A Remark on Differentiability of the Pressure Functional

Aernout van Enter

Institute for Theoretical Physics
University of Groningen
Groningen, The Netherlands

Boguslaw Zegarlinski

Department of Mathematics
Imperial College
London, UK

Abstract: We give a short review of results on equilibrium description and description by stochastic dynamics for spin systems on a lattice. We remark also that some coercive inequalities for the generators of stochastic dynamics, as e.g. the Logarithmic Sobolev inequality, can be used in a direct and natural way to prove strong differentiability properties of the pressure functional for lattice spin systems with multiparticle interactions at high temperatures. Motivated by this, we exhibit also a class of examples of multiparticle interactions which do not belong to the space $\mathbb{B}_2$ of spin interactions, but for which the Gibbs measures exist and are unique at high temperatures.

1. An Introduction.

In recent years great progress has been made in understanding the connection between the equilibrium description and the description by stochastic dynamics of lattice spin systems, see e.g. [**SZ1-3**], [**Z1-3**], [**MO1,2**], [**LY**], [**La**]. The method used there has been based on an indirect application of some inequalities involving a Gibbs measure and the Dirichlet form of a generator associated to a related stochastic dynamics. In the present paper we would like to show that one can use them also in a direct and natural way to obtain useful information about the strong (or Fréchet) differentiability of the pressure functional. To explain our motivation and results we would like first to recall some known facts. We need also to introduce some notation necessary to describe spin systems on a lattice $\Gamma \equiv \mathbb{Z}^d$. Let $\mathcal{F}$ be the family of all finite subsets of the lattice and let $\mathcal{F}_0$ be a countable exhaustion of Γ, i.e. an increasing sequence of finite sets invading all the lattice. A configuration space of a spin system is by definition a product space $(\Omega, \Sigma) \equiv (\mathbf{S}, \mathcal{B}_{\mathbf{S}})^\Gamma$, where the single spin space $\mathbf{S}$ is either a finite set or a (smooth, compact, connected) Riemannian manifold and $\mathcal{B}_{\mathbf{S}}$ denotes the Borel σ-algebra of subsets in $\mathbf{S}$. The interaction of the spins is described by the interaction potential $\Phi \equiv \{\Phi_X\}_{X\in\mathcal{F}}$, where to a finite subset X of the lattice we associate a (real) continuous function $\Phi_X(\omega) \equiv \Phi_X(\omega_X)$, which depends only on the spins in the set X. We will assume that the interactions are translation invariant in the sense that $\Phi_X(T_{\mathbf{j}}\omega) = \Phi_{X+\mathbf{j}}$, where $(T_{\mathbf{j}}\omega)_{\mathbf{i}} \equiv \omega_{\mathbf{i}-\mathbf{j}}$. It is convenient to classify the interactions using the following norms

$$||\Phi||_{\mathbf{g}} \equiv \sum_{\substack{X\in\mathcal{F}\\ X\ni 0}} \mathbf{g}(X)\cdot ||\Phi_X||_u \tag{0.1}$$

where $\mathbf{g}$ is some positive translation invariant function on $\mathcal{F}$, and $||\cdot||_u$ denotes the supremum norm. The corresponding Banach spaces are denoted by $\mathbb{B}_{\mathbf{g}}$. In particular if $\mathbf{g}(X) = |X|^{n-1}$ we will denote the corresponding space by $\mathbb{B}_n$. Obviously we have $\mathbb{B}_{n+1} \subset \mathbb{B}_n$, for any $n \in \mathbb{Z}^+$.

It is known (see e.g.[**R2,3**]) that the following pressure functional is well defined, continuous and convex on $\mathbb{B}_0$

$$p(\Phi) \equiv \lim_{v\mathcal{F}} p_\Lambda(\Phi) \tag{0.2}$$

where the finite volume pressure $p_\Lambda(\Phi)$ is given by

$$p_\Lambda(\Phi) \equiv \frac{1}{|\Lambda|} \log \mu_0^\Lambda e^{-H_\Lambda(\Phi)} \tag{0.3}$$

with μ_0^Λ denoting the restriction of the free measure μ_0,(which is defined as the product of uniform probability measures on $(\mathbf{S}, \mathcal{B}_\mathbf{S})$), to the coordinates in the finite set Λ, and

$$H_\Lambda(\Phi) \equiv \sum_{X \subset \Lambda} \Phi_X \tag{0.4}$$

and $\lim_{v\mathcal{F}}$ denotes the limit with a van Hove sequence $\mathrm{v}\mathcal{F} \equiv \{\Lambda_n \in \mathcal{F}\}_{n \in I\!N}$ of finite sets invading all the lattice Γ. Then an equilibrium state μ_Φ of the spin system with an interaction $\Phi \in I\!B_0$ is defined as a tangent functional to the pressure p at the point Φ, i.e. for any $\Psi \in I\!B_0$ we have

$$p(\Phi + \Psi) \geq p(\Phi) - \mu_\Phi(\mathbf{A}_\Psi) \tag{0.5}$$

where we have set

$$\mathbf{A}_\Psi \equiv \mathbf{A}(\Psi) \equiv \sum_{\substack{X \in \mathcal{F} \\ X \ni 0}} \frac{1}{|X|} \Psi_X . \tag{0.6}$$

It satisfies also the following variational principle

$$p(\Phi) = \sup\{-s(\nu) - \nu(\mathbf{A}_\Phi) : \nu \in I\!M_I\} = -s(\mu_\Phi) - \mu_\Phi(\mathbf{A}_\Phi) \tag{0.7}$$

where $s(\nu)$ denotes the entropy of a translation invariant probability measure $\nu \in I\!M_I$ defined by

$$s(\nu) \equiv \lim_{v\mathcal{F}} \mu_0^\Lambda \left(f_\Lambda \log f_\Lambda\right) \tag{0.8}$$

if the Radon-Nikodym derivative

$$f_\Lambda \equiv \frac{d\nu^\Lambda}{d\mu_0^\Lambda}$$

of the restriction of the involved probability measures to the σ-algebra generated by the spins in the set Λ is finite (and equals $+\infty$ otherwise).

It is known, [**DvE**], [**I2,3,4**], [**IP**], [**So**], [**W**], [**vEFS**], that the pressure functional on the space $I\!B_0$ has some pathological properties from the point of view of physical applications. In particular the correspondence between the interaction potentials and equilibrium states can be very weird, (even if one takes into account the physical equivalence of the interactions) in the sense that almost any measure can an equilibrium measure for various non-equivalent interactions. Moreover the pressure functional is nowhere Fréchet differentiable in $I\!B_0$. This means in particular that there is no high temperature regime. There does not exist a ball around the origin in the space $I\!B_0$, such that for each interaction in this ball there exists a unique equilibrium measure. At low temperatures all the spaces $I\!B_n$ are too big, in the sense that in all of them the Gibbs phase rule is generically violated and the pressure is not Fréchet differentiable at any phase coexistence point in any subspace of finite codimension. A slightly better situation one finds in spaces $I\!B_\mathbf{g}$ defined with $\mathbf{g}$ such that $diam(X) \cdot \mathbf{g}(X) \to_{diam(X) \to \infty} \infty$, where, whenever $p(\Phi)$ is Gateaux differentiable at Φ, it is also Fréchet differentiable at this point (and the set of such potentials is a dense G_δ set in $I\!B_\mathbf{g}$),[**Ph**]. For the Gibbs rule to be valid one needs stronger conditions of the type $\sum_{X \ni 0} diam(X) \cdot \mathbf{g}(X) < \infty$, [**vE**], [**Pa2**].

The potentials from the space $I\!B_1$ are called Gibbsian. For $\Phi \in I\!B_1$ and any $\Lambda \in \mathcal{F}$, $\omega \in \Omega$ we can define a local Gibbs measure $\mu_{\Phi,\Lambda}^\omega \equiv \mu_\Lambda^\omega$ as follows

$$\mu_\Lambda^\omega(f) \equiv \delta_\omega \left(\frac{\mu_0^\Lambda(e^{-U_\Lambda} f)}{\mu_0^\Lambda(e^{-U_\Lambda})} \right) \tag{0.9}$$

where δ_ω is the Dirac measure concentrated at the configuration $\omega \in \Omega$ and the interaction energy $U_\Lambda \equiv U_\Lambda(\Phi)$ for the volume $\Lambda \in \mathcal{F}$ is given by

$$U_\Lambda(\Phi) \equiv \sum_{X \cap \Lambda \neq \emptyset} \Phi_X \tag{0.10}$$

A Gibbs measure for an interaction $\Phi \in \mathbb{B}_1$ is defined as a solution of the Dobrushin-Lanford-Ruelle equations

$$\mu\left(\mu^{\cdot}_{\Phi,\Lambda} f\right) \;=\; \mu(f) \tag{\textbf{DLR}}$$

Since our configuration space Ω is compact and for every $\Lambda \in \mathcal{F}$ the map $\omega \longmapsto \mu^{\omega}_{\Phi,\Lambda}(\,\cdot\,)$ is continuous in the weak topology, the set of solutions of the (**DLR**) is nonempty. On physical grounds one would expect that if the interaction is sufficiently small, then the solution should be unique. *Unfortunately a result of this generality remains still unknown.* A better situation is observed in the subspaces ${\mathbb{B}_1}^{(spin)}$ and ${\mathbb{B}_1}^{(gas)}$ defined respectively by

$${\mathbb{B}_1}^{(spin)} \equiv \{\Phi \in \mathbb{B}_1 : \forall X \in \mathcal{F} \exists J_X \in \mathbb{R} \quad \Phi_X(\omega) = J_X \cdot \sigma_X\} \tag{0.11}$$

and

$${\mathbb{B}_1}^{(gas)} \equiv \{\Phi \in \mathbb{B}_1 : \forall X \in \mathcal{F} \exists j_X \in \mathbb{R} \quad \Phi_X(\omega) = j_X \cdot \mathbf{n}_X\} \tag{0.12}$$

where $\sigma_X \equiv \prod_{\mathbf{i}\in X} \sigma_{\mathbf{i}}$, with $\sigma_{\mathbf{i}} \equiv \sigma_{\mathbf{i}}(\omega) \equiv \sigma_{\mathbf{i}}(\omega_{\mathbf{i}})$ is an affine function of the coordinate $\omega_{\mathbf{i}}$, called the spin at site $\mathbf{i} \in \Gamma$, and $n_X \equiv \prod_{\mathbf{i}\in X} n_{\mathbf{i}}$ is defined with the occupation number variable $n_{\mathbf{i}} \equiv \frac{1}{2}(1+\sigma_{\mathbf{i}})$. It is known, [**GM**], [**GMR**], [**I1**], [**HS2**], that in both these spaces, when the interaction potential is sufficiently small, the Gibbs measure is not only unique, but it is also analytic with respect to the (finite dimensional) changes of the potential in a small neighborhood. (A result of [**DM2**] shows that in the general spaces $\mathbb{B}_n$ one can not have analyticity even for small potentials, i.e. even at high temperatures; see also [**vEF**] for discussion. This answers Ruelle's question [**R1**] in the negative. But one always has analyticity at high temperatures in the space $\mathbb{B}_{exp}$ defined with $\mathbf{g}(X) \equiv \exp(\alpha|X|)$ for some $\alpha > 0$, [**DM1**], [**I1**], [**Pa**].)
An interesting general condition for the uniqueness of the Gibbs measure has been introduced in [**D1**] and since then has been known as *the Dobrushin uniqueness condition*; for nice expositions of the Dobrushin theory see also [**Fö**], [**L**], [**Ge**], [**S**]. It has been shown in [**G1**] that this condition alone is sufficient for the pressure functional on the space $\mathbb{B}_2$ to be twice continuously differentiable in the weak sense at a point Φ, provided that

$$||\Phi||_2' \equiv \sum_{\substack{X\in\mathcal{F}\\ X\ni 0}} (|X|-1)\cdot ||\Phi_X||_u < 1 \tag{0.13}$$

Let us stress that this fact is true without any specific assumptions about the type of interactions or the single spin space. The inequality (0.13) is sufficient for the Dobrushin uniqueness condition to be true, but not necessary. Below we will give an explicit example of a class of interactions in $\mathbb{B}_1 \setminus \mathbb{B}_2$ for which the Dobrushin condition remains true. Since the proof of Gross relies in fact only on the Dobrushin condition and not on the inequality (0.13), our examples extend the region of applicability of his result.
Let us mention that the result of Gross obviously implies that the pressure functional is once Fréchet differentiable in $\mathbb{B}_2$ for the potentials for which the Dobrushin uniqueness condition is true. In general it does not imply the twice Fréchet differentiability of the pressure under this condition. The differentiability property should be uniform in all directions, which is more than Gross states.
The nice result of [**G1**] has been later extended in [**Pr**], where the author has shown that the pressure functional is n-times continuously differentiable on the space $\mathbb{B}_n$, $n \in \mathbb{N}$, $n \geq 2$, respectively, at high temperatures (small interactions).
Again the author in his proofs has in fact used a slightly weaker condition than the inequality (0.13) plus the corresponding $\mathbb{B}_n$-type conditions and we believe that his results remains true also for our examples; (compare [**G2**]).
(Moreover under the conditions of [**Pr**] one should have $n-1$ Fréchet differentiability.)
Let us come now to the description of lattice sytems involving stochastic dynamics. One defines a stochastic dynamics starting by introducing a Markov pre-generator $\mathcal{L}$ defined on the set of (smooth) local functions

(i.e. dependent only on a finite number of coordinates), which is dense in the space of continuous functions, by the following formula

$$\mathcal{L}f \equiv \sum_{\mathbf{j}} \mathcal{L}_{\mathbf{j}} f \tag{0.14}$$

with the local generators $\mathcal{L}_{\mathbf{j}}$ defined in the discrete case (when $\mathbf{S}$ is a finite set) as the local spin-flip operators

$$\mathcal{L}_{\mathbf{j}} f(\omega) \equiv \mathcal{L}_{\mathbf{j}}^{Y} f(\omega) \equiv \mu_{Y+\mathbf{j}}^{\omega} f - f(\omega) \tag{0.15}$$

with some set $Y \in \mathcal{F}$ and in the continuous case (when $\mathbf{S}$ is the Riemannian manifold) as the local diffusion operators

$$\mathcal{L}_{\mathbf{j}} f \equiv \Delta_{\mathbf{j}} f - \nabla_{\mathbf{j}} U_{\mathbf{j}} \cdot \nabla_{\mathbf{j}} f \tag{0.16}$$

where $\Delta_{\mathbf{j}}$ and $\nabla_{\mathbf{j}}$ denote the Laplace-Beltrami and the gradient operators, respectively, with respect to coordinate $\omega_{\mathbf{j}}$. Given $Y \in \mathcal{F}$, the corresponding spin flip generator will be denoted later on by $\mathcal{L}^{Y}$. A pre-generator introduced in this way satisfies the following detailed balance condition with respect to any Gibbs measure corresponding to the same potential

$$\mu f_1 \cdot \mathcal{L} f_2 = \mu \mathcal{L} f_1 \cdot f_2 \tag{0.17}$$

(A similar property holds with the operators obtained by replacing the corresponding local operators $\mathcal{L}_{\mathbf{j}}$ by $A_{\mathbf{j}}\mathcal{L}_{\mathbf{j}}$ with positive functions $A_{\mathbf{j}}$ which are independent of the coordinates $\omega_{X+\mathbf{j}}$, or by taking a convex linear combination of operators defined in this way.) Traditionally the generators of the spin flip dynamics have been also introduced by the formula

$$\mathcal{L}_{\mathbf{j}} f(\omega) \equiv \alpha_{\mathbf{j}}(\omega) \partial_{\mathbf{j}} f(\omega) \tag{0.18}$$

with $\partial_{\mathbf{j}}$ being the discrete gradient operator defined by

$$\partial_{\mathbf{j}} f(\omega) \equiv f(\omega) - \mu_0^{\{\mathbf{j}\}}(f) \tag{0.19}$$

and rate coefficients $\alpha_{\mathbf{j}}$ which are independent of the $\omega_{\mathbf{j}}$. In fact, in the literature concerning the construction of the stochastic dynamics, see e.g. [**Su1,2**], [**Li1**], [**HS1**], one usually formulates the conditions for a pre-generator $\mathcal{L}$ to be extendible to a Markov generator in terms of the rate coefficients. It is well - known that in terms of the interaction potential Φ a sufficient condition for a pre-generator $\mathcal{L} \equiv \mathcal{L}_{\Phi}$ as defined above to be extendible to the generator of a Markov semigroup P_t, $t \geq 0$ *on the space of continuous functions* $\mathcal{C}(\Omega)$ is the following

$$\sup_{\mathbf{i}} \sum_{\mathbf{j}} ||\nabla_{\mathbf{j}} \nabla_{\mathbf{i}} U_{\mathbf{i}}||_u < \infty \tag{0.20}$$

and similarly for the discrete case with the gradients replaced by their discrete counterparts. In particular one can see that for discrete spins the condition (0.20) is satisfied if $\Phi \in \mathbb{B}_2$, (although as we will see later the condition (0.20) is much better). If we restrict ourselves to spin interactions, then the best result in this case one can find in [**HS2**], where the stochastic dynamics has been constructed for potentials $\Phi \in \mathbb{B}_1{}^{(spin)}$ satisfying the bound

$$||\Phi||_1 = \sum_{\substack{X \in \mathcal{F} \\ X \ni 0}} |J_X| < \frac{\pi}{4} \tag{0.21}$$

For general interactions $\Phi \in \mathbb{B}_1$ the existence of the stochastic dynamics on the space of continuous functions remains an open problem and at the moment the best that one can have for a general Gibbsian potential is a stochastic dynamics in $L_p(\mu_\Phi)$, $p \in [1, \infty]$, for a corresponding Gibbs state μ_Φ. (This follows from the fact that each of our pre-generators is given on a dense domain in $L_2(\mu_\Phi)$ as a nonpositive and symmetric operator. Thus it always admit a trivial Friedrichs extension and using the definition of our pre-generator one can see that this extension generates a Markov semigroup in $L_2(\mu_\Phi)$ which extends uniquely to the Markov semigroups in any $L_p(\mu_\Phi)$.) As a symmetric pre-generator may admit many different extensions, this trivial construction is highly not satisfactory. In the non-reversible case a condition on the rate functions similar to $\mathbb{B}_2$ is enough for existence, while (see an example of Gray, [**Li1**] p.53), a $\mathbb{B}_1$ like condition is known to

be not enough. Whether in the reversible case every interaction in $I\!\!B_1$ give rise to a stochastic dynamics as described above seems to be open.
The literature concerning the ergodicity problem, i.e. the question whether or not for a stochastic dynamics P_t defined with respect to some interaction we have a return to equilibrium

$$P_t f \longrightarrow_{t\to\infty} \mu f \tag{0.22}$$

with μ being a unique Gibbs measure for the same interaction, is very vast. Let us mention here only some selected results; for a more comprehensive list of references consult [**Li1**], [**Li2**]. The first sufficient condition for strong ergodicity (with convergence to equilibrium in the uniform norm) for the single spin flip dynamics corresponding to potentials of finite range has been given in [**D2**]. It naturally extended the uniqueness condition of the author for the equilibrium description. Later this result has been generalized in [**Su1**] to include long range potentials as well as multispin flip generators, [**Su2**] ([**Li1**]). All these results when applied to a lattice spin system implied the uniqueness of the P_t-invariant measure and, since every Gibbs measure with the same potential is P_t-invariant, also the uniqueness of the Gibbs measure. (Let us mention that in dimensions $d \geq 3$ it is still an open problem whether or not every P_t-invariant measure is a Gibbs measure; for the case $d \leq 2$ see [**HS3**].)
An interesting use of the stochastic dynamics to prove analyticity properties of spin systems at high temperatures has been made in [**HS2**].
A nice extension of these ergodicity results has been obtained in [**AH**]. These authors have shown that if a local specification satisfies the Dobrushin-Shlosman uniqueness condition, [**DS1**], in a box X, (which is an extension of the Dobrushin uniqueness condition), then every stochastic dynamics with a generator $\mathcal{L}^Y$, defined for a box $Y \in \mathcal{F}$, $X \subseteq Y$, is strongly ergodic. This implies that also any other stochastic dynamics with a generator $\mathcal{L}^Z$, $Z \in \mathcal{F}$ has a spectral gap in $L_2(\mu)$, where μ is the corresponding unique Gibbs measure, (although it says nothing about their strong ergodicity properties). Let us mention that in connection to this there was a conjecture, that if a generator of a stochastic dynamics as discussed above has a spectral gap in $L_2(\mu)$ for some Gibbs measure corresponding to a given interaction, then this measure must be unique. We discuss this point later in connection with the Fréchet differentiability. Since the Dobrushin-Shlosman condition applies to lower temperatures than the original Dobrushin uniqueness condition, this method gave a broader range of applications than the previous ergodicity conditions. However one knows that if the temperature is lowered, but when we still remain above the critical point, to satisfy the Dobrushin-Shlosman uniqueness condition one would have to take larger and larger cubes $X \equiv X(\beta)$ with $|X(\beta)| \to \infty$ when the inverse temperature approaches the critical value. This means that by that method we cannot get anything about the strong decay to equilibrium for any stochastic dynamics $P_t^Y \equiv e^{t\mathcal{L}^Y}$ with any fixed box $Y \in \mathcal{F}$. A somewhat weaker version of Dobrushin-Shlosman theory, applying only to volumes built up from "fat cubes", gives the ergodicity in the uniform norm all the way to T_c in various models. Whether the original Dobrushin-Shlosman theory applies in this region is an open problem for the Ising model, but is certainly not true for Potts models [**vEFK**]. (If additionally one knows that the system is ferromagnetic, the dynamic problem is solvable; cf. [**MO1**] for an extension of the arguments by Holley [**H**]).
To overcome these problems Holley and Stroock, [**HS4**], have invented a very elegant strategy based on the hypercontractivity property of the semigroup P_t, which means the following property

$$||P_t f||_{L_q(\mu)} \leq ||f||_{L_2(\mu)} \tag{0.23}$$

with $q \equiv q(t) \equiv 1 + e^{\frac{2}{c}t}$, $c \in (0,\infty)$ and μ a P_t-invariant measure. It is known that every Markov semigroup is contractive in any $L_p(\mu)$ space and the property (0.23) should be regarded as rather peculiar as it says that the semigroup is contractive from the $L_2(\mu)$ space to the $L_q(\mu)$ space with $q \to \infty$ very rapidly as time increases. To explain why the hypercontractivity property helps to solve the strong ergodicity problem for all dynamics P_t^Y, $Y \in \mathcal{F}$, knowing it for one of them, let us mention that, as shown in [**G3**], this property is equivalent to the following Logarithmic Sobolev inequality with a coefficient $c \in (0,\infty)$

$$\mu f \log f \leq 2c\mu\left(f^{\frac{1}{2}}(-\mathcal{L}^X f^{\frac{1}{2}})\right) \tag{LS}$$

for positive functions f with $\mu f = 1$ for which the right hand side is finite. Now we observe that the right hand side of the Logarithmic Sobolev inequality depends on a Markov generator through its Dirichlet form.

Thus, if it is proven for one generator $\mathcal{L}^X$, it automatically implies a similar inequality for any other spin flip generator $\mathcal{L}^Y$, $Y \in \mathcal{F}$, since all their Dirichlet forms are equivalent. (Similar arguments have been used in [**AH**] to get the spectral gap property for all the spin flip generators once it is proven for one of them associated to a large box.)

The first proof of **LS** has been obtained in [**CS**] for the Gibbs measures of continuous spin systems on a lattice at high temperatures by application of the Bakry - Emery criterion. (The authors considered only finite range interactions, although no conceptual problems would arise for a more general case). A slight generalization of this result to similar systems one can find in [**DeS**]. A new approach to the **LS** based on an application of the Gibbs structure has been introduced in [**Z1-3**] and developed later in [**SZ1-3**], [**MO2**], [**LY**], [**La**]. In particular it has been shown in [**SZ2**] that, for systems with finite range interactions, the Logarithmic Sobolev inequalities for conditional measures $\mu^{\omega}_{\Phi,\Lambda}$ with a coefficient c independent of the volume Λ and external configuration ω are equivalent to *the complete analyticity* of Dobrushin and Shlosman, [**DS2,3**], i.e. the analyticity of the map $\Phi \longmapsto \mu^{\omega}_{\Phi,\Lambda}$ with the radius of analyticity independent of the set Λ and the configuration ω. (Let us remind the reader that complete analyticity is the strongest analyticity property possible. It can fail when we are close to the critical region, although the infinite volume Gibbs measures as well as the conditional measures for some "fat" sets can retain their analyticity properties.) The proof of the equivalence of **LS** for all conditional measures and the complete analyticity property involved a rather complicated route through the corresponding strong ergodicity property of the finite volume stochastic dynamics.

In our short note we would like to show that one can use **LS** in a direct and natural way to prove the strong (Fréchet) differentiability of the pressure functional for spin systems with multiparticle interactions. This is done in the next section. Later we discuss the connection of the spectral gap of the generator with the strong differentiability. Finally at the end we give also an example of an interesting class of multiparticle interactions contained in $\mathbb{B}_1 \setminus \mathbb{B}_2$, for which we can prove **LS** as well as Dobrushin uniqueness.

1. High Temperature Differentiability via Logarithmic Sobolev Inequalities.

As the Logarithmic Sobolev inequality can be viewed as an estimate on the relative entropy, it is natural to expect that it should have some thermodynamic consequences. In this section we discuss this matter in more detail. ¿From now on we will consider only translation invariant interactions with potentials having finite $|| \cdot ||_1$ norm and if necessary (in the continuous case) sufficiently smooth. We begin by noting that every Markov generator discussed before has a Dirichlet form equivalent to the one defined as an expectation with respect to the same Gibbs measure of the following square of the gradient

$$|\nabla f|^2 \equiv \sum_{\mathbf{j}\in\Gamma} |\nabla_{\mathbf{j}} f|^2 \tag{1.1}$$

in the case of continuous spins and with $\partial_{\mathbf{j}}$ replacing $\nabla_{\mathbf{j}}$ in the discrete case. Later we will need also to consider the corresponding Dirichlet forms for the finite volume Gibbs measures. In this case we will have to consider the square of a finite volume gradient $\nabla_\Lambda f$ defined similarly as in (1.1), but with the summation over $\mathbf{j}$'s restricted to the set Λ. For a given interaction Φ we define a finite volume pressure $p^{\omega}_{\Phi,\Lambda}$ as follows

$$p^{\omega}_{\Lambda}(\Phi) \equiv \frac{1}{|\Lambda|} \log \delta_\omega \mu_0^\Lambda e^{-U_\Lambda(\Phi)} \tag{1.2}$$

We have the following simple fact.

Lemma 1.1

Suppose that for an interaction $\Phi + \Psi$, we have with some constant $c_{\Phi+\Psi} \in (0, \infty)$, independent of Λ and ω,

$$\mu^{\omega}_{\Phi+\Psi,\Lambda} f \log f \leq 2 c_{\Phi+\Psi} \mu^{\omega}_{\Phi+\Psi,\Lambda} |\nabla_\Lambda f^{\frac{1}{2}}|^2 \tag{1.3}$$

for every positive and normalized function f for which the right hand side is finite. Then the following inequality is true

$$0 \leq p^\omega_\Lambda(\Phi+\Psi) - p^\omega_\Lambda(\Phi) - \mu^\omega_{\Phi,\Lambda}(\frac{1}{|\Lambda|}\sum_{\mathbf{j}\in\Lambda}\mathbf{A}_{\mathbf{j},\Lambda}(\Psi)) \leq \frac{1}{2}e^{2||\Psi||_1}c_{\Phi+\Psi}\mu^\omega_{\Phi,\Lambda}\frac{1}{|\Lambda|}\sum_{\mathbf{j}\in\Lambda}|\sum_{\substack{X\cap\Lambda\neq\emptyset\\ X\ni\mathbf{j}}}\nabla_{\mathbf{j}}\Psi_X|^2 \tag{1.4}$$

where

$$\mathbf{A}_{\mathbf{j},\Lambda}(\Psi) \equiv -\sum_{\substack{X\cap\Lambda\neq\emptyset\\ X\ni\mathbf{j}}}\frac{1}{|X|}\Psi_X \tag{1.5}$$

In the continuous case the factor $e^{2||\Psi||_1}$ can be omitted.

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Proof: The lower bound is a simple consequence of the convexity of the finite volume pressure. To prove the upper bound we use the Logarithmic Sobolev inequality for the measure $\mu^\omega_{\Phi+\Psi,\Lambda}$ with the function

$$f = \frac{e^{+U_\Lambda(\Psi)}}{\mu^\omega_{\Phi+\Psi,\Lambda}e^{+U_\Lambda(\Psi)}} \tag{1.6}$$

to get in the continuous case

$$|\Lambda|\cdot p^\omega_\Lambda(\Phi+\Psi) - |\Lambda|\cdot p^\omega_\Lambda(\Phi) - \mu^\omega_{\Phi,\Lambda}(-U_\Lambda(\Psi)) \leq \frac{1}{2}c_{\Phi+\Psi}\mu^\omega_{\Phi+\Psi,\Lambda}|\nabla_\Lambda U_\Lambda(\Psi)|^2 \tag{1.7}$$

In the discrete case, because of peculiar features of the discrete differential, we get an extra factor $e^{2||\Psi||_1}$ on the right hand side. Now using the representation

$$U_\Lambda(\Psi) = \sum_{\mathbf{j}\in\Lambda}\left(\sum_{\substack{X\cap\Lambda\neq\emptyset\\ X\ni\mathbf{j}}}\frac{1}{|X\cap\Lambda|}\Psi_X\right) \equiv -\sum_{\mathbf{j}\in\Lambda}\mathbf{A}_{\mathbf{j},\Lambda}(\Psi) \tag{1.8}$$

and dividing both sides of (1.7) by the volume of Λ, we get the right hand side bound in (1.3). This ends the proof of the lemma.

$\diamondsuit$

Now we note that for (translation invariant) interactions in $\mathbb{B}_1$, we have

$$\lim_{v\mathcal{F}} p^\omega_\Lambda(\Phi+\Psi) = p(\Phi+\Psi) \tag{1.9}$$

and similarly with the interaction Φ, independently of the configuration ω. Also for any translation invariant Gibbs measure μ_Φ for the potential Φ, we get

$$\lim_{v\mathcal{F}}\mu_\Phi\left(\mu^\omega_{\Phi,\Lambda}(\frac{1}{|\Lambda|}\sum_{\mathbf{j}\in\Lambda}\mathbf{A}_{\mathbf{j},\Lambda}(\Psi))\right) = \lim_{v\mathcal{F}}\mu_\Phi\left(\frac{1}{|\Lambda|}\sum_{\mathbf{j}\in\Lambda}\mathbf{A}_{\mathbf{j},\Lambda}(\Psi)\right) = \mu_\Phi\mathbf{A}_0(\Psi) \equiv \mu_\Phi\mathbf{A}(\Psi) \tag{1.10}$$

and

$$\lim_{v\mathcal{F}}\mu_\Phi\left(\mu^\omega_{\Phi,\Lambda}\frac{1}{|\Lambda|}\sum_{\mathbf{j}\in\Lambda}|\sum_{\substack{X\cap\Lambda\neq\emptyset\\ X\ni\mathbf{j}}}\nabla_{\mathbf{j}}\Psi_X|^2\right) = \lim_{v\mathcal{F}}\mu_\Phi\left(\frac{1}{|\Lambda|}\sum_{\mathbf{j}\in\Lambda}|\sum_{\substack{X\cap\Lambda\neq\emptyset\\ X\ni\mathbf{j}}}\nabla_{\mathbf{j}}\Psi_X|^2\right) \equiv \mu_\Phi\mathbf{B}^2(\Psi) \tag{1.11}$$

with a function $\mathbf{B}$ defined by

$$\mathbf{B}^2(\Psi) \equiv | \sum_{\substack{X \in \mathcal{F} \\ X \ni 0}} \nabla_{\mathbf{0}} \Psi_X |^2 \tag{1.12}$$

Hence, averaging the inequality (1.4) with respect to the Gibbs measure μ_Φ and passing to the thermodynamic limit with the use of (1.9) - (1.12), we conclude with the following statement in which we use a Banach space of interactions $I\!H_1$ defined with some norm $||\Phi||_{I\!H_1}$ satisfying

$$||\Phi||_{I\!H_1} \geq \left(||\Phi||_1^2 + \mathbf{B}^2(\Phi)\right)^{\frac{1}{2}} \tag{1.13}$$

Theorem 1.2:
Suppose in some neighborhood $\mathcal{O}_\Phi \subset I\!H_1$ of an interaction Φ the finite volume measures $\{ \mu^\omega_{\Phi+\Psi,\Lambda} \}_{\Lambda \in \mathcal{F}_0, \omega \in \Omega}$, defined with some countable exhaustion $\mathcal{F}_0$, satisfy the Logarithmic Sobolev inequality with coefficients $c(\Phi + \Psi) \leq C(\mathcal{O}_\Phi)$, for some constant $C(\mathcal{O}_\Phi) \in (0, \infty)$ independent of $\Lambda \in \mathcal{F}_0$ and of the interactions from $\mathcal{O}_\Phi$. Then we have

$$0 \leq p(\Phi + \Psi) - p(\Phi) - \mu_\Phi \mathbf{A}(\Psi) \leq \frac{1}{2} e^{2||\Psi||_1} c_{\Phi+\Psi} \mu_\Phi \mathbf{B}^2(\Psi) \tag{1.14}$$

and therefore the pressure functional $I\!H_1 \ni \Xi \longmapsto p(\Xi)$ is **Fréchet differentiable** at the point Φ.

$\circ$

Proof: The proof of the Fréchet differentiability of the pressure functional at the point Φ clearly follows from the inequality (1.14), since by our assumption (1.13) we have

$$0 \leq p(\Phi + \Psi) - p(\Phi) - \mu_\Phi \mathbf{A}(\Psi) \leq \frac{1}{2} e^{2||\Psi||_1} C(\mathcal{O}_\Phi) ||\Psi||^2_{I\!H_1} \tag{1.15}$$

$\diamondsuit$

Let us remark that in case of discrete spins we have a crude bound

$$\mathbf{B}(\Phi) \leq 2||\Phi||_1 \tag{1.16}$$

We will show however that for some interactions the left hand side can be actually much smaller.
One could also slightly improve the above result, by imposing only an assumption of uniform **LS** for some $\omega \in \Omega$.

2. Spectral Gap and the High Temperature Differentiability.

In this section we discuss briefly the uniqueness hypothesis under the assumption of a spectral gap for the generator of the stochastic dynamics. Due to the already mentioned equivalence of the Dirichlet forms for various dynamics, it is sufficient to consider the following universal inequality, called *the spectral gap inequality*

$$m \cdot \mu(f, f)^2 \leq \mu |\nabla f|^2 \tag{\textbf{SG}}$$

with a constant $m \in (0, \infty)$ independent of the function f, and $\mu(f, f) \equiv \mu(f - \mu f)^2$. The assumption of a spectral gap is weaker than **LS** and in fact it is well known that if a probability measure μ satisfies **LS** with a coefficient $c \in (0, \infty)$, then it automatically satisfies **SG** with

$$m \geq \frac{1}{c} \tag{2.1}$$

Now let us consider the Taylor expansion to the first order with remainder for the finite volume pressure. (Without making explicit the boundary conditions), we have

$$0 \leq p_\Lambda(\Phi + \Psi) - p_\Lambda(\Phi) - \frac{1}{|\Lambda|} \mu_{\Phi,\Lambda}(-U_\Lambda(\Psi)) = \int_0^1 dt \int_0^t ds \, \frac{1}{|\Lambda|} \mu_{\Phi+s\Psi,\Lambda}(U_\Lambda(\Psi), U_\Lambda(\Psi)) \tag{2.2}$$

If we suppose that the measures $\mu_{\Phi+s\Psi,\Lambda}$ satisfy the Spectral Gap inequality with the corresponding spectral gap $m(\Phi+s\Psi,\Lambda)$, we get

$$p_\Lambda(\Phi+\Psi)-p_\Lambda(\Phi)-\frac{1}{|\Lambda|}\mu_{\Phi,\Lambda}(-U_\Lambda(\Psi))\leq\int_0^1 dt\int_0^t ds\ (m(\Phi+s\Psi,\Lambda))^{-1}\,\frac{1}{|\Lambda|}\mu_{\Phi+s\Psi,\Lambda}|\nabla_\Lambda U_\Lambda(\Psi)|^2 \quad (2.3)$$

One can arrange that the following limit exists and is equal to the corresponding expectation with a (translation invariant) Gibbs measure μ_Φ

$$\lim_{\mathcal{F}_0}\frac{1}{|\Lambda|}\mu_{\Phi,\Lambda}(-U_\Lambda(\Psi))=\mu_\Phi(\mathbf{A}(\Psi)) \quad (2.4)$$

Since we have

$$\limsup_{\mathcal{F}_0}\frac{1}{|\Lambda|}\mu_{\Phi+s\Psi,\Lambda}|\nabla_\Lambda U_\Lambda(\Psi)|^2\leq\mathbf{B}(\Psi)^2 \quad (2.5)$$

after passing to the thermodynamic limit we get

$$0\leq p(\Phi+\Psi)-p(\Phi)-\mu_\Phi(\mathbf{A}(\Psi))\leq\frac{1}{2}\left(\sup_{\mathcal{F}_0,s\in[0,1]}m(\Phi+s\Psi,\Lambda)^{-1}\right)\mathbf{B}(\Psi)^2 \quad (2.6)$$

Hence we conclude with the following

Theorem 2.2:

Suppose in some neighborhood $\mathcal{O}_\Phi\subset I\!\!H_1$ of an interaction Φ the finite volume measures $\{\,\mu^\omega_{\Phi+s\Psi,\Lambda}\,\}_{\Lambda\in\mathcal{F}_0,\omega\in\Omega}$, defined with some countable exhaustion $\mathcal{F}_0$, satisfy the Spectral Gap inequality with spectral gaps $m(\Phi+s\Psi,\Lambda)\geq M(\mathcal{O}_\Phi)$, for some constant $M(\mathcal{O}_\Phi)\in(0,\infty)$ independent of $\Lambda\in\mathcal{F}_0$ and independent of the interactions in $\mathcal{O}_\Phi$. Then we have

$$0\leq p(\Phi+\Psi)-p(\Phi)-\mu_\Phi(\mathbf{A}(\Psi))\leq\frac{1}{2}M(\mathcal{O}_\Phi)\mathbf{B}^2(\Psi) \quad (2.7)$$

and therefore the pressure functional $I\!\!H_1\ni\Xi\longmapsto p(\Xi)$ is **Fréchet differentiable** at the point Φ.

$\circ$

We see that the statement in the present case is similar to that when we have assumed **LS** . We believe that in fact **LS** may have stronger consequences (e.g. in some cases the analyticity with respect to the potentials in $I\!\!H_1$), although at the moment we do not see whether one could get them directly from it. (As we have already mentioned the uniform - in volume and external configurations - **LS** inequality is equivalent to Dobrushin-Shlosman complete analyticity.)

3. An Example of Interactions in $I\!\!B_1\setminus I\!\!B_2$.

In this section we would like to exhibit examples of some peculiar Gibbsian interactions which do not belong to $I\!\!B_2\cup I\!\!B_1{}^{(spin)}$, but for which we can show **LS** as well as the Dobrushin uniqueness condition. For this, let us consider a configuration space $\Omega\equiv I\!\!M^\Gamma$ defined with $I\!\!M=S^{N-1}$, where S^{N-1} denoting a unit sphere in the Euclidean space $I\!\!R^N$. If $N\geq2$, for two unit vectors σ and η representing the points in $I\!\!M$, the expression $\sigma\cdot\eta$ will denote their Euclidean scalar product.
Let $\Phi\equiv\{\Phi_X\}_{X\in\mathcal{F}}$ be an interaction on Ω given by

$$\Phi_X\equiv\rho_X\cdot cos\left(\frac{\sum_{\mathbf{j}\in X}\eta_\mathbf{j}\cdot\sigma_\mathbf{j}}{|X|^\alpha}\right) \quad (3.1)$$

with some unit vectors $\eta_{\mathbf{j}} \in I\!M$, $\alpha \in [0,\infty)$ and coefficients $\rho_X \equiv \rho(|X|) \in I\!R$ such that

$$\sup_{\mathbf{i}\in \mathbb{Z}^d} \sum_{\substack{X\in\mathcal{F}\\ X\ni \mathbf{i}}} |\rho_X| < \infty \tag{3.2}$$

and

$$\sup_{\mathbf{i}\in \mathbb{Z}^d} \sum_{\substack{X\in\mathcal{F}\\ X\ni \mathbf{i}}} |X|\cdot|\rho_X| = \infty \tag{3.3}$$

Then clearly we have

$$||\Phi||_1 = \sup_{\mathbf{i}\in \mathbb{Z}^d} \sum_{\substack{X\in\mathcal{F}\\ X\ni \mathbf{i}}} ||\Phi_X||_\infty < \infty \tag{3.4}$$

i.e. $\Phi \in I\!B_1$, but

$$||\Phi||_2 = \sup_{\mathbf{i}\in \mathbb{Z}^d} \sum_{\substack{X\in\mathcal{F}\\ X\ni \mathbf{i}}} |X|\cdot||\Phi_X||_\infty = \infty \tag{3.5}$$

i.e. $\Phi \notin I\!B_2$. Clearly similar properties are possessed by the interactions defined with the function $\cos(x)$ replaced by $\sin(x)$. Moreover for $\alpha < 1$ the interaction given above cannot be transformed into a Gibbsian interaction from the space ${I\!B_1}^{(spin)}$. This can be seen by simple computations similar to the one given for a special case (of e^{ix} type complex interaction) with $\alpha = \frac{1}{2}$ considered in [**vEF**], when discussing the example given in [**DM2**] of complex (Gibbsian) interactions corresponding to $\alpha = 0$, which define a spin system failing to be analytic even in the high temperature region.
We have the following result showing that the applicability of our method presented earlier extends far beyond the space $I\!B_2$.

Theorem 3.1:
Suppose that $N \geq 2$ and $\alpha \geq \frac{1}{2}$. Then there is $\beta_0 \in (0,\infty)$ such that for any $|\beta| < \beta_0$ the Gibbs measure corresponding to the potential $\beta\Phi$ satisfies **LS** .

∘

Proof: Since for $N \geq 3$ the unit spheres have positive Ricci curvature, we can use the method based on the criterion of [**BE**], (see also [**CS**]). Then we need only to show that there is a constant $C \in (0,\infty)$ such that

$$\sum_{\substack{\mathbf{i},\mathbf{j}\in \mathbb{Z}^d\\ l,k=1,..,N}} \left(\sum_X \nabla^l_{\mathbf{i}}\nabla^k_{\mathbf{j}}\Phi_X\right) \nabla^l_{\mathbf{i}} f \nabla^k_{\mathbf{j}} f \leq C|\nabla f|^2 \tag{3.6}$$

Since we have

$$\sum_{\substack{\mathbf{i},\mathbf{j}\in \mathbb{Z}^d\\ l,k=1,..,N}} \left(\sum_X \nabla^l_{\mathbf{i}}\nabla^k_{\mathbf{j}}\Phi_X\right) \nabla^l_{\mathbf{i}} f \nabla^k_{\mathbf{j}} f \leq \sum_{\substack{\mathbf{i}\in \mathbb{Z}^d\\ l=1,..,N}} \left(\sum_{\substack{\mathbf{j}\in \mathbb{Z}^d\\ k=1,..,N}} \sum_X |\nabla^l_{\mathbf{i}}\nabla^k_{\mathbf{j}}\Phi_X|\right) |\nabla^l_{\mathbf{i}} f|^2 \leq$$

$$\leq \sup_{\mathbf{i}',l'} \left(\sum_{\substack{\mathbf{j}\in \mathbb{Z}^d\\ k=1,..,N}} |\sum_X \nabla^{l'}_{\mathbf{i}'}\nabla^k_{\mathbf{j}}\Phi_X|\right) \sum_{\substack{\mathbf{i}\in \mathbb{Z}^d\\ l=1,..,N}} |\nabla^l_{\mathbf{i}} f|^2 \tag{3.7}$$

and

$$\sup_{\mathbf{i}',l'} \left(\sum_{\substack{\mathbf{j}\in \mathbb{Z}^d\\ k=1,..,N}} \sum_X |\nabla^{l'}_{\mathbf{i}'}\nabla^k_{\mathbf{j}}\Phi_X|\right) = \sup_{\mathbf{i}',l'} \left(\sum_{\substack{\mathbf{j}\in \mathbb{Z}^d\\ k=1,..,N}} \sum_{X\ni \mathbf{i},\mathbf{j}} |\nabla^{l'}_{\mathbf{i}'}\nabla^k_{\mathbf{j}}\Phi_X|\right) =$$

$$= \sup_{\mathbf{i}',l'} \left(\sum_{X\ni \mathbf{i}} \sum_{\substack{\mathbf{j}\in X \\ k=1,..,N}} |\nabla_{\mathbf{i}'}^{l'} \nabla_{\mathbf{j}}^{k} \Phi_X| \right) \leq N \sup_{\mathbf{i}',l'} \left(\sum_{X\ni \mathbf{i}} |X| \sup_{\mathbf{j},k} |\nabla_{\mathbf{i}'}^{l'} \nabla_{\mathbf{j}}^{k} \Phi_X| \right), \tag{3.8}$$

using the explicit form of the interaction Φ we get

$$\sup_{\mathbf{i}',l'} \left(\sum_{X\ni \mathbf{i}} |X| \sup_{\mathbf{j},k} |\nabla_{\mathbf{i}'}^{l'} \nabla_{\mathbf{j}}^{k} \Phi_X| \right) \leq \sup_{\mathbf{i}',l'} \left(\sum_{X\ni i} |X| \cdot |\rho_X| \sup_{\mathbf{j},k} \frac{|\eta_{\mathbf{i}'}^{l'} \eta_{\mathbf{j}}^{k}|}{(|X|^{\alpha})^2} \right) \tag{3.9}$$

Hence, provided that $\alpha \geq \frac{1}{2}$, the last term can be bounded by

$$\leq \sup_{\mathbf{i}} \sum_{X\ni \mathbf{i}} |\rho_X| = ||\Phi||_1 \ < \ \infty \tag{3.10}$$

This implies (3.6). Hence we conclude that for $\beta \in [0, \beta_0)$, with some sufficiently small $\beta_0 \in (0, \infty)$, the corresponding conditional expectations $\mu^{\omega}_{\beta\Phi,\Lambda}$ satisfy Logarithmic Sobolev inequalities with a coefficient independent of the volume Λ and the external configuration ω. The case $N = 2$ (for which one cannot apply the Bakry-Emery condition, because the corresponding Ricci curvature equals zero) follows by checking the condition (0.13) from [**Z3**]. This ends the proof of Theorem 3.1.

$\diamondsuit$

A similar result should be true also for discrete spins, (although the details of the proof might be much more complicated).

After we have checked that **LS** is true, a natural question arises whether or not one can also prove that the Dobrushin uniqueness condition is satisfied. As one could see in the above considerations only the condition on the second derivative of the interaction entered. It obviously looks similar to the sufficient condition for the Dobrushin uniqueness condition given in Chapter V of [**S**]. However, there the assumption that the interaction is affine in the spin variables, (i.e. that it belongs to a space $I\!B^{(spin)}$), has been explicitly used. We will show that in fact it is not essential and we have the following result.

Theorem 3.2:

Suppose that $N \geq 1$ and $\alpha \geq \frac{1}{2}$. Then there is $\beta_0 \in (0, \infty)$ such that for any $|\beta| < \beta_0$ the Gibbs measure corresponding to the potential $\beta\Phi$ satisfies the Dobrushin uniqueness condition.

$\circ$

Proof: Let

$$\gamma_{\mathbf{ik}} \equiv \sup \left\{ |\mu^{\sigma}_{\beta\Phi,\mathbf{i}}(f) - \mu^{\tilde{\sigma}}_{\beta\Phi,\mathbf{i}}(f)| \ : \ \sigma_{\mathbf{j}} = \tilde{\sigma}_{\mathbf{j}}, \ \text{for } \mathbf{j} \neq \mathbf{k}, \mathbf{i} \text{ and } ||f||_{Lip} = 1 \right\} \tag{3.11}$$

where $f \equiv f(\sigma_{\mathbf{i}})$ and $|| \cdot ||_{Lip}$ denotes the Lipschitz norm associated to a given metric $d(\cdot\, , \cdot)$ on $I\!M$, i.e. the norm defined by

$$||f||_{Lip} \equiv \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x,y)}$$

We want to show that there is $\beta_0 \in (0, \infty)$ such that for all $\beta \in (-\beta_0, +\beta_0)$ we have

$$\sup_{\mathbf{i}\in\Gamma} \sum_{\mathbf{k}\neq\mathbf{i}} \gamma_{\mathbf{ik}} \ < \ 1 \tag{\textbf{DU}}$$

To estimate $\gamma_{\mathbf{ik}}$ for a given $\mathbf{k} \neq \mathbf{i}$ let us define an interpolating potential

$$\Phi(s, \omega) \equiv s\Phi(\omega_{\Gamma\setminus\mathbf{k}} \bullet \sigma_{\mathbf{k}}) + (1 - s)\Phi(\omega_{\Gamma\setminus\mathbf{k}} \bullet \tilde{\sigma}_{\mathbf{k}}) \tag{3.12}$$

Now we have

$$|\mu^{\sigma}_{\beta\Phi,\mathbf{i}}(f) - \mu^{\tilde{\sigma}}_{\beta\Phi,\mathbf{i}}(f)| = |\int_0^1 ds \frac{d}{ds} \mu^{\sigma}_{\beta\Phi(s),\mathbf{i}}(f)| = |\beta \int_0^1 ds\, \mu^{\sigma}_{\beta\Phi(s),\mathbf{i}}(f, \frac{d}{ds} U_{\mathbf{i}}(\Phi(s)))| \tag{3.13}$$

Using the Schwartz inequality for the covariance we get

$$|\mu^{\sigma}_{\beta\Phi,\mathbf{i}}(f)-\mu^{\tilde\sigma}_{\beta\Phi,\mathbf{i}}(f)| \le |\beta|\int_0^1 ds\ \left(\mu^{\sigma}_{\beta\Phi(s),\mathbf{i}}(\frac{d}{ds}U_{\mathbf{i}}(\Phi(s))),\frac{d}{ds}U_{\mathbf{i}}(\Phi(s)))\right)^{\frac12}\cdot\left(\mu^{\sigma}_{\beta\Phi(s),\mathbf{i}}(f,f)\right)^{\frac12} \tag{3.14}$$

Now we note that for any function F we have

$$0\le \mu^{\sigma}_{\beta\Phi(s),\mathbf{i}}(F,F)=\frac12\mu^{\sigma}_{\beta\Phi(s),\mathbf{i}}\otimes\tilde\mu^{\sigma}_{\beta\Phi(s),\mathbf{i}}(F(\omega_{\mathbf{i}})-F(\tilde\omega_{\mathbf{i}}))^2\le\frac12 e^{4||\beta\Phi||_1}\mu_0\otimes\tilde\mu_0(F(\omega_{\mathbf{i}})-F(\tilde\omega_{\mathbf{i}}))^2 \tag{3.15}$$

where the tilded measure denotes the isomorphic copy of the untilded one. ¿From this one can see that, for any function f such that $||f||_{Lip}=1$, we have

$$\left(\mu^{\sigma}_{\beta\Phi(s),\mathbf{i}}(f,f)\right)^{\frac12}\le 2^{-\frac12}e^{2||\beta\Phi||_1}\left(\mu_0\otimes\tilde\mu_0(d(\omega_{\mathbf{i}},\tilde\omega_{\mathbf{i}})^2)\right)^{\frac12} \tag{3.16}$$

On the other hand since the free measure μ_0 satisfies **SG** with a mass gap $m_0>0$, we have

$$\mu^{\sigma}_{\beta\Phi(s),\mathbf{i}}(F,F)=\le m_0^{-1}e^{4||\beta\Phi||_1}\mu_0|\nabla_{\mathbf{i}}F|^2 \tag{3.17}$$

(respectively with a discrete gradient if the spins are discrete.) Applying this to the first factor in the integrand on the right hand side of (3.14), we get

$$\left(\mu^{\sigma}_{\beta\Phi(s),\mathbf{i}}(\frac{d}{ds}U_{\mathbf{i}}(\Phi(s))),\frac{d}{ds}U_{\mathbf{i}}(\Phi(s)))\right)^{\frac12}\le m_0^{-\frac12}e^{2||\beta\Phi||_1}\left(\mu_0|\nabla_{\mathbf{i}}\frac{d}{ds}U_{\mathbf{i}}(\Phi(s)))|^2\right)^{\frac12} \tag{3.18}$$

Finally, using the definition of the potential Φ, we observe that

$$|\nabla_{\mathbf{i}}\frac{d}{ds}U_{\mathbf{i}}(\Phi(s)))|\le\sum_{X\ni\mathbf{i},\mathbf{k}}|\rho_X|\cdot\frac{1}{|X|^{2\alpha}} \tag{3.19}$$

Combining (3.13)-(3.19), we obtain

$$\gamma_{\mathbf{i},\mathbf{k}}\le D\cdot\sum_{X\ni\mathbf{i},\mathbf{k}}|\rho_X|\cdot\frac{1}{|X|^{2\alpha}} \tag{3.20}$$

with the constant

$$D\equiv|\beta|2^{-\frac12}m_0^{-\frac12}e^{4||\beta\Phi||_1}\cdot\left(\mu_0\otimes\tilde\mu_0(d(\omega_{\mathbf{i}},\tilde\omega_{\mathbf{i}})^2)\right)^{\frac12} \tag{3.21}$$

Thus we have

$$\sum_{\mathbf{k}\neq\mathbf{i}}\gamma_{\mathbf{i},\mathbf{k}}\le D\cdot\sum_{\mathbf{k}\neq\mathbf{i}}\sum_{X\ni\mathbf{i},\mathbf{k}}|\rho_X|\cdot\frac{1}{|X|^{2\alpha}}=D\cdot\sum_{X\ni\mathbf{i}}\frac{|X|-1}{|X|^{2\alpha}}\cdot|\rho_X| \tag{3.22}$$

If $\alpha\ge\frac12$ we obtain

$$\sum_{\mathbf{k}\neq\mathbf{i}}\gamma_{\mathbf{i},\mathbf{k}}\le D\cdot||\Phi||_1 \tag{3.23}$$

As by our assumption we have $||\Phi||_1<\infty$ and $D\equiv D(|\beta|)\to0$, when $|\beta|\to0$, there is $\beta_0\in(0,\infty)$ such that for every $\beta\in(-\beta_0,+\beta_0)$ the right hand side of (3.23) is smaller that 1, i.e. **DU** is satisfied. This ends the proof of Theorem 3.2.

$\diamondsuit$

Remark Let us note that for the special case of $I\!\!M=\{-1,+1\}$ we could get better estimates with $D=|\beta|\cdot||\Phi||_1$.

$\circ$

Finally let us remark that for the potentials Φ defined with $\alpha \geq \frac{1}{2}$ one can construct the stochastic dynamics. Then **LS** proven above (together with an appropriate approximation property) should allow us to prove the exponential decay to equilibrium in the uniform norm (with the rate governed by the spectral gap of the generator) similarly as in [**SZ2**]. Alternatively, due to **DU**, one can use the method of [**AH**] (see also [**SZ2**]) to get a similar result. Thus at high temperatures we have again a very nice correspondence between dynamics and "equilibrium" uniqueness conditions for the spin systems described by the discussed interactions with $\alpha \geq \frac{1}{2}$.
For the moment we leave open the intriguing problem concerning the high temperature behavior of systems with $\alpha < \frac{1}{2}$ potentials. As one can expect from the form of the interactions Φ, it should connected to some interesting large deviations theory. We believe that solving this problem would be important for a better understanding of Gibbsian description, as well as its nonequilibrium counterpart, for lattice spin systems.

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