MAXIMUM ENTROPY PRINCIPLES FOR DISORDERED SPINS

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Summary. We transform nonstationary independent random fields with exponential
Radon-Nikodym factors and study the asymptotics of the transformed processes. As applications we deduce conditional limit theorems for such random fields, and we study a Curie-Weiss-type mean-field model of a quenched mixed magnetic crystal. This model has quenched site disorder and frustration but non-random coupling constants. We find a continuous phase transition with critical exponents equal to those of the classical Curie-Weiss theory.

1 Introduction

This paper is a sequel to the study of large deviations of nonstationary processes, or disordered spins, initiated in [Se1] and [Se2], though acquaintance with these earlier papers is not necessary for reading the present one. We transform the original process by a family of exponential Radon-Nikodym factors, prove that the collection of transformed processes is sequentially compact in a suitable sense, and characterize the limit points. The limit points are solutions to variational principles, hence the term 'maximum entropy'. In our case the original processes are independent and the modifying Radon-Nikodym factors are functions of the empirical distribution. Thus this is a way of introducing dependence among the previously independent variables.

In Section 2 we remind the reader of the setting in [Se1] and state the general theorems about compactness, limit points, and large deviations of the transformed processes. Proofs appear at the end of the paper in Section 5, and the middle sections 3 and 4 are devoted to two applications. Regarding the two tasks mentioned

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above, compactness and characterization of limit points, our results on the latter are far more complete than on the former. We use only rather soft large deviation techniques and consequently get convergence only in a very weak sense. It is clear that there is room for improvement here. Our methods do give precise information about the asymptotics of empirical averages, which is useful for studying models from statistical mechanics.

Section 3 generalizes the following well-known fact: A fair coin-tossing process, conditioned on consistently producing tails on no fewer than, say, 3/4 of the tosses, behaves asymptotically like a biased coin with probabilities $(\frac{1}{4}, \frac{3}{4})$. More precisely, if $P = (\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1)^{\otimes\mathbb{N}}$ is the law of an infinite sequence of fair coins and S_n is the number of tails in the first n tosses, then the conditioned laws

$$P\big(\cdot \,\big|\, S_n \geqq \tfrac{3}{4}n\big)$$

converge to $(\frac{1}{4}\delta_0 + \frac{3}{4}\delta_1)^{\otimes\mathbb{N}}$ as $n \to \infty$. This convergence holds not only in the weak topology but also in an information-theoretic sense as proved by Csiszár, who termed this phenomenon 'asymptotic quasi-independence' [Cs2]. We prove analogous results for nonstationary processes, which means that the coin does not have to be the same for each toss.

Section 4 presents an application to the statistical mechanics of quenched disorder. We study a mean-field model of a quenched mixed magnetic crystal where atoms of two different types, A and B, are randomly distributed on the lattice sites and their spins interact in a Curie-Weiss-type mean-field fashion. We characterize the limiting Gibbs measures and study the specific magnetization. The model exhibits a continuous phase transition, with nonzero spontaneous magnetization at low temperatures and with critical exponents equal to those of the classical Curie-Weiss theory. We find no "spin glass" phase where an overlap order parameter would be strictly positive without the accompanying spontaneous magnetization.

However, the interaction of the A- and B-atoms does produce interesting effects not visible in a pure A- or B-model. The parameters of the model are chosen so that a pure A-model is a ferromagnetic Curie-Weiss with critical temperature 1, and a pure B-model is an antiferromagnetic Curie-Weiss, hence stays in the paramagnetic phase at arbitrarily low temperatures and has no phase transition. But by strengthening the AB-interaction we can induce ferromagnetic ordering in the quenched model at arbitrarily high temperatures despite the resistance to alignment by the antiferromagnetic BB-coupling. And, surprisingly, the spontaneous magnetization of the B-atoms can be strictly greater than that of the A-atoms.

While the proofs of section 4 obviously depend on section 2, the results of section 4 are hopefully accessible to the reader independently of the rest of the paper.

To our knowledge, maximum entropy principles have not been studied previously for nonstationary processes. Our development owes much to the elegant treatment of uniformly ergodic Markov chains by Bolthausen and Schmock [BS], as will become clear to the reader who compares the arguments. For further related results see [B], [CC], [CCC], [HR], [M], [Sc], and their references. Thanks are due to an anonymous referee for several improvements in the final version of the paper.

2 The setting and the general results

Our setting is that of [Se1] with some changes in notation: At each site $\mathbf{i}$ of $\mathbb{Z}^d$ we have a spin $\sigma_{\mathbf{i}}$ with values in a Polish space $\mathcal{S}$, and $\Omega = \mathcal{S}^{\mathbb{Z}^d}$ denotes the space of spin configurations. $\mathbb{Z}^d$ acts on Ω by translations: $(\theta_{\mathbf{i}}\sigma)_{\mathbf{j}} = \sigma_{\mathbf{i}+\mathbf{j}}$. Additionally, there is a Polish space Σ of *quenched variables*, as they are called in the physics literature. (In [Se1] we called them simply *parameters.*) Elements of Σ are denoted by $\mathbf{x}$ and $\mathbf{y}$. We assume that $\mathbb{Z}^d$ acts on Σ by some homeomorphisms $\theta_{\mathbf{i}}$ that satisfy $\theta_{\mathbf{i}} \circ \theta_{\mathbf{j}} = \theta_{\mathbf{i}+\mathbf{j}}$. All infinite-volume limits are taken along the sequence of cubes $V_n = [-n, n]^d \cap \mathbb{Z}^d$.

We assume given a measurable map $\mathbf{x} \mapsto p^{\mathbf{x}}$ from Σ into the space $\mathcal{M}_1(\mathcal{S})$ of probability measures on $\mathcal{S}$ (all measurable structures are Borel and all topologies on measures are generated by bounded continuous functions), and form the product measure $P^{\mathbf{x}} = \otimes_{\mathbf{i}\in\mathbb{Z}^d} p^{\theta_{\mathbf{i}}\mathbf{x}}$. The goal is to prove asymptotic results for the spin process $(\sigma_{\mathbf{i}})$ in terms of the empirical distribution and empirical field, defined by

$$\mathbf{L}_n = \frac{1}{|V_n|}\sum_{\mathbf{i}\in V_n} \delta_{\sigma_{\mathbf{i}}}$$

and

$$\mathbf{R}_n = \frac{1}{|V_n|}\sum_{\mathbf{i}\in V_n} \delta_{\theta_{\mathbf{i}}\sigma}.$$

The laws of these two measure-valued random variables under the probabilities $P^{\mathbf{x}}$ were studied in [Se1]. Here we add some new ingredients: Assume given a measurable function $F : \mathcal{M}_1(\mathcal{S}) \to [-\infty, c]$, where c is a finite constant. Define probability measures $\gamma_n^{\mathbf{x}}$ on Ω by setting

$$\gamma_n^{\mathbf{x}}(f) = \frac{1}{Z_n^{\mathbf{x}}} P^{\mathbf{x}}\big(f\, e^{|V_n|\, F(\mathbf{L}_n)}\big) \tag{2.1}$$

for bounded Borel functions f on Ω, where $Z_n^{\mathbf{x}} = P^{\mathbf{x}}(e^{|V_n|\, F(\mathbf{L}_n)})$ is the appropriate normalizing factor. In case $P^{\mathbf{x}}(F(\mathbf{L}_n) = -\infty) = 1$, set $\gamma_n^{\mathbf{x}} = P^{\mathbf{x}}$. Define the measures $\zeta_n^{\mathbf{x}}$ by adding spatial averaging inside $\gamma_n^{\mathbf{x}}$:

$$\zeta_n^{\mathbf{x}}(f) = \gamma_n^{\mathbf{x}}(\mathbf{R}_n(f)) = \frac{1}{|V_n|}\sum_{\mathbf{i}\in V_n} \gamma_n^{\mathbf{x}}(f \circ \theta_{\mathbf{i}}).$$

Our motivation for studying these measures comes from two examples:

(1) Taking $F = \log \mathbf{1}_C$ for a set C amounts to studying the spin process conditioned on the events $\{\mathbf{L}_n \in C\}$. This is interesting when the event $\{\mathbf{L}_n \in C\}$ represents a deviation, that is, its $P^{\mathbf{x}}$-probability vanishes as $n \to \infty$.

(2) In statistical mechanical language, $\gamma_n^{\mathbf{x}}$ is a Gibbs measure for the spins $\sigma_{\mathbf{i}}$ located on the sites of V_n whose interaction energy is $-|V_n| F(\mathbf{L}_n(\sigma))$ and whose noninteracting state is described by the measure $P^{\mathbf{x}}$.

The goal of this section is to characterize the limit points of the measures $\gamma_n^{\mathbf{x}}$ and $\zeta_n^{\mathbf{x}}$ as $n \to \infty$. Sections 3 and 4 present applications corresponding to items (1) and (2) above. Proofs for this section are presented in Section 5.

We first develop the terminology and assumptions needed for the results. Throughout, π denotes an element of the space $\mathcal{M}_\Theta(\Sigma)$ of invariant probability measures on Σ. Most of the functions, sets, and measures that appear in our definitions and theorems depend on π, but to keep the notation simple this dependence is suppressed, as it is fairly easy to detect from the context. We say that $\mathbf{x}$ is *generic* for π if both

$$\pi = \lim_{n\to\infty} \frac{1}{|V_n|} \sum_{\mathbf{i}\in V_n} \delta_{\theta_{\mathbf{i}}\mathbf{x}} \tag{2.2}$$

and

$$\pi(\mathbf{y} : P^{\mathbf{y}} \in \,\cdot\,) = \lim_{n\to\infty} \frac{1}{|V_n|} \sum_{\mathbf{i}\in V_n} \delta_{P^{\theta_{\mathbf{i}}\mathbf{x}}} \tag{2.3}$$

hold in the weak topologies of probability measures on Σ and $\mathcal{M}_1(\Omega)$, respectively. The left-hand side in (2.3) is the law of the random measure $\mathbf{y} \mapsto P^{\mathbf{y}}$ under π. (2.3) follows from (2.2) in case $\mathbf{x} \mapsto p^{\mathbf{x}}$ (and hence $\mathbf{x} \mapsto P^{\mathbf{x}}$) is continuous. In general, π-a.e. $\mathbf{x}$ is generic for π whenever π is ergodic. In [Se1], conditions (2.2) and (2.3) were termed *quasiregularity* and *P-regularity*, respectively.

Given $\pi \in \mathcal{M}_\Theta(\Sigma)$, an entropy for $\nu \in \mathcal{M}_1(\mathcal{S})$ is defined by

$$K(\nu) = \sup_{f\in b\mathcal{B}^{\mathcal{S}}} \left\{ \nu(f) - \int_\Sigma \log p^{\mathbf{y}}(e^f)\, \pi(d\mathbf{y}) \right\}. \tag{2.4}$$

$\mathcal{B}^{\mathcal{S}}$ is the Borel field of $\mathcal{S}$, and $b\mathcal{A}$ denotes the space of bounded $\mathcal{A}$-measurable functions for any σ-field $\mathcal{A}$. The function K is related to the familiar *relative entropy* or *Kullback-Leibler information* in the following way: Define the probability measure φ on $\mathcal{S} \times \Sigma$ by $\varphi(ds, d\mathbf{y}) = p^{\mathbf{y}}(ds)\, \pi(d\mathbf{y})$. By Theorem 2.13 of [Se1],

$$K(\nu) = \inf_\rho H(\rho \,|\, \varphi), \tag{2.5}$$

where the infimum is over $\rho \in \mathcal{M}_1(\mathcal{S} \times \Sigma)$ with marginals ν and π, and the relative entropy $H(\rho \,|\, \varphi)$ is defined by

$$H(\rho \,|\, \varphi) = \begin{cases} \rho\big(\log \frac{d\rho}{d\varphi}\big) & \text{if } \rho \ll \varphi, \\ \infty & \text{otherwise.} \end{cases}$$

The upper and lower semicontinuous regularizations of F are defined by

$$F^u(\nu) = \inf_{G:G \text{ open}, G\ni\nu} \left\{ \sup_{\rho:\rho\in G} F(\rho) \right\}$$

and

$$F_\ell(\nu) = \sup_{G:G \text{ open}, G\ni\nu} \left\{ \inf_{\rho:\rho\in G} F(\rho) \right\}.$$

To get results for unbounded measurable functions F, we need the following technical assumption. It is automatically satisfied if F is bounded and continuous.

Assumption A. $P^{\mathbf{x}}(F(\mathbf{L}_n) = -\infty) < 1$ *for large enough n, and*

$$r_1 \equiv \inf_{\nu\in\mathcal{M}_1(\mathcal{S})} \{K(\nu) - F^u(\nu)\} = \inf_{\nu\in\mathcal{M}_1(\mathcal{S})} \{K(\nu) - F_\ell(\nu)\} < \infty. \tag{2.6}$$

An important special case is that where the measure-valued random variables $p^{\theta_{\mathbf{i}}\mathbf{x}}$, $\mathbf{i} \in \mathbb{Z}^d$, are i.i.d. This leads us to formulate the following sharpening of Assumption A:

Assumption B. $\Sigma = \mathcal{X}^{\mathbb{Z}^d}$ *for a Polish space $\mathcal{X}$, $\mathbb{Z}^d$ acts on Σ by translations, $p^{\mathbf{x}} = p^{\mathbf{x}_{\mathbf{0}}}$ so that $\mathbf{x} \mapsto p^{\mathbf{x}}$ is $\mathcal{B}_0^\Sigma$-measurable, and $\pi = \pi_o^{\otimes\mathbb{Z}^d}$ for some $\pi_o \in \mathcal{M}_1(\mathcal{X})$. Furthermore, assume* (2.6) *and that $P^{\mathbf{x}}(F(\mathbf{L}_n) = -\infty) < 1$ for large enough n, π-a.s.*

The notation $\mathcal{B}_n^\Sigma$ above denotes the σ-field of Σ generated by the $\mathcal{X}$-valued coordinate variables $(\mathbf{x}_{\mathbf{i}} : \mathbf{i} \in V_n)$.

Since $H(\,\cdot\,|\,\varphi)$ is lower semicontinuous, strictly convex, and has compact sublevel sets (see [DS] or [V]), (2.5) has a unique minimizer ρ whenever $K(\nu) < \infty$. For such a ν we denote this unique minimizer by ψ_ν. Let $\psi_\nu^{\mathbf{y}}$ denote a conditional distribution of ψ_ν on $\mathcal{S}$, given $\mathbf{y} \in \Sigma$. This associates to ν a π-a.s. unique map $\mathbf{y} \mapsto \psi_\nu^{\mathbf{y}}$ from Σ into $\mathcal{M}_1(\mathcal{S})$ that satisfies $\nu = \int \psi_\nu^{\mathbf{y}}\, \pi(d\mathbf{y})$ and

$$K(\nu) = \int H(\psi_\nu^{\mathbf{y}}\,|\,p^{\mathbf{y}})\, \pi(d\mathbf{y}). \tag{2.7}$$

With this map we define the product measure $\Psi_\nu^{\mathbf{y}}(d\sigma) = \otimes_{\mathbf{i}\in\mathbb{Z}^d}\psi_\nu^{\theta_{\mathbf{i}}\mathbf{y}}(d\sigma_{\mathbf{i}})$ on Ω and the invariant measure $\Psi_\nu(d\sigma, d\mathbf{y}) = \Psi_\nu^{\mathbf{y}}(d\sigma)\, \pi(d\mathbf{y})$ on $\Omega \times \Sigma$, with marginal $j_\infty(\nu) \in \mathcal{M}_\Theta(\Omega)$ given by

$$j_\infty(\nu) = \int \Psi_\nu^{\mathbf{y}}\, \pi(d\mathbf{y}). \tag{2.8}$$

Under Assumption B, let $\varphi_o(ds, dx) = p^x(ds)\, \pi_o(dx)$ on $\mathcal{S} \times \mathcal{X}$. We can replace (2.5) by

$$K(\nu) = \inf_\rho H(\rho\,|\,\varphi_o), \tag{2.9}$$

where the infimum is now over $\rho \in \mathcal{M}_1(\mathcal{S}\times\mathcal{X})$ with marginals ν and π_o. (Justification follows in Section 5.) Denote the unique minimizer again by ψ_ν. Now we can take $\psi_\nu^{\mathbf{y}} = \psi_\nu^{\mathbf{y}_{\mathbf{0}}}$ to be $\mathcal{B}_0^\Sigma$-measurable and $\Psi_\nu^{\mathbf{y}}(d\sigma) = \otimes_{\mathbf{i}\in\mathbb{Z}^d}\psi_\nu^{\mathbf{y}_{\mathbf{i}}}(d\sigma_{\mathbf{i}})$. It is obvious that $j_\infty(\nu)$ and Ψ_ν are i.i.d.

Let

$$\mathcal{K}_1 = \{\nu \in \mathcal{M}_1(\mathcal{S}) : K(\nu) - F^u(\nu) = r_1\}. \tag{2.10}$$

Assumption A and the properties of K given in Theorem 2.6 of [Se1] imply that $\mathcal{K}_1$ is a nonempty compact set. It will turn out that j_∞ is a homeomorphism from $\mathcal{K}_1$ onto a set $\mathcal{K}_\infty$ of invariant measures on Ω, and $\nu \mapsto \Psi_\nu^{\cdot}$ is continuous on $\mathcal{K}_1$ in an appropriate sense. In particular, the integral in (2.12) below makes sense.

We are ready to state our main result, Theorem 2.11. For background on the last sentence of the theorem, see [Al] on exchangeable measures and their mixing measures.

2.11. Theorem. *Let* $\mathbf{x}$ *be generic for* π *and assume Assumption A. Then the set* $\{\zeta_n^{\mathbf{x}}\}_{n=1}^{\infty}$ *is tight. Let* Q *be any limit point. Then* Q *is shift-invariant, and there is a probability measure* Λ *on* $\mathcal{K}_1$ *such that*

$$Q = \int_{\mathcal{K}_1} j_\infty(\nu)\, \Lambda(d\nu), \tag{2.12}$$

and Λ *is a limit point of the laws* $\gamma_n^{\mathbf{x}}(\mathbf{L}_n \in \cdot\,)$*. Under Assumption B* Q *is exchangeable with mixing measure* Λ*.*

2.13. Remark. In case π is ergodic, subsequences of the two sequences of laws appearing in the theorem converge together. More precisely, for any subsequence $\{n_j\}$ the following are equivalent (still assuming Assumption A and that $\mathbf{x}$ is generic for π):

(1) There exists a limit $\Pi = \lim_{j\to\infty} \gamma_{n_j}^{\mathbf{x}}(\mathbf{R}_{n_j} \in \cdot\,)$ in the weak topology of $\mathcal{M}_1(\mathcal{M}_1(\Omega))$.

(2) There exists a limit $\Lambda = \lim_{j\to\infty} \gamma_{n_j}^{\mathbf{x}}(\mathbf{L}_{n_j} \in \cdot\,)$ in the weak topology of $\mathcal{M}_1(\mathcal{M}_1(\mathcal{S}))$.

(3) There exists a limit $Q = \lim_{j\to\infty} \zeta_{n_j}^{\mathbf{x}}$ in the weak topology of $\mathcal{M}_1(\Omega)$.

When this happens, $\Pi = \Lambda(j_\infty \in \cdot\,)$, and (2.12) is the ergodic decomposition of Q. In particular, Π and Λ are uniquely determined by Q.

More precise information on the convergence of the $\zeta_n^{\mathbf{x}}$ seems to depend on the particular model at hand. The Hewitt-Savage 0-1 law implies that, under an i.i.d. π, any convergence that can be seen on a set of positive π-probability happens almost everywhere:

2.14. Proposition. *Assume Assumption B. Fix a subsequence* $\{n_j\}$*. Then there is a Borel function* $g : \Sigma \times \mathcal{M}_1(\Omega) \to \{0,1\}$ *such that* $g(\mathbf{x}, Q) = 1$ *iff* Q *is a limit point of* $\{\zeta_{n_j}^{\mathbf{x}}\}_{j=1}^{\infty}$*. For each fixed* Q*,* $\mathbf{x} \mapsto g(\mathbf{x}, Q)$ *is invariant under finite permutations of the coordinates of* $\mathbf{x}$*, hence* π*-a.s. constant. There is a Borel set* $\Sigma_0 \subset \Sigma$ *such that* $\pi(\Sigma_0) = 1$ *and for any* $\mathbf{x}, \mathbf{y} \in \Sigma_0$*,* $\{\zeta_{n_j}^{\mathbf{x}}\}_{j=1}^{\infty}$ *and* $\{\zeta_{n_j}^{\mathbf{y}}\}_{j=1}^{\infty}$ *have the same limit points.*

Next we discuss the convergence of the measures $\gamma_n^{\mathbf{x}}$. It turns out fruitful to consider the maps $\gamma_n^{\cdot} : \mathbf{y} \mapsto \gamma_n^{\mathbf{y}}$ instead of individual measures. Let $\mathcal{M}_\pi^{\cdot}$ be the space of π-a.s. defined Borel-measurable maps $\varrho^{\cdot} : \mathbf{y} \mapsto \varrho^{\mathbf{y}}$ from Σ into $\mathcal{M}_1(\Omega)$. We put a Polish topology on $\mathcal{M}_\pi^{\cdot}$ by identifying $\varrho^{\cdot}$ with the measure $\pi \otimes \varrho^{\cdot} \equiv \varrho^{\mathbf{y}}(d\sigma)\, \pi(d\mathbf{y}) \in \mathcal{M}_1(\Omega \times \Sigma)$. In other words, this topology is generated by the seminorms

$$p_f(\varrho^{\cdot}) = \Big| \int_\Sigma \varrho^{\mathbf{y}}(f^{\mathbf{y}})\, \pi(d\mathbf{y}) \Big|,$$

where f is a bounded continuous function on $\Omega \times \Sigma$ and $f^{\mathbf{y}}$ is its $\mathbf{y}$-section. Write $\pi\varrho^{\cdot}$ for the marginal of $\pi \otimes \varrho^{\cdot}$, $\pi\varrho^{\cdot} = \int \varrho^{\mathbf{y}}\, \pi(d\mathbf{y})$. Let us say $\varrho^{\cdot}$ is invariant if $\varrho^{\theta_{\mathbf{i}}\mathbf{y}} = \varrho^{\mathbf{y}} \circ \theta_{-\mathbf{i}}$ for all $\mathbf{i}$, π-a.s.

A particular class of elements of $\mathcal{M}_\pi^{\cdot}$ are the maps $\Psi_\nu^{\cdot} : \mathbf{y} \mapsto \Psi_\nu^{\mathbf{y}}$ associated to $\nu \in \mathcal{K}_1$. We will show later that the map $(\nu, \mathbf{y}) \mapsto \Psi_\nu^{\mathbf{y}}$ is jointly measurable on $\mathcal{K}_1 \times \Sigma$, so the integral below is legitimate. In this new notation $j_\infty(\nu) = \pi\Psi_\nu^{\cdot}$ and $\Psi_\nu = \pi \otimes \Psi_\nu^{\cdot}$.

2.15. Theorem. *Assume Assumption B. Then* $\{\gamma_n^{\cdot}\}_{n=1}^{\infty}$ *is relatively compact. Let* $\gamma^{\cdot}$ *be any limit point. Then* $\gamma^{\cdot}$ *is invariant. There is a unique measure* Λ *on* $\mathcal{K}_1$ *such that*

$$\gamma^{\cdot} = \int_{\mathcal{K}_1} \Psi_{\nu}^{\cdot}\, \Lambda(d\nu), \tag{2.16}$$

and Λ *is a limit point of the laws* $\pi\gamma^{\cdot}(\mathbf{L}_n \in \cdot\,)$.

An analogue of Remark 2.13 holds, but we leave the details to the reader. As one would expect, under suitable circumstances a limit point Q of Theorem 2.11 is an expectation of a limit point $\gamma^{\cdot}$ of Theorem 2.15:

2.17. Proposition. *Assume Assumption B. Suppose that for some fixed subsequence* $\{n_j\}$ *and* $Q \in \mathcal{M}_{\Theta}(\Omega)$, $\zeta_{n_j}^{\mathbf{x}} \to Q$ *as* $j \to \infty$ *for all* $\mathbf{x}$ *on a set of positive* π*-measure. Then* $\zeta_{n_j}^{\mathbf{x}} \to Q$ *for* π*-a.e.* $\mathbf{x}$*, and there exists a* $\gamma^{\cdot} \in \mathcal{M}_{\pi}^{\cdot}$ *such that* $\gamma_{n_j}^{\cdot} \to \gamma^{\cdot}$ *as* $j \to \infty$*. Furthermore, the uniquely determined measures* Λ *of* (2.12) *and* (2.16) *coincide, and* $Q = \pi\gamma^{\cdot}$.

Underlying these results is a large deviation theory for the laws $\gamma_n^{\mathbf{x}}(\mathbf{R}_n \in \cdot\,)$ to which we now turn. We need to extend the entropy defined by (2.4) to $Q \in \mathcal{M}_{\Theta}(\Omega)$: Set

$$K_n(Q) = \sup_{f \in b\mathcal{B}_n^{\Omega}} \left\{ Q(f) - \int_{\Sigma} \log P^{\mathbf{y}}(e^f)\, \pi(d\mathbf{y}) \right\}, \tag{2.18}$$

and

$$k(Q) = \lim_{n\to\infty} \frac{1}{|V_n|} K_n(Q). \tag{2.19}$$

$\mathcal{B}_n^{\Omega}$ is the σ-field on Ω generated by the spins in the cube V_n. The limit in (2.19) exists by Theorem 2.8 in [Se1]. By Theorems 2.1 and 2.2 of [Se1], K and k are the rate functions that govern the large deviations of the laws $P^{\mathbf{x}}(\mathbf{L}_n \in \cdot\,)$ and $P^{\mathbf{x}}(\mathbf{R}_n \in \cdot\,)$, respectively, whenever $\mathbf{x}$ is generic for π. Let $Q_0 \in \mathcal{M}_1(\mathcal{S})$ denote the single-spin marginal of a probability measure $Q \in \mathcal{M}_{\Theta}(\Omega)$. Let

$$r_\infty = \inf_{Q \in \mathcal{M}_{\Theta}(\Omega)} \{k(Q) - F^u(Q_0)\},$$

and for all $Q \in \mathcal{M}_1(\Omega)$, set

$$I_\infty(Q) = k(Q) - F^u(Q_0) - r_\infty$$

in case Q is invariant and $I_\infty(Q) = \infty$ otherwise. Let

$$\mathcal{K}_\infty = \{Q \in \mathcal{M}_{\Theta}(\Omega) : k(Q) - F^u(Q_0) = r_\infty\} = \{I_\infty = 0\}$$

and

$$\mathcal{K}_\infty^{\cdot} = \{\varrho^{\cdot} \in \mathcal{M}_{\pi}^{\cdot} : k(\pi\varrho^{\cdot}) - F^u((\pi\varrho^{\cdot})_0) = r_\infty\}.$$

2.20. Lemma. *Suppose $r_1 < \infty$. Then $r_1 = r_\infty$. I_∞ is a tight rate function, meaning that I_∞ is a lower semicontinuous function from $\mathcal{M}_1(\Omega)$ into $[0, \infty]$ with compact sublevel sets $\{I_\infty \leqq b\}$ for all real b.*

$\mathcal{K}_\infty$ and $\mathcal{K}_\infty^{\cdot}$ are nonempty and compact in their respective topologies. Finally, j_∞ is a homeomorphism from $\mathcal{K}_1$ onto $\mathcal{K}_\infty$, whose inverse is the projection $Q \mapsto Q_0$, and $\nu \mapsto \Psi_\nu^{\cdot}$ is a homeomorphism from $\mathcal{K}_1$ onto $\mathcal{K}_\infty^{\cdot}$, with inverse $\varrho^{\cdot} \mapsto (\pi\varrho^{\cdot})_0$.

Let $\overline{E}$ and E° denote the weak closure and interior, respectively, of a set E of probability measures. Define

$$\underline{I}(Q) = k(Q) - F_\ell(Q_0) - r_1$$

for $Q \in \mathcal{M}_\Theta(\Omega)$ and $\underline{I}(Q) = \infty$ for noninvariant Q.

2.21. Theorem. *Assume Assumption A and that $\mathbf{x}$ is generic for π. Then for any Borel set $E \subset \mathcal{M}_1(\Omega)$,*

$$\begin{aligned} -\inf_{Q \in E^\circ} \underline{I}(Q) &\le \liminf_{n\to\infty} \frac{1}{|V_n|} \log \gamma_n^{\mathbf{x}}(\mathbf{R}_n \in E) \\ &\le \limsup_{n\to\infty} \frac{1}{|V_n|} \log \gamma_n^{\mathbf{x}}(\mathbf{R}_n \in E) \le -\inf_{Q \in \overline{E}} I_\infty(Q). \end{aligned} \tag{2.22}$$

If $I_\infty = \underline{I}$ so that both the upper and the lower bound are given by the same function (this happens whenever F is continuous), we say that a large deviation principle (LDP) holds, or that I_∞ is a rate function for the laws $\gamma_n^{\mathbf{x}}(\mathbf{R}_n \in \cdot\,)$. It then follows that the laws $\gamma_n^{\mathbf{x}}(\mathbf{L}_n \in \cdot\,)$ satisfy a LDP with rate function $I_1(\nu) = K(\nu) - F^u(\nu) - r_1$, and $\mathcal{K}_1 = \{I_1 = 0\}$.

We close this section with two simple examples to get some feeling for the content of the theorems. They show how measures of the Gibbsian type can arise in our framework.

2.23. Example. The simplest nontrivial example is $F(\nu) = \nu(f)$ for some $f \in C_b(\mathcal{S})$. Define $\mu_1^{\mathbf{x}}, \mu_1 \in \mathcal{M}_1(\mathcal{S})$ by $d\mu_1^{\mathbf{x}} = p^{\mathbf{x}}(e^f)^{-1} e^f \, dp^{\mathbf{x}}$ and $\mu_1 = \int \mu_1^{\mathbf{x}} \, \pi(d\mathbf{x})$. Then $\mathcal{K}_1 = \{\mu_1\}$ and the unique limit point $\gamma^{\cdot}$ of $\gamma_n^{\cdot}$ is defined by $\gamma^{\mathbf{x}} = \bigotimes_{\mathbf{i}} \mu_1^{\theta_{\mathbf{i}}\mathbf{x}}$. Indeed, $\gamma_n^{\mathbf{x}}$ and $\gamma^{\mathbf{x}}$ coincide on $\mathcal{B}_n^\Omega$.

2.24. Example. To generalize the previous example, let us study the limit points of the laws

$$\gamma_n^{\mathbf{x}}(d\sigma) = \frac{1}{Z_n^{\mathbf{x}}} \exp\left\{ |V_n| \, g\left(\frac{1}{|V_n|} \sum_{\mathbf{i} \in V_n} f(\sigma_{\mathbf{i}}) \right) \right\} P^{\mathbf{x}}(d\sigma) \tag{2.25}$$

for some $f \in C_b(\mathcal{S})$ and $g \in C_b(\mathbb{R})$. To apply Theorem 2.15, set $F(\nu) = g(\nu(f))$. For $\beta \in \mathbb{R}$ let $d\mu_\beta^{\mathbf{x}} = p^{\mathbf{x}}(e^{\beta f})^{-1} e^{\beta f} \, dp^{\mathbf{x}}$ and $\mu_\beta = \int \mu_\beta^{\mathbf{x}} \, \pi(d\mathbf{x})$. Let $a(\mathbf{x}) = \operatorname{ess\,inf} f$ and $b(\mathbf{x}) = \operatorname{ess\,sup} f$ with respect to $p^{\mathbf{x}}$-measure. Since $\mu_\beta^{\mathbf{x}}(f) \to a(\mathbf{x})$ as $\beta \to -\infty$, a is measurable, and similarly is b. By dominated convergence applied to $\mu_\beta(f)$ as $\beta \to \pm\infty$ it makes sense to adopt the convention $\mu_\infty(f) = \pi(b)$ and $\mu_{-\infty}(f) = \pi(a)$, and this is consistent with setting

$$\mu_\infty^{\mathbf{x}} = p^{\mathbf{x}}(\,\cdot \mid f = b(\mathbf{x})) \qquad \text{and} \qquad \mu_{-\infty}^{\mathbf{x}} = p^{\mathbf{x}}(\,\cdot \mid f = a(\mathbf{x})) \tag{2.26}$$

whenever these measures are well-defined. Assume

$$\pi\{\mathbf{x} : f \text{ is } p^{\mathbf{x}}\text{-a.s. constant}\} < 1.$$

(Otherwise $\mathcal{K}_1 = \{p\}$ where $p = \int p^{\mathbf{x}}\, \pi(d\mathbf{x})$ and we are done.) Then

$$\frac{d}{d\beta}\mu_\beta(f) = \int \left(\mu_\beta^{\mathbf{x}}(f^2) - \mu_\beta^{\mathbf{x}}(f)^2\right) \pi(d\mathbf{x}) > 0$$

for real β, so by dominated convergence $\beta \mapsto \mu_\beta(f)$ is continuous and strictly increasing from $[-\infty, \infty]$ onto $[\pi(a), \pi(b)]$. To compute the entropy $K(\mu_\beta)$, Cor. 3.1 in [Cs1] implies that $\rho = \pi \otimes \mu_\beta^{\cdot}$ is the minimizer for $\nu = \mu_\beta$ in (2.5), hence by the conditional entropy formula ((4.4.8) in [DS] or (10.2) in [V])

$$K(\mu_\beta) = \int H(\mu_\beta^{\mathbf{x}} \,|\, p^{\mathbf{x}})\, \pi(d\mathbf{x}).$$

Claim. Suppose $\pi(a) \leqq c \leqq \pi(b)$ and assume $\inf_{\nu(f)=c} K(\nu) < \infty$. Then the unique minimizer of $K(\nu)$ subject to $\nu(f) = c$ is $\nu = \mu_\beta$, where β is the unique number in $[-\infty, \infty]$ satisfying $\mu_\beta(f) = c$.

To prove the Claim, suppose first that $c = \pi(a)$. We wish to argue that the existence of a measure λ such that $\lambda(f) = c$ and $K(\lambda) < \infty$ implies that the measures $\mu_{-\infty}^{\mathbf{x}}$ in (2.26) are well-defined π-a.s. From (2.7), $\psi_\lambda^{\mathbf{x}} \ll p^{\mathbf{x}}$ and consequently $\psi_\lambda^{\mathbf{x}}(f) \geqq a(\mathbf{x})$ for π-a.e. $\mathbf{x}$. But then $\int \psi_\lambda^{\mathbf{x}}(f)\, \pi(d\mathbf{x}) = c = \pi(a)$ forces $\psi_\lambda^{\mathbf{x}}(f = a(\mathbf{x})) = 1$ π-a.s. It follows that $p^{\mathbf{x}}(f = a(\mathbf{x})) > 0$ π-a.s. so the measures $\mu_{-\infty}^{\mathbf{x}}$ are well-defined. On the other hand, we just saw that $\psi_\lambda^{\mathbf{x}}(f = a(\mathbf{x})) = 1$ π-a.s., hence

$$H(\mu_{-\infty}^{\mathbf{x}} \,|\, p^{\mathbf{x}}) \leqq H(\psi_\lambda^{\mathbf{x}} | \, p^{\mathbf{x}}) \quad \pi\text{-a.s.}$$

(This is an elementary fact: For any probability measure α and event E such that $\alpha(E) > 0$, $\rho = \alpha(\cdot \,|\, E)$ uniquely minimizes $H(\rho \,|\, \alpha)$ over ρ satisfying $\rho(E) = 1$.) Integrating the inequality against π gives $K(\mu_{-\infty}) \le K(\lambda)$ and proves the Claim for the case $c = \pi(a)$. The case $c = \pi(b)$ is handled similarly.

Now let $\pi(a) < c < \pi(b)$ and λ again as above. In particular β is finite. By the finite-volume variational principle of classical statistical mechanics [Is, p. 46],

$$H(\psi_\lambda^{\mathbf{x}} | \, p^{\mathbf{x}}) - \beta\psi_\lambda^{\mathbf{x}}(f) \geqq H(\mu_\beta^{\mathbf{x}} | \, p^{\mathbf{x}}) - \beta\mu_\beta^{\mathbf{x}}(f)$$

for π-a.a. $\mathbf{x}$ with equality iff $\psi_\lambda^{\mathbf{x}} = \mu_\beta^{\mathbf{x}}$. Integrating against π gives

$$K(\lambda) - \beta\lambda(f) \geqq K(\mu_\beta) - \beta\mu_\beta(f)$$

with equality iff $\lambda = \mu_\beta$. This proves the Claim.

It follows from the Claim that the elements of $\mathcal{K}_1$ must be μ_β's for some values of β: For suppose $\lambda \in \mathcal{K}_1$, that is, $\nu = \lambda$ minimizes $K(\nu) - g(\nu(f))$. Let $c = \lambda(f)$. Then $c = \int \psi_\lambda^{\mathbf{x}}(f)\, \pi(d\mathbf{x}) \in [\pi(a), \pi(b)]$ by the argument used above, and the Claim forces $\lambda = \mu_\beta$ with β chosen so that $\mu_\beta(f) = c$. Consequently $\mathcal{K}_\infty^{\cdot}$ consists of elements $\gamma_\beta^{\cdot}$ defined by $\gamma_\beta^{\mathbf{x}} = \bigotimes_{\mathbf{i}} \mu_\beta^{\theta_{\mathbf{i}}\mathbf{x}}$ for each μ_β in $\mathcal{K}_1$. By Theorem 2.15, the possible limit points of the measures (2.25) are mixtures of the type

$$\int_{[-\infty,\infty]} \gamma_\beta^{\mathbf{x}} \, \Lambda(d\beta).$$

(Recall though that Theorem 2.15 does not say anything about the limiting behavior of individual measures $\gamma_n^{\mathbf{x}}$ but only of the maps $\mathbf{x} \mapsto \gamma_n^{\mathbf{x}}$.)

Further conclusions depend on the specific g in question. For example, if g is concave and differentiable on $[\pi(a), \pi(b)]$, then

$$\mathcal{K}_1 = \begin{cases} \{\mu_\infty\} & \text{if } \beta < g'(\mu_\beta(f)) \text{ for all } \beta, \\ \{\mu_{-\infty}\} & \text{if } \beta > g'(\mu_\beta(f)) \text{ for all } \beta, \\ \{\mu_{\beta_o}\} & \text{otherwise, where } \beta_o = g'(\mu_{\beta_o}(f)) \text{ uniquely determines } \beta_o. \end{cases}$$

3 Conditional limit theorems

In this section we investigate how nonstationary independent random fields behave under conditioning. Such a field is specified by a configuration $\boldsymbol{\mu} = (\boldsymbol{\mu}_{\mathbf{i}} : \mathbf{i} \in \mathbb{Z}^d)$ of probability measures on $\mathcal{S}$. Put the product measure $P^{\boldsymbol{\mu}} = \otimes_{\mathbf{i}\in\mathbb{Z}^d} \boldsymbol{\mu}_{\mathbf{i}}$ on Ω so that the spins $(\sigma_{\mathbf{i}})$ become independent random variables with laws $\mathcal{L}(\sigma_{\mathbf{i}}) = \boldsymbol{\mu}_{\mathbf{i}}$. Let $\Sigma = \mathcal{M}_1(\mathcal{S})^{\mathbb{Z}^d}$ be the space of measure configurations $\boldsymbol{\mu}$.

Let $\pi \in \mathcal{M}_\Theta(\Sigma)$ with marginal $\pi_o \in \mathcal{M}_1(\mathcal{M}_1(\mathcal{S}))$. As in Section 2 let $\varphi_o(ds, d\mu) = \mu(ds)\, \pi_o(d\mu)$, and set $p = \int \mu \, \pi_o(d\mu)$. Throughout this section, C denotes a convex subset of $\mathcal{M}_1(\mathcal{S})$ that satisfies

$$\inf_{\nu \in \overline{C}} K(\nu) = \inf_{\nu \in C^\circ} K(\nu) < \infty. \tag{3.1}$$

The first equality in (3.1) is not a vacuous requirement as one can see by constructing simple examples. Assuming that $K(\lambda) < \infty$ for some $\lambda \in C^\circ$, (3.1) holds if

(i) $\overline{C} = \overline{C^\circ}$

and

(ii) there exists a convex open set G in the ambient space of all signed Borel measures, topologized weakly by $C_b(\mathcal{S})$, such that $G \cap \mathcal{M}_1(\mathcal{S}) = C^\circ$.

Under (i) and (ii) $t\mu + (1-t)\lambda \in C^\circ$ for any $\mu \in \overline{C}$ and $0 \leqq t < 1$, and then (3.1) follows from K's convexity.

Requirement (i) is trivially true for convex sets with nonempty interior in a locally convex topological vector space, but not so in the relative topology of $\mathcal{M}_1(\mathcal{S})$. The convexity requirement of (ii) is also a genuine restriction. Sets that satisfy (i) and (ii) include open and closed δ-fattenings of a given convex set A, for $\delta > 0$, and intersections of finitely many half-spaces of the form $\{\nu \in \mathcal{M}_1(\mathcal{S}) : \nu(f) < c\}$ and $\{\nu \in \mathcal{M}_1(\mathcal{S}) : \nu(f) \leqq c\}$ for $f \in C_b(\mathcal{S})$.

The strict convexity of relative entropy and (2.5) imply that K is strictly convex, hence by (3.1) there is a unique $\nu_* \in \overline{C}$ such that $K(\nu_*) = \inf_{\nu \in C} K(\nu)$. It is for this uniqueness that we require C's convexity. The construction done in Section 2 gives a π_o-a.s. unique $\mathcal{M}_1(\mathcal{S}) \to \mathcal{M}_1(\mathcal{S})$ map $\mu \mapsto \psi^\mu$ such that $\nu_* = \int \psi^\mu \, \pi_o(d\mu)$ and $K(\nu_*) = \int H(\psi^\mu | \, \mu) \, \pi_o(d\mu)$. Set $\Psi^{\boldsymbol{\mu}} = \otimes_{\mathbf{i} \in \mathbb{Z}^d} \psi^{\boldsymbol{\mu}_\mathbf{i}}$.

3.2. Theorem. *Whenever $\boldsymbol{\mu}$ is generic for π,*

$$\lim_{n\to\infty} P^{\boldsymbol{\mu}}\Big(\frac{1}{|V_n|} \sum_{\mathbf{i} \in V_n} f \circ \theta_\mathbf{i} \,\Big|\, \mathbf{L}_n \in C\Big) = \int \Psi^{\boldsymbol{\mu}}(f) \, \pi(d\boldsymbol{\mu})$$

for any $f \in C_b(\Omega)$. If π is i.i.d., the limiting measure is $\nu_*^{\otimes \mathbb{Z}^d}$.

3.3. Theorem. *Suppose π is i.i.d. Then $P^\cdot(\,\cdot \,|\, \mathbf{L}_n \in C)$ converges to $\Psi^\cdot$ in the sense that*

$$\lim_{n\to\infty} \int P^{\boldsymbol{\mu}}(f(\,\cdot\,, \boldsymbol{\mu}) \,|\, \mathbf{L}_n \in C) \, \pi(d\boldsymbol{\mu}) = \iint f(\sigma, \boldsymbol{\mu}) \, \Psi^{\boldsymbol{\mu}}(d\sigma) \, \pi(d\boldsymbol{\mu})$$

for any bounded continuous function f on $\Omega \times \Sigma$.

Proofs. To make the connection with Section 2, let $p^{\boldsymbol{\mu}} = \boldsymbol{\mu}_\mathbf{0}$ and define F on $\mathcal{M}_1(\mathcal{S})$ by $F(\mu) = \log \mathbf{1}_C(\mu)$. Then $\gamma_n^\cdot = P^\cdot(\,\cdot \,|\, \mathbf{L}_n \in C)$ and $\zeta_n^{\boldsymbol{\mu}} = P^{\boldsymbol{\mu}}(\mathbf{R}_n(\,\cdot\,) \,|\, \mathbf{L}_n \in C)$. Assumption A of Section 2 is automatically satisfied under (3.1): $F^u = \log \mathbf{1}_{\overline{C}}$ and $F_\ell = \log \mathbf{1}_{C^\circ}$, so (2.6) is equivalent to (3.1). And

$$P^{\boldsymbol{\mu}}(F(\mathbf{L}_n) > -\infty) = P^{\boldsymbol{\mu}}(\mathbf{L}_n \in C) \geqq P^{\boldsymbol{\mu}}(\mathbf{L}_n \in C^\circ),$$

which is eventually > 0, for by the LDP of Theorem 2.3 in [Se2] and (3.1),

$$\liminf_{n\to\infty} \frac{1}{|V_n|} \log P^{\boldsymbol{\mu}}(\mathbf{L}_n \in C^\circ) \geqq -\inf_{\nu \in C^\circ} K(\nu) > -\infty.$$

Now $\mathcal{K}_1 = \{\nu_*\}$, so Theorem 2.11 gives Theorem 3.2 and Theorem 2.15 gives Theorem 3.3. □

Let us make some comparisons with the expected process. Suppose π is i.i.d. and that $\boldsymbol{\mu}$ is generic for π (only (2.2) needs to be checked now). The expected process has law $P = \int P^{\boldsymbol{\mu}} \, \pi(d\boldsymbol{\mu}) = p^{\otimes \mathbb{Z}^d}$. Assume that C satisfies (3.1) not only for K but also for the function $H(\,\cdot \,|\, p)$ to guarantee that the laws $P(\,\cdot \,|\, \mathbf{L}_n \in C)$ converge. On the first level of description, the empirical measures $\mathbf{L}_n$ and $\mathbf{R}_n$ behave very much alike under P and $P^{\boldsymbol{\mu}}$. For example $\mathbf{L}_n \to p$ a.s. under both $P^{\boldsymbol{\mu}}$ and P, and p is the unique zero for both the rate function K of $P^{\boldsymbol{\mu}}(\mathbf{L}_n \in \,\cdot\,)$ and the rate function $H(\,\cdot \,|\, p)$ of $P(\mathbf{L}_n \in \,\cdot\,)$. Only the rate of convergence may differ: $K(\nu) \geqq H(\nu \,|\, p)$ for all ν (see Theorem 3.9 in [Se2]).

However, $P(\,\cdot \,|\, \mathbf{L}_n \in C)$ is not an average of the laws $P^{\boldsymbol{\mu}}(\,\cdot \,|\, \mathbf{L}_n \in C)$, and we can no longer expect a common limit. We have $P(\,\cdot \,|\, \mathbf{L}_n \in C) \to \alpha_*^{\otimes \mathbb{Z}^d}$ where α_*

minimizes $H(\alpha\,|\,p)$ over $\alpha \in \overline{C}$. There is no a priori reason why α_* and ν_* should coincide, and in fact it is easy to construct examples where this does not happen.

To clarify the distinction, let us reformulate these minimization problems on the bigger space $\mathcal{S} \times \mathcal{M}_1(\mathcal{S})$: Let ψ_* minimize $H(\rho\,|\,\varphi_o)$ over probability measures ρ on $\mathcal{S} \times \mathcal{M}_1(\mathcal{S})$ with $\rho_{\mathcal{S}} \in \overline{C}$ and $\rho_{\mathcal{M}_1(\mathcal{S})} = \pi_o$, and let η_* minimize $H(\rho\,|\,\varphi_o)$ over ρ with $\rho_{\mathcal{S}} \in \overline{C}$. Then ν_* and α_* are the $\mathcal{S}$-marginals of ψ_* and η_*, respectively, and depending on the particular case, may or may not coincide. In Example 3.5 below $\nu_* = \alpha_*$ but $\psi_* \neq \eta_*$, so $\psi_* = \eta_*$ is a stricter requirement. Here is the precise sense:

3.4. Proposition. *The following are equivalent:*

(1) $\psi_* = \eta_*$.
(2) $\nu_* = \alpha_*$ *and* $K(\nu_*) = H(\alpha_*\,|\,p)$.
(3) $\pi_o = \int \varphi_o^s\, \alpha_*(ds)$, *where* φ_o^s *is a conditional distribution of* φ_o *on* $\mathcal{M}_1(\mathcal{S})$, *given* $s \in \mathcal{S}$.
(4) $(\eta_*)_{\mathcal{M}_1(\mathcal{S})} = \pi_o$.

Proof. (1)$\Longleftrightarrow$(4) is immediate since η_* minimizes entropy without the constraint $(\eta_*)_{\mathcal{M}_1(\mathcal{S})}$
$= \pi_o$. The minimizing η_* must be $\eta_*(ds, d\mu) = \alpha_*(ds)\, \varphi_o^s(d\mu)$ (look at the conditional entropy formula, (4.4.8) in [DS] or (10.2) in [V]) and consequently $H(\alpha_*|\,p) = H(\eta_*|\,\varphi_o)$. Hence (3)$\Longleftrightarrow$(4), and together with $K(\nu_*) = H(\psi_*|\,\varphi_o)$ this gives (1)$\Longrightarrow$(2). Let ψ_*^s be a conditional distribution of ψ_* on $\mathcal{M}_1(\mathcal{S})$, given $s \in \mathcal{S}$. (2) and the conditional entropy formula give

$$
\begin{aligned}
K(\nu_*) &= H(\nu_*|\,p) + \int H(\psi_*^s|\,\varphi_o^s)\, \nu_*(ds) \\
&= H(\alpha_*|\,p) + \int H(\psi_*^s|\,\varphi_o^s)\, \alpha_*(ds),
\end{aligned}
$$

hence (2) again forces $\int H(\psi_*^s|\,\varphi_o^s)\, \alpha_*(ds) = 0$ and we conclude that

$$
\psi_* = \nu_*(ds)\, \psi_*^s(d\mu) = \alpha_*(ds)\, \varphi_o^s(d\mu) = \eta_*. \quad \square
$$

For another point of comparison with the expected process, notice that the measure $P(\,\cdot\,|\,\mathbf{L}_n \in C)$ not only has identical marginals $P(d\sigma_{\mathbf{i}}|\,\mathbf{L}_n \in C)$ for $\mathbf{i} \in V_n$, but also this marginal itself lies in $\overline{C}$. Here is the easy argument: For any $f \in b\mathcal{B}^{\mathcal{S}}$,

$$
P(f(\sigma_{\mathbf{i}})\,|\,\mathbf{L}_n \in C) = P(\mathbf{L}_n(f)\,|\,\mathbf{L}_n \in C) \leqq \sup_{\nu \in C} \nu(f),
$$

which suffices by the separation theorems of locally convex spaces. The first equality above relies on invariance under permutations which we do not have in our setting. We can make a similar claim only on the average, that is,

$$
\int P^{\boldsymbol{\mu}}(d\sigma_{\mathbf{i}}\,|\,\mathbf{L}_n \in C)\, \pi(d\boldsymbol{\mu}) \in \overline{C}.
$$

This follows since

$$
P^{\tau\boldsymbol{\mu}}(f(\sigma_{\mathbf{i}})\,|\,\mathbf{L}_n \in C) = P^{\boldsymbol{\mu}}(f(\sigma_{\tau\mathbf{i}})\,|\,\mathbf{L}_n \in C)
$$

for any permutation τ on V_n, and π is invariant under permutations (still assuming π i.i.d.), so

$$\int P^{\boldsymbol{\mu}}(f(\sigma_{\mathbf{i}}) \mid \mathbf{L}_n \in C)\, \pi(d\boldsymbol{\mu}) = \int P^{\boldsymbol{\mu}}(\mathbf{L}_n(f) \mid \mathbf{L}_n \in C)\, \pi(d\boldsymbol{\mu}) \leqq \sup_{\nu \in C} \nu(f).$$

In particular, it is not clear how a result corresponding to Csiszár's 'convergence in information' could be derived in our setting (see p. 790 in [Cs2]).

To close this section we return to the coin-tossing example mentioned in the introduction, this time with two different coins.

3.5. Example. Let $\frac{1}{2} < a < 1$ and suppose we have two biased coins, with probability measures $\alpha = (1-a, a)$ and $\beta = (a, 1-a)$ on the outcome space $\mathcal{S} = \{0, 1\}$. We choose a sequence of coins independently, each time picking an α-coin with probability z and a β-coin with probability $1-z$, and once the choices are made, flip the coins to run our coin-tossing process. Suppose $1 > t > m(z) \equiv za + (1-z)(1-a)$, so that the long run frequency $m(z)$ of tails ($\equiv$ the value $1 \in \mathcal{S}$) is below t, for almost every realization of the coin choices. Given a typical sequence $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3, \dots)$ of coin choices ($\mu_i = \alpha$ or β), how does the coin-tossing process behave upon conditioning on $S_n \geqq nt$?

To be precise, we have $\pi_o = z\delta_\alpha + (1-z)\delta_\beta$, the unique zero of K is $p = (1 - m(z), m(z))$, $C = \{\nu : \nu(1) \geqq t\}$, and by strict convexity of K, $\nu_* = (1-t, t)$. We need to find ψ^α and ψ^β to produce the π_o-a.s. defined map $\mu \mapsto \psi^\mu$ that satisfies $\nu_* = \int \psi^\mu\, \pi_o(d\mu)$ and $K(\nu_*) = \int H(\psi^\mu | \, \mu)\, \pi_o(d\mu)$. We look for it in the form $d\psi^\mu = \mu(f)^{-1} f\, d\mu$, for whenever $d\psi = f \otimes g\, d\varphi_o$ with $g(\mu) = \mu(f)^{-1}$, ψ must be the minimizer in (2.9) by Cor. 3.1 in [Cs1]. So define a function f on $\mathcal{S}$ by $f(0) = 1$ and $f(1) = \theta$ for a number θ to be determined, and set $v(\theta) = \int \psi^\mu(1)\, \pi_o(d\mu)$. Then

$$v(\theta) = z\frac{a\theta}{1-a+a\theta} + (1-z)\frac{(1-a)\theta}{a+(1-a)\theta},$$

a strictly increasing continuous function of θ with $v(0) = 0$ and $v(\infty) = 1$. Let $\theta = \theta(t) \in (0, \infty)$ be the unique number satisfying $v(\theta) = t = \nu_*(1)$; then ψ^μ is the map we want.

This answers our question: The coin-tossing sequence $(\mu_1, \mu_2, \mu_3, \dots)$, conditioned on $S_n \geqq nt$, behaves asymptotically like the sequence $(\psi^{\mu_1}, \psi^{\mu_2}, \psi^{\mu_3}, \dots)$, where the new coins have probabilities

$$\psi^\alpha = \left(\frac{1-a}{1-a+a\theta}, \frac{a\theta}{1-a+a\theta} \right)$$

and

$$\psi^\beta = \left(\frac{a}{a+(1-a)\theta}, \frac{(1-a)\theta}{a+(1-a)\theta} \right).$$

Note that ψ^α and ψ^β do depend on z, namely through $\theta(t) = \theta(t, z)$. But for any z it turns out that $\alpha(1) < \psi^\alpha(1)$, so perhaps somewhat surprisingly the lowest entropy is achieved by making the ψ^α-coin even more biased than the α-coin. (To see this observe that $v(1) = m(z)$ so $\theta(t) > 1$.) Both $\partial\psi^\alpha(1)/\partial z < 0$ and $\partial\psi^\beta(1)/\partial z < 0$, so picking the α-coin more and more often decreases the expectations of the new coins.

We claimed above (see the paragraph preceding Proposition 3.4) that in this example $\alpha_* = \nu_*$ but $\psi_* \neq \eta_*$. That $\alpha_* = \nu_*$ follows from the one-dimensionality of the example: Both $H(\,\cdot\,|\,p)$ and K are strictly convex functions with common global minimum at p, so over C they must both be minimized by ν_*. In particular, the process $P^{\boldsymbol{\mu}}(\mathbf{R}_n(\,\cdot\,)\,|\,S_n \geqq nt)$ and the conditioned expected process $p^{\otimes\mathbb{N}}(\,\cdot\,|\,S_n \geqq nt)$ both converge to $\nu_*^{\otimes\mathbb{N}}$. To see that $\psi_* \neq \eta_*$ take $z = \frac{1}{2}$ again and compute

$$(\eta_*)_{\mathcal{M}_1(\mathcal{S})} = \int \varphi_o^s\,\nu_*(ds) = t(a\delta_\alpha + (1-a)\delta_\beta) + (1-t)((1-a)\delta_\alpha + a\delta_\beta) \neq \pi_o.$$

4 A mean-field mixed magnetic crystal

Next we apply our theory to a mean-field version of the $\mathrm{A}_p\mathrm{B}_{1-p}$ model of a quenched mixed magnetic crystal. The $\mathrm{A}_p\mathrm{B}_{1-p}$ model was studied by Aharony [Ah] and the mean-field model by Luttinger [Lu]. Our discussion is closer to the analyses of the Curie-Weiss model given by Ellis [E] and Orey [Or]. The quenched model is in fact a generalization of the Curie-Weiss, and turns out to have the same critical behavior.

We imagine two types of atoms, A and B, located on the positive integer sites $i = 1, 2, 3, \dots$ and each possessing a ± 1-valued spin variable σ_i. For each site we flip a coin to place an A-atom with probability p and a B-atom with probability $1-p$, where $0 < p < 1$ is a fixed parameter of the model. The A- and B-atoms have opposite types of magnetic ordering: An AA-pair interacts ferromagnetically with coupling constant 1 and a BB-pair interacts antiferromagnetically with coupling constant -1. A mixed AB-pair interacts with coupling constant $J > 0$ which is another parameter of the model. Let $x_i = 1$ or 0 according to whether site i is occupied by an A- or a B-atom. This gives the quenched variable $\mathbf{x} = (x_i)_{i=1}^{\infty}$.

The Hamiltonian or interaction energy of n spins $\sigma = (\sigma_1, \dots, \sigma_n)$ in the sample $\mathbf{x}$ with external field h is

$$H_n^{\mathbf{x}}(\sigma) = -\frac{1}{2n} \sum_{1 \leqq i,j \leqq n} \sigma_i\sigma_j \left[x_i x_j - (1-x_i)(1-x_j) + J(x_i(1-x_j) + x_j(1-x_i))\right] - h \sum_{1 \leqq i \leqq n} \sigma_i.$$

This Hamiltonian is of the mean-field type in the sense that every spin interacts with every other spin and the strength of the interaction is independent of the distance separating the spins. The model has *frustration* in that no spin configuration can simultaneously satisfy all the bonds: In an ABB-triple, the J-coupling tends to align the B spins with the A-spin, but the antiferromagnetic BB-coupling works against aligning the BB-pair. (Frustration requires that $x_1, \dots, x_n$ not be identical, which happens for large enough n, a.s.) We study the thermodynamic equilibrium of the spins for a fixed realization $\mathbf{x}$ of the occupation coin flips. This situation is called *quenched disorder* in the physics literature.

Let $\mathcal{S} = \{\pm 1\}$ be the single spin space and $\mathcal{X} = \{0, 1\}$. The space of spin configurations is $\Omega = \mathcal{S}^{\mathbb{N}}$ and the space of quenched variables $\Sigma = \mathcal{X}^{\mathbb{N}}$. The a priori measure of a single spin is $\lambda_o = (\delta_{-1} + \delta_{+1})/2$ on $\mathcal{S}$, the coin-tossing measure on $\mathcal{X}$ is $\pi_o = p\delta_1 + (1-p)\delta_0$, and we put the products $\lambda = \lambda_o^{\otimes\mathbb{N}}$ and $\pi = \pi_o^{\otimes\mathbb{N}}$ on Ω and Σ, respectively. Once an inverse temperature $\beta > 0$ is specified, the Gibbs measure $\mu_n^{\mathbf{x}}$ gives probability

$$\mu_n^{\mathbf{x}}(\sigma) = \frac{1}{Z_n^{\mathbf{x}}} e^{-\beta H_n^{\mathbf{x}}(\sigma)} \lambda_n(\sigma) \tag{4.1}$$

to the configuration $\sigma = (\sigma_1, \dots, \sigma_n)$. Here λ_n is the restriction of λ to n spins.

We shall first describe the results for this model and then present the proofs at the end. The limiting Gibbs measures are determined by the magnetizations of A- and B-atoms, or mathematically speaking, by the expected values of the spins. Let z_1 denote the expected spin of an A-atom and z_0 of a B-atom. The equilibrium values $(z_0, z_1) = (z_0(\beta, h), z_1(\beta, h))$ that minimize free energy are among the solutions of

$$\begin{aligned} s_1 = \xi_1(s_0) &\equiv \frac{1}{\beta J p}\left[\frac{1}{2}\log\frac{1+s_0}{1-s_0} + \beta(1-p)s_0 - \beta h\right] \\ s_0 = \xi_0(s_1) &\equiv \frac{1}{\beta J(1-p)}\left[\frac{1}{2}\log\frac{1+s_1}{1-s_1} - \beta p s_1 - \beta h\right] \end{aligned} \tag{4.2}$$

for (s_0, s_1) in $[-1, 1] \times [-1, 1]$. These equations reveal a phase transition at the critical temperature

$$\beta_c = \beta_c(p, J) = \frac{1 - 2p + (1 + 4J^2 p(1-p))^{1/2}}{2(1+J^2)p(1-p)}. \tag{4.3}$$

These are the equilibrium solutions of (4.2) for $\beta > 0$ and real h:

- $\beta \leqq \beta_c$, $h = 0$: The unique solution $(z_0, z_1) = (0, 0)$.
- $\beta > \beta_c$, $h = 0$: $\{(-z_0, -z_1), (z_0, z_1)\}$, where (z_0, z_1) is the unique solution in $(0, 1) \times (0, 1)$. The solution $(0, 0)$ does not qualify as an equilibrium.
- $\beta > 0$, $h > 0$: The unique solution (z_0, z_1) in $(0, 1) \times (0, 1)$. Other solutions do not qualify as equilibria.
- $\beta > 0$, $h < 0$: The unique solution (z_0, z_1) in $(-1, 0) \times (-1, 0)$. Other solutions do not qualify as equilibria.

It is clear from (4.2) that $(z_0(\beta, -h), z_1(\beta, -h)) = (-z_0(\beta, h), -z_1(\beta, h))$ for $h \neq 0$. Next we define measures $\mu^{\mathbf{x}}$ on Ω determined by the spin expectations, for a given occupation variable $\mathbf{x} = (x_i)_{i=1}^{\infty}$:

$\beta \leqq \beta_c$ or $h \neq 0$: (*The uniqueness case*) Define $\psi^x \in \mathcal{M}_1(\mathcal{S})$ by

$$\psi^x = \frac{1 - z_x}{2}\delta_{-1} + \frac{1 + z_x}{2}\delta_{+1} \tag{4.4}$$

for $x = 0, 1$ and then $\mu^{\mathbf{x}} = \otimes_{i=1}^{\infty} \psi^{x_i}$. Note that $\mu^{\mathbf{x}} = \lambda$ for all $\mathbf{x}$ in case $\beta \leqq \beta_c$ and $h = 0$.

$\beta > \beta_c$ and $h = 0$: (*The nonuniqueness case*) Set $\mu^{\mathbf{x}}_{\pm} = \otimes_{i=1}^{\infty} \psi^{x_i}_{\pm}$, where $\psi^{x}_{\pm} \in \mathcal{M}_1(\mathcal{S})$ are defined by

$$\psi^{x}_{\pm} = \frac{1 \mp z_x}{2}\delta_{-1} + \frac{1 \pm z_x}{2}\delta_{+1} \tag{4.5}$$

for $x = 0, 1$. Then put $\mu^{\mathbf{x}} = (\mu^{\mathbf{x}}_{+} + \mu^{\mathbf{x}}_{-})/2$.

This defines an element $\mu^{\cdot} = \mu^{\beta,h,\cdot}$ of $\mathcal{M}^{\cdot}_{\pi}$. The connection to the z_x is clear: For example under $\mu^{\mathbf{x}}_{+}$ the expected spin at site i is z_1 if the site is occupied by an A-atom and z_0 if the site is occupied by a B-atom. Recall the Polish topology on $\mathcal{M}^{\cdot}_{\pi}$ defined in Section 2 after Proposition 2.14.

4.6. Theorem. *For all p, J, β, and h, $\mu^{\cdot} = \lim \mu^{\cdot}_n$ as $n \to \infty$.*

Considering $\mu^{\mathbf{x}}$ as the equilibrium state of the spins with occupations $\mathbf{x}$, we can say the following: At high temperature ($\beta \leqq \beta_c$) and zero field ($h = 0$) we see a completely random state where the spins are independent and identically distributed, even independently of the type of atom (each marginal of $\mu^{\mathbf{x}}$ equals λ_o), with no preference for $+$ or $-$. The state ordered by a positive field ($h > 0$) has a positive magnetization (expected spin $\psi^{x_i}(s) = z_{x_i} > 0$ at each site), although the spins remain independent and there is lack of long range order in the sense that the magnetization (z_1 or z_0) at site i varies with the type (A or B) of the occupant of i. At $\beta > \beta_c$, $h = 0$ we have a ferromagnetically ordered state where all the spins simultaneously obey either $\mu^{\mathbf{x}}_{+}$ or $\mu^{\mathbf{x}}_{-}$, and in both cases they tend to align themselves, without the force of an external field.

The phase transition is also reflected in the large deviations of empirical averages, and the zeroes of the rate functions indicate where the mass concentrates in the limit $n \to \infty$. Let I_∞ be the rate function of the laws $\mu^{\mathbf{x}}_n(\mathbf{R}_n \in \cdot)$. Recall the definition $\pi\mu^{\cdot} \equiv \int \mu^{\mathbf{y}}\pi(d\mathbf{y})$ from Section 2. Then we have

$$I_\infty(Q) = 0 \text{ iff } \begin{cases} Q = \pi\mu^{\cdot} & \text{in case } \beta \leqq \beta_c \text{ or } h \neq 0, \\ Q = \pi\mu^{\cdot}_{+} \text{ or } Q = \pi\mu^{\cdot}_{-} & \text{in case } \beta > \beta_c \text{ and } h = 0. \end{cases} \tag{4.7}$$

The uniqueness/nonuniqueness dichotomy is similarly evident in the rate function S of the laws $\mu^{\mathbf{x}}_n(\frac{1}{n}\sum_1^n \sigma_i \in \cdot\,)$ of the average block spin. Put $z(\beta, h) = pz_1(\beta, h) + (1-p)z_0(\beta, h)$.

$$S(t) = 0 \text{ iff } \begin{cases} t = z(\beta, h) & \text{in case } \beta \leqq \beta_c \text{ or } h \neq 0, \\ t = z(\beta, 0) \text{ or } t = -z(\beta, 0) & \text{in case } \beta > \beta_c \text{ and } h = 0. \end{cases} \tag{4.8}$$

The *specific magnetization* is defined by

$$m(\beta, h) = \lim_{n\to\infty} \frac{1}{n}\,\mu^{\mathbf{x}}_n\Big(\sum_{i=1}^{n} \sigma_i\Big),$$

a limit which exists π-a.s. and is given by

$$\begin{gathered} m(\beta, 0) = 0 \text{ for all } \beta > 0, \text{ and} \\ m(\beta, h) = z(\beta, h) \text{ for all } \beta > 0 \text{ and } h \neq 0. \end{gathered} \tag{4.9}$$

The spin-flip symmetry of the Hamiltonian forces $\mu_n^{\mathbf{x}}(\sigma_i) = 0$ if $h = 0$. So in the absence of a symmetry-breaking field, the magnetization $m(\beta, 0)$ does not see the phase transition as our model does not incorporate boundary conditions. As functions of h, the quantities $z_1(\beta, h)$, $z_0(\beta, h)$, and $m(\beta, h)$ are strictly increasing. Thus the limits

$$m(\beta, +) = \lim_{h \searrow 0} m(\beta, h) \quad \text{and} \quad m(\beta, -) = \lim_{h \nearrow 0} m(\beta, h)$$

exist, and we shall prove that

$$m(\beta, \pm) = \pm z(\beta, 0). \tag{4.10}$$

From this we see that $m(\beta, +) = 0 = m(\beta, 0)$ if $\beta \leqq \beta_c$ but $m(\beta, +) > 0 = m(\beta, 0)$ if $\beta > \beta_c$. Thus at low temperatures our model exhibits *spontaneous magnetization*, another aspect of the phase transition.

Suppose for the moment that $h \geqq 0$. Since the A-atoms interact ferromagnetically among themselves and the B-atoms antiferromagnetically, it seems likely that the $+$-tendency of the A-atoms is stronger than that of the B-atoms, that is, $z_1 \geqq z_0$. This turns out not to be necessarily the case, and even more surprisingly, the cut-off point depends only on p and J and not at all on β and h, provided we are in the regime of nontrivial ($\neq (0, 0)$) solutions:

4.11. Proposition. *Suppose $\beta > \beta_c$ and $h = 0$, or $h > 0$. Then*

$$\begin{aligned} z_1(\beta, h) > z_0(\beta, h) &\text{ iff } p \leqq 1/2 \text{ or } J < (2p-1)^{-1}, \\ z_1(\beta, h) = z_0(\beta, h) &\text{ iff } p > 1/2 \text{ and } J = (2p-1)^{-1}, \\ z_1(\beta, h) < z_0(\beta, h) &\text{ iff } p > 1/2 \text{ and } J > (2p-1)^{-1}. \end{aligned}$$

Our final result looks at the behavior of some physical quantities near the critical temperature. A basic fact is that the phase transition is continuous, in that as $\beta \searrow \beta_c$, $m(\beta, \pm) \to 0 = m(\beta_c, \pm) = m(\beta_c, 0)$. This is a consequence of the next lemma.

4.12. Lemma. *The quantities $z_0(\beta, 0)$ and $z_1(\beta, 0)$ are continuous functions of β, and for $\beta \geqq \beta_c$ strictly increasing. In particular they decrease to 0 as $\beta \searrow \beta_c$.*

The zero-field specific heat $C_{h=0}$ is defined by

$$C_{h=0}(\beta) = -\beta^2 \frac{\partial u}{\partial \beta}(\beta, 0),$$

where the specific energy

$$u(\beta, h) = \lim_{n \to \infty} \frac{1}{n} \mu_n^{\mathbf{x}}(H_n^{\mathbf{x}})$$

exists as a π-a.s. constant limit. We will argue that u is a continuous function of (z_0, z_1), hence a continuous function of β at fixed $h = 0$, with $u(\beta, 0) = 0$ for $\beta \leqq \beta_c$ and $u(\beta, 0) < 0$ for $\beta > \beta_c$.

A *critical exponent* c is defined as follows: We write $f \sim x^c$ as $x \searrow 0$ if

$$\lim_{x \searrow 0} \frac{\log f(x)}{\log x} = c.$$

4.13. Theorem. *We have the following critical behavior, for $x = 0, 1$:*

$$z_x(\beta, 0) \sim (\beta - \beta_c)^{1/2} \quad \text{as } \beta \searrow \beta_c, \tag{4.14}$$

$$z_x(\beta_c, h) \sim h^{1/3} \quad \text{as } h \searrow 0, \text{ and} \tag{4.15}$$

$$\frac{\partial z_x}{\partial h}(\beta, 0) \sim |\beta - \beta_c|^{-1} \quad \text{as } \beta \to \beta_c^{\pm}. \tag{4.16}$$

$$C_{h=0} \text{ has a discontinuity at } \beta = \beta_c. \tag{4.17}$$

We close with some remarks before turning to the proofs.

4.18. Remark. These exponents agree with the classical Curie-Weiss theory of magnetism (see Section 4-6 in [T]).

4.19. Remark. In (4.16) above $z_x(\beta, h)$ is not continuous as h passes through 0 if $\beta > \beta_c$, so in that case the derivative is one-sided, i.e. $\partial z_x / \partial h(\beta, 0^{\pm})$.

4.20. Remark. The physical theory of spin glasses [FH] leads one to study the asymptotics of the overlap $\frac{1}{n}\sum_1^n \sigma_i \tau_i$, where τ is an independent copy of σ, and to look for a state where the overlap has nonzero limit points while $m(\beta, +) = 0$. But our model does not exhibit such a state, as can be verified by identifying the zeroes of the rate function of the laws $\mu_n^{\mathbf{x}} \otimes \mu_n^{\mathbf{x}}\big(\frac{1}{n}\sum_1^n \sigma_i\tau_i \in \cdot\,\big)$.

4.21. Remark. We assumed $0 < p < 1$ and $J > 0$ throughout, but the expression (4.3) for β_c gives correct information also about the limiting cases. Existence of these limits is guaranteed by $\partial\beta_c/\partial p < 0$ and $\partial\beta_c/\partial J < 0$.

1. In the case $p = 0$ (all atoms are of type B) we have an antiferromagnetic Curie-Weiss model whose only limiting Gibbs measure is easily seen to be λ for all temperatures. Accordingly $\lim_{p\searrow 0}\beta_c = \infty$.
2. The opposite case $p = 1$ is the classical Curie-Weiss model, whose $\beta_c = 1$ [E], and we have $\lim_{p\nearrow 1}\beta_c = 1$.
3. If we set $J = 0$ the A- and B-atoms equilibrate independently of each other, which is reflected mathematically in a decoupling of the equations (4.2). The B-atoms always choose λ and the A-atoms behave as a dilute Curie-Weiss model with $\beta_c = p^{-1}$. Correspondingly $\lim_{J\searrow 0}\beta_c = p^{-1}$.

4.22. Remark. The constant J influences the model in seemingly contradictory ways. On the one hand, β_c decays like J^{-1} as $J \nearrow \infty$ by (4.3), so we can produce a phase transition at arbitrarily high temperatures (recall that temperature $\sim \beta^{-1}$) by taking J large enough. An intuitively plausible explanation is that increasing J increases the effective coupling of the A-atoms, since each AA-pair is coupled not only directly but also via each B-atom. Thus the ordering effect dominates the entropy effect already at higher temperatures. In fact (4.29) shows that for any given β and p, we are in the ferromagnetically ordered state whenever

$$J^2 > \frac{(1-\beta p)(1+\beta(1-p))}{\beta^2 p(1-p)}.$$

On the other hand, J inhibits magnetization by the external field. For a fixed $\beta < \beta_c$, $\partial z_1/\partial h$ decays like J^{-1} for large J. This effect is natural since the antiferromagnetic BB-couplings resist the aligning effect of the field.

4.23. Remark. Keeping β fixed, we can vary p and parametrize the phase transition in terms of a critical density $p_c = p_c(\beta, J)$ of A-atoms, so that the model is in the ferromagnetic state whenever $p > p_c$. (But note that $\beta > 1$ is always required for the ferromagnetic state.) For the critical exponent one finds $z_x \sim (p - p_c)^{1/2}$ as $p \searrow p_c$. We leave the details to the reader.

To prove our results we apply the theory of Section 2 with the following twist: Since the quenched variable $\mathbf{x}$ appears explicitly in the Hamiltonian, we must work with the skew model which includes the quenched variable as a deterministic component: The map $\mathbf{x} \mapsto P^{\mathbf{x}}$ is defined by $P^{\mathbf{x}} = \lambda \otimes \delta_{\mathbf{x}}$, so $P^{\mathbf{x}}$ is a measure on $\Omega \times \Sigma$. Define the function F for probability measures ν on $\mathcal{S} \times \mathcal{X}$ by

$$F(\nu) = \frac{\beta}{2}\big[-\nu(s)^2 + 2(1+J)\nu(s)\nu(sx) - 2J\nu(sx)^2\big] + \beta h \nu(s),$$

where we write (s,x) for a generic element of $\mathcal{S}\times\mathcal{X}$, $\nu(s)$ for the integral $\int s\,\nu(ds,dx)$, and similarly for $\nu(sx)$. Then $-\beta H_n^{\mathbf{x}}(\sigma) = nF(\overline{\mathbf{L}}_n)$ where $\overline{\mathbf{L}}_n = n^{-1}\sum_1^n \delta_{(\sigma_i, x_i)}$ is the empirical distribution of the skew model, and $\gamma_n^{\mathbf{x}} = \mu_n^{\mathbf{x}} \otimes \delta_{\mathbf{x}}$ for the measure $\gamma_n^{\mathbf{x}}$ defined by (2.1).

To find the minimizing measures in $\mathcal{K}_1$ we need only consider ν with $\mathcal{X}$-marginal π_o, by (5.7). Let us parametrize such a measure ν by its conditional spin expectations:

$$s_0 = \nu(s \mid x = 0) = \text{ expected spin of a B-atom,}$$
$$s_1 = \nu(s \mid x = 1) = \text{ expected spin of an A-atom.}$$

The parameter (s_0, s_1) ranges over the square $[-1,1]\times[-1,1]$. In these terms $\nu(s) = ps_1 + (1-p)s_0$ and $\nu(sx) = ps_1$, so

$$F(\nu) = \frac{\beta}{2}\big[p^2 s_1^2 - (1-p)^2 s_0^2 + 2Jp(1-p)s_0 s_1\big] + \beta h\big[ps_1 + (1-p)s_0\big]. \tag{4.24}$$

Entropy is given by

$$\begin{aligned} K(\nu) = {}& \frac{1-p}{2}[(1-s_0)\log(1-s_0) + (1+s_0)\log(1+s_0)] \\ & + \frac{p}{2}[(1-s_1)\log(1-s_1) + (1+s_1)\log(1+s_1)]. \end{aligned} \tag{4.25}$$

Set $G(\nu) = K(\nu) - F(\nu)$. Physically G corresponds to free energy. We wish to argue that the minimum of G is taken uniquely at $(z_0(\beta,h), z_1(\beta,h))$ if $h \neq 0$ or $\beta \leqq \beta_c$ and at $\pm(z_0(\beta,h), z_1(\beta,h))$ in the case $h = 0$, $\beta > \beta_c$. $\partial G/\partial s_x = 0$ iff $s_{1-x} = \xi_{1-x}(s_x)$ for $x = 0,1$, where the functions ξ_x are defined in (4.2). ξ_1 is strictly increasing, so it has a well-defined inverse η_1 on $[-1,1]$: $s_0 = \eta_1(s_1)$ iff $s_1 = \xi_1(s_0)$. (The reader is advised to sketch the graphs of ξ_0 and η_1 in the square $[-1,1]\times[-1,1]$.) We have $\partial G/\partial s_0 = 0$ iff $s_0 = \eta_1(s_1)$. Since $\partial^2 G/\partial s_0^2 > 0$ on $[-1,1]$, the minima of G are necessarily on the graph $s_0 = \eta_1(s_1)$. Set $g(s_1) = G(\eta_1(s_1), s_1)$. Differentiating gives

$$g'(s_1) = \beta J p(1-p)[\xi_0(s_1) - \eta_1(s_1)], \tag{4.26}$$

and we can read off the minima:

Case I: $h = 0$. Now $\xi_0(0) = \eta_1(0) = 0$, the graph of η_1 is increasing and crosses the square from left to right, and $\lim_{s_1 \to \pm 1} \xi_0(s_1) = \pm\infty$. Furthermore, on $[0,1]$ η_1' is strictly decreasing but ξ_0' strictly increasing, and vice versa on $[-1,0]$. From this it follows that if

$$\eta_1'(0) \leqq \xi_0'(0) \tag{4.27}$$

g takes its unique minimum at 0, whereas if

$$\eta_1'(0) > \xi_0'(0) \tag{4.28}$$

g takes its minima at $\pm z_1$ for the unique positive number z_1 that satisfies $\xi_0(z_1) = \eta_1(z_1)$. (4.28) is equivalent to

$$\beta^2(1 + J^2)p(1-p) + \beta(2p-1) - 1 > 0. \tag{4.29}$$

This quadratic has a unique positive root β_c given by (4.3), so (4.27) and (4.28) are equivalent to $\beta \leqq \beta_c$ and $\beta > \beta_c$, respectively.

Case II: $h \neq 0$. Suppose first that $h > 0$. Compared to case I this amounts to shifting the graph of $s_0 = \eta_1(s_1)$ up and the graph of $s_0 = \xi_0(s_1)$ down. Thus the graphs of η_1 and ξ_0 no longer intersect at $(0,0)$. But there is a unique point of intersection (z_1, z_0) in $(0,1) \times (0,1)$ where the graph of ξ_0 crosses η_1 from below to above, and hence z_1 is a local minimum of g. There can be other local minima with either $s_0 < 0$ or $s_1 < 0$, but it is easy to check directly from (4.24)–(4.25) that $G(s_0, s_1) > G(|s_0|, |s_1|)$ unless both $s_0 > 0$ and $s_1 > 0$. Thus the global minimum of G is attained at (z_0, z_1). The case $h < 0$ follows from this because $K(\nu)$ and $F_{h=0}(\nu)$ are invariant under the spin flip $(s_0, s_1) \mapsto (-s_0, -s_1)$.

We have found the measures in $\mathcal{K}_1$ in terms of the conditional expected spins (z_0, z_1) specified earlier after (4.3). We emphasize again that the elements of $\mathcal{K}_1$ are measures on $\mathcal{S} \times \mathcal{X}$, and consequently the measures $j_\infty(\nu)$ for $\nu \in \mathcal{K}_1$ live on $\Omega \times \Sigma$:

- $\beta \leqq \beta_c$, $h = 0$: $\mathcal{K}_1 = \{\lambda_o \otimes \pi_o\}$ and $j_\infty(\lambda_o \otimes \pi_o) = \lambda \otimes \pi$.
- $\beta > \beta_c$, $h = 0$: $\mathcal{K}_1 = \{\nu_+, \nu_-\}$, where $\nu_\pm(ds, dx) = \psi^x_\pm(ds)\,\pi_o(dx)$ (recall definition (4.5)), and $j_\infty(\nu_\pm) = \Psi_\pm \equiv \pi \otimes \mu^{\cdot}_\pm$.
- $\beta > 0$, $h \neq 0$: $\mathcal{K}_1 = \{\nu\}$, where $\nu(ds, dx) = \psi^x(ds)\,\pi_o(dx)$ (recall definition (4.4)), and $j_\infty(\nu) = \Psi \equiv \pi \otimes \mu^{\cdot}$.

The next step is to prove that the measures $\zeta^{\mathbf{x}}_n = \gamma^{\mathbf{x}}_n(\overline{\mathbf{R}}_n(\,\cdot\,))$ converge, where $\overline{\mathbf{R}}_n$ is the empirical field of the skew model,

$$\overline{\mathbf{R}}_n = \frac{1}{n}\sum_{i=0}^{n-1} \delta_{(\theta_i\sigma, \theta_i\mathbf{x})}.$$

Only in the case $\beta > \beta_c$, $h = 0$ can we have more than one limit point. The possible limit points are $t\Psi_+ + (1-t)\Psi_-$ for $0 \leqq t \leqq 1$. By the spin-flip symmetry of the Hamiltonian

$$\zeta^{\mathbf{x}}_n(\sigma_1) = \frac{1}{n}\gamma^{\mathbf{x}}_n\Big(\sum_{i=1}^{n} \sigma_i\Big) = 0$$

for all n and $\mathbf{x}$. A straightforward computation gives

$$t\Psi_+(\sigma_1) + (1-t)\Psi_-(\sigma_1) = (2t-1)((1-p)z_0 + pz_1)$$

which must equal 0, so $t = 1/2$ since $(1-p)z_0 + pz_1 > 0$. Theorem 2.11 now yields

4.30. Lemma. *For all* $\mathbf{x}$ *generic for* π*,*

$$\lim_{n\to\infty} \zeta_n^{\mathbf{x}} = \begin{cases} \lambda \otimes \pi & \text{in case } h = 0 \text{ and } \beta \leqq \beta_c, \\ (\Psi_+ + \Psi_-)/2 & \text{in case } h = 0 \text{ and } \beta > \beta_c, \\ \Psi & \text{in case } h \neq 0. \end{cases}$$

Proof of Theorem 4.6. Apply Theorem 2.15, Proposition 2.17, and Lemma 4.30. □

Proof of (4.7) and (4.8). Ω-marginals of the measures $j_\infty(\nu)$, $\nu \in \mathcal{K}_1$, specified above give the zeroes of the rate function I_∞. The contraction $Q \mapsto Q(\sigma_1)$ gives the zeroes of the rate function S. □

(4.9) follows for $h = 0$ from the spin-flip symmetry and for $h \neq 0$ from (4.8).

4.31. Lemma.

(1) $z_0(\beta, h)$ *and* $z_1(\beta, h)$ *are strictly increasing functions of* h*.*
(2) $z_0(\beta, 0)$ *and* $z_1(\beta, 0)$ *are strictly increasing functions of* $\beta \geqq \beta_c$*.*

Proof. (1) It suffices to consider $h \geqq 0$. Fix β, let $z_x^h = z_x(\beta, h)$, and write ξ_x^h for the functions defined in (4.2) to specify the h. Let $0 \leqq k < h$. Then

$$\xi_x^h = \xi_x^k + (k-h)/Jp < \xi_x^k. \tag{4.32}$$

In particular, $\xi_1^k(\eta_1^k(s_1)) = s_1 = \xi_1^h(\eta_1^h(s_1)) < \xi_1^k(\eta_1^h(s_1))$, so by the monotonicity of ξ_1^k

$$\eta_1^k < \eta_1^h. \tag{4.33}$$

Now let $t \geqq z_1^h$. Since $(\eta_1^h)'$ decreases and $(\xi_0^h)'$ increases on $[0,1)$, $(\xi_0^h)'(z_1^h) > (\eta_1^h)'(z_1^h)$ because $g''(z_1^h) > 0$ (recall (4.26)), and $\eta_1^h(z_1^h) = \xi_0^h(z_1^h)$, we have $\eta_1^h(t) \leqq \xi_0^h(t)$. This together with (4.32) and (4.33) gives $\eta_1^k(t) < \xi_0^k(t)$, which implies that t cannot equal z_1^k. Since this holds for all $t \geqq z_1^h$, it must be that $z_1^k < z_1^h$. (4.33) and the monotonicity of η_1^h then give $z_0^k = \eta_1^k(z_1^k) < \eta_1^h(z_1^h) = z_0^h$.

(2) Upon replacing k and h by β' and β such that $\beta_c \leqq \beta' < \beta$, the above argument can be repeated with obvious changes. □

Proof of (4.10). Part (1) of Lemma 4.31 implies that the magnetization $m(\beta, h)$ is a strictly increasing function of h. We may let $h \searrow 0$ and $h \nearrow 0$ in the equations $z_x^h = \xi_x^h(z_{1-x}^h)$, $x = 0, 1$, and since the limits of z_x^h satisfy the equations for $h = 0$, they are the solutions $(z_0(\beta, 0), z_1(\beta, 0))$ and $(-z_0(\beta, 0), -z_1(\beta, 0))$. □

Proof of Proposition 4.11. Assuming $\beta > \beta_c$ or $h > 0$ implies that $z_0, z_1 > 0$. Use (4.2) and the fact that $\log(1+t)/(1-t)$ is strictly increasing: $z_1 > z_0$ implies

$$\begin{aligned} \beta J p z_1 - \beta(1-p) z_0 + \beta h &= \frac{1}{2} \log \frac{1+z_0}{1-z_0} \\ &< \frac{1}{2} \log \frac{1+z_1}{1-z_1} = \beta J(1-p) z_0 + \beta p z_1 + \beta h \\ \implies \quad & \beta p(J-1) z_1 < \beta(1+J)(1-p) z_0 < \beta(1+J)(1-p) z_1 \\ \implies \quad & J(2p-1) < 1, \end{aligned}$$

so either $p \leqq 1/2$ or $J < (2p-1)^{-1}$. Similar reasoning shows that $z_1 < z_0$ forces $J(2p-1) > 1$ and $z_1 = z_0$ leads to $J(2p-1) = 1$. $\square$

Proof of Lemma 4.12. Since $z_x(\beta, 0) = 0$ for $\beta \leqq \beta_c$ and the strict increasingness for $\beta \geqq \beta_c$ was proved in Lemma 4.31, we only need to show that, if $\beta \searrow \beta'$ or $\beta \nearrow \beta'$, $\beta' \geqq \beta_c$, then $\lim_{\beta \to \beta'} z_x(\beta, 0) = z_x(\beta', 0)$. But this is immediate upon passing to the limit in the equations (4.2). $\square$

Proof of Theorem 4.13. To save space we only outline the arguments here and leave out the tedious calculations. To prove (4.14)–(4.15), apply the expansion

$$\frac{1}{2}\log\frac{1+s}{1-s} = s + \frac{s^3}{3} + O(s^5) \tag{4.34}$$

to (4.2). To prove (4.16) differentiate

$$z_x(\beta, h) = \xi_x(\beta, h, \xi_{1-x}(\beta, h, z_x(\beta, h))) \tag{4.35}$$

implicitly with respect to h at $h = 0$.

(4.17) Let us first verify the remarks on specific energy stated before Theorem 4.13. Since $-\beta H_n^{\mathbf{x}} = nF(\overline{\mathbf{L}}_n)$, specific energy is given by

$$u(\beta, h) = -\beta^{-1} \lim_{n\to\infty} \gamma_n^{\mathbf{x}}(F(\overline{\mathbf{L}}_n)),$$

hence it is clear from the LDP's (use Theorem 2.21 and (4.8)) that

$$u(\beta, h) = -\beta^{-1} F(z_0(\beta, h), z_1(\beta, h)).$$

(In the case $\beta > \beta_c$, $h = 0$ the additional fact $F(z_0, z_1) = F(-z_0, -z_1)$ is needed, for the limit above is $-\beta^{-1}\big[F(z_0, z_1) + F(-z_0, -z_1)\big]/2$.) The continuity of $u(\beta, 0)$ comes from Lemma 4.12, and for $\beta \leqq \beta_c$, $u(\beta, 0) = -\beta^{-1} F(0, 0) = 0$. For $\beta > \beta_c$ and $x = 0$ and 1, the partial derivatives $\partial z_x / \partial\beta$ exist by the implicit function theorem applied to (4.35). Since (z_0, z_1) minimizes $K - F$,

$$\frac{\partial F}{\partial z_x}(z_0, z_1) = \frac{\partial K}{\partial z_x}(z_0, z_1) > 0,$$

where the inequality follows by direct computation from (4.25) since now $z_x > 0$. From all this

$$\begin{aligned}\frac{\partial u}{\partial \beta} = \frac{\partial}{\partial\beta}(-\beta^{-1}F) &= -\beta^{-1}\frac{\partial F}{\partial z_0}\frac{\partial z_0}{\partial \beta} - \beta^{-1}\frac{\partial F}{\partial z_1}\frac{\partial z_1}{\partial\beta} \\ &= -\beta^{-1}\frac{\partial K}{\partial z_0}\frac{\partial z_0}{\partial \beta} - \beta^{-1}\frac{\partial K}{\partial z_1}\frac{\partial z_1}{\partial\beta} < 0,\end{aligned} \tag{4.36}$$

where the second equality follows from (4.24) and the last inequality from Lemma 4.12. This shows that $u(\beta, 0) < 0$ for $\beta > \beta_c$.

We now know that $C_{h=0}(\beta) = 0$ for $\beta < \beta_c$. We will show that $\lim_{\beta\searrow\beta_c} C_{h=0} > 0$ to establish the discontinuity claimed in (4.17). For $\beta > \beta_c$, start with (4.36) and use $z_0 = \xi_0(z_1)$ to get

$$\beta^{-1}C_{h=0} = \left(\frac{\partial K}{\partial z_1} + \frac{\partial K}{\partial z_0}\frac{\partial \xi_0}{\partial z_1}\right)\frac{\partial z_1}{\partial \beta} + \frac{\partial K}{\partial z_0}\frac{\partial \xi_0}{\partial \beta} \equiv D(\beta) + \frac{\partial K}{\partial z_0}\frac{\partial \xi_0}{\partial \beta}.$$

The last term on the right vanishes as $\beta \searrow \beta_c$ so we ignore it. Differentiate (4.36) with respect to β, solve for $\partial z_1/\partial\beta$, and insert this in $D(\beta)$. Do the other derivatives in $D(\beta)$ directly from (4.2) and (4.25). Now apply (4.34) repeatedly to estimate various parts of $D(\beta)$ to conclude that $\lim_{\beta\searrow\beta_c} D(\beta) > 0$. $\Box$

5 Proofs of the general theorems

We start by proving the large deviation results presented at the end of Section 2. Let $\mathcal{E}_n$ denote the σ-field on $\Omega\times\Sigma$ generated by $\mathcal{B}_n^\Omega$ and $\mathcal{B}^\Sigma$. Define $\Phi \in \mathcal{M}_\Theta(\Omega\times\Sigma)$ by $\Phi(d\sigma, d\mathbf{y}) = P^{\mathbf{y}}(d\sigma)\,\pi(d\mathbf{y})$, and let $\Gamma \in \mathcal{M}_\Theta(\Omega\times\Sigma)$ have Σ-marginal π, or $\Gamma_\Sigma = \pi$ for short. Then by Theorem 2.13 in [Se1], the specific entropy

$$h^{\mathcal{E}}(\Gamma\,|\,\Phi) = \lim_{n\to\infty}\frac{1}{|V_n|}H_{\mathcal{E}_n}(\Gamma\,|\,\Phi)$$

exists, and $k(Q) = \inf_\Gamma h^{\mathcal{E}}(\Gamma\,|\,\Phi)$ where the infimum is over Γ with marginals Q and π. Recall the probability measures $\Psi_\nu^{\cdot}$ and $j_\infty(\nu)$ on Ω and Ψ_ν on $\Omega\times\Sigma$, defined in (2.8) and on the line above.

5.1. Lemma. $K(\nu) = k(j_\infty(\nu)) = h^{\mathcal{E}}(\Psi_\nu|\Phi)$ *for* $\nu \in \mathcal{M}_1(\mathcal{S})$ *such that* $K(\nu) < \infty$.

Proof. Since entropies of product measures are sums of the entropies of the factors, we get

$$\begin{aligned} H_{\mathcal{E}_n}(\Psi_\nu\,|\,\Phi) &= \int H_{\mathcal{E}_n}(\Psi_\nu^{\mathbf{y}}\,|\,P^{\mathbf{y}})\,\pi(d\mathbf{y}) \\ &= |V_n|\int H(\psi_\nu^{\mathbf{y}}\,|\,p^{\mathbf{y}})\,\pi(d\mathbf{y}) \\ &= |V_n|\,H(\psi_\nu\,|\,\varphi). \end{aligned}$$

The first and last equalities follow from the conditional entropy formula ((4.4.8) in [DS] or (10.2) in [V]). Hence

$$k(j_\infty(\nu)) \le h^{\mathcal{E}}(\Psi_\nu\,|\,\Phi) = H(\psi_\nu\,|\,\varphi) = K(\nu) \le k(j_\infty(\nu)),$$

where the last inequality follows from the contraction principle $K(\nu) = \inf_{Q_0=\nu} k(Q)$. $\Box$

Order $\mathbb{Z}^d$ lexicographically, and let $W^- = \{\mathbf{i}\in\mathbb{Z}^d : \mathbf{i} < \mathbf{0}\}$ be the past of the origin. For $\Gamma \in \mathcal{M}_\Theta(\Omega\times\Sigma)$, let $\Gamma_0^{\sigma_{W^-},\mathbf{y}}$ be a conditional distribution of the spin $\sigma_{\mathbf{0}}$ at the origin under Γ, given the past $\sigma_{W^-} = \{\sigma_{\mathbf{i}} : \mathbf{i}\in W^-\}$ and $\mathbf{y}$. Analogously to (3.13) in [Fö], we have the representation

$$h^{\mathcal{E}}(\Gamma\,|\,\Phi) = \int H(\Gamma_0^{\sigma_{W^-},\mathbf{y}}\,|\,p^{\mathbf{y}})\,\Gamma(d\sigma_{W^-}, d\mathbf{y})$$

whenever $\Gamma_\Sigma = \pi$.

5.2. Lemma. *Suppose $k(Q) < \infty$. Then $k(Q) > k(j_\infty(Q_0))$ unless $Q = j_\infty(Q_0)$.*

Proof. Pick $\Gamma \in \mathcal{M}_\Theta(\Omega \times \Sigma)$ with marginals Q and π so that $k(Q) = h^{\mathcal{E}}(\Gamma \,|\, \Phi)$. Let $\gamma \in \mathcal{M}_1(\mathcal{S} \times \Sigma)$ be the $\mathcal{E}_0$-marginal of Γ. Let ψ_{Q_0} be the minimizer for Q_0 in (2.5).

$$
\begin{aligned}
k(Q) = h^{\mathcal{E}}(\Gamma \,|\, \Phi) &= \int H(\Gamma_0^{\sigma_{W^-},\mathbf{y}} \,|\, p^{\mathbf{y}})\, \Gamma(d\sigma_{W^-}, d\mathbf{y}) \\
&\geq \int H\Big(\int \Gamma_0^{\sigma_{W^-},\mathbf{y}}\, \Gamma^{\mathbf{y}}(d\sigma_{W^-}) \,\Big|\, p^{\mathbf{y}}\Big)\, \pi(d\mathbf{y}) = \int H(\gamma^{\mathbf{y}} \,|\, p^{\mathbf{y}})\, \pi(d\mathbf{y}) = H(\gamma \,|\, \varphi) \\
&\geq K(Q_0) = k(j_\infty(Q_0)).
\end{aligned}
$$

The first inequality above follows from convexity, the second from (2.5). Forcing the second inequality to be an equality gives $\gamma = \psi_{Q_0}$. Equality in the first inequality together with the strict convexity of relative entropy implies that $\Gamma_0^{\sigma_{W^-},\mathbf{y}} = \gamma^{\mathbf{y}}$ Γ-a.s. In other words, $\sigma_{\mathbf{0}}$ is independent of the past σ_{W^-}, given $\mathbf{y}$, and by shift-invariance this independence holds for all spins. We get $\Gamma^{\mathbf{y}}(d\sigma) = \bigotimes_{\mathbf{i}} \psi_{Q_0}^{\theta_{\mathbf{i}}\mathbf{y}}(d\sigma_{\mathbf{i}})$, from which $Q = \int \Gamma^{\mathbf{y}}\, \pi(d\mathbf{y}) = j_\infty(Q_0)$. $\square$

Proof of part of Lemma 2.20.

$$
\begin{aligned}
r_\infty &= \inf_Q \{k(Q) - F^u(Q_0)\} \\
&= \inf_Q \{k(j_\infty(Q_0)) - F^u(Q_0)\} \\
&= \inf_Q \{K(Q_0) - F^u(Q_0)\} = r_1.
\end{aligned}
$$

$\mathcal{K}_\infty = j_\infty(\mathcal{K}_1)$ is now immediate from Lemma 5.2. I_∞ is lower semicontinuous by its definition, so from $\{I_\infty \leq b\} \subset \{k \leq r_1 + b + c\}$ we see that $\{I_\infty \leq b\}$ is a closed subset of a compact set, hence itself compact. Taking $b = 0$ shows that $\mathcal{K}_\infty$ is compact. Since $j_\infty(\nu)_0 = \nu$, the projection $Q \mapsto Q_0$ is a continuous one-to-one map of the compact set $\mathcal{K}_\infty$ onto $\mathcal{K}_1$, hence its inverse j_∞ is a homeomorphism. The remaining statement of Lemma 2.20 about $\mathcal{K}^{\boldsymbol{\cdot}}_\infty$ will be proved in Lemma 5.9 below where it will follow naturally from considerations in the skew model setting. $\square$

Proof of Theorem 2.21. The inequalities follow from the LDP of Theorem 2.1 in [Se1]: Apply Lemmas 2.1.7 and 2.1.8 of [DS] to the upper semicontinuous functions $\log \mathbf{1}_{\overline{E}}(Q) + F^u(Q_0)$ and $F^u(Q_0)$ and to the lower semicontinuous functions $\log \mathbf{1}_{E^\circ}(Q) + F_\ell(Q_0)$ and $F_\ell(Q_0)$, defined for $Q \in \mathcal{M}_1(\Omega)$. Use (2.6). $\square$

Proof of Theorem 2.11. Since $\mathcal{K}_\infty = \{I_\infty = 0\}$ is compact, the upper bound of (2.22), applied to the complements of open neighborhoods of $\mathcal{K}_\infty$, implies that the laws $\gamma_n^{\mathbf{x}}(\mathbf{R}_n \in \cdot\,)$ are tight. Consequently so are the measures $\zeta_n^{\mathbf{x}} = \gamma_n^{\mathbf{x}}(\mathbf{R}_n(\,\cdot\,))$. Suppose $\zeta_{n_j}^{\mathbf{x}} \to Q$ as $j \to \infty$. By passing to a subsequence, we may assume that $\gamma_{n_j}^{\mathbf{x}}(\mathbf{R}_{n_j} \in \cdot\,) \to \Pi$. By the upper bound of (2.22) $\Pi(\mathcal{K}_\infty) = 1$, and so

$$
Q = \int_{\mathcal{K}_\infty} M\, \Pi(dM). \tag{5.3}
$$

Let $\Lambda(d\nu) = \Pi(M_0 \in d\nu)$ be the image of Π under the projection $M \mapsto M_0$. Since $M = j_\infty(M_0)$ for $M \in \mathcal{K}_\infty$, (5.3) implies (2.12). Since $\mathbf{L}_n$ is the projection of $\mathbf{R}_n$, $\gamma^{\mathbf{x}}_{n_j}(\mathbf{L}_{n_j} \in \cdot\,) \to \Lambda$. The last statement of the theorem follows from (2.12) because the $j_\infty(\nu)$'s are i.i.d. under Assumption B. $\square$

Proof of Remark 2.13. Since the three sequences in question are tight, it suffices to show that limit points Q, Π, and Λ of these sequences determine each other uniquely. Q and Λ are obviously determined by Π as its images. Conversely, since j_∞ is the inverse of the projection, Λ determines Π through $\Pi = \Lambda(j_\infty \in \cdot\,)$. Assuming π ergodic guarantees that all the measures Ψ_ν are ergodic. In particular, $\mathcal{K}_\infty$ is a set of ergodic measures. Thus (5.3) is the ergodic decomposition of Q, and Q determines Π by the uniqueness of the ergodic decomposition. $\square$

For the remainder of this section we work under Assumption B. The first thing to check is that permuting finitely many coordinates of $\mathbf{x}$ does not affect the limiting behavior of $\zeta^{\mathbf{x}}_n$. So suppose τ is a permutation of V_k for some k, and set $(\tau\mathbf{x})_{\mathbf{i}} = \mathbf{x}_{\tau(\mathbf{i})}$ for $\mathbf{i} \in V_k$, $(\tau\mathbf{x})_{\mathbf{i}} = \mathbf{x}_{\mathbf{i}}$ for $\mathbf{i} \in V_k^c$. Let $f \in b\mathcal{B}^\Omega_m$ for some m.

5.4. Lemma. $\lim_{n\to\infty} |\zeta^{\mathbf{x}}_n(f) - \zeta^{\tau\mathbf{x}}_n(f)| = 0.$

Proof. Letting τ act on Ω the same way it acts on Σ, it is clear that $P^{\tau\mathbf{x}} = P^{\mathbf{x}} \circ \tau^{-1}$. If $n > k$, $F(\mathbf{L}_n)$ is invariant under τ, and consequently

$$\zeta^{\tau\mathbf{x}}_n(f) = P^{\tau\mathbf{x}}(\mathbf{R}_n(f)\, e^{|V_n|\,F(\mathbf{L}_n)})/Z^{\tau\mathbf{x}}_n = P^{\mathbf{x}}(\mathbf{R}_n(f) \circ \tau \,\cdot\, e^{|V_n|\,F(\mathbf{L}_n)})/Z^{\mathbf{x}}_n.$$

Now notice that $f(\theta_{\mathbf{i}}\tau(\sigma)) = f(\theta_{\mathbf{i}}\sigma)$ whenever $\mathbf{i} + V_m \cap V_k = \emptyset$. The fraction of sites in V_n for which this condition fails vanishes as $n \to \infty$, hence

$$\lim_{n\to\infty} \|\mathbf{R}_n(f) \circ \tau - \mathbf{R}_n(f)\| = 0. \quad \square$$

Proof of Proposition 2.14. Without loss of generality we prove the proposition for the original sequence $\{n\}$. To construct g, let $\{B_j\}$ be a countable basis for the topology of $\mathcal{M}_1(\Omega)$ with the property that only finitely many B_j have diameter $\geq \varepsilon$ for any fixed $\varepsilon > 0$. Set

$$g(\mathbf{x}, Q) = \limsup_{j\to\infty} \big[\mathbf{1}_{B_j}(Q) \,\cdot\, \limsup_{n\to\infty} \mathbf{1}_{B_j}(\zeta^{\mathbf{x}}_n)\big].$$

Clearly $g(\mathbf{x}, Q)$ is either 0 or 1, and it is 1 precisely when there are infinitely many B_j that both contain Q and are visited infinitely often by $\zeta^{\mathbf{x}}_n$. Since diam $B_j \to 0$, this is equivalent to saying that Q is a limit point of $\{\zeta^{\mathbf{x}}_n\}_{n=1}^\infty$. The previous lemma then implies that $g(\mathbf{x}, Q) = g(\tau\mathbf{x}, Q)$ for all finite permutations τ.

To construct Σ_0, employ the functions

$$h_j(\mathbf{x}) = \lim_{n\to\infty} \left[\max_{k\geq n} \mathbf{1}_{B_j^{(1/n)}}(\zeta^{\mathbf{x}}_k)\right].$$

Here $B_j^{(1/n)}$ is a $1/n$-fattening of B_j. So for $\mathbf{x}$ and $\mathbf{y}$ such that $\{\zeta^{\mathbf{x}}_n\}_{n=1}^\infty$ and $\{\zeta^{\mathbf{y}}_n\}_{n=1}^\infty$ are tight, $h_j(\mathbf{x}) = 1$ iff $\{\zeta^{\mathbf{x}}_n\}_{n=1}^\infty$ has a limit point in $\overline{B}_j$, and $\{\zeta^{\mathbf{x}}_n\}_{n=1}^\infty$ and $\{\zeta^{\mathbf{y}}_n\}_{n=1}^\infty$

have the same limit points iff $h_j(\mathbf{x}) = h_j(\mathbf{y})$ for all j. From the above lemma again we know that h_j is invariant under finite permutations. Set

$$\Sigma_0 = \left\{ \mathbf{x} : \; h_j(\mathbf{x}) = \int h_j \, d\pi \text{ for all } j \right\}.$$

An application of the Hewitt-Savage 0-1 law concludes the proof of the proposition. $\square$

Our next goal is Theorem 2.15 about the convergence of the maps $\gamma_n^{\cdot}$. First a simple general lemma about preserving tightness under averaging: Let Z and W be Polish spaces, $\kappa \in \mathcal{M}_1(W)$, and $w \mapsto \rho_n^w$, $n = 1, 2, 3, \ldots$, a collection of measurable maps from W into $\mathcal{M}_1(Z)$. Define $\mu_n \in \mathcal{M}_1(Z)$ by $\mu_n = \int \rho_n^w \, \pi(dw)$.

5.5. Lemma. *If $\{\rho_n^w\}_{n=1}^\infty$ is tight for κ-a.e. w, then $\{\mu_n\}_{n=1}^\infty$ is also tight.*

Proof. Let $\{z_j\}$ be a countable dense subset of Z, and let

$$A_{k,\ell} = \bigcup_{j=1}^{\ell} \overline{B}_{1/k}(z_j),$$

where $\overline{B}_{1/k}(z_j)$ is the closed $1/k$-radius ball around z_j. Let $\varepsilon > 0$ and find $\varepsilon_k > 0$ such that $2\sum_k \varepsilon_k < \varepsilon$. Let W_0 be the set of $w \in W$ for which $\{\rho_n^w\}_{n=1}^\infty$ is tight. For each $w \in W_0$ and k find $\ell(w,k)$ such that $\inf_n \rho_n^w(A_{k,\ell(w,k)}) > 1 - \varepsilon_k$ and $w \mapsto \ell(w,k)$ is measurable. Let

$$W_{k,\ell} = \{w \in W_0 : \ell(w,k) = \ell\}.$$

For each k find $m(k)$ such that $\kappa(\cup_{\ell=1}^{m(k)} W_{k,\ell}) > 1 - \varepsilon_k$. Then for all k and n,

$$\begin{aligned}
\mu_n(A_{k,m(k)}) =& \sum_{\ell=1}^{\infty} \int_{W_{k,\ell}} \rho_n^w(A_{k,m(k)}) \, \kappa(dw) \\
\geq& \sum_{\ell=1}^{m(k)} \int_{W_{k,\ell}} (1-\varepsilon_k) \, \kappa(dw) \\
\geq& (1-\varepsilon_k)^2 \geq 1 - 2\varepsilon_k.
\end{aligned}$$

Now set $A = \cap_k A_{k,m(k)}$, and deduce $\mu_n(A) > 1 - \varepsilon$ for all n. $A \subset \cup_{j=1}^{m(k)} \overline{B}_{1/k}(z_j)$ for all k, so A is compact by Lemma 3.1 on p. 29 of [Pa]. $\square$

A further useful technical twist is periodization: Given $\sigma \in \Omega$, define $\sigma^{(n)} \in \Omega$ by $\sigma_{\mathbf{i}}^{(n)} = \sigma_{\mathbf{i}}$ for $\mathbf{i} \in V_n$ and by requiring $\sigma_{\mathbf{i}+(2n+1)\mathbf{j}}^{(n)} = \sigma_{\mathbf{i}}^{(n)}$ for all $\mathbf{i}$ and $\mathbf{j}$. Using this we form the stationary version of the empirical process:

$$\mathbf{R}_n^s = \frac{1}{|V_n|} \sum_{\mathbf{i} \in V_n} \delta_{\theta_{\mathbf{i}} \sigma^{(n)}}.$$

Under Assumption B we can perform the same operation also on quenched variables.

5.6. Lemma. *Let $f \in b\mathcal{B}^\Omega$, $g \in b\mathcal{B}^\Sigma$, and $\mathbf{i} \in \mathbb{Z}^d$.*

(1) $\displaystyle\int f(\theta_{\mathbf{i}}\sigma^{(n)})\, P^{\mathbf{y}^{(n)}}(d\sigma) = \int f(\sigma^{(n)})\, P^{\theta_{\mathbf{i}}\mathbf{y}^{(n)}}(d\sigma)$.

(2) $\displaystyle\int g(\theta_{\mathbf{i}}\mathbf{y}^{(n)})\, \pi(d\mathbf{y}) = \int g(\mathbf{y}^{(n)})\, \pi(d\mathbf{y})$.

(3) *If $f \in b\mathcal{B}^\Omega_n$, then* $\displaystyle\frac{1}{|V_n|}\sum_{\mathbf{i}\in V_n} \gamma_n^{\theta_{\mathbf{i}}\mathbf{y}^{(n)}}(f) = \gamma_n^{\mathbf{y}^{(n)}}(\mathbf{R}_n^s(f))$.

Proof. (1) It suffices to consider functions of the form $f(\sigma) = \prod_{\mathbf{j}\in W} f_{\mathbf{j}}(\sigma_{\mathbf{j}})$ for a finite set W. Write $\mathbf{i} \equiv \mathbf{j}$ if $\mathbf{i} - \mathbf{j} \in (2n+1)\mathbb{Z}^d$. Then

$$\begin{aligned}\int f(\theta_{\mathbf{i}}\sigma^{(n)})\, P^{\mathbf{y}^{(n)}}(d\sigma) &= \int \prod_{\mathbf{j}} f_{\mathbf{j}}(\sigma^{(n)}_{\mathbf{i}+\mathbf{j}})\, P^{\mathbf{y}^{(n)}}(d\sigma) = \prod_{\mathbf{k}\in V_n} \int \prod_{\mathbf{j}\,:\,\mathbf{j}+\mathbf{i}\equiv\mathbf{k}} f_{\mathbf{j}}\, dp^{\mathbf{y}_{\mathbf{k}}} \\ &= \prod_{\mathbf{l}\in V_n} \int \prod_{\mathbf{j}\,:\,\mathbf{j}\equiv\mathbf{l}} f_{\mathbf{j}}\, dp^{\mathbf{y}^{(n)}_{\mathbf{l}+\mathbf{i}}} = \prod_{\mathbf{l}\in V_n} \int \prod_{\mathbf{j}\,:\,\mathbf{j}\equiv\mathbf{l}} f_{\mathbf{j}}\, dp^{(\theta_{\mathbf{i}}\mathbf{y}^{(n)})_{\mathbf{l}}} \\ &= \int f(\sigma^{(n)})\, P^{\theta_{\mathbf{i}}\mathbf{y}^{(n)}}(d\sigma).\end{aligned}$$

(2) is a special case of (1).

For (3), note that $\mathbf{L}_n(\sigma) = \mathbf{L}_n(\sigma^{(n)}) = \mathbf{L}_n(\theta_{\mathbf{i}}\sigma^{(n)})$ and that $f(\sigma) = f(\sigma^{(n)})$ for $f \in b\mathcal{B}^\Omega_n$. If $\gamma_n^{\mathbf{y}^{(n)}} = P^{\mathbf{y}^{(n)}}$, then also $\gamma_n^{\theta_{\mathbf{i}}\mathbf{y}^{(n)}} = P^{\theta_{\mathbf{i}}\mathbf{y}^{(n)}}$, and (3) follows from (1). Otherwise use (1) to get

$$\begin{aligned}\gamma_n^{\theta_{\mathbf{i}}\mathbf{y}^{(n)}}(f) =& P^{\theta_{\mathbf{i}}\mathbf{y}^{(n)}}(f\, e^{|V_n|\, F(\mathbf{L}_n)})/Z_n^{\theta_{\mathbf{i}}\mathbf{y}^{(n)}} \\ =& \int f(\theta_{\mathbf{i}}\sigma^{(n)})\, e^{|V_n|\, F(\mathbf{L}_n(\sigma))}\, P^{\mathbf{y}^{(n)}}(d\sigma)/Z_n^{\mathbf{y}^{(n)}} = \int f(\theta_{\mathbf{i}}\sigma^{(n)})\, \gamma_n^{\mathbf{y}^{(n)}}(d\sigma).\end{aligned}$$

Now average over $\mathbf{i} \in V_n$ to conclude. □

We now introduce the skew model associated with the setting of Assumption B: The quenched variable is adjoined to the process as a deterministic component by defining the measures $\overline{p}^{\mathbf{x}} = p^{\mathbf{x_0}} \otimes \delta_{\mathbf{x_0}}$ and $\overline{P}^{\mathbf{x}} = P^{\mathbf{x}} \otimes \delta_{\mathbf{x}}$ on $\overline{\mathcal{S}} = \mathcal{S} \times \mathcal{X}$ and $\overline{\Omega} = \Omega \times \Sigma$, respectively. This way we create a new setting, notationally distinguished by overbars, that satisfies the same assumptions as the original model. By Theorem 3.3 in [Se1], the entropy of $\overline{\nu} \in \mathcal{M}_1(\overline{\mathcal{S}})$ is given by

$$\overline{K}(\overline{\nu}) = \begin{cases} H(\overline{\nu} \,|\, \varphi_o) & \text{if } \overline{\nu} \text{ has marginal } \pi_o, \\ \infty & \text{otherwise.} \end{cases} \tag{5.7}$$

(In the setting of Section 3 in [Se1], $U = \{\mathbf{0}\}$ because $\mathbf{x} \mapsto \overline{p}^{\mathbf{x}}$ is $\mathcal{B}_0^\Sigma$-measurable by Assumption B.) Write $\overline{\nu}_{\mathcal{S}}$ for the $\mathcal{S}$-marginal of a probability measure $\overline{\nu}$ on $\overline{\mathcal{S}}$. We get the LDP of the original model by a contraction to the marginal [V, Remark 1, p. 5], hence by the uniqueness of the rate function

$$K(\nu) = \inf_{\overline{\nu}:\overline{\nu}_{\mathcal{S}}=\nu} \overline{K}(\overline{\nu}), \tag{5.8}$$

which together with (5.7) justifies (2.9).

Define $\overline{F} : \mathcal{M}_1(\overline{\mathcal{S}}) \to [-\infty, c]$ by $\overline{F}(\overline{\nu}) = F(\overline{\nu}_{\mathcal{S}})$. Write $\overline{\mathbf{L}}_n$ and $\overline{\mathbf{R}}_n$ for the skew empirical distribution and field, so for example

$$\overline{\mathbf{R}}_n = \frac{1}{|V_n|} \sum_{\mathbf{i} \in V_n} \delta_{(\theta_{\mathbf{i}}\sigma, \theta_{\mathbf{i}}\mathbf{x})}.$$

Let $\overline{\gamma}_n^{\mathbf{x}}$ be defined as in (2.1), and $\overline{\zeta}_n^{\mathbf{x}}(\,\cdot\,) = \overline{\gamma}_n^{\mathbf{x}}(\overline{\mathbf{R}}_n(\,\cdot\,))$. Since $\overline{F}(\overline{\mathbf{L}}_n)$ does not depend on the Σ-valued coordinate, we have $\overline{\gamma}_n^{\mathbf{x}} = \gamma_n^{\mathbf{x}} \otimes \delta_{\mathbf{x}}$. The set

$$\overline{\mathcal{K}}_1 = \{\overline{\nu} \in \mathcal{M}_1(\overline{\mathcal{S}}) : \overline{K}(\overline{\nu}) - F^u(\overline{\nu}_{\mathcal{S}}) = \overline{r}_1\}$$

and its infinite-volume counterpart $\overline{\mathcal{K}}_\infty$ are defined as before, and they are related by the homeomorphism $\overline{\jmath}_\infty : \overline{\mathcal{K}}_1 \to \overline{\mathcal{K}}_\infty$.

5.9. Lemma.

(1) $\overline{\mathcal{K}}_\infty = \{\Psi_\nu : \nu \in \mathcal{K}_1\}$ *and the map* $\nu \mapsto \Psi_\nu$ *is a homeomorphism from* $\mathcal{K}_1$ *to* $\overline{\mathcal{K}}_\infty$.

(2) $\mathcal{K}^{\boldsymbol{\cdot}}_\infty = \{\Psi^{\boldsymbol{\cdot}}_\nu : \nu \in \mathcal{K}_1\}$ *and the map* $\nu \mapsto \Psi^{\boldsymbol{\cdot}}_\nu$ *is a homeomorphism from* $\mathcal{K}_1$ *to* $\mathcal{K}^{\boldsymbol{\cdot}}_\infty$.

(3) *The map* $(\nu, \mathbf{y}) \mapsto \Psi_\nu^{\mathbf{y}}$ *is jointly measurable on* $\mathcal{K}_1 \times \Sigma$.

Proof. (1) Since

$$\overline{r}_1 = \inf_{\nu \in \mathcal{M}_1(\mathcal{S})} \bigl\{ \inf_{\overline{\nu}_{\mathcal{S}} = \nu} \overline{K}(\overline{\nu}) - F^u(\nu) \bigr\},$$

it is clear from (2.9), (5.7), and the uniqueness of ψ_ν that $\overline{r}_1 = r_1$ and $\overline{\mathcal{K}}_1 = \{\psi_\nu : \nu \in \mathcal{K}_1\}$. Since $\mathcal{K}_1 \ni \nu \mapsto \psi_\nu \in \overline{\mathcal{K}}_1$ is the inverse of a one-to-one continuous map (the projection) on a compact set, it is a homeomorphism. By the part of Lemma 2.20 already proved, for (1) we need to show that $\overline{\jmath}_\infty(\psi_\nu) = \Psi_\nu$. But this is really only a matter of seeing through the formalities: Suppose $\overline{\nu}_{\mathcal{X}} = \pi_o$. Strictly speaking the measure $\overline{\psi}_{\overline{\nu}}$ associated to $\overline{\nu}$ by (2.9) lives on $\overline{\mathcal{S}} \times \mathcal{X} = \mathcal{S} \times \mathcal{X} \times \mathcal{X}$, but by (5.7) we can neglect the unnecessary extra $\mathcal{X}$-factor and identify $\overline{\psi}_{\overline{\nu}}$ with $\overline{\nu}$ itself. Under this identification the kernel $\overline{\psi}_{\overline{\nu}}^{x}$ associated to $\overline{\nu}$ becomes just the conditional distribution $\overline{\nu}^x$ of $\overline{\nu}$ on $\mathcal{S}$, given $x \in \mathcal{X}$, and then $\overline{\jmath}_\infty(\overline{\nu}) = \bigotimes_{\mathbf{i}} \overline{\nu}^{\mathbf{y}_{\mathbf{i}}}(d\sigma_{\mathbf{i}})\, \pi(d\mathbf{y})$. Taking $\overline{\nu} = \psi_\nu$ then gives $\overline{\jmath}_\infty(\psi_\nu) = \Psi_\nu$.

There is also an indirect argument via entropy: By Lemma 5.1, (5.7), and (3.6) in [Se1],

$$\overline{k}(\overline{\jmath}_\infty(\psi_\nu)) = \overline{K}(\psi_\nu) = H(\psi_\nu | \varphi_o) = K(\nu) = h^{\mathcal{E}}(\Psi_\nu | \Phi) = \overline{k}(\Psi_\nu).$$

Hence by Lemma 5.2 $\overline{\jmath}_\infty(\psi_\nu) = \Psi_\nu$.

(2) By Lemma 5.1 and (3.6) in [Se1], $k(\pi\Psi^{\boldsymbol{\cdot}}_\nu) = h^{\mathcal{E}}(\pi \otimes \Psi^{\boldsymbol{\cdot}}_\nu | \Phi) = \overline{k}(\pi \otimes \Psi^{\boldsymbol{\cdot}}_\nu)$, hence it is clear that the homeomorphism $\varrho^{\boldsymbol{\cdot}} \mapsto \pi \otimes \varrho^{\boldsymbol{\cdot}}$ from $\mathcal{M}^{\boldsymbol{\cdot}}_\pi$ into $\mathcal{M}_1(\Omega \times \Sigma)$ restricts to a homeomorphism from $\mathcal{K}^{\boldsymbol{\cdot}}_\infty$ onto $\overline{\mathcal{K}}_\infty$. (2) now follows from (1).

(3) Let $(\overline{\nu}, x) \mapsto q(\overline{\nu}, x)$ be a jointly measurable conditional distribution map on $\mathcal{M}_1(\overline{\mathcal{S}}) \times \mathcal{X}$, that is, $x \mapsto q(\overline{\nu}, x)$ is a version of $\overline{\nu}^x$. (Since the conditioning σ-field

$\mathcal{B}^{\mathcal{X}}$ is countably generated, $q(\overline{\nu}, x)$ can be defined by the martingale convergence theorem.) Write $\Psi_\nu^{\mathbf{y}} = \bigotimes_{\mathbf{i}} q(\psi_\nu, \mathbf{y}_{\mathbf{i}})$ and recall that $\nu \mapsto \psi_\nu$ is continuous. $\square$

Proof of Theorem 2.15. By Theorem 2.11 applied to the skew model and Lemma 5.5, the measures $\pi\overline{\zeta}_n^{\cdot} = \int \overline{\zeta}_n^{\mathbf{y}}\, \pi(d\mathbf{y})$ are tight. Thus the relative compactness of $\{\gamma_n^{\cdot}\}_{n=1}^{\infty}$ will follow from proving

$$\lim_{n\to\infty} \left| \int \gamma_n^{\mathbf{y}}(f^{\mathbf{y}})\, \pi(d\mathbf{y}) - \int \overline{\zeta}_n^{\mathbf{y}}(f)\, \pi(d\mathbf{y}) \right| = 0$$

for any $f \in b\mathcal{B}_m^{\Omega\times\Sigma}$, where the σ-field $\mathcal{B}_m^{\Omega\times\Sigma}$ on $\overline{\Omega}$ is generated by $(\sigma_{\mathbf{i}}, \mathbf{x}_{\mathbf{i}} : \mathbf{i} \in V_m)$. So fix such an f and let $n > m$. In the following calculation, use $\overline{\gamma}_n^{\mathbf{y}} = \gamma_n^{\mathbf{y}} \otimes \delta_{\mathbf{y}}$, the fact that $\overline{\gamma}_n^{\mathbf{y}}(f) = \overline{\gamma}_n^{\mathbf{y}^{(n)}}(f)$ by $\mathcal{B}_n^{\Sigma}$-measurability, Lemma 5.6(2) and (3), again $\mathcal{B}_n^{\Sigma}$-measurability, then the fact that

$$\lim_{n\to\infty} \|\overline{\mathbf{R}}_n(f) - \overline{\mathbf{R}}_n^s(f)\| = 0,$$

and finally the definition of $\overline{\zeta}_n^{\mathbf{y}}$.

(5.10)
$$\begin{aligned}
&\int \gamma_n^{\mathbf{y}}(f^{\mathbf{y}})\, \pi(d\mathbf{y}) = \int \overline{\gamma}_n^{\mathbf{y}}(f)\, \pi(d\mathbf{y}) = \int \overline{\gamma}_n^{\mathbf{y}^{(n)}}(f)\, \pi(d\mathbf{y}) \\
&= \int \frac{1}{|V_n|} \sum_{\mathbf{i}\in V_n} \overline{\gamma}_n^{\theta_{\mathbf{i}}\mathbf{y}^{(n)}}(f)\, \pi(d\mathbf{y}) = \int \overline{\gamma}_n^{\mathbf{y}^{(n)}}(\overline{\mathbf{R}}_n^s(f))\, \pi(d\mathbf{y}) = \int \overline{\gamma}_n^{\mathbf{y}}(\overline{\mathbf{R}}_n^s(f))\, \pi(d\mathbf{y}) \\
&= \int \overline{\gamma}_n^{\mathbf{y}}(\overline{\mathbf{R}}_n(f))\, \pi(d\mathbf{y}) + o(1) = \int \overline{\zeta}_n^{\mathbf{y}}(f)\, \pi(d\mathbf{y}) + o(1).
\end{aligned}$$

This proves the relative compactness of $\{\gamma_n^{\cdot}\}_{n=1}^{\infty}$.

Suppose $\gamma_{n_j}^{\cdot} \to \gamma^{\cdot}$ as $j \to \infty$. Applying the upper bound of (2.22) to the skew model shows that $\overline{\gamma}_n^{\mathbf{x}}(\overline{\mathbf{R}}_n \in U^c) \to 0$ for any open neighborhood U of $\overline{\mathcal{K}}_\infty$, hence the laws $\overline{\gamma}_n^{\mathbf{x}}(\overline{\mathbf{R}}_n \in \cdot)$ are tight for π-a.e. $\mathbf{x}$. An application of Lemma 5.5 and a passage to a subsequence if necessary imply that $\pi\overline{\gamma}_{n_j}^{\cdot}(\overline{\mathbf{R}}_{n_j} \in \cdot) \to \overline{\Pi}$ as $j \to \infty$, for some probability measure $\overline{\Pi}$ on $\mathcal{M}_\Theta(\Omega\times\Sigma)$, and by dominated convergence $\overline{\Pi}(\overline{\mathcal{K}}_\infty) = 1$. The equality of the first and penultimate term in (5.10) gives, upon passing to the limit along $\{n_j\}$,

$$\int_\Sigma \gamma^{\mathbf{y}}(f^{\mathbf{y}})\, \pi(d\mathbf{y}) = \int_{\mathcal{M}_\Theta(\Omega\times\Sigma)} \mu(f)\, \overline{\Pi}(d\mu). \tag{5.11}$$

Letting $\Lambda = \lim_{j\to\infty} \pi\gamma_{n_j}^{\cdot}(\mathbf{L}_{n_j} \in \cdot)$, Lemma 5.9(1) implies $\overline{\Pi} = \Lambda(\nu : \Psi_\nu \in \cdot)$. We can rewrite (5.11) as

$$\int \gamma^{\mathbf{y}}(f^{\mathbf{y}})\, \pi(d\mathbf{y}) = \int \Psi_\nu(f)\, \Lambda(d\nu) = \iint \Psi_\nu^{\mathbf{y}}(f^{\mathbf{y}})\, \Lambda(d\nu)\, \pi(d\mathbf{y}). \tag{5.12}$$

This tells us that Λ is the unique measure that represents the exchangeable $\pi \otimes \gamma^{\cdot}$ as a mixture of the i.i.d. measures Ψ_ν over $\nu \in \mathcal{K}_1$. This also gives (2.16), and the invariance of $\gamma^{\cdot}$ follows from (2.16). $\square$

Proof of Proposition 2.17. The a.s. convergence $\zeta^{\mathbf{x}}_{n_j} \to Q$ follows from Proposition 2.14. Let Λ be the measure appearing in the decomposition (2.12) of Q. Let $\gamma^{\cdot}$ be a limit point of $\{\gamma^{\cdot}_{n_j}\}$ realized along a further subsequence $\{n'_j\}$, with mixing measure Λ' in its decomposition (2.16). Take $f \in b\mathcal{B}^{\Omega}_m$ in (5.10) and (5.12) to see that

$$\begin{aligned}\int j_\infty(\nu)\,\Lambda(d\nu) = Q = \lim_{j\to\infty} \pi\zeta^{\cdot}_{n'_j} &= \lim_{j\to\infty} \pi\gamma^{\cdot}_{n'_j} = \pi\gamma^{\cdot} \\ &= \iint \Psi^{\mathbf{y}}_{\nu}\,\Lambda'(d\nu)\,\pi(d\mathbf{y}) = \int j_\infty(\nu)\,\Lambda'(d\nu).\end{aligned}$$

The above decomposition of Q is unique because the $j_\infty(\nu)$'s are i.i.d., so $\Lambda' = \Lambda$. The relatively compact sequence $\{\gamma^{\cdot}_{n_j}\}$ must converge since it has a unique limit point. $\square$

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