Almost sure quasilocality in the random cluster model

Charles-Edouard Pfister

Département de Mathématiques, Ecole Polytechnique Fédérale de Lausanne

CH-1015 Lausanne Switzerland

Koen Vande Velde*

Instituut voor Theoretische Fysica, K.U. Leuven

Celestijnenlaan 200D, B-3001 Leuven Belgium

Abstract

We investigate the Gibbsianness of the random cluster measures $\mu^{q,p}$ and $\tilde{\mu}^{q,p}$, obtained as the infinite-volume limit of finite volume measures with free and wired boundary conditions. For $q > 1$, the measures are not Gibbs measures, but it turns out that the conditional distribution on one edge, given the configuration outside that edge is almost surely quasilocal.

keywords: random cluster model, non-Gibbs states,
quasilocality of conditional distributions.

*Aspirant N.F.W.O. Belgium

1 Introduction.

In recent years, it has become apparent (see [1] and references therein) that not all states of physical interest for statistical mechanics are Gibbs measures [2]. Examples can be found in renormalization group theory, where applying renormalization group transformations to Gibbs states may lead out of the class of Gibbs measures [1]. Other examples come from the theory of interacting particle systems, where nonreversible processes may have non-Gibbsian stationary states [3]. A general theory of non-Gibbsian states is not available. All one can do for the moment is investigate particular models and try to classify the examples one has of non-Gibbsian states. One way of classification is the following. Gibbs states verify the property of quasilocality. This means that conditional expectations of local events are continuous functions of the configuration one conditions on. Non-Gibbsian states (if they are not non-Gibbsian for the reason that there are some constraints or hard-core interactions) lack this quasilocality property. A way of classification is therefore to look how large the set is on which quasilocality fails [4]. In many cases, this seems very difficult [5, 4].
In this paper we investigate the free and wired random cluster measures with $q > 1$ and $0 < p < 1$. We show that we have almost sure quasilocality of conditional distributions, but no quasilocality everywhere. At the same time it makes rigorous the common wisdom that the random cluster model has nonlocal features.
The paper is organized as follows. In section 2 we present the model and state the main result of the paper. In section 3 we construct regular conditional distributions for our measures. In section 4 a proof of almost sure quasilocality is given.

2 Model and main result.

We consider the square lattice $\mathbb{Z}^2$. The set of edges of $\mathbb{Z}^2$ is denoted by $\mathbb{Z}^2_*$. The model under consideration is the random cluster model. The configuration space is $\Omega = \{1,0\}^{\mathbb{Z}^2_*}$. On Ω we put the product topology and its Borel σ-field $\mathcal{F}$. Configurations are denoted by ω, η or ζ. The value of ω at $e \in \mathbb{Z}^2_*$ is ω_e; the restriction of ω to $\Lambda \subset \mathbb{Z}^2_*$ is $\omega_\Lambda := \{\omega_e : e \in \Lambda\}$. To each edge $e \in \mathbb{Z}^2_*$ we assign a variable χ_e with values in $\{0,1\}$, $\chi_e(\omega) := \omega_e$; $\chi_e(\omega) = 1$ declares the edge *open* and $\chi_e(\omega) = 0$ if the edge is *closed.* Two sites x, y are said to be *connected* if there is a finite path via open edges from x to y. A *cluster* is a maximal set of connected sites. We fix $0 < p < 1$ and $q > 1$. For the construction of the free boundary condition state, we start with a finite set of edges $\Lambda \subset \mathbb{Z}^2_*$. We define a probability measure μ_Λ by its weights:

$$\mu^{q,p}_\Lambda(\eta) := \begin{cases} \frac{1}{Z_\Lambda} p^{N_1(\eta_\Lambda)}(1-p)^{N_0(\eta_\Lambda)} q^{c(\eta)} & \text{if } \eta_e = 0 \text{ outside } \Lambda \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

Z_Λ is a normalization constant. $N_1(\eta_\Lambda)$ is the number of open edges of η in Λ; $N_0(\eta_\Lambda)$ is the number of closed edges of η in Λ; $c(\eta)$ is the number of clusters of η.
We proceed in a similar way to construct the wired state $\tilde{\mu}^{q,p}$. The finite volume measures $\tilde{\mu}^{q,p}_\Lambda$ are defined by

$$\tilde{\mu}^{q,p}_\Lambda(\eta) := \begin{cases} \frac{1}{Z_\Lambda} p^{N_1(\eta_\Lambda)}(1-p)^{N_0(\eta_\Lambda)} q^{c(\eta)} & \text{if } \eta_e = 1 \text{ outside } \Lambda \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The following results are well known [6].

Lemma 1 *1. $\mu^{q,p}_\Lambda$ and $\tilde{\mu}^{q,p}_\Lambda$ have the FKG property, i.e. for any increasing functions g, h on Ω*

$$\begin{aligned} \mu^{q,p}_\Lambda(gh) &\geq \mu^{q,p}_\Lambda(g)\mu^{q,p}_\Lambda(h) \\ \tilde{\mu}^{q,p}_\Lambda(gh) &\geq \tilde{\mu}^{q,p}_\Lambda(g)\tilde{\mu}^{q,p}_\Lambda(h) \end{aligned} \tag{2.3}$$

2. For any increasing function g on Ω,

$$\begin{aligned} \mu^{q,p}_\Lambda(g) &\leq \mu^{q,p}_{\Lambda'}(g) \\ \tilde{\mu}^{q,p}_\Lambda(g) &\geq \tilde{\mu}^{q,p}_{\Lambda'}(g), \end{aligned} \tag{2.4}$$

if $\Lambda \subset \Lambda'$.

3. The weak limits $\mu^{q,p} := \lim_\Lambda \mu^{q,p}_\Lambda$ and $\tilde{\mu}^{q,p} := \lim_\Lambda \tilde{\mu}^{q,p}_\Lambda$ exist.

It is also well known that there exists a critical value $p_c(q)$ for p above which there is percolation in the state $\mu^{q,p}$ and below which there is absence of percolation. By percolation is meant the almost sure existence of an infinite connected cluster. Off the critical point, it is believed that $\mu^{q,p} = \tilde{\mu}^{q,p}$ and that there is thus a unique state for the model (because free and wired boundary conditions are extremal in FKG sense). It can be proven rigorously for $q = 1, 2, \ldots$ by the existing connection with the Potts model (see [6] for more information on this). Thus $\mu^{q,p}$ and $\tilde{\mu}^{q,p}$ can only differ at $p_c(q)$, and they indeed do at high integer values of q where one can make the connection with a first order transition for the Potts model. For notational convenience we will drop the superscript q, p unless explicitly needed.

Definition 1 *We call a function g on Ω quasilocal at η iff for any $\epsilon > 0$, there exists a finite region Λ_ϵ such that*

$$\sup_{\substack{\zeta: \\ \zeta_{\Lambda_\epsilon} = \eta_{\Lambda_\epsilon}}} |g(\zeta) - g(\eta)| < \epsilon. \tag{2.5}$$

The following result is well known [7, 8, 1].

Lemma 2 *Let $\mathcal{F}^e$ be the σ-algebra of events* not *depending on ω_e. The measure ρ on Ω is a Gibbs measure iff for every edge e there exists a version $\pi_e(\chi_e|\cdot)$ of $\mathbb{E}_\rho[\chi_e|\mathcal{F}^e]$ satisfying*

1. $0 < \pi_e(\chi_e|\cdot) < 1$
2. *$\pi_e(\chi_e|\cdot)$ is quasilocal at every $\eta \in \Omega$.*

We can now state the main result of the paper.

Theorem 1 *Let μ and $\tilde{\mu}$ be the probability measures constructed above.*

1. *μ is not a Gibbs measure. No version of $\mathbb{E}_\mu[\chi_e|\mathcal{F}^e]$ is quasilocal everywhere.*
2. *There exists a version of $\mathbb{E}_\mu[\chi_e|\mathcal{F}^e]$ which is quasilocal μ-a.s..*

The same is true for $\tilde{\mu}$.

3 Conditional probabilities.

Let us now construct a version $\pi_e(\chi_e|\cdot)$ of the conditional probability $\mathbb{E}_\mu[\chi_e|\mathcal{F}^e]$. Let Λ be a finite subset of $\mathbb{Z}^2_*$ and $\Lambda^c := \mathbb{Z}^2_* \backslash \Lambda$. We denote by 0 the configuration in which every edge is closed, and by 1 the configuration in which every edge is open. We define $\pi_e(\chi_e|\eta)$ for all $\eta = \eta_\Lambda 0_{\Lambda^c}$ as the conditional expectation of χ_e with respect to μ_Λ; since those configurations are dense in Ω we then extend the definition by a limiting procedure. (The proof of the existence of the limit is given after the next Lemma.) Thus,

$$\pi_e(\chi_e|\eta_\Lambda 0_{\Lambda^c}) := \mu_\Lambda(\chi_e|\eta_{\Lambda\backslash\{e\}} 0_{\Lambda^c}) \tag{3.1}$$
$$\pi_e(\chi_e|\eta) := \lim_\Lambda \pi_e(\chi_e|\eta_\Lambda 0_{\Lambda^c}). \tag{3.2}$$

Similarly we construct a version $\tilde{\pi}_e(\chi_e|\cdot)$ of $\mathbb{E}_{\tilde{\mu}}[\chi_e|\mathcal{F}^e]$.

$$\tilde{\pi}_e(\chi_e|\eta_\Lambda 1_{\Lambda^c}) := \tilde{\mu}_\Lambda(\chi_e|\eta_{\Lambda\backslash\{e\}} 1_{\Lambda^c}) \tag{3.3}$$
$$\tilde{\pi}_e(\chi_e|\eta) := \lim_\Lambda \tilde{\pi}_e(\chi_e|\eta_\Lambda 1_{\Lambda^c}). \tag{3.4}$$

Lemma 3 *Let $V \ni e$ be a finite subset of $\mathbb{Z}^2_*$. For a $\mathcal{F}^e$-measurable function g*

$$\mu_V(\pi_e(\chi_e|\cdot)g) = \mu_V(\chi_e g) \tag{3.5}$$
$$\tilde{\mu}_V(\tilde{\pi}_e(\chi_e|\cdot)g) = \tilde{\mu}_V(\chi_e g) \tag{3.6}$$

Proof:

$$\begin{aligned}
\mu_V(\pi_e(\chi_e|\cdot)g) &= \int \mu_V(d\omega)\pi_e(\chi_e|\omega)g(\omega) \\
&= \int \mu_V(d\omega)\pi_e(\chi_e|\omega_V 0_{V^c})g(\omega_V 0_{V^c}) \\
&= \int \mu_V(d\omega)\mathbb{E}_{\mu_V}[\chi_e|\mathcal{F}^e]g(\omega) \\
&= \mu_V(\chi_e g)
\end{aligned}$$

The proof of the second statement is similar. ∎

Let us compute now more explicitly our expression for $\pi_e(\chi_e|\eta)$. Denote by η^0 the configuration equal to η off e and equal to 0 on e and by η^1 the configuration equal to η off e and equal to 1 on e. Now

$$\pi_e(\chi_e|\eta_\Lambda 0_{\Lambda^c}) = \frac{\mu_\Lambda(\eta^1)}{\mu_\Lambda(\eta^1) + \mu_\Lambda(\eta^0)}. \tag{3.7}$$

It is thus clear that

$$\pi_e(\chi_e|\eta_\Lambda 0_{\Lambda^c}) = \begin{cases} \frac{p}{p+q(1-p)} & \text{if } c(\eta^0_\Lambda 0_{\Lambda^c}) = c(\eta^1_\Lambda 0_{\Lambda^c}) + 1 \\ \\ p & \text{if } c(\eta^0_\Lambda 0_{\Lambda^c}) = c(\eta^1_\Lambda 0_{\Lambda^c}). \end{cases} \tag{3.8}$$

The event where $c(\eta^0_\Lambda 0_{\Lambda^c}) = c(\eta^1_\Lambda 0_{\Lambda^c}) - 1$ is impossible. Indeed, since isolated lattice sites are also counted as clusters, the creation of an open edge cannot increase the number of clusters.

For any finite Λ we say that two sites x and y are connected inside Λ for the configuration η if they are connected for the configuration $\eta_\Lambda 0_{\Lambda^c}$. Similarly, we say that two points are connected outside Λ for η if they are connected for $\eta_\Lambda 1_{\Lambda^c}$. Let e be the edge with sites x and y as endpoints. Define the following events:

$$E^i_{\sqcap,\Lambda} := \{x \text{ is connected to } y \text{ inside } \Lambda\},$$

and

$$E^i_\sqcap := \bigcup_\Lambda E^i_{\sqcap,\Lambda}.$$

Similarly, define

$$E^o_{\sqcap,\Lambda} := \{x \text{ is connected to } y \text{ outside } \Lambda\},$$

and

$$E^o_\sqcap := \left(\bigcup_\Lambda E^i_{\sqcap,\Lambda}\right) \bigcup \left(\bigcap_\Lambda E^o_{\sqcap,\Lambda}\right).$$

Let $\chi^i_\sqcap$ denote the indicator function of $E^i_\sqcap$. Denote with $\chi^o_\sqcap$ the indicator function of $E^o_\sqcap$. Then

$$c(\eta^0_\Lambda 0_{\Lambda^c}) = c(\eta^1_\Lambda 0_{\Lambda^c}) + 1 \quad \text{iff } \eta^0 \notin E^i_{\sqcap,\Lambda}$$

$$c(\eta^0_\Lambda 0_{\Lambda^c}) = c(\eta^1_\Lambda 0_{\Lambda^c}) \qquad \text{iff } \eta^0 \in E^i_{\sqcap,\Lambda}.$$

Now we want to take the limit $\Lambda \nearrow \infty$. If $\eta^0 \in E^i_\sqcap$ there has to exist some finite volume Λ such that $\eta^0 \in E^i_{\sqcap,\Lambda}$. Therefore, for each such η the limit over Λ exists and

$$\begin{aligned} \pi_e(\chi_e|\eta) &= \frac{p}{p+q(1-p)}\left(1-\chi^i_\sqcap(\eta^0)\right) + p\chi^i_\sqcap(\eta^0) \\ &= \frac{p}{p+q(1-p)} + \left(p - \frac{p}{p+q(1-p)}\right)\chi^i_\sqcap(\eta^0). \end{aligned} \tag{3.9}$$

It is clear that $\pi_e(\chi_e|\cdot)$ is a nonlocal, nonnegative and increasing function since $q > 1$. In a similar way

$$\tilde{\pi}_e(\chi_e|\eta_\Lambda 1_{\Lambda^c}) = \begin{cases} \frac{p}{p+q(1-p)} & \text{if } c(\eta^0_\Lambda 1_{\Lambda^c}) = c(\eta^1_\Lambda 1_{\Lambda^c}) + 1 \\ p & \text{if } c(\eta^0_\Lambda 1_{\Lambda^c}) = c(\eta^1_\Lambda 1_{\Lambda^c}). \end{cases} \tag{3.10}$$

It is again clear then that

$$c(\eta^0_\Lambda 1_{\Lambda^c}) = c(\eta^1_\Lambda 1_{\Lambda^c}) + 1 \quad \text{iff } \eta^0 \notin E^o_{\sqcap,\Lambda}$$

$$c(\eta^0_\Lambda 1_{\Lambda^c}) = c(\eta^1_\Lambda 1_{\Lambda^c}) \qquad \text{iff } \eta^0 \in E^o_{\sqcap,\Lambda}.$$

For $\eta^0 \in E^i_\sqcap$ we already know that we can take the limit $\Lambda \nearrow \infty$. Thus now take $\eta \in \Omega$, $\eta^0 \in E^o_\sqcap$ but $\eta^0 \notin E^i_\sqcap$. This η is such that $\eta^0 \in E^o_{\sqcap,\Lambda}$ for all finite Λ. Thus,

$$\tilde{\pi}_e(\chi_e|\eta_\Lambda 1_{\Lambda^c}) = p.$$

The limit over Λ thus exists and equals p. It is then clear that

$$\tilde{\pi}_e(\chi_e|\eta) = \frac{p}{p+q(1-p)} + \left(p - \frac{p}{p+q(1-p)}\right)\chi^o_\sqcap(\eta^0). \tag{3.11}$$

The expressions for $\pi_e(\chi_e|\cdot)$ and $\tilde{\pi}_e(\chi_e|\cdot)$ differ thus on the set of configurations η such that in η^0 both x and y are connected to infinity, but not to each other.
The following Lemma is an adaptation of Lemma 3.1 in [4].

Lemma 4 *Let g be a monotone bounded function. If*

$$g(\omega) = \lim_{\Lambda} g(\omega_\Lambda 0_{\Lambda^c}) \tag{3.12}$$

then

$$\mu(g) = \lim_{\Lambda} \mu_\Lambda(g). \tag{3.13}$$

If

$$g(\omega) = \lim_{\Lambda} g(\omega_\Lambda 1_{\Lambda^c}) \tag{3.14}$$

then

$$\tilde{\mu}(g) = \lim_{\Lambda} \tilde{\mu}_\Lambda(g). \tag{3.15}$$

Proof:
We just prove (3.13) for monotone increasing functions. The proof of the other statements is similar. We denote $g_\Lambda(\omega) := g(\omega_\Lambda 0_{\Lambda^c})$. Since g is increasing, $g_\Lambda \leq g_{\Lambda'}$ for $\Lambda \subset \Lambda'$. Let $M \subset \Lambda \subset N$, $|N| < \infty$.

$$\mu_\Lambda(g_M) \leq \mu_\Lambda(g_\Lambda) = \mu_\Lambda(g) \leq \mu_N(g_\Lambda). \tag{3.16}$$

The left inequality is due to the above remark, the right one is due to Lemma 1 since g_Λ is monotone increasing. Since g_Λ is a local function, we can take the limit over N, and get (Lemma 1)

$$\mu_\Lambda(g_M) \leq \mu_\Lambda(g) \leq \sup_N \mu_N(g_\Lambda) = \mu(g_\Lambda). \tag{3.17}$$

By hypothesis $\lim_\Lambda g_\Lambda = g$; by the monotone convergence theorem we obtain

$$\mu(g_M) \leq \liminf_{\Lambda} \mu_\Lambda(g) \leq \limsup_{\Lambda} \mu_\Lambda(g) \leq \mu(g). \tag{3.18}$$

Finally taking the limit over M yields

$$\mu(g) \leq \lim_{\Lambda} \mu_\Lambda(g) \leq \mu(g). \tag{3.19}$$

■

Proposition 1 $\pi_e(\chi_e|\cdot)$ *is a version of* $\mathbb{E}_\mu[\chi_e|\mathcal{F}^e]$.

Proof: It suffices to prove that for any nonnegative increasing local $\mathcal{F}^e$-measurable function g

$$\mu(\chi_e g) = \mu(\pi_e(\chi_e|\cdot)g). \tag{3.20}$$

Because both χ_e and g are local we have from weak convergence and Lemma 3 that

$$\begin{aligned} \mu(\chi_e g) &= \lim_{\Lambda} \mu_\Lambda(\chi_e g) \\ &= \lim_{\Lambda} \mu_\Lambda(\pi_e(\chi_e|\cdot)g). \end{aligned} \tag{3.21}$$

Now $\pi_e(\chi_e|\cdot)$ is nonnegative and increasing and so is g. Then so is $\pi_e(\chi_e|\cdot)g$. By definition $\pi_e(\chi_e|\omega) = \lim_\Lambda \pi_e(\chi_e|\omega_\Lambda 0_{\Lambda^c})$; since g is local the same is true for $\pi_e(\chi_e|\cdot)g$. We can therefore apply Lemma 4 which guarantees the convergence of $\mu_\Lambda(\pi_e(\chi_e|\cdot)g)$ to $\mu(\pi_e(\chi_e|\cdot)g)$. This concludes the proof. ■

In the same manner we prove that $\tilde{\pi}_e(\chi_e|\cdot)$ is a version of $\mathbb{E}_{\tilde{\mu}}[\chi_e|\mathcal{F}^e]$.

4 Almost sure quasilocality.

We now investigate the quasilocality properties of $\pi_e(\chi_e|\cdot)$. We follow [1] section 4.5.3. Because $\pi_e(\chi_e|\cdot)$ is only one version of $\mathbb{E}_\mu[\chi_e|\mathcal{F}^e]$, to prove non-quasilocality of $\mathbb{E}_\mu[\chi_e|\mathcal{F}^e]$, we have to show that no function that equals $\pi_e(\chi_e|\cdot)$ μ-a.s. is quasilocal everywhere. We say then that $\pi_e(\chi_e|\cdot)$ displays an essential non-quasilocality. Because μ gives nonzero probability to any open set, it suffices to investigate the function $\pi_e(\chi_e|\cdot)$ on a neighborhood of η, to search for essential non-quasilocality of $\pi_e(\chi_e|\cdot)$ at η. A neighborhood of η is constructed in the following way. Fix a finite set Λ, then

$$\mathcal{N}_\Lambda(\eta) := \{\eta \in \Omega : \zeta_\Lambda = \eta_\Lambda\} \tag{4.1}$$

is a neighborhood of η.

Proposition 2 *No version of* $\mathbb{E}_\mu[\chi_e|\mathcal{F}^e]$ *is quasilocal everywhere.*

Proof:
The proof is given in [1]. We repeat it here for the sake of completeness and because it gives more insight in the properties of $\pi_e(\chi_e|\cdot)$.
Let $\Lambda_n := [-n,n]^2$. Let η be the configuration which sets $\eta_f := 1$ on parallel rays running from x and y to infinity, perpendicular to the edge e, and sets $\eta_f = 0$ on all other edges. We choose two subsets of $\mathcal{N}_{\Lambda_n}(\eta)$: $\mathcal{N}^1_{\Lambda_n}(\eta)$ in which an open edge in $\Lambda_{n+1}\backslash\Lambda_n$ connects the two rays, $\mathcal{N}^0_{\Lambda_n}(\eta)$ in which all edges of $\Lambda_{n+1}\backslash\Lambda_n$ are closed, so the parallel rays cannot be connected no matter what the configuration ouside Λ_{n+1} is. Now for all $\zeta^{1,n} \in \mathcal{N}^1_{\Lambda_n}(\eta)$, $\zeta^{2,n} \in \mathcal{N}^0_{\Lambda_n}(\eta)$

$$\pi_e(\chi_e|\zeta^{1,n}) - \pi_e(\chi_e|\zeta^{2,n}) = p - \frac{p}{p+q(1-p)} > 0, \tag{4.2}$$

uniformly in n.
Since $\mathcal{N}^1_{\Lambda_n}(\eta)$ and $\mathcal{N}^0_{\Lambda_n}(\eta)$ carry positive μ-measure, it follows that no function that equals

$\pi_e(\chi_e|\cdot)$ μ-a.s. can be quasilocal at η. ∎

We set out now to find how large the set of configurations is, where $\pi_e(\chi_e|\cdot)$ exhibits essential non-quasilocality. It turns out that this set is rather small in measure-theoretical sense.

Lemma 5 *μ-a.s. there exists no more than one infinite cluster. The same holds for $\tilde{\mu}$.*

Proof:
As a consequence of the Burton-Keane uniqueness Theorem [9], the result follows from translation invariance of μ and the so-called finite-energy property, i.e.

$$0 < \mathbb{E}_\mu[\chi_e|\mathcal{F}^e] < 1, \ \mu\text{-a.s.}. \tag{4.3}$$

But this condition is easily verified from expression (3.8) for $\pi_e(\chi_e|\cdot)$ and Proposition 1. The same reasoning applies to $\tilde{\mu}$. ∎

Proposition 3 *The function $\pi_e(\chi_e|\cdot)$ is quasilocal μ-a.s..*

Proof:
According to Lemma 5, the set Ω_1 of configurations where there is no infinite cluster or a unique infinite cluster carries full measure. Take thus any $\eta \in \Omega_1$. Suppose first that η has no infinite cluster. Then there exists some finite set Λ, $e \in \Lambda$, for which no site in Λ is connected to Λ^c. But then $\pi_e(\chi_e|\eta) = \pi_e(\chi_e|\zeta)$ whenever η and ζ agree inside Λ. This proves locality of $\pi_e(\chi_e|\cdot)$ at such η.
Now take any configuration $\eta \in \Omega_1$ that has a unique infinite cluster. If not both x and y are connected to infinity, then again there exists a finite set Λ such that $\pi_e(\chi_e|\eta) = \pi_e(\chi_e|\zeta)$ whenever η and ζ agree inside Λ. So suppose that both x and y are connected to infinity. Because of the uniqueness of the infinite cluster, there exists now a finite set Λ such that x and y are connected by a path of open edges within Λ. In that case again $\pi_e(\chi_e|\zeta) = \pi_e(\chi_e|\eta)$ for all ζ that agree with η inside Λ. Hence $\pi_e(\chi_e|\cdot)$ is quasilocal for all configurations $\eta \in \Omega_1$. ∎

Because both μ and $\tilde{\mu}$ have no more than one infinite cluster it is now clear from expressions (3.9) and (3.11) that μ-a.s. and $\tilde{\mu}$-a.s. $\pi_e(\chi_e|\cdot) = \tilde{\pi}_e(\chi_e|\cdot)$. It follows that $\pi_e(\chi_e|\cdot)$ is a version of both $\mathbb{E}_\mu[\chi_e|\mathcal{F}^e]$ and $\mathbb{E}_{\tilde{\mu}}[\chi_e|\mathcal{F}^e]$ and that $\pi_e(\chi_e|\cdot)$ is also quasilocal $\tilde{\mu}$-a.s..

Acknowledgment: KVV thanks the EPFL in Lausanne, where part of the work was done, for kind hospitality.

References

[1] A.C.D. van Enter, R. Fernández and A.D. Sokal: *Regularity properties and pathologies of position-space renormalization transformations: scope and limitations of Gibbsian theory*, J. Stat. Phys. **72**, 879 (1993)

[2] H.-O. Georgii: *Gibbs measures and phase transitions*, de Gruyter, Berlin (1988)

[3] J.L. Lebowitz, R.H. Schonmann: *Pseudo-free energies and large deviations for non-Gibbsian FKG measures*, Prob. Th. Rel. Fields **77**, 49 (1988)

[4] R. Fernández and C.-E. Pfister, *Non-quasilocality of projections of Gibbs measures*, EPFL preprint.

[5] C. Maes, K. Vande Velde: *Defining relative energies for the projected Ising measure*, Helv. Phys. Acta **65**, 1055 (1992)

[6] M. Aizenman, J.T. Chayes, L. Chayes and C.M. Newman: *Discontinuity of the magnetization in one-dimensional* $1/|x-y|^2$ *Ising and Potts models*, J. Stat. Phys. **50**, 1 (1988).

[7] W.G. Sullivan: *Potentials for almost Markovian random fields*, Commun. Math. Phys. **33**, 61 (1973)

[8] O.K. Kozlov: *Gibbs description of a system of random variables*, Probl. Inform. Transmiss. **10**, 258 (1974)

[9] R.M. Burton, M. Keane: *Density and uniqueness in percolation*, Commun. Math. Phys. **121**, 501 (1989)