Phase Uniqueness and Correlation Length in Diluted-Field Ising Models

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November 18, 1994

Abstract

The Diluted-Field Ising Model, a random nonnegative field ferromagnetic model, is shown to have a unique Gibbs measure with probability 1 when the field mean is positive. Our methods involve comparisons with ordinary uniform field Ising models. They yield as a corollary a way of obtaining spontaneous magnetization through the application of a vanishing random magnetic field. We also establish existence and nonrandomness of the correlation length of this model, defined as $(\lim_{n\to\infty} -(1/n)\log\langle\sigma_{\mathbf{0}};\sigma_{\boldsymbol{n}}\rangle)^{-1}$, where $\boldsymbol{n}$ is the site on the first coordinate axis at distance n from the origin and $\langle\sigma_{\mathbf{0}};\sigma_{\boldsymbol{n}}\rangle$ is the origin to $\boldsymbol{n}$ two point truncated correlation function,and derive an upper bound for it in terms of the correlation length of an ordinary nonrandom model with uniform field related to the field distribution of the diluted model.

Mathematics Subject Classification 1991: 60K35, 82B44.
Key words and phrases: Phase uniqueness, exponential decay of correlations, spontaneous magnetization, disordered systems, diluted systems.

1 Introduction

In this work we introduce what we call the Diluted-Field Ising Model (DFIM) which is a quenched random field ferromagnetic pair interaction Ising model in $\{-1,+1\}^{\mathbb{Z}^d}$ with the the i.i.d. fields having a distribution which is concentrated on the nonnegative extended real numbers. That is, $\mathbf{h} \equiv \{h_i,\, i \in \mathbb{Z}^d\}$, the fields of the model, form a family of i.i.d. random variables with $\Pr(h_i \geq 0) = 1$.

We adopt this terminology in order to distinguish this model from the usual Random Field Ising Model (RFIM), for which the fields distribution support also negative values, usually having mean zero. Another reason is that the randomness in the DFIM is closer in nature to that in diluted-site and bond models (although some features of the DFIM, e.g. absence of first order phase transition — see Theorem 1 below —, differ from those of these other diluted models), so much so that techniques used in analyzing site and bond diluted models work for the DFIM as well ([2]).

Important somewhat recent works in the RFIM are those of Bricmont and Kupiainen ([3]) and Aizenman and Wehr ([4]) who established respectively the existence of first order phase transition in at least three dimensions and its absence in dimension two.

Dreifus, Klein and Perez recently discussed the RFIM ([5]) in high field, no condition on the mean, and, improving on work by Beretti ([6]), derived results similar to ours for the DFIM (see Theorems 1 and 2 below). Their argument needs the condition that the sites with field zero do not percolate, which appears to be necessary in general. In the context of the DFIM, although there has been some consideration that percolation might also play a role, indeed it does not. The competing heuristics prevails which says that the volume contribution of the diluted field, no matter how small its distribution, as long as it has positive mean, dominates the boundary effects.

Our approach is to put rigor in the (physical) intuition that macroscopically the DFIM should behave as an ordinary Ising model with uniform field related to the diluted field mean (or, more precisely, distribution).

By means of correlation inequalities leading to convexity properties of correlations as functions of $\mathbf{h}$ and the use of Jensen's inequalities, we arrive at comparison inequalities between the DFIM and ordinary uniform field Ising models from which we get the results for the DFIM.

A corollary of our argument provides alternative 'random' ways to spontaneously magnetize an Ising system below the critical temperature other than the usual 'deterministic' way. One of them goes as follows. As in the usual case, turn on an external field h but *only* for sites chosen at random each with positive probability p. Do not turn any field on for the sites not chosen. Then turn the external field down slowly. Another way is assign to each site a uniform in [0,1] random variable independent of the other sites. If for a given site its uniform random variable is less than a positive number p, then turn on an external field h for this site. Do not turn any external field on if the random variable is greater than p. Then make p go to zero. In both ways (and in any combination of them), the spontaneous magnetization achieved is the same attained by the usual way of turning on the external field h for *all* sites and then turning it off slowly.

Let μ denote the common field mean, i.e. $\mu = E(h_i)$. Of course then $\mu \geq 0$. Our main results are stated below. The first one establishes insensitivity to boundary conditions of the (infinite volume) magnetization at site i (the *minus* and *plus* cases, denoted $\langle \sigma_i \rangle^-$ and $\langle \sigma_i \rangle^+$ respectively, are well defined by ferromagnetism).

Theorem 1 *If $\mu > 0$, then $\langle \sigma_i \rangle^- = \langle \sigma_i \rangle^+$ with probability 1.*

Remark 1.1 *By ferromagetism, the above result implies a.s. uniqueness of the infinite volume Gibbs state (which of course nevertheless depends on* **h***). By the same reason, there is also no extra magnetization gained or lost by turning on an extra positive or negative uniform field and subsequently turning it off slowly.*

The next result concerns the correlation length of the DFIM and its relation to the fields distribution. Let $\boldsymbol{n} = (n, 0, \dots, 0)$ be the site on the first coordinate axis of $\mathbb{Z}^d$ at distance n from the origin. We define the correlation length ξ by

$$\xi^{-1} = \lim_{n \to \infty} -\frac{1}{n} \log \langle \sigma_{\mathbf{0}}; \sigma_{\boldsymbol{n}} \rangle, \tag{1.1}$$

when the limit exists (otherwise replace lim by lim inf).

Theorem 2 *If $E(-\log \langle \sigma_{\mathbf{0}}; \sigma_{\mathbf{1}} \rangle) < \infty$, then with probability 1 the limit in (1.1) exists, is finite and nonrandom. Moreover,*

$$\xi \leq \xi(\delta),$$

where $\xi(a)$, $a \in$ ˚; denotes the correlation length of the Ising model with uniform field a and δ is a function of the distribution of **h** *which is positive whenever μ is positive.*

In the next section we present some basic auxiliary results ingredient to the proofs of the above theorems which appear in Section 3. In the concluding Section 4 we discuss briefly the case of field distributions including negative values. The methods which were successful in the diluted-field case are seen to go only halfway in establishing uniqueness of the Gibbs measure, stopping short of saying anything about the correlation length. The extension of Theorem 2 to other diluted models is considered.

Acknowledgements. This paper originated from discussions with H. v. Dreifus and A. C. Paiva, whom we gladly thank. We benefitted also from discussions with D. H. U. Marchetti and R. H. Schonmann. This research is part of FAPESP "Projeto Temático" Grant number 90/3918-5. Partially supported by CNPq.

2 Correlation Inequalities and Convexity

Consider the (ferromagnetic) Ising model in $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$ with Hamiltonian H given by

$$-H(\sigma) = \sum_{\langle ij \rangle} \sigma_i \sigma_j + \sum_i h_i \sigma_i, \tag{2.1}$$

where the first sum is taken, as notation indicates, over nearest neighbor sites, each pair appearing only once. The fields $\mathbf{h} = \{h_i,\, i \in \mathbb{Z}^d\}$ are in principle any extended real numbers.

Let $\langle \cdot \rangle$ as usual denote the expectation with respect to the Ising measure

$$\langle f \rangle = \frac{1}{Z} \sum_{\sigma} f(\sigma) e^{-\beta H}, \tag{2.2}$$

where β is a positive real parameter which is usually interpreted as the inverse of the temperature, Z is the usual normalization factor

$$Z = \sum_{\sigma} e^{-\beta H}, \tag{2.3}$$

and f is a local function.

The quantities to be studied here include the magnetization at site i, $\langle\sigma_i\rangle$, the truncated two point function at sites i and j,

$$\langle\sigma_i;\sigma_j\rangle \equiv \langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle$$

and the truncated three point function at sites i, j and k,

$$\langle\sigma_i;\sigma_j;\sigma_k\rangle \equiv \langle\sigma_i\sigma_j\sigma_k\rangle - \langle\sigma_i\sigma_j\rangle\langle\sigma_k\rangle - \langle\sigma_i\sigma_k\rangle\langle\sigma_j\rangle - \langle\sigma_j\sigma_k\rangle\langle\sigma_i\rangle + 2\langle\sigma_i\rangle\langle\sigma_j\rangle\langle\sigma_k\rangle.$$

We keep the volume and boundary dependence of $\langle\cdot\rangle$ implicit for awhile. The volume is in principle finite, but the thermodynamic limit will be taken eventually.

The following *GHS* type inequalities will be useful for us. Their proof can be found in [1]. Let us consider the duplicated Ising model (σ,τ) in Ω^2 with two independent copies of the model considered initially. Define the transformed variables

$$t_i = \frac{1}{\sqrt{2}}(\sigma_i+\tau_i) \quad q_i = \frac{1}{\sqrt{2}}(\sigma_i-\tau_i), \quad i\in\mathbb{Z}^d.$$

For a subset A of $\mathbb{Z}^d$ (with multiplicity of elements allowed), let t_A denote the product $\prod_{i\in A} t_i$ and similarly for q.

Proposition 2.1 *For any* $\mathbf{h}$*,*

$$\langle q_A\rangle \geq 0. \tag{2.4}$$

For any $\mathbf{h}$ *such that* $h_l\geq 0$*, for all* l*,*

$$\langle q_A t_B\rangle \geq 0, \tag{2.5}$$

$$\langle q_A t_B\rangle - \langle q_A\rangle\langle t_B\rangle \leq 0, \tag{2.6}$$

$$\langle q_A q_B\rangle - \langle q_A\rangle\langle q_B\rangle \geq 0. \tag{2.7}$$

We have the following consequent results, respectively of (2.4) (making $A=\{i,j\}$) and (2.6) (making $A=\{i,j\}$ and $B=\{k\}$).

Corollary 2.1 *For any* $\mathbf{h}$*,* i *and* j*,*

$$\langle\sigma_i;\sigma_j\rangle \geq 0. \tag{2.8}$$

For any $\mathbf{h}$ *such that* $h_l\geq 0$*, for all* l*,* i*,* j *and* k*,*

$$\langle\sigma_i;\sigma_j;\sigma_k\rangle \leq 0. \tag{2.9}$$

The following is a corollary to (2.5) and (2.7).

Corollary 2.2 *For any* $\mathbf{h}$ *such that* $h_l \geq 0$*, for all* l*,* i*,* j *and* k*,*

$$\langle \sigma_i ; \sigma_j \rangle \geq \frac{1}{2} \langle \sigma_i ; \sigma_k \rangle \langle \sigma_j ; \sigma_k \rangle. \tag{2.10}$$

Proof:

¿From $(q_k + t_k)^2 = 2$, we have

$$\begin{aligned} 2\langle q_i q_j \rangle &= \langle q_i (q_k + t_k)^2 q_j \rangle = \langle q_i q_k^2 q_j \rangle + 2 \langle q_i q_j q_k t_k \rangle + \langle q_i q_j t_k^2 \rangle \\ &\geq \langle q_i q_k^2 q_j \rangle \geq \langle q_i q_k \rangle \langle q_j q_k \rangle, \end{aligned}$$

where the two inequalities are due respectively to (2.5) and (2.7). The proof is concluded by the observation that $\langle \sigma_i ; \sigma_j \rangle = \langle q_i q_j \rangle$ for all pair of sites i and j. $\square$

Consider now $\langle \sigma_i \rangle(\mathbf{h})$ and $\langle \sigma_i ; \sigma_j \rangle(\mathbf{h})$ as functions of $\mathbf{h}$. The above inequalities imply the following properties of these functions, which are well known and will be of later use.

Proposition 2.2

$\langle \sigma_i \rangle(\mathbf{h})$ *is (coordinatewise) nondecreasing in* $\{\mathbf{h} : h_i \in$; for all $i\}$.
$\langle \sigma_i ; \sigma_j \rangle(\mathbf{h})$ *is (coordinatewise) nonincreasing in* $\{\mathbf{h} : h_i \geq 0, \text{ for all } i\}$.
$\langle \sigma_i \rangle(\mathbf{h})$ *is (coordinatewise) concave in* $\{\mathbf{h} : h_i \geq 0, \text{ for all } i\}$.

Proof:

$\frac{\partial}{\partial h_k} \langle \sigma_i \rangle(\mathbf{h}) = \beta \langle \sigma_i ; \sigma_k \rangle \geq 0$, by (2.8), proves the first assertion.
$\frac{\partial}{\partial h_k} \langle \sigma_i ; \sigma_j \rangle(\mathbf{h}) = \beta \langle \sigma_i ; \sigma_j ; \sigma_k \rangle \leq 0$, by (2.9), proves the second one.
$\frac{\partial^2}{\partial h_k^2} \langle \sigma_i \rangle(\mathbf{h}) = \beta^2 \langle \sigma_i ; \sigma_k ; \sigma_k \rangle \leq 0$, by (2.9), proves the third one. $\square$

Next we derive convexity properties of $\langle \sigma_i \rangle(\mathbf{h})$ and $\langle \sigma_i ; \sigma_j \rangle(\mathbf{h})$ when $\mathbf{h}$ is restricted to a hyperretangle.

For a and b two real numbers such that $a < b$ define

$$H_{ab} = \{\mathbf{h} : a \leq h_i \leq b, \text{for all } i\}.$$

Let η and ζ be two functions from $[a,b]$ to $[a,b]$ defined as

$$\begin{aligned}
\eta(x) &= \eta_{ab}(x) = \frac{1}{2\beta}\log\left[1-(1-e^{-2\beta(b-a)})\frac{x-a}{b-a}\right]^{-1}+a,\\
\zeta(x) &= \zeta_{ab}(x) = \frac{1}{2\beta}\log\left[1+(e^{2\beta(b-a)}-1)\frac{x-a}{b-a}\right]+a.
\end{aligned}$$

They have the following properties which are easy to check. Both are strictly increasing, $\eta(a)=\zeta(a)=a$, $\eta(b)=\zeta(b)=b$ and $\eta(x)<x<\zeta(x)$ in (a,b). Moreover

$$\eta''(x)-2\beta(\eta'(x))^2=0, \tag{2.11}$$

$$\zeta''(x)+2\beta(\zeta'(x))^2=0, \tag{2.12}$$

in (a,b).

Further define $\boldsymbol{\eta}(\mathbf{h})=\{\eta(h_i)\}$ and $\boldsymbol{\zeta}(\mathbf{h})=\{\zeta(h_i)\}$ and the functions of $\mathbf{h}$

$$\phi_i(\mathbf{h}) = \langle\sigma_i\rangle(\boldsymbol{\eta}(\mathbf{h})), \tag{2.13}$$

$$\lambda_i(\mathbf{h}) = \langle\sigma_i\rangle(\boldsymbol{\zeta}(\mathbf{h})), \tag{2.14}$$

$$\psi_i(\mathbf{h}) = \langle\sigma_i;\sigma_j\rangle(\boldsymbol{\eta}(\mathbf{h})). \tag{2.15}$$

They have the following properties.

Proposition 2.3 *For any a and b, $\phi_i(\mathbf{h})$ is (coordinatewise) convex and $\lambda_i(\mathbf{h})$ is (coordinatewise) concave in H_{ab}.*

If $a\geq 0$, then $\psi_i(\mathbf{h})$ is (coordinatewise) concave in H_{ab}.

Proof:

$$\frac{\partial^2}{\partial h_k^2}\langle\sigma_i\rangle(\boldsymbol{\eta}(\mathbf{h})) = [\eta''(h_k)-2\beta\langle\sigma_k\rangle(\eta'(h_k))^2]\langle\sigma_i;\sigma_k\rangle\geq 0, \tag{2.16}$$

as $\eta''(x)-2\beta(\eta'(x))^2=0$, using (2.8) and $\langle\sigma_k\rangle\leq 1$.

$$\frac{\partial^2}{\partial h_k^2}\langle\sigma_i\rangle(\boldsymbol{\zeta}(\mathbf{h})) = [\zeta''(h_k)-2\beta\langle\sigma_k\rangle(\zeta'(h_k))^2]\langle\sigma_i;\sigma_k\rangle\leq 0, \tag{2.17}$$

as $\zeta''(x)+2\beta(\zeta'(x))^2=0$, using (2.8) and $\langle\sigma_k\rangle\geq -1$.

$$\begin{aligned}
\frac{\partial^2}{\partial h_k^2}\langle\sigma_i;\sigma_j\rangle(\boldsymbol{\eta}(\mathbf{h})) &= [\eta''(h_k)-2\beta\langle\sigma_k\rangle(\eta'(h_k))^2]\langle\sigma_i;\sigma_j;\sigma_k\rangle &(2.18)\\
&- 2\langle\sigma_i;\sigma_k\rangle\langle\sigma_j;\sigma_k\rangle\leq 0, &(2.19)
\end{aligned}$$

as $\eta''(x)-2\beta(\eta'(x))^2=0$, using (2.8), (2.9) and $\langle\sigma_k\rangle\leq 1$. □

3 Proofs of Main Results

We now use the results of the previous section to give proofs to Theorems 1 and 2.

For Λ any subset of $\mathbb{Z}^d$ and δ any real number, let $\mathbf{h}_\Lambda$ denote $\{h_i,\, i \in \Lambda\}$ and δ_Λ denote $\{\delta\}^\Lambda$.

Proof of Theorem 1:

We consider the conditional expectations of $\langle\sigma_i\rangle^-$ and $\langle\sigma_i\rangle^+$ with respect to the field configuration inside a finite volume Λ of $\mathbb{Z}^d$, denoted by $E\left[\langle\sigma_i\rangle^-|\mathbf{h}_\Lambda\right]$ and $E\left[\langle\sigma_i\rangle^+|\mathbf{h}_\Lambda\right]$, respectively.

By the Martingale Convergence Theorem, with probability 1

$$\langle\sigma_i\rangle^+ = \lim_{\Lambda\nearrow\mathbb{Z}^d} E\left[\langle\sigma_i\rangle^+|\mathbf{h}_\Lambda\right] \tag{3.1}$$

and similarly for $\langle\sigma_i\rangle^-$.

We treat first the case that the distribution of $\mathbf{h}$ is supported in a finite interval $[a,b]$. By Propositions 2.2 and 2.3, using Jensen's inequalities, we have

$$E\left[\langle\sigma_i\rangle^+|\mathbf{h}_\Lambda\right] \;\leq\; \langle\sigma_i\rangle^+(\mathbf{h}_\Lambda, \Delta_{\Lambda^c}), \tag{3.2}$$

$$E\left[\langle\sigma_i\rangle^-|\mathbf{h}_\Lambda\right] \;\geq\; \langle\sigma_i\rangle^-(\mathbf{h}_\Lambda, \delta_{\Lambda^c}), \tag{3.3}$$

where $\Delta = \mu$ and $\delta = \eta(\mu)$. Both are strictly positive since μ is.

By the absence of first order phase transition for models with uniform nonzero field (outside of a finite volume) ([7]), the right hand side of both inequalities do not depend on boundary conditions and will be denoted respectively $\langle\sigma_i\rangle(\mathbf{h}_\Lambda, \Delta_{\Lambda^c})$ and $\langle\sigma_i\rangle(\mathbf{h}_\Lambda, \delta_{\Lambda^c})$.

We consider now finite volume (Γ) approximants to those quantities denoted respectively by $\langle\sigma_i\rangle(\mathbf{h}_\Lambda, \Delta_{\Gamma\setminus\Lambda})$ and $\langle\sigma_i\rangle(\mathbf{h}_\Lambda, \delta_{\Gamma\setminus\Lambda})$. One then has (by the Fundamental Theorem of Calculus)

$$\langle\sigma_i\rangle(\mathbf{h}_\Lambda, \Delta_{\Gamma\setminus\Lambda}) - \langle\sigma_i\rangle(\mathbf{h}_\Lambda, \delta_{\Gamma\setminus\Lambda}) = \beta\int_\delta^\Delta \sum_{j\in\Gamma\setminus\Lambda} \langle\sigma_i;\sigma_j\rangle(\mathbf{h}_\Lambda, x_{\Gamma\setminus\Lambda})\,dx. \tag{3.4}$$

By the FKG inequality ([7]), (3.2) and (3.3)

$$0 \leq \langle\sigma_i\rangle^+ - \langle\sigma_i\rangle^- \leq \liminf_{\Lambda\nearrow\mathbb{Z}^d}\,\lim_{\Gamma\nearrow\mathbb{Z}^d} \beta\int_\delta^\Delta \sum_{j\in\Gamma\setminus\Lambda} \langle\sigma_i;\sigma_j\rangle(\mathbf{h}_\Lambda, x_{\Gamma\setminus\Lambda})\,dx. \tag{3.5}$$

Now Fatou's Lemma, Propositions 2.2 and 2.3 and Jensen's inequality imply

$$
\begin{aligned}
0 \le E(\langle\sigma_i\rangle^+ - \langle\sigma_i\rangle^-) &\le \liminf_{\Lambda\nearrow\mathbb{Z}^d} \lim_{\Gamma\nearrow\mathbb{Z}^d} \beta \int_\delta^\Delta \sum_{j\in\Gamma\setminus\Lambda} E(\langle\sigma_i;\sigma_j\rangle(\mathbf{h}_\Lambda, x_{\Gamma\setminus\Lambda}))\,dx \\
&\le \liminf_{\Lambda\nearrow\mathbb{Z}^d} \lim_{\Gamma\nearrow\mathbb{Z}^d} \beta \int_\delta^\Delta \sum_{j\in\Gamma\setminus\Lambda} \langle\sigma_i;\sigma_j\rangle(\delta_\Lambda, x_{\Gamma\setminus\Lambda})\,dx \\
&= \liminf_{\Lambda\nearrow\mathbb{Z}^d} \lim_{\Gamma\nearrow\mathbb{Z}^d} (\langle\sigma_i\rangle(\delta_\Lambda, \Delta_{\Gamma\setminus\Lambda}) - \langle\sigma_i\rangle(\delta_\Gamma)).
\end{aligned}
$$

The right hand side is zero by the absence of first order phase transition in the model with uniform positive field (δ).

A final recourse to FKG yields the result in this first case.

For the general case, let $\mathbf{h}_\Lambda^{(n)} = \{\min(h_i, n)\}$, $n \ge 1$, and $\delta = \eta_{01}(E[h_i^{(1)}])$, which is positive. Equation 3.3 and FKG combine to give, with probability 1

$$
\langle\sigma_i\rangle^- = \lim_{\Lambda\nearrow\mathbb{Z}^d} \langle\sigma_i\rangle(\mathbf{h}_\Lambda, \delta_{\Lambda^c}). \tag{3.6}
$$

We then only have to show that $\langle\sigma_i\rangle^+$ equals the right hand side . But this is a consequence of the inequality

$$
E\left[\langle\sigma_i\rangle(\mathbf{h}_\Lambda^{(n)}, \Delta_{\Gamma\setminus\Lambda}) - \langle\sigma_i\rangle(\mathbf{h}_\Lambda^{(n)}, \delta_{\Gamma\setminus\Lambda})\right] \le \langle\sigma_i\rangle(\delta_\Lambda, \Delta_{\Gamma\setminus\Lambda}) - \langle\sigma_i\rangle(\delta_\Gamma), \tag{3.7}
$$

with $\Delta > \delta$, which is proved as (3.6) using the monotonicity of $\langle\sigma_i;\sigma_j\rangle(\mathbf{h})$ (Proposition 2.2) as an extra ingredient. Just take the limits $n\uparrow\infty$, $\Delta\uparrow\infty$, $\Gamma\nearrow\mathbb{Z}^d$ and $\Lambda\nearrow\mathbb{Z}^d$ (in this order) together with the Dominated Convergence Theorem and FKG on the left hand side and the absence of first order phase transition in the uniform positive field regime on right hand side to get the result. $\square$

Remark 3.1 *As anticipated in the introduction, it follows from the above proof that one can spontaneously magnetize a ferromagnetic Ising system below its critical temperature by first selecting randomly with arbitrary uniform positive probability the sites to be submitted to a constant field, the non-selected sites receiving no direct field, and then turning the field off slowly. This is because the expected magnetization of the system is bounded below by* *that of an ordinary system with positive uniform field at the same temperature. Another way, by the same reason, is to assign to each site a standard uniform random variable independent of those of the other sites and turn on an external field h for this site if its random variable is less than a positive number p, leaving it with no direct field if the random variable is bigger than p. Then make p go to zero.*

Proof of Theorem 2:

Let $\mathcal{L}_{i,j}$ denote $-\log\frac{1}{2}\langle\sigma_i;\sigma_j\rangle$. By Corollary 2.2 the following holds

$$\mathcal{L}_{\mathbf{0},\boldsymbol{n}+\boldsymbol{m}} \leq \mathcal{L}_{\mathbf{0},\boldsymbol{m}} + \mathcal{L}_{\boldsymbol{m},\boldsymbol{n}+\boldsymbol{m}}. \tag{3.8}$$

This together with the ergodicity of $\mathbf{h}$ and the finite expectation hypothesis allows the application of Kingman's Subadditive Ergodic Theorem so that we have

$$\begin{aligned} \xi^{-1} &= \lim_{n\to\infty}\frac{1}{n}\mathcal{L}_{\mathbf{0},\boldsymbol{n}} = \lim_{n\to\infty}\frac{1}{n}E(\mathcal{L}_{\mathbf{0},\boldsymbol{n}}) \qquad (3.9)\\ &= \lim_{n\to\infty}\frac{1}{n}E(-\log\langle\sigma_{\mathbf{0}};\sigma_{\boldsymbol{n}}\rangle). \qquad (3.10)\end{aligned}$$

To prove the last assertion of the theorem (with a somewhat optimal δ) we apply Propositions 2.2 and 2.3 and Jensen to get

$$E(-\log\langle\sigma_{\mathbf{0}};\sigma_{\boldsymbol{n}}\rangle(\mathbf{h})) \geq E(-\log\langle\sigma_{\mathbf{0}};\sigma_{\boldsymbol{n}}\rangle(\mathbf{h}^{(m)}) \geq -\log\langle\sigma_{\mathbf{0}};\sigma_{\boldsymbol{n}}\rangle(\boldsymbol{\delta}^{(m)}), \tag{3.11}$$

for all m and n, with $\mathbf{h}^{(m)}$ as defined prior to (3.6) and $\boldsymbol{\delta}^{(m)} = \eta_{0m}(E[h_i^{(m)}])$. We can take then $\delta = \sup_m \boldsymbol{\delta}^{(m)}$, so that

$$\xi^{-1} \geq \lim_{n\to\infty} -\frac{1}{n}\log\langle\sigma_{\mathbf{0}};\sigma_{\boldsymbol{n}}\rangle(\delta) = \xi^{-1}(\delta).$$

(Notice that the above limit exists by subadditivity [1].) □

Remark 3.2 *A special case where the condition $E(-\log\langle\sigma_{\mathbf{0}};\sigma_{\mathbf{1}}\rangle) < \infty$ is not met is when $\Pr(h_i = \infty) > 0$, for then $\langle\sigma_{\mathbf{0}};\sigma_{\mathbf{1}}\rangle = 0$ with positive probability. It is clear that in this situation one can find with probability 1 a (random) subsequence (n_k) along which $\langle\sigma_{\mathbf{0}};\sigma_{\boldsymbol{n_k}}\rangle = 0$, so that with probability 1*

$$\limsup_{n\to\infty} -\frac{1}{n}\log\langle\sigma_{\mathbf{0}};\sigma_{\boldsymbol{n}}\rangle = \infty.$$

[1] We note that the limit $\lim_{n\to\infty} -\frac{1}{n}\log E(\langle\sigma_{\mathbf{0}};\sigma_{\boldsymbol{n}}\rangle))$ also exists by subadditivity, which is a consequence of Proposition 2.2, Corollary 2.2 and the Harris-FKG inequality.

4 Extensions and Open Questions

We use this section to briefly discuss some extensions and limitations of the previous arguments and methods to other related models, basically the RFIM with field distribution including negative values and diluted-bond ferromagnetic models.

For the RFIM with both positive and negative fields, we do not have monotonicity nor convexity of the truncated two point function (at least not in the form of Propositions 2.2 and 2.3), but we still have them for the magnetization. This is sufficient for the following result.

Let $\langle\sigma_i\rangle^x$ denote the limit

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} \langle\sigma_i\rangle(\mathbf{h}_\Lambda, x_{\Lambda^c}), \tag{4.1}$$

where x is any extended real number, when the limit exists

Proposition 4.1 *Suppose the distribution of h_i is supported in a finite interval $[a,b]$. Then the limit in (4.1) exists for $x \leq \delta = \eta_{ab}(\mu)$ and $x \geq \Delta = \zeta_{ab}(\mu)$ and*

$$\begin{aligned} \langle\sigma_i\rangle^- &= \langle\sigma_i\rangle^x, \ \textit{for } x \leq \delta, & (4.2)\\ \langle\sigma_i\rangle^+ &= \langle\sigma_i\rangle^x, \ \textit{for } x \geq \Delta. & (4.3) \end{aligned}$$

Proof:

Use FKG and (3.1), (3.2), (3.3) as in (3.6), which are all valid for any finite a and b $\square$

Together with the FKG ordering, this proposition implies that the Gibbs measure with minus boundary conditions equals all with x *boundary conditions* for $x \leq \delta$ and the *plus* Gibbs measure equals all with x *boundary conditions* for $x \geq \Delta$.

In the case of Theorem 1, we proceeded to prove that $\langle\sigma_i\rangle^\delta = \langle\sigma_i\rangle^\Delta$ by using the convexity properties of the truncated two point function, which do not hold if $a < 0 < b$.

Notice that if μ is sufficiently close to b, then $\delta > 0$ and the *minus* Gibbs measure equals one with positive field (δ) boundary condition. This could be an evidence that $\langle\sigma_i\rangle^\delta = \langle\sigma_i\rangle^\Delta$ and so that $\langle\sigma_i\rangle^- = \langle\sigma_i\rangle^+$, yielding uniqueness. But this seems to be a problem not amenable to our methods.

Another issue raised by our methods is that the functions η (and ζ) depend on β in a way that $\eta \downarrow 0$ as $\beta \uparrow \infty$, making the positive field of the uniform field model dominated by the DFIM not uniform with temperature, which probably should not be the case.

We turn now briefly to the issue of the existence and nonrandomness of the correlation length for *diluted-bond Ising models*, which are quenched random Ising models with

$$-H(\sigma) = \sum_{\langle ij \rangle} J_{ij}\sigma_i\sigma_j, \tag{4.4}$$

where $\{J_{ij}, \langle ij \rangle\}$ is a family of i.i.d. non-negative random variables. We note that Theorem 2 holds in this situation, as well as its proof. It is worth mentioning that the more common dilution variables (the J_{ij}'s), for which $\Pr(J_{ij} = 0)$ is positive, do not satisfy the finite expectation hypothesis. Another limitation is we do not have a convexity property of $\langle \sigma_i; \sigma_j \rangle$ as a function of $\{J_{ij}\}$ from which to get bounds for ξ.

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