The fuzzy Potts model.

Christian Maes* and Koen Vande Velde†
Instituut voor Theoretische Fysica
K.U. Leuven
Celestijnenlaan 200D
B-3001 Leuven (Belgium)
e–mail: Christian.Maes@fys.kuleuven.ac.be
e–mail: Koen.VandeVelde@fys.kuleuven.ac.be

Abstract

We consider the ferromagnetic q-state Potts model on the d-dimensional lattice $\mathbb{Z}^d$, $d \geq 2$. Suppose that the Potts variables $(\rho_x,\ x \in \mathbb{Z}^d)$ are distributed in one of the q low temperature phases. Suppose that $n \neq 1, q$ divides q. Partitioning the single site state space into n equal parts $K_1, \ldots, K_n$, we obtain a new random field $\sigma = (\sigma_x,\ x \in \mathbb{Z}^d)$ by defining fuzzy variables $\sigma_x = \alpha$ if $\rho_x \in K_\alpha,\ \alpha = 1, \ldots, n$. We investigate the state induced on these fuzzy variables. First we look at the conditional distribution of ρ_x given all values $\sigma_y, y \in \mathbb{Z}^d$. We find that below the critical temperature all versions of this conditional distribution are non-quasilocal on a set of configurations which carries positive measure. Then we look at the conditional distribution of σ_x given all values $\sigma_y, y \neq x$. If the system is not at the critical temperature of a first order phase transition, there exists a version of this conditional distribution that is almost surely quasilocal.

keywords: Potts model, quasilocality, fuzzy variables, random cluster model.
International Classification for Physics numbers: 0250, 0520, 0550.

*Onderzoeksleider N.F.W.O. Belgium
†Aspirant N.F.W.O. Belgium
Research supported by EC grant CHRX-CT93-0411

1 Introduction.

The q-state Potts model is a generalization of the Ising model coinciding with it for $q = 2$. It shows a number of well appreciated features sometimes similar (e.g. in its stochastic-geometric representation) and sometimes quite dissimilar (e.g. in the nature of the phase transition for large q) to the Ising model.
One of the popular applications of the Potts model is in the theory of Gibbs sampling and image restoration. It is then sometimes assumed a priori that the undistorted multi-color image is distributed according to the Potts Hamiltonian.
In this paper we are interested in the (distorted) two-color (or more generally n-color) image obtained from the original one in lower resolution. This new 'fuzzy' image is composed of variables that may for example be Ising-like (two possible values per site) but their joint distribution, as inherited from the original variables, may be quite different from the Ising model. In fact, it may be the case that the induced measure is not a Gibbs measure for any quasilocal interaction. A similar example was treated in [1], where the original model was the massless harmonic crystal and the fuzzy variables were binary, specifying at each site the sign of the Gaussian spins.
To be more precise, we take any one of the q low temperature phases of the Potts model. For $n \neq 1, q$ we investigate the distribution induced by it on the variables σ_x specifying to which one of n families the Potts variable belongs at each site x. We are interested in some conditional distributions for this model and especially in the question of quasilocality. First we look at the distribution of the value of the Potts variable at some site x when we are given to which family each Potts variable belongs. We find that below the critical temperature all versions of the conditional distribution are almost surely non-quasilocal. It means that the expectation value of a Potts variable given in which family each variable is, is very sensitive to the knowledge in what family the variables far away are. This is not only relevant for image restoration: from a statistical mechanics point of view it is interesting to see that this partial information one is given about the system does not block the phase transition. It indicates a form of robustness of the phase transition.
Then we look at the conditional distribution of σ_x given all values $\sigma_y, y \neq x$. If the system is not at the critical temperature of a first order phase transition, there exists a version of this conditional distribution that is almost surely quasilocal. This does not say that the induced measure is a Gibbs measure, but it comes close to that [2, 3]. For a Gibbs measure there exists a version of this conditional distribution that is quasilocal everywhere [4, 5, 2].
The following Section contains the model and the main results. Section 3 reviews the relation of the Potts model with the random cluster model and extends this to the fuzzy Potts model. Section 4 is devoted to the proof of the main results.

2 Model and main results.

In the q-state Potts model on the lattice $\mathbb{Z}^d, d \geq 2$, one first assigns to each lattice site x a Potts variable ρ_x with uniform a priori distribution in the single site state space $K = \{1, \dots, q\}$. The energy of a configuration $\rho = (\rho_x, x \in \mathbb{Z}^d)$ is formally given by

$$\mathcal{H}(\rho) = -J \sum_{\langle x,y \rangle} \delta_{\rho_x, \rho_y}, \tag{2.1}$$

where $J > 0$ is the nearest neighbor coupling and the sum is over all nearest neighbor pairs. $\delta_{a,b}$ ($= 1$ if $a = b$ and $= 0$ if $a \neq b$) is the Kronecker-delta.
Let $\Lambda_N = [-N, N]^d \cap \mathbb{Z}^d$ be a cubic region centered around the origin o. Let $\mathbb{1}$ be the Potts configuration in which $\mathbb{1}_x = 1, x \in \mathbb{Z}^d$. The Potts measure at inverse temperature $\beta \geq 0$ with $\mathbb{1}$ boundary conditions outside Λ_N is the probability measure $P^1_{N,\beta}$ on $K^{\mathbb{Z}^d}$ giving weight

$$P^1_{N,\beta}(\rho) = \frac{1}{Z^1_{N,\beta}} \exp\{-\beta[\mathcal{H}(\rho) - \mathcal{H}(\mathbb{1})]\} \tag{2.2}$$

to a configuration ρ coinciding with $\mathbb{1}$ outside V_N and $P^1_{N,\beta}(\rho) = 0$ otherwise. We refer to [6] for the discussion of the infinite volume limit $\lim_{N\uparrow\infty} P^1_{N,\beta} = P^1_\beta$. We call this limiting measure the one-phase, because it is an extremal translation invariant measure and because there is a critical value $\beta_c = \beta_c(q, J)$ so that for all $\beta > \beta_c$ (and for large q also at β_c) $P^1_\beta[\rho_o = 1] > 1/q$. Moreover in this regime there is, with P^1_β probability one, an infinite nearest neighbor connected cluster of sites on which the Potts variables take the value 1 [7].
Suppose now that there is an integer $n \neq 1, q$ dividing q. We divide the single site state space K into n disjoint parts $K_1, \dots, K_n$, each containing q/n elements. This allows us to define fuzzy variables

$$\sigma_x = \alpha \text{ if } \rho_x \in K_\alpha \text{ for } \alpha = 1, \dots, n. \tag{2.3}$$

We can thus write $\rho_x = (\sigma_x, \tau_x)$, where $\sigma_x \in \{1, \dots, n\}$ indicates in which family or class ρ_x is and $\tau_x \in \{1, \dots, q/n\}$. For convenience we assume that $\sigma_x = \tau_x = 1$ corresponds to $\rho_x = 1$. Obviously any continuous function of $\sigma(\rho) = \sigma = (\sigma_x, x \in \mathbb{Z}^d)$ is a continuous function of the Potts configuration ρ and we can consider the infinite volume measure Q^1_β on $\Omega = \{1, \dots, n\}^{\mathbb{Z}^d}$ defined by

$$Q^1_\beta(A) = P^1_\beta(B)$$

for any cylinder set A corresponding to the σ variables and

$$B \equiv \{\rho \in K^{\mathbb{Z}^d} | \sigma(\rho) \in A\}$$

is a cylinder set for the ρ variables.
For a configuration $\eta \in \Omega$ define $\eta_\Lambda = \{\eta_x : x \in \Lambda\}$. Let us denote by $\mathcal{F}_\sigma$ the σ-algebra of events depending only on the vales of the σ variables. $\mathcal{F}^x_\sigma$ denotes the sub-σ-algebra of events in $\mathcal{F}_\sigma$ not depending on σ_x. We recall the definition of pointwise quasilocality [3].

Definition 1 *We call a real-valued function g on Ω quasilocal at η iff for any $\epsilon > 0$, there exists a finite region $\Lambda \subset \mathbb{Z}^d$ such that*

$$\sup_{\substack{\zeta\in\Omega: \\ \zeta_\Lambda=\eta_\Lambda}} |g(\zeta) - g(\eta)| < \epsilon. \tag{2.4}$$

Main Results*(Propositions 1 and 2 below)*

1. *If $\beta > \beta_c$, then all versions of $P^1_\beta(\delta_{\tau_x,1}|\mathcal{F}_\sigma)$ are non-quasilocal on a set of configurations with positive Q^1_β-measure.*

2. *If $\beta \neq \beta_c$, then there exists a version of $Q^1_\beta(\delta_{\sigma_x,1}|\mathcal{F}^x_\sigma)$ that is Q^1_β-a.s. quasilocal .*

3 Connection with the random cluster model.

We review here the connection between Potts and random cluster model [8, 6] and extend it to the fuzzy Potts model.
Put $p = 1 - e^{-\beta J}$. As is well known, if $\rho = \mathbb{1}$ ouside Λ_N,

$$P^1_{N,\beta}(\rho) = \frac{1}{Z^1_N}\sum_\omega (1-p)^{N_0(\omega)} p^{N_1(\omega)} \prod_{\substack{e=\langle xy\rangle: \\ \omega_e=1}} \delta_{\rho_x,\rho_y}, \tag{3.1}$$

where ω is a configuration on the edges $e = \langle xy\rangle$ with either x or y in Λ_N. To each edge e we assign a variable $\omega_e = 1, 0$. $\omega_e = 1$ declares the edge open and $\omega_e = 0$ declares the edge closed. $N_1(\omega)$ is the number of open edges in ω. $N_0(\omega)$ is the number of closed edges in ω. The partition function Z^1_N is equal to

$$Z^1_N = \sum_\omega p^{N_1(\omega)}(1-p)^{N_0(\omega)} q^{c_1(\omega)}, \tag{3.2}$$

where $c_1(\omega)$ counts the number of clusters of ω not connected to Λ^c_N. Two sites x, y are said to be connected inside a set $B \subset \Lambda$ if there is a path via open edges from x to y inside B. A cluster is a maximal set of connected sites. The partition function Z^1_N is the partition function of the wired random cluster measure μ^1_N, defined by

$$\mu^1_N(\omega) = \begin{cases} \frac{1}{Z^1_N} p^{N_1(\omega)}(1-p)^{N_0(\omega)} q^{c_1(\omega)} & , \text{ if } \omega_e = 1 \text{ for } e \subset \Lambda^c_N \\ 0 & , \text{ otherwise.} \end{cases} \tag{3.3}$$

Consider now $P^0_{N,\beta}$, the measure in Λ_N with free boundary conditions. Then in (3.1) we only sum over configurations ω on the edges $\langle xy\rangle \subset \Lambda_N$ and in (3.2) we must replace $c_1(\omega)$

by $c_0(\omega)$ which counts all connected clusters of ω. The partition function Z^0_N is thus also the partition function of the free random cluster measure μ^0_N, defined by

$$\mu^0_N(\omega) = \begin{cases} \frac{1}{Z^0_N} p^{N_1(\omega)}(1-p)^{N_0(\omega)} q^{c_0(\omega)} & , \text{ if } \omega_e = 0 \text{ for } e \cap \Lambda^c_N \neq \emptyset \\ \\ 0 & , \text{ otherwise.} \end{cases} \tag{3.4}$$

We say that a function g on a set $\{0,1\}^B$ is increasing if $g(\omega) \geq g(\omega')$ whenever $\omega'_e = 1$ implies $\omega_e = 1$ for $\omega, \omega' \in \{0,1\}^B$, and all $e \in B$. The following results are well known [6, 9].

Lemma 1 *1. μ^0_N and μ^1_N have the FKG property, i.e. for increasing functions g, h*

$$\mu^*_N(gh) \geq \mu^*_N(g)\mu^*_N(h) \tag{3.5}$$

2. For any increasing function g

$$\begin{aligned} \mu^0_M(g) &\leq \mu^0_N(g) \\ \mu^1_M(g) &\geq \mu^1_N(g), \end{aligned} \tag{3.6}$$

if $M < N$.

3. The weak van Hove limits $\mu^0 := \lim_N \mu^0_N$ and $\mu^1 := \lim_N \mu^1_N$ exist.

It is also well known that there exists a critical value $p_c(q)$ for p above which there is percolation in the state μ^*, $* = 1, 0$, and below which there is absence of percolation. By percolation is meant the almost sure existence of an infinite cluster. Off the critical point, $\mu^0 = \mu^1$ and there is thus a unique state for the model (because free and wired boundary conditions are extremal in FKG sense). Thus μ^0 and μ^1 can only differ at $p_c(q)$, and indeed they do at high values of q where one can make the connection with a first order transition for the Potts model.
Expectations of observables in the Potts model can be expressed via the random cluster model as follows: for a function g on the Potts configurations ρ_{Λ_N}:

$$P^*_{N,\beta}(g) = \mu^*_N\left(\mathbb{E}^*_{N,q}(g)\right) \tag{3.7}$$

where $* = 0, 1$ and $\mathbb{E}^*_{N,q}$ is the average over the Potts variables where the Potts variables are constrained to the same value within the same cluster and uniformly distributed over the q possible values for the different clusters not connected to Λ^c_N. Thus, with $\rho_{\Lambda^c_N} = \mathbb{1}_{\Lambda^c_N}$

$$\begin{aligned} \mathbb{E}^0_{N,q}(g)(\omega) &= q^{-c_0(\omega)} \sum_{\rho_{\Lambda_N}} g(\rho_{\Lambda_N}) \prod_{\substack{e=\langle xy\rangle \subset \Lambda_N: \\ \omega_e = 1}} \delta_{\rho_x,\rho_y} \\ \mathbb{E}^1_{N,q}(g)(\omega) &= q^{-c_1(\omega)} \sum_{\rho_{\Lambda_N}} g(\rho_{\Lambda_N}) \prod_{\substack{e=\langle xy\rangle \cap \Lambda_N \neq \emptyset: \\ \omega_e = 1}} \delta_{\rho_x,\rho_y} \end{aligned} \tag{3.8}$$

The following relations follow from (3.7) (for a proof see [6]).

$$P_\beta^1\left(\frac{1}{q-1}(q\delta_{\rho_x,1}-1)\right) = \mu^1(x \leftrightarrow \infty) \tag{3.9}$$

$$P_\beta^*\left(\frac{1}{q-1}(q\delta_{\rho_x,\rho_y}-1)\right) = \mu^*(x \leftrightarrow y). \tag{3.10}$$

Here, $x \leftrightarrow y$ stands for the event that x is connected to y and $x \leftrightarrow \infty$ means that for any finite set Λ x is connected to the complement of Λ.
All the above is also valid for Potts models on edge-diluted lattices (which we will use later on), since we have not used the specific structure of the lattice. It means that in the above p must be replaced by the appropriate $p_e = 1 - e^{-\beta J_e}$ with $J_e = J$ or $J_e = 0$. The following result will be useful (for a proof see [6, 9]).

Lemma 2 *Let $\nu_N^*, \hat{\nu}_N^*$ be two possibly inhomogeneous free or wired random cluster measures ($* = 1$ or 0) corresponding to coupling constants $J_e, \hat{J}_e$ respectively for the edges e. If for all edges e $J_e \geq \hat{J}_e$, then for all increasing functions g:*

$$\nu_N^*(g) \geq \hat{\nu}_N^*(g). \tag{3.11}$$

It is easy to see that for the fuzzy Potts model, with $\sigma_{\Lambda_N^c} = \mathbb{1}_{\Lambda_N^c}$

$$\begin{aligned} Q_{N,\beta}^*(\sigma) &= \frac{1}{Z_N^*}\sum_\tau \prod_{\langle xy\rangle}\left((1-p) + p\delta_{\sigma_x,\sigma_y}\delta_{\tau_x,\tau_y}\right) \\ &= \frac{1}{Z_N^*}\sum_\omega (1-p)^{N_0(\omega)}p^{N_1(\omega)}(q/n)^{c_*(\omega)}\prod_{e:\omega_e=1}\delta_{\sigma_x,\sigma_y}. \end{aligned} \tag{3.12}$$

The difference with the full Potts model resides solely in the factor $(q/n)^{c_*(\omega)}$.
For a function g on Ω, measurable with respect to $\mathcal{F}_\sigma$, we then have that

$$Q_{N,\beta}^*(g) = \mu_N^*\left(\mathbb{E}_{N,n}^*(g)\right). \tag{3.13}$$

4 Conditional expectations.

We want to investigate the conditional distribution $P_\beta^1(\delta_{\tau_x,1}|\mathcal{F}_\sigma)$. Looking at the Hamiltonian (2.1), we see that a fixation of the fuzzy field σ in a particular configuration η has the following effect. For nearest neighbor sites x, y we have that

- if $\eta_x = \eta_y$, there appears an interaction term δ_{τ_x,τ_y}

- if $\eta_x \neq \eta_y$, there is no interaction between τ_x and τ_y.

A particular fixation of σ thus produces a diluted (because certain edges are just cut) q/n-state Potts model with coupling J and inverse temperature β for the τ variables. Let us denote by $P^{\eta,1}_{N,\beta}$ the Potts measure on the volume Λ_N diluted in the above way via the configuration η, but with 1 boundary conditions. It depends thus only on η_{Λ_N}. Let $P^{\eta,1}_{\beta} = \lim_N P^{\eta,1}_{N,\beta}$ (the existence of the limit is standard).

Lemma 3

$$P^1_\beta(\delta_{\tau_x,1}) = \int Q^1_\beta(d\eta) P^{\eta,1}_\beta(\delta_{\tau_x,1}) \tag{4.1}$$

$$P^1_\beta(f) = \int Q^1_\beta(d\eta) P^{\eta,1}_\beta(f), \tag{4.2}$$

where f is the two-point function δ_{τ_x,τ_y} or f is a (finite) product of such two-point functions.

Proof:
We only prove the first statement. The proof of the second part is analogous. Take $N > M$ and let $\partial\Lambda_M$ be the exterior boundary of Λ_M. Then

$$\begin{aligned}
&\int Q^1_{N,\beta}(d\eta) P^{\eta,1}_{M,\beta}(\delta_{\tau_x,1}) = \int Q^1_{N,\beta}(d\eta)\mu^{\eta,1}_M(x \leftrightarrow \partial\Lambda_M)\frac{q-n}{q} + \frac{n}{q} \\
&\leq \int Q^1_{M,\beta}(d\eta)\mu^{\eta,1}_M(x \leftrightarrow \partial\Lambda_M)\frac{q-n}{q} + \frac{n}{q} = \int Q^1_{M,\beta}(d\eta) P^{\eta,1}_{M,\beta}(\delta_{\tau_x,1}) \\
&= P^1_{M,\beta}(\delta_{\tau_x,1}).
\end{aligned} \tag{4.3}$$

Here, $\mu^{\eta,1}_M$ is the wired state on the lattice diluted via η. The first equality is an application of (3.9) to the q/n-state Potts model. The inequality holds because of Lemma 2. On the other hand,

$$\begin{aligned}
&\int Q^1_{N,\beta}(d\eta) P^{\eta,1}_{M,\beta}(\delta_{\tau_x,1}) = \int Q^1_{N,\beta}(d\eta)\mu^{\eta,1}_M(x \leftrightarrow \partial\Lambda_M)\frac{q-n}{q} + \frac{n}{q} \\
&\geq \int Q^1_{N,\beta}(d\eta)\mu^{\eta,1}_N(x \leftrightarrow \partial\Lambda_N)\frac{q-n}{q} + \frac{n}{q} = \int Q^1_{N,\beta}(d\eta) P^{\eta,1}_{N,\beta}(\delta_{\tau_x,1}) \\
&= P^1_{N,\beta}(\delta_{\tau_x,1}),
\end{aligned} \tag{4.4}$$

because of Lemma 1. The result follows by taking limits over N and M. ■

Setting $F^1_\beta(\eta) = P^{\eta,1}_\beta(\delta_{\tau_x,1})$, which is well defined for all $\eta \in \Omega$, (4.1) says that F^1_β is a version of $P^1_\beta(\delta_{\tau_x,1}|\mathcal{F}_\sigma)$.
We now state our first main result:

Proposition 1 *If $\beta > \beta_c$, then all versions of $P^1_\beta(\delta_{\tau_x,1}|\mathcal{F}_\sigma)$ are non-quasilocal on a set of configurations in Ω of positive Q^1_β-measure.*

Proof:
If $\beta > \beta_c$, then there exists $\epsilon > 0$

$$P^1_\beta(\delta_{\tau_x,1}) - \frac{n}{q} = \mu^1(x \leftrightarrow \infty)\frac{q-n}{q} = \left(P^1_\beta(\delta_{\rho_x,1}) - \frac{1}{q}\right)\frac{q-n}{q-1} > \epsilon. \tag{4.5}$$

The first equality is an instance of formula (3.13). Hence, using Lemma 3 we find then that for η in a set $\mathcal{B}$ of positive Q^1_β-measure

$$P^{\eta,1}_\beta(\delta_{\tau_x,1}) - \frac{n}{q} > \epsilon. \tag{4.6}$$

Now for each configuration $\eta \in \mathcal{B}$, we build one new configuration η^N according to the following prescription.

$$\begin{aligned} (\eta^N)_x &= \eta_x \qquad \text{if} \qquad x \in \Lambda_N \qquad \text{or} \qquad x \in (\Lambda_N \cup \partial\Lambda_N)^c \\ &\neq \eta_y \qquad \text{if} \qquad \langle x, y\rangle, x \in \partial\Lambda_N \qquad \text{and} \qquad y \in \Lambda_N \end{aligned}$$

This configuration η^N cuts all τ bonds connecting Λ_N to the rest of the lattice, so conditioning on η^N leaves us with a diluted q/n-state Potts model on the finite volume Λ_N with free boundary conditions. Therefore,

$$P^{\eta^N,1}_\beta(\delta_{\tau_x,1}) - \frac{n}{q} = 0. \tag{4.7}$$

And so, for all configurations $\eta \in \mathcal{B}$, uniformly in N,

$$P^{\eta,1}_\beta(\delta_{\tau_x,1}) - P^{\eta^N,1}_\beta(\delta_{\tau_x,1}) = F^1_\beta(\eta) - F^1_\beta(\eta^N) > \epsilon. \tag{4.8}$$

It follows that the function F^1_β is non-quasilocal on $\mathcal{B}$. But since F^1_β is only one version of $P^1_\beta(\delta_{\tau_x,1}|\mathcal{F}_\sigma)$, we must show that this non-quasilocality does not disappear if one changes the expression for this conditional probability on a set of zero measure.
Since $Q^1_\beta(\mathcal{B}) > 0$, it follows from the positivity of the finite-set conditional probabilities of Q^1_β that also the set

$$\mathcal{B}_N = \{\eta^N \in \Omega | \eta \in \mathcal{B}\} \tag{4.9}$$

carries positive Q^1_β-measure. Suppose now that G^1_β is another version of $P^1_\beta(\delta_{\tau_x,1}|\mathcal{F}_\sigma)$. Then there exists a set $\mathcal{C} \in \Omega$, with $Q^1_\beta(\mathcal{C}) = 1$ such that $G^1_\beta(\eta) = F^1_\beta(\eta)$ for all $\eta \in \mathcal{C}$. Again due to the positivity of finite-set conditional probabilities the sets

$$\begin{aligned} \mathcal{A} &= \{\eta \in \mathcal{B} \cap \mathcal{C} | \eta^N \in \mathcal{B}_N \cap \mathcal{C}\} \\ \mathcal{A}_N &= \{\eta^N \in \mathcal{B}_N \cap \mathcal{C} | \eta \in \mathcal{B} \cap \mathcal{C}\} \end{aligned}$$

have positive Q^1_β-measure. The sets $\mathcal{A}$ and $\mathcal{A}_N$ are constructed in such a way that for every $\eta \in \mathcal{A}$ the configuration η^N is a member of $\mathcal{A}_N$. Thus for $\eta \in \mathcal{A}$ and uniformly in N

$$G^1_\beta(\eta) - G^1_\beta(\eta^N) > \epsilon. \tag{4.10}$$

Thus also G^1_β is non-quasilocal on a set of positive Q^1_β-measure. This concludes the proof. ■

We now investigate the conditional expectation $Q^1_\beta(\delta_{\sigma_x,1}|\mathcal{F}^x_\sigma)$. This conditional probability is related to $P^1_\beta(\delta_{\tau_x,1}|\mathcal{F}_\sigma)$ in the following way. For Q^1_β-a.e. $\eta \in \Omega$

$$\begin{aligned} & Q^1_\beta(\delta_{\sigma_x,1}|\mathcal{F}^x_\sigma)(\eta) = \\ & \frac{1}{Z_\eta} P^{\eta,1}_\beta \left(\exp[\beta J \sum_{y:\langle xy\rangle} (\delta_{\eta_y,1} - \delta_{\eta_y,\eta_x})\delta_{\tau_y,\tau_x}] \right) = \\ & \frac{1}{Z_\eta} P^{\eta,1}_\beta \left(\prod_{y:\langle xy\rangle} \left\{ 1 + \left(\exp[\beta J(\delta_{\eta_y,1} - \delta_{\eta_y,\eta_x})] - 1 \right) \delta_{\tau_y,\tau_x} \right\} \right). \end{aligned} \tag{4.11}$$

The factor Z_η is a normalization factor depending on η. We can get rid of it by considering a ratio of conditional probabilities. For $k \in \{2, \ldots, n\}$

$$\begin{aligned} & \frac{Q^1_\beta(\delta_{\sigma_x,1}|\mathcal{F}^x_\sigma)(\eta)}{Q^1_\beta(\delta_{\sigma_x,k}|\mathcal{F}^x_\sigma)(\eta)} = \\ & \frac{P^{\eta,1}_\beta \left(\prod_{y:\langle xy\rangle} \left\{ 1 + \left(\exp[\beta J(\delta_{\eta_y,1} - \delta_{\eta_y,\eta_x})] - 1 \right) \delta_{\tau_y,\tau_x} \right\} \right)}{P^{\eta,1}_\beta \left(\prod_{y:\langle xy\rangle} \left\{ 1 + \left(\exp[\beta J(\delta_{\eta_y,k} - \delta_{\eta_y,\eta_x})] - 1 \right) \delta_{\tau_y,\tau_x} \right\} \right)}. \end{aligned} \tag{4.12}$$

Choosing $\eta_x = k$ we get that for Q^1_β-a.e. η

$$\begin{aligned} & \frac{Q^1_\beta(\delta_{\sigma_x,1}|\mathcal{F}^x_\sigma)(\eta)}{Q^1_\beta(\delta_{\sigma_x,k}|\mathcal{F}^x_\sigma)(\eta)} = \\ & P^{\eta,1}_\beta \left(\prod_{y:\langle xy\rangle} \left\{ 1 + \left(\exp[\beta J(\delta_{\eta_y,1} - \delta_{\eta_y,k})] - 1 \right) \delta_{\tau_y,\tau_x} \right\} \right). \end{aligned} \tag{4.13}$$

Observe that the above expression depends only on the expectation of δ_{τ_x,τ_y} and products of such two-point functions. Therefore, this expression will not be sensitive to the occurence of a second order phase transition in the diluted τ-system, but only to the occurence of a first order transition. We need the following lemma.

Lemma 4

$$P^0_\beta(f) = \int Q^0_\beta(d\eta) P^{\eta,0}_\beta(f), \tag{4.14}$$

where f is the two-point function δ_{τ_x,τ_y} or f is a (finite) product of such two-point functions.

Proof:
Take $f(\tau) = \delta_{\tau_x,\tau_y}$ (the rest is analogous). We use the fact that $P^0_\beta = \lim_N P^D_{N,\beta}$ where the superscript D stands for disordered boundary conditions. It means that we take a

configuration outside Λ_N in which at each site the Potts variable differs from all of its neighbors. We then proceed as in Lemma 3 to get for $N > M$ (with x,y in Λ_M)

$$\int Q^D_{N,\beta}(d\eta) P^{\eta,0}_{M,\beta}(\delta_{\tau_x,\tau_y}) = \int Q^D_{N,\beta}(d\eta)\mu^{\eta,0}_M(x \leftrightarrow y)\frac{q-n}{q} + \frac{n}{q}$$
$$\leq \int Q^D_{N,\beta}(d\eta)\mu^{\eta,0}_N(x \leftrightarrow y)\frac{q-n}{q} + \frac{n}{q} = \int Q^D_{N,\beta}(d\eta) P^{\eta,0}_{N,\beta}(\delta_{\tau_x,\tau_y}). \tag{4.15}$$

The inequality holds because of Lemma 1. On the other hand,

$$\int Q^D_{N,\beta}(d\eta) P^{\eta,0}_{M,\beta}(\delta_{\tau_x,1}) = \int Q^D_{N,\beta}(d\eta)\mu^{\eta,0}_M(x \leftrightarrow y)\frac{q-n}{q} + \frac{n}{q}$$
$$\geq \int Q^D_{M,\beta}(d\eta)\mu^{\eta,0}_M(x \leftrightarrow y)\frac{q-n}{q} + \frac{n}{q} = \int Q^D_{M,\beta}(d\eta) P^{\eta,0}_{M,\beta}(\delta_{\tau_x,\tau_y}), \tag{4.16}$$

because of Lemma 2. The result follows now easily by taking limits over N and M, and from the observation that, since δ_{τ_x,τ_y} is a local function, for every $\epsilon > 0$, there exists M large enough such that

$$|P^{\eta,0}_{M,\beta}(\delta_{\tau_x,\tau_y}) - P^{\eta,D}_{M,\beta}(\delta_{\tau_x,\tau_y})| < \epsilon. \tag{4.17}$$

■

Lemma 5 *Let $P^{\eta,1}_\beta(\delta_{\tau_x,\tau_y}) = P^{\eta,0}_\beta(\delta_{\tau_x,\tau_y})$ for all neighboring sites $x, y \in \mathbb{Z}^d$. Then the two states $P^{\eta,1}_\beta$ and $P^{\eta,0}_\beta$ also agree on (finite) products of these two-point functions.*

Proof:
¿From the relation between Potts and random cluster model, it is easy to calculate that for $e = \langle xy \rangle$

$$\mu^{\eta,1}_N(\delta_{\omega_e,1}) = pP^{\eta,1}_{N,\beta}(\delta_{\tau_x,\tau_y})$$
$$\mu^{\eta,0}_N(\delta_{\omega_e,1}) = pP^{\eta,0}_{N,\beta}(\delta_{\tau_x,\tau_y}) \tag{4.18}$$

Thus, because of weak convergence, it follows from the assumption that $\mu^{\eta,0}(\delta_{\omega_e,1}) = \mu^{\eta,1}(\delta_{\omega_e,1})$. It is then a simple application [10] of the FKG inequality in Lemma 1 that the two states are equal. Since expectations of products of two-point functions δ_{τ_x,τ_y} in the Potts model can be expressed as expectations of local functions in the random cluster model, the result follows. ■

We come now to our second main result:

Proposition 2 *There exists a version of $Q^1_\beta(\delta_{\sigma_x,1}|\mathcal{F}^x_\sigma)$ that is Q^1_β-a.s. quasilocal if $\beta \neq \beta_c$.*

Proof:
Away from the first order phase transition for the q-state Potts model, we have that

$$\begin{aligned} &P^1_\beta(\delta_{\tau_x,\tau_y}) - P^0_\beta(\delta_{\tau_x,\tau_y}) \\ &= \left(\mu^1(x \leftrightarrow y) - \mu^0(x \leftrightarrow y)\right) \frac{q-n}{q} = \\ &\left(P^1_\beta(\delta_{\rho_x,\rho_y}) - P^0_\beta(\delta_{\rho_x,\rho_y})\right) \frac{q-n}{q-1} = 0. \end{aligned} \tag{4.19}$$

The first equality is an instance of formula (3.13). Using Lemmas 3,4 we have that

$$\begin{aligned} &P^1_\beta(\delta_{\tau_x,\tau_y}) - P^0_\beta(\delta_{\tau_x,\tau_y}) = \\ &\int P^1_\beta(d\eta) \left\{P^{\eta,1}_\beta(\delta_{\tau_x,\tau_y}) - P^{\eta,0}_\beta(\delta_{\tau_x,\tau_y})\right\} \\ &+ \int P^1_\beta(d\eta) P^{\eta,0}_\beta(\delta_{\tau_x,\tau_y}) - \int P^0_\beta(d\eta) P^{\eta,0}_\beta(\delta_{\tau_x,\tau_y}). \end{aligned} \tag{4.20}$$

The sum of the last two terms in the above equation is nonnegative by Lemma 2. Using the fact that $P^{\eta,1}_\beta(\delta_{\tau_x,\tau_y}) - P^{\eta,0}_\beta(\delta_{\tau_x,\tau_y})$ is nonnegative by Lemma 1, we find then that Q^1_β-a.s.

$$P^{\eta,1}_\beta(\delta_{\tau_x,\tau_y}) - P^{\eta,0}_\beta(\delta_{\tau_x,\tau_y}) = 0. \tag{4.21}$$

For a configuration η define η^N_1 as the configuration that agrees with η inside Λ_N and is identically 1 outside. η^N_D is then the configuration that agrees with η inside Λ_N and is disordered outside. Now by Lemma 2 we have that for η, η' that agree inside Λ_N

$$\begin{aligned} &\sup_{\substack{\eta' \in \Omega \\ \eta'_{\Lambda_N} = \eta_{\Lambda_N}}} : |P^{\eta,1}_\beta(\delta_{\tau_x,\tau_y}) - P^{\eta',1}_\beta(\delta_{\tau_x,\tau_y})| = \\ &\sup_{\substack{\eta' \in \Omega \\ \eta'_{\Lambda_N} = \eta_{\Lambda_N}}} |\mu^{\eta,1}_\beta(x \leftrightarrow y) - \mu^{\eta',1}_\beta(x \leftrightarrow y)| \leq \\ &\mu^{\eta^N_1,1}_\beta(x \leftrightarrow y) - \mu^{\eta^N_D,1}_\beta(x \leftrightarrow y), \end{aligned} \tag{4.22}$$

which as $N \uparrow \infty$ converges to

$$\begin{aligned} &\mu^{\eta,1}_\beta(x \leftrightarrow y) - \mu^{\eta,0}_\beta(x \leftrightarrow y) = \\ &P^{\eta,1}_\beta(\delta_{\tau_x,\tau_y}) - P^{\eta,0}_\beta(\delta_{\tau_x,\tau_y}) = 0 \end{aligned} \tag{4.23}$$

for Q^1_β-a.e. η. Because of Lemma 5, we can apply the same argument to products of δ_{τ_x,τ_y}. This then proves the almost sure quasilocality of the right hand side of expression (4.13). This is enough to garantuee that the version of $Q^1_\beta(\delta_{\sigma_x,1}|\mathcal{F}^x_\sigma)$ constructed in (4.11) is Q^1_β-a.s. quasilocal. ∎

Acknowledgment: KVV thanks C.-E. Pfister for useful discussions.

References

[1] J.L. Lebowitz, C. Maes, J. Stat. Phys. **59**, 39 (1987)

[2] A.C.D. van Enter, R. Fernández and A.D. Sokal, J. Stat. Phys. **72**, 879 (1993)

[3] R. Fernández and C.E. Pfister, *Non-quasilocality of projections of Gibbs measures*, EFPL preprint

[4] W.G. Sullivan, Commun. Math. Phys. **33**, 61 (1973)

[5] O.K. Kozlov, Probl. Inform. Transmiss. **10**, 258 (1974)

[6] M. Aizenman, J.T. Chayes, L. Chayes and C.M. Newman, J. Stat. Phys. **50**, 1 (1988)

[7] Giacomin, J.L. Lebowitz, C. Maes, *Agreement percolation and phase coexistence in some Gibbs systems*, preprint KUL-TF-94/19, Leuven

[8] C.M. Fortuin, Physica A **59**, 393 (1972)

[9] G. Grimmet, *The stochastic random-cluster process and the uniqueness of random-cluster measures*, Research report no.94-8, Statistical Laboratory, Cambridge (1994)

[10] J. Fröhlich, C.-E. Pfister, Commun. Math. Phys. **109**, 493 (1987)