \frac{1}{3}\sum_{i=1}^{3}\phi_i$$

we get:

$$\bar{R}_\phi(\xi) = \xi - \phi$$

therefore:

$$\begin{aligned} \bar{R}_\phi : C \times S^1 &\longrightarrow C \times S^1 \\ (\vec{\rho}, \xi) &\longrightarrow (\vec{\rho}, \xi - \phi) \end{aligned} \tag{34}$$

Now we have to consider the action of reflections $S_2$. We give the action $S$ on $\tilde{Q}$ and then we want to find the action $\bar{S}$ on $C \times S^1$ such that:

$$\tilde{F} \circ S = \bar{S} \circ \tilde{F}$$

Let $\vec{u}$, $\vec{v}$ be two vectors in $\mathbb{R}^2$, then the vector $S_{\vec{v}}(\vec{u})$ is the reflection of $\vec{u}$ by $\vec{v}$ and is defined by

$$S_{\vec{v}} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$S_{\vec{v}}(\vec{u}) = 2\frac{(\vec{u}, \vec{v})}{\|\vec{v}\|}\vec{v} - \vec{u} \tag{35}$$

if $\vec{v} = (v_1\ v_2)$ then $S$ has a matrix expression which is

$$S_{\vec{v}} = \frac{1}{\|\vec{v}\|^2}\begin{pmatrix} v_1^2 - v_2^2 & 2v_1v_2 \\ 2v_1v_2 & v_2^2 - v_1^2 \end{pmatrix}$$

where we denote with $(.,.)$ the natural scalar product in $\mathbb{R}^2$.

It is easy to see that

$$\det S_{\vec{v}} = -1$$

and

$$S_{\vec{v}} \cdot S_{\vec{v}} = id$$

Define:

$$R_{\vec{u},\vec{v}} \doteq S_{\vec{u}} \cdot S_{\vec{v}}$$

then $R_{\vec{u},\vec{v}} \in SO(2)$. Equivalently we can say that one can generate all $S_2$ by a reflection and $SO(2)$. This fact will be very useful in what follows.

The explicit expression for $R_{\vec{u},\vec{v}}$:

$$R_{\vec{u},\vec{v}} = \frac{1}{\|\vec{v}\|^2\|\vec{u}\|^2} \begin{pmatrix} (\vec{u},\vec{v})^2 - (\vec{u} \wedge \vec{v})^2 & (\vec{u},\vec{v})^2(\vec{u} \wedge \vec{v})^2 \\ -(\vec{u},\vec{v})^2(\vec{u} \wedge \vec{v})^2 & (\vec{u},\vec{v})^2 - (\vec{u} \wedge \vec{v})^2 \end{pmatrix}$$

If we denote with $\theta$ the angle between $\vec{u}$ and $\vec{v}$ we find:

$$R_{\vec{u},\vec{v}} = R_{2\theta} = \frac{1}{\|\vec{v}\|^2} \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$$

and if we take $\vec{v} = (\|\vec{v}\| \cos\alpha \ \|\vec{v}\| \sin\alpha)$ then

$$S_{\vec{v}} = \begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$$

On $\tilde{Q}$ we consider the following action:

$$S_{\vec{v}}(\vec{q}) \doteq (S_{\vec{v}}(\vec{q}_1) \ S_{\vec{v}}(\vec{q}_2) \ S_{\vec{v}}(\vec{q}_3))$$

with

$$\vec{q} = (\vec{q}_1 \ \vec{q}_2 \ \vec{q}_3) \in \tilde{Q}$$

and $\vec{v}$ a direction in $\tilde{Q}$. Since $S_{\vec{v}}$ acts linearly the condition $\sum_i \vec{q}_i = 0$ is fulfilled.

We can now define the action of $S_2$ on $C \times S^1$

$$\bar{S}_\alpha : C \times S^1 \longrightarrow C \times S^1$$

from the definition of $\vec{\rho}$ we have:

$$\bar{S}_\alpha(\rho_i) = \|S_{\vec{v}}(\vec{q}_j) - S_{\vec{v}}(\vec{q}_k)\| = \|\vec{q}_j - \vec{q}_k\| = \rho_i$$

as we have done in the case of $SO(2)$ using the fact that:

$$S_{\vec{v}}\left(\begin{array}{c}\cos\phi_i\\ \sin\phi_i\end{array}\right)=\left(\begin{array}{c}\cos(2\alpha-\phi_i)\\ \sin(2\alpha-\phi_i)\end{array}\right)$$

using (33) we obtain:

$$\bar{S}_\alpha(\phi_i)=2\alpha-\phi_i$$

and remembering the definition of $\xi$:

$$\xi=\frac{1}{3}\sum_{i=1}^{3}\phi_i$$

we get:

$$\bar{S}_\alpha(\xi)=2\alpha-\xi$$

therefore:

$$\begin{array}{rcl}\bar{S}_\alpha : C\times S^1 & \longrightarrow & C\times S^1\\ (\vec{\rho},\xi) & \longrightarrow & (\vec{\rho},2\alpha-\xi)\end{array} \tag{36}$$

The reduced configuration manifold is:

$$C\times S^1/SO(2)$$

Since $SO(2)\simeq S^1$ acts trivially on $C$ we have:

$$C\times S^1/SO(2)\simeq C\times\{0\}\doteq\tilde{C}$$

where $\{0\}$ is a single point (an origin) in $S^1$.

The reduction from $\tilde{Q}$ to $\tilde{C}$ can be described by the following commuting diagram:

$$\begin{array}{ccc}\tilde{Q} & \stackrel{\tilde{F}}{\rightarrow} & C\times S^1\\ & \stackrel{p}{\searrow} & \stackrel{\pi}{\downarrow}\\ & & \tilde{C}\end{array}$$

$\pi$ is the projection on the quotient, and $p$ is defined by:

$$p\doteq\pi\circ\tilde{F}$$

**Proposition 4.1**

*The map $p$ is a double covering. Points $\tilde{Q}$ related by reflection through a generic direction in $\mathbb{R}^2$ have the same image under $p$. The branching set is the set of the collinear configurations.*

**Proof**

In $\tilde{Q}$, $p$ is locally defined by

$$\begin{aligned} \rho_i &= \sqrt{(q_j^1 - q_k^1)^2 + (q_j^2 - q_k^2)^2} \quad \textit{cyclic } i\ j\ k = 1\ 2\ 3 \\ \xi &= 0 \end{aligned} \tag{37}$$

with

$$\frac{\sum_{i=1}^3 m_i q_i^1}{M} = 0 \qquad \frac{\sum_{i=1}^3 m_i q_i^2}{M} = 0$$

The rank of this map is 3 for all configurations with $\vec{q}_i \neq \vec{q}_j \quad i \neq j$ and for all configurations that are not collinear. Collinear configurations will play a central role, they are going to be the ramification point of the map $p$. Outside them is locally a smooth surjection.

For a generic configuration $\vec{q} \in \tilde{Q}$ we represent the action of $O(2)$ as:

$$S_{\vec{v}} \cdot R_\phi(\vec{q}) = (S_{\vec{v}} \cdot R_\phi(\vec{q}_1), S_{\vec{v}} \cdot R_\phi(\vec{q}_2), S_{\vec{v}} \cdot R_\phi(\vec{q}_3))$$

The definitions of rotoreflection depends on the order of the operations. We have to observe that, reflections are conjugate by rotations:

$$R_\phi \cdot S_{\vec{v}} = S_{\vec{v}} \cdot R_{\phi_1}$$

such that the quotient by $S^1$ is independent on the order of the operations.

Now by means of the previous definitions of the action of $O(2)$ on $C \times S^1$ we see that: taken $\vec{q} \in \tilde{Q}$

$$\tilde{F}(S_{\vec{v}} \cdot R_\phi(\vec{q})) = (\bar{S}_\alpha \cdot \bar{R}_\phi)(\tilde{F}(\vec{q})) = (\bar{S}_\alpha \cdot \bar{R}_\phi)(\vec{\rho}, \xi) = (\vec{\rho}, 2\alpha + \phi - \xi)$$

but

$$\pi(\vec{\rho}, \xi) = \{\vec{\rho}\} \times \{0\} \quad \forall \ (\vec{\rho}, \xi) \in C \times S^1$$

hence

$$(\pi \circ \tilde{F})(S_{\vec{v}} \cdot R_{\phi}(\vec{q})) = (\pi \circ \tilde{F})(R_{\phi}(\vec{q})) = (\pi \circ \tilde{F})(\vec{q}) = \{\vec{\rho}\} \times \{0\}$$

for all $\vec{q} \in \tilde{Q}, \quad \vec{v} \in \mathbb{R}^2 \quad \phi \in S^1$.

¿From this and the regularity property of $p$ follows that: given $c^* \in \tilde{C}$, $c^* = \vec{\rho}^* \times \{0\}$ there is $\vec{q} \in \tilde{Q}$ such that:

$$\rho_i^* = \|\vec{q}_j - \vec{q}_k\| \quad i, j, k \ \ cyclic$$

$\vec{q}$ is determined up to a rotation and a rotoreflection; then we have:

$$p^{-1}(c^*) = \bigcup_{\phi \in S^1} \{R_{\phi}(\vec{q})\} \cup \bigcup_{\phi \in S^1} \{S_{\vec{v}} \cdot R_{\phi}(\vec{q})\} \tag{38}$$

with $\vec{v} \in \mathbb{R}^2$ fixed and where the second union is disjoint. We can choose this vector in arbitrary manner since the two following sets are equal:

$$A_{\vec{v}_1} = \bigcup_{\phi \in S^1} \{S_{\vec{v}_1} \cdot R_{\phi}(\vec{q})\}$$

$$A_{\vec{v}_2} = \bigcup_{\phi \in S^1} \{S_{\vec{v}_2} \cdot R_{\phi}(\vec{q})\}$$

where $\vec{q} \in \tilde{Q}$ fixed, and $\vec{v}_1, \ \vec{v}_2 \in \mathbb{R}^2$ are generically chosen. In fact: we can always take $\phi_1 \in S^1$ such that

$$S_{\vec{v}_1} \cdot R_{\phi}(\vec{q}) = S_{\vec{v}_1} \cdot R_{-\phi_1} \cdot R_{\phi_1} \cdot R_{\phi}(\vec{q}) = S_{\vec{v}_2} \cdot R_{\phi_1 + \phi}(\vec{q})$$

the first term belongs to $A_{\vec{v}_1}$ and the last belongs to $A_{\vec{v}_2}$ since $\phi + \phi_1$ is always reached in the union so the two sets are equal.

The two sets

$$\bigcup_{\phi \in S^1} \{R_{\phi}(\vec{q})\}, \quad \bigcup_{\phi \in S^1} \{S_{\vec{v}} \cdot R_{\phi}(\vec{q})\}$$

are coincident only when the configuration $\vec{q}$ is collinear.

In the collinear case there exists $\phi \in S^1$ such that:

$$R_{\phi}(\vec{q}) = \lambda \vec{v} \quad \lambda > 0$$

and then

$$S_{\vec{v}}(R_{\phi}(\vec{q})) = \lambda \vec{v}$$

therefore the two sets are coincident; the collinear configurations form the branching point of the map $p$.

Always using the regularity property of $p$ for each neighborhood $U_c$ of $c \in \tilde{C}$ there exists $V_{\vec{q}}$ in $\tilde{Q}$ such that

$$p(\vec{q}) = c$$

$$p\left(\bigcup_{\phi \in S^1} \{R_\phi(V_{\vec{q}})\}\right) = U_c$$

and

$$p\left(\bigcup_{\phi \in S^1} \{S_{\vec{v}} \cdot R_\phi(V_{\vec{q}})\}\right) = U_c$$

and then

$$p^{-1}(U_c) = \bigcup_{\phi \in S^1} \{R_\phi(V_{\vec{q}})\} \cup \bigcup_{\phi \in S^1} \{S_{\vec{v}} \circ R_\phi(V_{\vec{q}})\}$$

therefore $p$ is a double covering @.

# 5 Reduced Action functional

In this section we construct the new Action functional. We have shown that $\tilde{Q}$ is a double copy of $\tilde{C}$ glued on their boundaries. We will work on trajectories which do not lie on the collinear set. In order to distinguish $\Gamma_+$ from $\Gamma_-$ we introduce a parameter $z$ which takes value in $\{-1, 0, +1\}$ and consider the map:

$$\begin{array}{ccc} \tilde{Q} & \longrightarrow & \tilde{C} \times \{+1\} \cup \partial\tilde{C} \times \{0\} \cup \tilde{C} \times \{-1\} \doteq \bar{C} \\ (\vec{q}_i \quad i = 1,2,3) & \longrightarrow & (\rho_1, \rho_2, \rho_3, z) \doteq \zeta \end{array} \tag{39}$$

The parameter $z$ which takes value 0 on $\partial\tilde{C}$ and $\pm 1$ on the two sheets. The operation $\sigma$ which is the image of the action of $S_2$ in $\tilde{C}$ acts as follows:

$$\sigma(\vec{\rho}, z) = (\vec{\rho}, -z)$$

and we call it "reflection".

It is easy to see that with these definitions collinear configurations $(\vec{\rho}, 0)$ are fixed points of $\sigma$.

We now define:

$$\tilde{\Lambda}_T^a = \{\zeta(t) = (\vec{\rho}(t), z(t)) \ \ / \ \ \zeta(t+T/2) = \sigma\zeta(t)\} \tag{40}$$

$$\tilde{\Lambda}_T^s = \{\zeta(t) = (\vec{\rho}(t), z(t)) \ \ / \ \ \zeta(t+T/2) = \zeta(t)\} \tag{41}$$

so the space of $T$-periodic, $H^1$-functions with values in $\bar{C}$

$$H^1_{\bar{C}} \doteq H^1([0,T], \bar{C}) \doteq \Big\{(\vec{\rho}, z) \in \bar{Q} \ \ / \ \ \rho_i \in H^1([0,T], \mathbb{R}_+) \ \ i = 1,2,3\Big\}$$

can be decomposed into

$$H^1_{\bar{C}} = H^1_{\bar{C}} \cap \tilde{\Lambda}_T^s \oplus H^1_{\bar{C}} \cap \tilde{\Lambda}_T^a \tag{42}$$

We can now define the Action functional for trajectories in $\tilde{Q}$. Let us recal the reduced Lagrangian:

$$L(\vec{\rho}, z) = \sum_{ij}^{3} B_{ij}(\vec{\rho})\dot{\rho}_i\dot{\rho}_j + \sum_{i,j,k} \frac{m_i m_j}{\rho_k}$$

Let $\zeta(t) = (\vec{\rho}(t), z(t))$ be a regular trajectory. We have noticed that either $z(t) = 0 \ \ \forall t$ or there are $0 < t_1 < .... < t_n < T$ such that $z(t_i) = 0\ ,\ z(t) \neq 0 \ \ with \ \ t \neq t_i \quad \lim_{\epsilon\to 0} z(t_i+\epsilon)z(t_i-\epsilon) = -1$. If $z(t) = 0 \ \ \forall t$ we define the Action functional as:

$$A_T\left[\zeta(t)\right] = \int_0^T dt L(\vec{\rho}, \dot{\vec{\rho}}, 0)$$

where $\rho_i + \rho_j = \rho_k$ for some $i, j, k$.

If $z(t) \neq 0$, we define the Action functional to be the sum of the Action functional restricted to the trajectories in the open interval $(t_i, t_{i+1})$. One has:

$$A_T\left[\zeta(t)\right] = \sum_{i=0}^{N} \int_{t_i}^{t_{i+1}} dt L(\vec{\rho}, \dot{\vec{\rho}}, z_i)$$

where $z_i = z(t) \ \ with \ \ t \in (t_i, t_{i+1})$.

Since $L$ does not depend on $z$ one has finally:

$$A_T\left[\vec{\rho}(t)\right] = \int_0^T dt \left\{ \sum_{ij}^{3} B_{ij}(\vec{\rho})\dot{\rho}_i\dot{\rho}_j + {\sum_{i,j,k}}' \frac{m_i m_j}{\rho_k} \right\} \tag{43}$$

defined in $H^1_{\bar{C}}$.

In the next section we will consider collision trajectories. For collision trajectories, let $\{\tau_k\}_k$ be a sequence of collision times. Between two collision times the trajectory is regular and so we can use the previous definition, hence:

$$A_T\left[\zeta(t)\right] = \lim_{\epsilon\to 0}\sum_k \int_{\tau_{k+1}-\epsilon}^{\tau_k+\epsilon} dt L(\vec{\rho},\dot{\vec{\rho}})$$

The behaviour at collision garantees that the limit exists. We have to observe that the sum over the collision times might diverge when the collision times form an infinite sequence. We shall therefore use the (43) also for collision solutions. In conclusion (43) describes the Action functional for trajectories in $H^1_{\bar{C}}$.

# 6 Collision orbits

In this section we prove that collision solutions are not minima for the Action functional. The proof is based on a local analysis near the collision orbit and in particular on the asymptotic behaviour of the orbits at the collision. We know that collision orbit are asymptotic to Central Configurations [24], [23]. Let us consider triple-collision, assume $t = 0$ to be the collision time. In the space $Q$ we have the following representation of the collision orbit:

$$\vec{q}_i^{\,c}(t) = (\vec{c}_i + \vec{\gamma}_i(t))t^{2/3}$$

with $\|\vec{\gamma}_i(t)\| = O(t^d) \quad 0 < d < 1$, $\{\vec{c}_i\}$ is a Central Configuration.

In the space $\bar{Q}$ we have:

$$\rho_i^c(t) = (c_i + \gamma_i(t))t^{2/3}$$

where $\gamma_i$ and $c_i$ are functions of $\vec{\gamma}_i$ and $\vec{c}_i$, with $\gamma_i(t) = O(t^d) \quad 0 < d < 1$. In the next Proposition we will use the definition of the Action functional given above. In the Three-body there are Equilateral Central Configurations and Collinear Central Configurations: in the first case there can be only a finite number of times at which the trajectory is in a collinear configuration, while in the second case there may be an infinite denumerable sequence of such times converging to the collision time. As we have seen in both cases we can use the Action functional previous defined.

**Proposition 6.1**

*Collision solutions are not minima for the Action functional (43)* $A_T[.]$.

**Proof**

Remark: In the proof we consider only triple-collisions but it is easy to see that the proof works for double-collisions as well.

Let

$$\vec{\rho}_c(t) = \{\rho_i^c(t)\}_{i=1}^3 \qquad \vec{g} = \{w_i g(t)\}_{i=1}^3$$

where for all $i = 1, 2, 3$ $w_i \in \mathbb{R}^+$ with:

$$w_i + w_j \geq w_k \qquad \textit{cyclic permutations}$$

and

$$g(t) = \begin{cases} 1 & |t| \leq \epsilon_0 \\ 0 & |t| \geq 2\epsilon_0 \\ \frac{2\epsilon_0 - |t|}{\epsilon_0} & \epsilon_0 \leq |t| \leq 2\epsilon_0 \end{cases}$$

with $\epsilon_0 \in \mathbb{R}^+$ We consider variations:

$$\rho_i^c(t) = (r + \gamma_i(t))t^{2/3} \to (r + \gamma_i(t))t^{2/3} + w_i \epsilon g(t) \tag{44}$$

with $\epsilon \in \mathbb{R}^+$.

Estimating:

$$A_T\left[\vec{\rho}_c(t) + \epsilon\vec{g}\right] - A_T\left[\vec{\rho}_c(t)\right] = \int_0^1 ds \frac{d}{ds} A_T\left[\vec{\rho}_c(t) + s\epsilon\vec{g}\right] \tag{45}$$

this becomes equal to:

$$\int_0^1 ds \int_0^T dt \sum_{jkl} \frac{\partial B_{jk}}{\partial \rho_l} g_l(t)\epsilon(\rho_j^c + s\epsilon g_j(t))(\rho_k^c + s\epsilon g_k(t)) + 2\epsilon\dot{g}_k(\dot{\rho}_j^c + s\epsilon\dot{g}_j(t))B_{jk} - \sum_{jkl}{}' \frac{\epsilon g_k m_j m_l}{(\rho_k^c + s\epsilon g_k)^2}$$

Denoting with $(.,.)$ the natural scalar product in $\mathbb{R}^3$, let us consider the second term due to the kinetic energy:

$$\int_0^1 ds \int_0^T dt 2 \sum_{jk} \epsilon \dot{g}_k(\dot{\rho}_j^c + s\epsilon\dot{g}_j(t))B_{jk} = \int_0^1 ds \int_0^T dt 2\epsilon(B\dot{\vec{g}}, \dot{\vec{\rho}}_c + s\epsilon\dot{\vec{g}})$$

¿From Proposition 3.1 one obtains:

$$\left| 2a_2\epsilon \int_0^1 ds \int_0^T dt 2\epsilon(B\dot{\vec{g}}, \dot{\vec{\rho}}_c + s\epsilon\dot{\vec{g}}) \right| \leq C_1(\epsilon/\epsilon_0) + C_2(\epsilon/\epsilon_0)^2$$

where the second inequality and the constants $C_1$ , $C_2 > 0$ are determined by the explicit form of $\dot{\vec{\rho}}$ and $\dot{\vec{g}}$.

To estimate the first term due to kinetic energy we need again to use the homogeneity properties of the matrix $B$:

Let us fix:

$$x = \frac{\rho_2}{\rho_1} \qquad y = \frac{\rho_3}{\rho_1}$$

then:

$$B_{jk}(\rho_1, \rho_2, \rho_3) = B_{jk}(x, y)$$

hence:

$$\frac{\partial B_{jk}}{\partial \rho_l} = \frac{\partial B_{jk}}{\partial x}\frac{\partial x}{\partial \rho_l} + \frac{\partial B_{jk}}{\partial y}\frac{\partial y}{\partial \rho_l}$$

Computing explicitly the derivatives $\frac{\partial x}{\partial \rho_l}$ and $\frac{\partial y}{\partial \rho_l}$ we have that:

$$\frac{\partial B_{jk}}{\partial \rho_l}\epsilon g_l = O(\epsilon)$$

for example:

$$\frac{\partial B_{jk}}{\partial \rho_1} = \frac{\partial B_{jk}}{\partial x}\frac{\partial x}{\partial \rho_1}g_1\epsilon + \frac{\partial B_{jk}}{\partial y}\frac{\partial y}{\partial \rho_1}g_1\epsilon$$

which is equal to:

$$-\frac{\partial B_{jk}}{\partial x} \cdot \frac{(c_2+\gamma_2)t^{2/3} + s\epsilon g_2}{((c_1+\gamma_1)t^{2/3} + s\epsilon g_1)^2}\epsilon g_1 - \frac{\partial B_{jk}}{\partial y} \cdot \frac{(c_3+\gamma_3)t^{2/3} + s\epsilon g_3}{((c_1+\gamma_1)t^{2/3} + s\epsilon g_1)^2}\epsilon g_1 = O(\epsilon)$$

Hence there exist $C_3$ , $C_4$ , $C_5 > 0$ such that:

$$\left| \int_0^1 ds \int_0^T dt \sum_{jkl} \frac{\partial B_{jk}}{\partial \rho_l} g_l(t)\epsilon(\rho_j^c + s\epsilon g_j(t))(\rho_k^c + s\epsilon g_k(t)) \right| \le C_3(\epsilon/\epsilon_0) + C_4(\epsilon/\epsilon_0)^2 + C_4(\epsilon/\epsilon_0)^3$$

We now evaluate the terms due to the potential. They wil turn out to be the leading terms in $\epsilon$.

Each of such terms has the form:

$$I_i = \int_0^1 ds \int_0^T dt \frac{\epsilon g_i(t)}{((c_i+\gamma_2)t^{2/3} + s\epsilon g_i(t))^2} = I_i^a + I_i^b$$

$$I_i^a = \int_0^1 ds \int_0^{\epsilon_0} dt \frac{\epsilon}{((c_i+\gamma_2)t^{2/3} + s\epsilon g_i(t))^2}$$

$$I_i^b = \int_0^1 ds \int_{\epsilon_0}^{2\epsilon_0} dt \frac{2\epsilon/\epsilon_0(2\epsilon_0 - t)}{((c_i + \gamma_2)t^{2/3} + sw_i\epsilon/\epsilon_0(2\epsilon_0 - t))^2}$$

it is easy to prove:

$$I_i^b = O(\epsilon)$$

For $I_i^a$ we need some computations: let

$$\epsilon\tau = t^{2/3}$$

so

$$dt = (3/2)\epsilon^{3/2}\tau^{1/2}d\tau$$

$$I_i^a = \int_0^1 ds \int_0^{\epsilon_0^{2/3}/\epsilon} d\tau \frac{(3/2)\epsilon^{1/2}\tau^{1/2}}{((c_i + \gamma_i(\tau))\tau + sw_i)^2}$$

we define now the following two integrals:

$$I_i^\infty = \int_0^1 ds \int_0^\infty d\tau \frac{(3/2)\epsilon^{1/2}\tau^{1/2}}{((c_i + \gamma_i(\tau))\tau + sw_i)^2}$$

and

$$I_i^0 = \int_0^1 ds \int_0^\infty d\tau \frac{(3/2)\epsilon^{1/2}\tau^{1/2}}{(c_i\tau + w_i s)^2}$$

we are going to show that there exist $C_6$ , $C_7 > 0$ $\alpha > 1/2$ such that:

$$|I_i^a - I_i^\infty| \le C_6\epsilon^\alpha \tag{46}$$

$$|I_i^\infty - I_i^0| \le C_7\epsilon^\alpha \tag{47}$$

thus there exists $C_8 > 0$ such that:

$$|I_i^a - I_i^0| \le C_8\epsilon^\alpha \tag{48}$$

but since we find that:

$$I_i^0 = C_9\epsilon^{1/2} \qquad C_9 > 0 \tag{49}$$

we conclude: there exist $C_{10}$ , $C_{11} > 0$ $\beta > 1/2$ such that:

$$A_T\left[\vec{\rho}_c(t) + \epsilon\vec{g}\right] - A_T\left[\vec{\rho}_c(t)\right] \le C_{10}\epsilon^\beta - C_{11}\epsilon^{1/2}$$

hence for $\epsilon$ small enough we have a negative variation for the Action functional which implies that collision solutions cannot be a minimum. We have to prove (46), (47), (49).

Proof of (46):

using the explicit form of $I_i^a$ ed $I_i^\infty$:

$$|I_i^a - I_i^\infty| = \left| \int_0^1 ds \int_{\epsilon_0^{2/3}/\epsilon}^\infty d\tau \frac{(3/2)\epsilon^{1/2}\tau^{1/2}}{((c_i+\gamma_i(\tau))\tau + w_i s)^2} \right|$$

the integrand goes as $\tau^{-(3/2+2d)}$ with $0<d<1$ so it is integrable and then:

$$|I_i^a - I_i^\infty| \le \epsilon^{1/2} c(\epsilon) C_{13}(\epsilon_0)$$

with $c(\epsilon) = O(\epsilon)$.

Proof of (47):

$$|I_i^\infty - I^0| = \int_0^1 ds \int_0^\infty d\tau (3/2)\epsilon^{1/2}\tau^{1/2} \left[ \frac{2\tau\gamma_i(\tau)(\tau + sw_i) + \tau^2\gamma_i(\tau)^2}{(\tau+sw_i)^2((c_i+\gamma_i(\tau))\tau + sw_i)^2} \right]$$

actually $\gamma_i(\tau) = \gamma_i(\epsilon^{3/2}\tau^{3/2})$ and it is infinitesimal in its argument, so we have:

$$|I_i^\infty - I_i^0| \le \epsilon^{1/2} c_1(\epsilon) C_{14}$$

with $c_1(\epsilon) = O(\epsilon)$ and for $\epsilon$ small enough and $C_{14} > 0$.

Proof of (49):

$$I_i^0 = \int_0^1 ds \int_0^\infty d\tau \frac{(3/2)\epsilon^{1/2}\tau^{1/2}}{(c_i\tau + sw_i)^2}$$

this integral can be explicitly computed, and it is equal to:

$$c_i^{1/2}(3/2)\epsilon^{1/2} \int_0^1 ds \left[ -\frac{\tau^{1/2}}{\tau + sw_i} + (sw_i)^{-1/2} \arctan\sqrt{\frac{\tau}{sw_i}} \right]_0^\infty = O(\epsilon^{1/2})$$

so we have concluded our proof.@

# 7 Study of the Action functional with $J = 0$ $\vec{P} = 0$ fixed

In section 5 we have seen that the reduced Action on $\bar{Q}$ has the form:

$$A_T\left[\vec{\rho}(t), z(t)\right] = \int_0^T dt \left\{ \sum_{ij}^3 B_{ij}(\vec{\rho})\dot{\rho}_i\dot{\rho}_j + \sum_{i,j,k} \frac{m_i m_j}{\rho_k} \right\} \tag{50}$$

Before studying (50) let us recall the procedure one follows to study the unreduced Action functional. In $q$'s coordinates the unreduced Action functional has form:

$$A^*_T[q] = \int_0^T dtL = \int_0^T dt \sum_{i=1}^3 m_i \frac{\|\dot{\vec{q}}_i\|^2}{2} + \sum_{i,j,i\neq j}^3 \frac{m_i m_j}{\|\vec{q}_i - \vec{q}_j\|} \tag{51}$$

This functional is studied on the space

$$\Lambda_T = \left\{ \vec{q}_i \in H^1([0,T];\mathbb{R}^2) \ \ / \ \ \vec{q}_i(t) = \vec{q}_i(t+T); \ \ \int_0^T dt \sum_{i,j,i\neq j}^3 \frac{m_i m_j}{\|\vec{q}_i - \vec{q}_j\|} < \infty \right\} \tag{52}$$

Split $\Lambda_T$ into

$$\Lambda_T = \Lambda_T^s \oplus \Lambda_T^a \tag{53}$$

where

$$\Lambda_T^a = \left\{ \vec{q}_i \in H^1([0,T];\mathbb{R}^2) \ \ / \ \ \vec{q}_i(t) = -\vec{q}_i(t+T/2) \right\} \tag{54}$$

$$\Lambda_T^s = \left\{ \vec{q}_i \in H^1([0,T];\mathbb{R}^2) \ \ / \ \ \vec{q}_i(t) = \vec{q}_i(t+T/2) \right\} \tag{55}$$

Let us restrict $A^*_T$ on $\Lambda^a_T$; on this set the Poincaré Inequality holds:

$$\int_0^T dt \|\dot{\vec{q}}_i\|^2 \geq \frac{1}{T} \int_0^T dt \|\vec{q}_i\|^2$$

so there exists $b > 0$ such that:

$$A^*_T[q] \geq \int_0^T dt \left\{ \sum_{i=1}^3 m_i b \left( \frac{\|\dot{\vec{q}}_i\|^2}{2} + \frac{\|\vec{q}_i\|^2}{2} \right) + \sum_{i,j,i\neq j}^3 \frac{m_i m_j}{\|\vec{q}_i - \vec{q}_j\|} \right\} \geq b\|q\|^2_{H^1}$$

$A^*$ is coercive on $\Lambda^a_T$ and the sub-levels are compact in $L^2([0,T])$ and hence $A^*$ has a minimum $q^*$. The minimum is not a collision solution as is proved in [23]. At the minimum the functional is of class $C^1$. By the symmetry of the Lagrangian for $q^* \in \Lambda^a_T$ :

$$D_s A^*_T(q^*) = 0$$

where $D_s$ is the differential along directions $\Lambda^s_T$. Therefore

$$DA^*_T(q^*) = 0$$

i.e. $q^*$ is a solution of the Euler-Lagrange equations, so it is a $T$-periodic solution without collisions.

In sections 4 and 5 we described the coordinates of $\bar{C}$ and we gave the definitions of $\tilde{\Lambda}_T^a$ and $\tilde{\Lambda}_T^s$. These sets will play a role in $\bar{C}$ similar to (54) and (55). Let us recall the definitions:

$$\tilde{\Lambda}_T^a = \{\zeta(t) = (\vec{\rho}(t), z(t)) \ \ / \ \ \zeta(t+T/2) = \sigma\zeta(t)\}$$

$$\tilde{\Lambda}_T^s = \{\zeta(t) = (\vec{\rho}(t), z(t)) \ \ / \ \ \zeta(t+T/2) = \zeta(t)\}$$

Now since $A_T$ is invariant under $\sigma$, if $p \in \tilde{\Lambda}_T^a$ then:

$$DA_T(p)(\zeta) = 0 \qquad \forall \zeta \in \tilde{\Lambda}_T^s$$

so, $p \in \tilde{\Lambda}_T^a$ is a critical point, if and only if is a critical point of $A_T$ in $\tilde{\Lambda}_T^a$.

**Proposition 7.1**

*Let be $\zeta(t) = (\vec{\rho}(t), z(t)) \in \tilde{\Lambda}_T^a$. The following inequality holds:*

$$\int_0^T dt \sum_{ij}^3 B_{ij}(\vec{\rho})\dot{\rho}_i\dot{\rho}_j = \int_0^T dt (B(\vec{\rho})\dot{\vec{\rho}}, \dot{\vec{\rho}}) \geq \frac{4a_1}{T} \sup_{s\in[0,T]} \min_{i,j,k}{}' \left\{(\rho_i(s) + \rho_j(s) - \rho_k(s))^2\right\} \quad (56)$$

Since $B(\vec{\rho})$ is bounded away from zero $\vec{\rho} \in H^1([0,T], \tilde{C})$ the "sup" of the r.h.s. is attained in $[0,T]$. Without loss of generality we assume that is attained in 0 and we denote the last term in (56) by

$$\frac{4a_1}{T}\Gamma(\vec{\rho}(0))$$

**Proof**

To prove the proposition let us consider the trajectory in the interval $[0,\tau]$ with $\tau < T/2$ the first instant at which the configuration is collinear. Observe that from the definition of $\tilde{\Lambda}_T^a$ if there is such a time $\tau$ then $\tau + T/2$ is another *collinear* time.

¿From Proposition 3.1 we have that there exists $a_1 > 0$ such that

$$K = \int_0^\tau dt (B(\vec{\rho})\dot{\vec{\rho}}, \dot{\vec{\rho}}) \geq a_1 \int_0^\tau dt ||\dot{\vec{\rho}}||^2$$

for each $\rho_i$ by using the Fundamental theorem of the Calculus we can write

$$|\rho_i(\tau) - \rho_i(0)| \leq \int_0^\tau dt |\dot{\rho}_i|$$

then by Schwartz inequality

$$|\rho_i(\tau) - \rho_i(0)| \leq \sqrt{\tau} \left\{\int_0^\tau dt |(\dot{\rho}_i)^2|\right\}^{1/2}$$

so

$$|\rho_i(\tau) - \rho_i(0)|^2 \leq \tau \int_0^\tau dt|(\dot{\rho}_i)^2|$$

and hence

$$K \geq \sum_{i=1}^{3} a_1 \frac{[\rho_i(\tau) - \rho_i(0)]^2}{\tau} \tag{57}$$

Let us now put for simplicity

$$\rho_i(\tau) = x_i \;\; collinear \;\; configuration$$

and

$$\rho_i(0) = y_i \;\; non - degenerate \;\; triangular \;\; configuration$$

with $i = 1, 2, 3$; obviously we have to choose one of the following condition $x_i + x_j = x_k$ cyclic permutation of $i, j, k$, the three possibilities in the min argument depends on this choice. Let us consider the case $x_1 + x_2 = x_3$, then we seek the minimum of the auxiliary function

$$f(x_1, x_2, x_3) = (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_1 + x_2 - y_3)^2$$

we consider $f$ as function of $x_i$. With very elementary computations we find that $f$ has only one minimum whose value is

$$\frac{(y_1 + y_2 - y_3)^2}{3}$$

hence we can say that

$$K \geq a_1 \frac{(y_1 + y_2 - y_3)^2}{3\tau} \tag{58}$$

Now if we take account of the choice $x_1 + x_2 = x_3$ we see that to get the thesis we only should repeat computations with the other three possible choices.

We consider now the case of a $T$ periodic trajectory. We know that there are times $t_1 \in (0, T/2)$ $t_2 \in (3T/4, T)$ at which the configuration is collinear. In this case we can write:

$$\begin{aligned} K &= \int_0^T dt(B(\vec{\rho})\dot{\vec{\rho}}, \dot{\vec{\rho}}) = \int_0^{t_1} dt(B(\vec{\rho})\dot{\vec{\rho}}, \dot{\vec{\rho}}) + \int_{t_1}^{T/2} dt(B(\vec{\rho})\dot{\vec{\rho}}, \dot{\vec{\rho}}) \;\; + \\ &\quad + \int_{T/2}^{t_2} dt(B(\vec{\rho})\dot{\vec{\rho}}, \dot{\vec{\rho}}) + \int_{t_2}^{T} dt(B(\vec{\rho})\dot{\vec{\rho}}, \dot{\vec{\rho}}) \end{aligned} \tag{59}$$

as we have done above, we obtain:

$$K \geq a_1 \sum_{i=1}^{3} \frac{[\rho_i(t_1)-\rho_i(0)]^2}{t_1} + a_1 \sum_{i=1}^{3} \frac{[\rho_i(T/2)-\rho_i(t_1)]^2}{(T/2-t_1)} + \\ + a_1 \sum_{i=1}^{3} \frac{[\rho_i(t_2)-\rho_i(T/2)]^2}{(t_2-T/2)} + a_1 \sum_{i=1}^{3} \frac{[\rho_i(T)-\rho_i(t_2)]^2}{(T-t_2)} \tag{60}$$

by (58) we get:

$$K \geq a_1 \frac{T/2}{3(T/2-t_1)t_1} \Gamma(\vec{\rho}(0)) + a_1 \frac{T/2}{3(T-t_2)(t_2-T/2)} \Gamma(\vec{\rho}(T))$$

so that:

$$K \geq a_1 \left[ \frac{4}{3T/2} + \frac{4}{3T} \right] \Gamma(\vec{\rho}(0))$$

hence

$$K \geq \frac{4a_1}{T} \Gamma(\vec{\rho}(0)) \tag{61}$$

Thus we have proved the thesis **@**.

The r.h.s. of (56) does not bound the $L^2$ norm of $\vec{\rho}$, therefore $A_T$ is not coercive. One has in fact:

**Proposition 7.2**

*For any $k \geq 0$ the set*

$$S_k = \left\{ \zeta \in \tilde{\Lambda}_T^a \cap H_{\tilde{C}}^1 \;\;/\;\; A_T[\zeta(t)] \leq k \right\}$$

*are non compact in the $L^2$-topology.*

**Proof**

For each $k \geq 0$ we exhibit a sequence in the sublevel which does not converge. By the preceeding Proposition we find easily that

$$A_T[\vec{\rho}(t)] \geq \frac{4a_1}{T} \Gamma(\vec{\rho}(0))$$

Consider the following sequences in $n \in \mathbb{N}$, $v_i \in \mathbb{R}$:

$$\rho_i^{(n)}(t) = \rho_i^{(n)}(0) + v_i t \quad t \in [0, T/2] \quad i = 1,2,3$$

$$\rho_i^{(n)}(t) = \rho_i^{(n)}(0) + v_i(T/2 - t) \quad t \in [T/2, T] \quad i = 1,2,3$$

which are in $\tilde{\Lambda}_T^a$

$$\rho_1^{(n)}(0) = n + a \quad \rho_2^{(n)}(0) = n + a \quad \rho_3^{(n)}(0) = 2n + a \quad a > 0$$

those will be collinear at $t_* = \frac{a}{v_3 - v_2 - v_1}$ (we choose $v_i$'s such that $T/2 = t_*$)

$$\rho_3^{(n)}(t_*) = \rho_1^{(n)}(t_*) + \rho_2^{(n)}(t_*)$$

It is easy to see that

$$\frac{4a_1}{T}\Gamma(\vec{\rho}(0)) = \frac{4a_1}{T}a$$

we chose $a$ so that

$$\frac{4a_1}{T}a = k$$

the sequence stays always in the level $A_T = k$.

But with a simple computation we can see that:

$$\|\rho_i^{(n)}\|_{L^2}^2 \geq cn^2$$

for some $c > 0$ so that no convergent sub sequence exists.@

Let us now define the following set:

$$M_c \doteq \left\{ \zeta \in \tilde{\Lambda}_T^a \cap H_{\bar{C}}^1 \;\; such \;\; that : \;\; \sup_{s \in [0,T]} g(\vec{\rho}(s)) \geq c \right\}$$

with $c > 0$ and where:

$$g(\vec{\rho}(s)) \doteq \frac{1}{\|\vec{\rho}(s)\|} \min_{i,j,k}{}' \left\{ (\rho_i(s) + \rho_j(s) - \rho_k(s))^2 \right\}$$

$$\|\vec{\rho}(s)\| = \sqrt{\sum_{i=1}^{3} (\rho_i)^2}$$

We show now that for any $c \; , \; k > 0$

$$M_{c,k} \doteq M_c \cap S_k$$

is compact in $L^2([0,T], \bar{C})$.

**Proposition 7.3**

*Given $k, b \in \mathbb{R}^+$,the set $M_{c,k} = M_c \cap S_k$ is compact in the topology of $L^2([0,T], \bar{C})$.*

**Proof**

Let us define the set:

$$B_{c,k} \doteq \{ \zeta \in M_{c,k} \; / \; \sup_{s \in [0,T]} \|\vec{\rho}(s)\| \leq 3\sqrt{Tk/a_1} \}$$

where $a_1$ is is the lower bound in Proposition 3.1; then:

$$M_{c,k} = B_{c,k} \cup (B_{c,k})^c$$

$B_{c,k}$ is compact in $L^2([0,T],\bar{C})$ since $\int_0^T \|\vec{\rho}\|^2 dt$ is uniformly bounded for all $\zeta(s) \in B_{c,k}$. We must prove that $(B_{c,k})^c$ is compact.

If $\zeta \in (B_{c,k})^c$ then $\sup_s \|\vec{\rho}(s)\| > 3\sqrt{Tk/a_1}$ implies

$$\sup_s \|\vec{\rho}(s)\| - \inf_s \|\vec{\rho}(s)\| \leq \int_0^T \left(\frac{d}{ds}\|\vec{\rho}(s)\|\right) ds \leq \sqrt{T}\left\{\int_0^T \left(\frac{d}{ds}\|\vec{\rho}(s)\|\right)^2 ds\right\}^{1/2}$$

by means of Proposition 3.1 we get

$$\sup_s \|\vec{\rho}(s)\| - \inf_s \|\vec{\rho}(s)\| \leq \sqrt{T/a_1}\left\{\int_0^T (B(\vec{\rho}(s))\dot{\vec{\rho}}(s), \dot{\vec{\rho}}(s))ds\right\}^{1/2} \leq \sqrt{Tk/a_1}$$

so:

$$\inf_s \|\vec{\rho}(s)\| \geq \sup_s \|\vec{\rho}(s)\| - \sqrt{Tk/a_1} = 2\sqrt{Tk/a_1} \tag{62}$$

hence

$$\frac{1}{T}\int_0^T \|\vec{\rho}(s)\|dt \leq \sup_s \|\vec{\rho}(s)\| \leq C_1 \inf_s \|\vec{\rho}(s)\| \tag{63}$$

for $C_1 > 0$ big enough.

We can then use the polar coordinates and by Proposition 7.1 we get:

$$k \geq A_T(\vec{\rho}) \geq \frac{a_1}{2}\int_0^T \left(\frac{d}{dt}\|\vec{\rho}(s)\|\right)^2 dt + \frac{a_1}{2}\int_0^T \|\vec{\rho}(s)\|^2(\dot{\vec{\Sigma}}, \dot{\vec{\Sigma}})dt + \frac{a_1}{2T}\sup_{s\in[0,T]} \|\vec{\rho}(s)\| g(\vec{\rho}(s))$$

where:

$$\vec{\Sigma} \doteq \frac{\vec{\rho}}{\|\vec{\rho}\|}$$

Now we have to notice that:

$$\sup_{s\in[0,T]} \|\vec{\rho}(s)\| g(\vec{\rho}(s)) \geq \inf_{s\in[0,T]} \|\vec{\rho}(s)\| \sup_{s\in[0,T]} g(\vec{\rho}(s))$$

¿From (62) and (63) we obtain:

$$k \geq A_T(\vec{\rho}) \geq \frac{a_1}{2}\int_0^T \left(\frac{d}{dt}\|\vec{\rho}(s)\|\right)^2 dt + \frac{ca_1}{2TC_1}\int_0^T \|\vec{\rho}(s)\|dt + 2Tk\int_0^T (\dot{\vec{\Sigma}}, \dot{\vec{\Sigma}})dt \tag{64}$$

Let be $r(s) \doteq \|\vec{\rho}(s)\|$, let $(r_n(s), \vec{\Sigma}_n(s))$ be a sequence of functions in $M_{c,k}$ with

$$r_n(t) = \tilde{r}_n(t) + \bar{r}_n \qquad with \;\; \bar{r}_n \doteq \frac{\int_0^T dt r_n(t)}{T}, \qquad \int_0^T dt \tilde{r}_n(t) = 0$$

moreover

$$\|\vec{\Sigma}_n(t)\| = 1 \qquad \forall \; t, n$$

Since all the elements of (64) are positive definite, we have:

$$k \geq \frac{a_1}{2} \int_0^T (\dot{\tilde{r}}_n)^2 dt \qquad k \geq \frac{c a_1}{2 T C_1} \bar{r}_n \qquad 1 \geq 2T \int_0^T (\dot{\vec{\Sigma}}_n, \dot{\vec{\Sigma}}_n) dt$$

The sequence $\bar{r}_n$ admits therefore a convergent subsequence. Since $S^2$ is compact and Poincaré's inequality holds for zero mean functions, sequences $(\bar{r}_n(s), \vec{\Sigma}_n(s))$ have convergent subsequences. Therefore $(B_{c,k})^c$ is compact in $L^2([0,T], \bar{C})$ @.

Now the functional $A_T$ is continuous hence it attains a minimum on each $M_{c,k}$. Let $M^0_{c,k}$ be the interior of the set considered:

$$M^0_c \doteq \left\{ \zeta \in \tilde{\Lambda}^a_T \cap H^1_{\bar{C}} \;\; such \;\; that: \;\; \sup_{s \in [0,T]} g(\vec{\rho}(s)) > c \right\}$$

$$M^0_{c,k} = M^0_c \cap S_k$$

If for some $c, k$ the minimum on $M_{c,k}$ is attained in $M^0_{c,k}$ then $DA_T = 0$ at the minimum. This would give a *non-collision T-periodic orbits with zero total angular momentum* since we have verified that collision-solutions cannot be minima. A better understanding of the homology structure of $M_{c,k}$ is needed before being able to prove that there are $c, k$ such that there are minima in the interior of the set $M_{c,k}$.

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# References


[1] J.Moser *Regularization of Kepler problem and averaging method on a manifold* Comm.on Pure and Appl. Math. **23** 609 (1970)

[2] S.Smale *Topology and Mechanics* I, II Inventiones Math. **10** (1970)

[3] J.Moser, C.L.Siegel *Lectures on Celestial Mechanics* Springer- Verlag (1971)

[4] R.Easton *Regularization of vector field by surgery* J.of Diff. Eq. **10** 92 (1971)

[5] R.Easton *Some topology of 3-body problems* J.of Diff. Eq. **10** 371 (1971)

[6] R.McGehee *Triple collision in the collinear 3-body problem* Inventiones Math. **27** 191 (1974)

[7] R.Devaney *Non regularizability of the anisotropic Kepler problem* J.of Diff.Eq. **29** 253 (1978)

[8] R.Devaney *Collision orbits in anisotropic Kepler problem* Inventiones Math. **45** 221 (1978)

[9] R.Devaney *Singularities in classical mechanical systems* in *Ergodic theory and dynamical systems* Birkauser (1981)

[10] J.Gerver *A possible model for a singularity without collision in the 5-body problem* J. of Diff. Eq. **52** 76 (1984)

[11] D.Saari *The manifold structure for collision and hyperbolic- parabolic orbits in N-body problems* J. of Diff. Eq. **55** 300 (1984)

[12] E.Lacomba, J.Llibre *Transversal ejection-collision orbits for the restricted problem and the Hill's problem with applications* J. of Diff. Eq. **74** 69 (1988)

[13] R.Moeckel *Orbits of the 3-body problem which pass close to triple collision* Amer.J.Math. **103,6** 70 (1981)

[14] R.Moeckel *Orbits near triple collision in the 3-body problem* Ind.U.Math J. **32,2** 221 (1983)

[15] R.Moeckel *Chaotic dynamics near triple collision* Arch. Rational Mech. and Analysis **107** (1989)

[16] R.Moeckel *On central configurations* Math. Zeitschrift **205** 499 (1990)

[17] D.Saari *Symmetry in N-particle systems* Contemporary Mathematics **81** (1988)

[18] D.Saari *A visit to newtonian N-body problem via elementary complex variables* Amer. Math. Monthly **97** 105 (1991)

[19] Lacomba, Simo' *Regularization of simultaneous binary collisions in the N-body problem* J.of Diff. Eq. **98** 21 (1992)

[20] Pham Mau Quan *Geometrie du problem de Kepler: courbure et geodesiques* C.R. Acad. Sc. Paris **291** 219 (1980)

[21] Pham Mau Quan *Regularisation riemmaniane de singularite d'equation differentielle* C.R. Acad. Sc. Paris **296** 241 (1983)

[22] B.Cordani *Conformal regularization of the Kepler problem* Comm. Math. Phys. **103** 403 (1986)

[23] G.F.Dell'Antonio *Finding non-collisional periodic solutions to a perturbed $N$-body Kepler problem* Preprint SISSA (1993)

[24] K.F.Sundman Acta. Sci. Fenn. **35**, 9, 1909

[25] E.R.Van Kampen, A.Wintner *On a symmetrical canonical reduction of the three-body problem* Amer. Jou. of Math. **59** 153 (1937)

[26] P. Blanchard, E. Brüning *Variational Methods in Mathematical Physics. A Unified Approach.* Springer-Verlag