Renormalizing Partial Differential Equations

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Abstract

We explain how to apply Renormalization Group ideas to the analysis of the long-time asymptotics of solutions of partial differential equations. We illustrate the method on several examples of nonlinear parabolic equations. We discuss many applications, including the stability of profiles and fronts in the Ginzburg-Landau equation, anomalous scaling laws in reaction-diffusion equations, and the shape of a solution near a blow-up point.

1 Introduction.

The development of a qualitative theory of infinite dimensional dynamical systems is a major scientific challenge. Such systems are expressed through (nonlinear) partial differential equations, and we shall concentrate on equations of the form

$$u_t = \Delta u + F(u, \nabla u, \nabla\nabla u). \tag{1}$$

where, $u(x,t) \in \mathbf{R}$ (or $\mathbf{C}$), $x \in \mathbf{R}^d$, $t \in \mathbf{R}^+$ and F is nonlinear or random. Usually, the following problems are considered:

1. Existence and regularity of the solution over a finite time interval.
2. Extension of the first problem to an infinite time interval.

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3. If the solution exists for all times, what is its asymptotic behaviour? How does the latter depend on the initial conditions? If the solution ceases to exist after a finite time, we may still ask about its behaviour near the point where it breaks down.

The first question is often easy to deal with. In particular, perturbation methods (small initial data or small nonlinearity) tend to work well over finite (or short) time intervals. However, naive perturbation theory does not, in general, yield an answer to the last two questions. A similar situation was encountered in the theory of critical phenomena and in quantum field theory: perturbation theory works well whenever there are enough cutoffs, but tends to diverge when the latter are removed. In order to solve these problems, the Renormalization Group (RG) method was developed and proved to be very useful. One of our goals is to develop RG ideas in the theory of PDE's. Here, the role of the cutoff is played by the finite time.

The RG method for nonlinear PDE's was developed by Goldenfeld, Oono and others [43]. Previously, Barenblatt [2] emphasized the role of self-similar solutions of these equations. The latter can be viewed as fixed points of certain RG transformations (see below). Independently, we used RG methods to deal with an equation like (1),with a linear but random F [9, 10, 11]. Here we shall explain the basic RG ideas in the framework of PDE's, which turn out to be very natural and easy, and we shall review some of the rigorous results obtained so far.

The general idea behind the RG approach is to solve the problem iteratively: first integrate the equations far away from the cutoff to be removed, i.e. here solve a finite time problem. Next, rescale the time and possibly the space and the u variables so as to produce a new problem similar to the original one. Finally, iterate and see what happens. The method will work when the new problem tends to be simpler than the original one (meaning, e.g., a weaker nonlinearity). Upon iteration, it will become simpler and simpler. Of course, this will happen only if we have chosen the scaling appropriately.

The underlying properties of the dynamical system that make the RG method work are universality and scaling. In the PDE framework, scaling amounts to the observation that many problems have solutions that behave asymptotically as

$$u(x,t) \sim t^{-\frac{\alpha}{2}} f^*(\frac{x}{t^\beta}) \tag{2}$$

Often such a limit law is *universal*: the numbers α, β and the function f^* will not depend on the initial conditions or even on the form of the equation. More precisely, pairs of initial data and equations will fall into universality classes, corresponding to given α, β and f^*. The fact that whole classes of problems may yield the same asymptotic behaviour is called universality. In fact, f^* will be a fixed point of the RG transformation, while the exponents α and β are determined by the choice of the scaling. The basin of attraction of that fixed point is the universality class. The RG will lead to a dynamical system picture, with an analysis of the stable, neutral and unstable manifolds of the fixed points.

The outline of the paper is as follows: first, we give an elementary definition of the RG transformation, starting from the (linear) heat equation. Then, we show how the RG flow acts on spaces of initial data and equations for various nonlinear perturbations of the heat equation, and for data that decay at infinity. In Section 3, we consider similar questions, but for data that do not decay at inifinity, which leads to the formation of patterns and fronts. Finally (Section 4), we apply the RG method to equations whose solutions blow-up in a finite time, and we give a list of possible profiles near a blow-up point. We end this review with a list of open problems.

2 The Renormalization Group.

2.1 The linear heat equation: Gaussian fixed points.

To explain the RG idea, let us start with the simplest equation, the linear heat equation:

$$u_t = u_{xx} \tag{1}$$

in one dimension, with integrable initial data, $u(x,0) = f(x) \in L^1(\mathbf{R})$. The solution is

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int dy \exp(-\frac{(x-y)^2}{4t}) f(y) dy \tag{2}$$

From this, it is easy to deduce that

$$\sqrt{t} u(\xi\sqrt{t}, t) \to \frac{e^{-\xi^2/4}}{\sqrt{4\pi}} \int f(y) dy \tag{3}$$

as $t \to \infty$ (pointwise and in $L^1(\mathbf{R})$). Let us reformulate this (trivial) result in the RG language. Notationally, it is useful to take the initial time equal to 1 instead of 0:

$$u(x,1) = f(x) \tag{4}$$

This will not change anything to the long-time asymptotics. Next, take a number $L > 1$ and define the renormalization map, acting on the initial data f as

$$(R_L f)(x) = L u(Lx, L^2) \tag{5}$$

where $u(\cdot, L^2)$ is the solution of (1) at time L^2. Thus, we solve the equation up to a finite time L^2, rescale the solution and take it as a new initial data. Here, R_L is explicit: in Fourier transform, $e^{t\Delta}$ multiplies by e^{-tk^2} and scaling x by L amounts to scaling k by $\frac{1}{L}$. Thus,

$$\widehat{R_L f}(k) = e^{-k^2(1-L^{-2})} \hat{f}(L^{-1}k) \tag{6}$$

(we solve (1) over a time interval of length $L^2 - 1$).

Observe that the set of multiples of

$$\widehat{f_0^*}(k) = e^{-k^2} \tag{7}$$

forms a line of fixed points of the RG map (6). Note also that R_L satisfies the semigroup property: Since L is arbitrary, we may replace it by L^n, and we have:

$$R_{L^n} = R_L^n. \tag{8}$$

This is simply because, if we define $u_L(x,t) = Lu(Lx, L^2t)$, then

$$(R_L f)(x) = u_L(x,1) \tag{9}$$

and u_L satisfies (1) again. Equation (1) is scale invariant.

Now, assume that our initial f belongs to the basin of attraction of the line of fixed points, i.e.

$$R_L^n f \to A f_0^* \tag{10}$$

as $n \to \infty$, for some A, in a suitable space. The basic observation is that this is equivalent to (3), by letting $t = L^{2n}$, $A = \hat{f}(0)$, and combining (5,8,10). Note that A depends on the initial data and that this dependence is the only trace left by $u(x,1)$ on the long-time asymptotics of the solution. This is why the latter is called universal.

To see what kind of functions belong to the basin of attraction of this line of fixed points, write formally $\hat{f}(k) = \hat{f}(0) + ck + \cdots$. Then, for k of order one, we have from the scaling

$$\widehat{R_L f}(k) = \hat{f}(0) f_0^*(k) + \mathcal{O}(L^{-1}) \tag{11}$$

and, for large k, we get contraction from the multiplication by $e^{-k^2(1-\frac{1}{L^2})}$ in (6).

To be more precise, we use below the following norm (inspired by [23]):

$$\|f\| = \sup_k (1+|k|^q)(|\hat{f}(k)| + |\hat{f}'(k)|) \tag{12}$$

on a space of functions with $\hat{f} \in C^1(R)$. We denote by $\mathcal{B}$ the corresponding Banach space. The value of q will be specified when we deal with nonlinear problems. For the moment, any $q \geq 0$ will do. Any function $f \in \mathcal{B}$ can be written in Fourier transform as

$$\hat{f}(k) = \hat{f}(0) e^{-k^2} + \hat{g}(k) \tag{13}$$

with $\hat{g}(0) = 0$. Then the convergence (10) with $A = \hat{f}(0)$ will follow from

Lemma 1 *There exists a constant C independent of L so that, if $g \in \mathcal{B}$ and $\hat{g}(0) = 0$,*

$$\|R_L g\| \leq C L^{-1} \|g\| \tag{14}$$

Proof

Write $\hat{g}(\frac{k}{L}) = \int_0^{k/L} \hat{g}'(p) dp$ so, $|\hat{g}(\frac{k}{L})| \leq \frac{|k|}{L} \|g\|$. The scaling (6) brings a factor of L^{-1} in the derivative and $|k|^n e^{-k^2}$ is bounded by a constant for any $n \geq 0$. □

Remark 1. At this point, we may comment on the choice of the constant L. It is arbitrary except that we take it large enough so that $CL^{-1} < 1$ in (14). This will be assumed, wherever needed, about any other numerical constant entering the proofs. We shall denote generic constants by C, even when they change in the same equation. The only important property of such constants is that they are independent of L.

Remark 2. Using (13,14) we can prove (10), i.e. (3) for $f \in \mathcal{B}$.

2.2 Nonlinear heat equations.

Let us now consider a less trivial example, namely

$$u_t = u_{xx} + \lambda u^p \tag{15}$$

where λ could be scaled away but is introduced for convenience. Throughout this Section, we work in one dimension (to simplify the notation), but the extension to higher dimensions is straightforward. There is no problem in proving the existence of solutions for any fixed time, provided λ is small enough, and we may define the renormalization group map as in (5). However, we do not have anymore the semigroup property (8), because the equation is not scale invariant. In fact, $u_L(x,t) = Lu(Lx, L^2t)$ satisfies

$$u_{Lt} = u_{Lxx} + \lambda L^{3-p} u_L^p \tag{16}$$

So, if we define R_L on the pair (f, λ) by

$$R_L(f, \lambda) = (u_L(x,1), \lambda L^{3-p}) \tag{17}$$

we recover the semigroup property (8), provided there is a set of pairs (f, λ) that is mapped into itself by R_L.

Now, consider $p > 3$. Then the line of fixed points $(Af_0^*, 0)$ of R_L is stable. So we may expect the long time behaviour of the solutions of (15) to be again given by Af_0^* for a suitable A. This is indeed the case. However, the constant A will not depend only on f, but also on p and λ.

We have the following result which is a variant of Theorem 1 in [12]. Fix $p \in \mathbf{N}, p > 3$ in (15) and fix $q > 1$ in (12); let $\mathcal{B}_1$ be the unit ball in the corresponding Banach space $\mathcal{B}$.

Theorem 1 *There exist an $\epsilon > 0$ such that, if $|\lambda| \le \epsilon$ and $f \in \mathcal{B}_1$, equation (15) with $u(x,1) = f(x)$ has a unique solution which satisfies, for some number $A = A(f, \lambda, p)$,*

$$\lim_{t\to\infty} \|\sqrt{t}u(\sqrt{t}\cdot, t) - Af_0^*(\cdot)\| = 0.$$

Remark 1. For related results on the $p > 3$ case, see [23, 35, 42, 48].
Remark 2. The convergence in $\mathcal{B}$ implies convergence both in L^1 and in L^∞.

Let us explain the main ideas of the proof (for details, see [12]). The problem is to "construct" the constant A which is not given by an explicit formula as in (3). This construction is done inductively.

First of all, the existence of the solution for finite times is rather trivial (we take λ small and $\|f\| \le 1$). Indeed, we can write

$$u(t) = e^{(t-1)\partial^2} f + \lambda \int_0^{t-1} ds e^{s\partial^2} u^p(t-s) \tag{18}$$

(leaving out the x dependence) and use a standard fixed point argument (see [12]; here we need only $q>1$, because there are no derivatives in the nonlinear term). Since the first term in (18) is the linear evolution, we may write

$$Rf = R_0 f + v \tag{19}$$

where R_0 denotes the linear RG map (5) (we suppress here the L dependence of R) and $\|v\|$ is $\mathcal{O}(\lambda)$. Now, write

$$f = A_0 f_0^* + g_0 \tag{20}$$

with $\hat{g}_0(0)=0$. This splitting of f is not preserved by R, as it was in the linear case. Thus, we shall have to change A at each step. Write

$$Rf = A_1 f_0^* + g_1 \tag{21}$$

where again $\hat{g}_1(0)=0$. From (19) and (20), we see that

$$A_1 = A_0 + \hat{v}(0), \tag{22}$$

since $\widehat{R_0 g_0}(0)=0$ (see (6)), and

$$g_1 = R_0 g_0 + v - \hat{v}(0) f_0^*. \tag{23}$$

Using Lemma 1 and $\|v\| = \mathcal{O}(\lambda)$, we have

$$A_1 = A_0 + \mathcal{O}(\lambda) \tag{24}$$

and

$$\begin{aligned} \|g_1\| &\leq CL^{-1}\|g_0\| + \mathcal{O}(\lambda) \\ &\leq L^{-(1-\delta)} \end{aligned} \tag{25}$$

for any $\delta>0$, provided λ is small enough, since $\|g_0\| = \mathcal{O}(1)$ for $f \in \mathcal{B}_1$.

Now iterate: under scaling λ becomes $\lambda_1 = \lambda L^{3-p}$, and after n steps

$$\lambda_n = \lambda L^{(3-p)n} \tag{26}$$

Thus, $\|v_n\|$ will be $\mathcal{O}(\lambda_n)$ for the n^{th} iteration, and, writing $R^n f = A_n f_0^* + g_n$, we have

$$A_{n+1} = A_n + \hat{v}_n(0) \tag{27}$$

$$g_{n+1} = R_L^0 g_n + v_n - \hat{v}_n(0) f_0^*, \tag{28}$$

with the bounds:

$$|A_{n+1} - A_n| \leq \mathcal{O}(\lambda_n) \tag{29}$$

$$\begin{aligned} \|g_{n+1}\| &\leq CL^{-1}\|g_n\| + \mathcal{O}(\lambda_n) \\ &\leq L^{-(1-\delta)n} \end{aligned} \tag{30}$$

(use (26) and $p-3 \geq 1$ since p is an integer).

Therefore there exists a constant A such that $A_n \to A$ as $n \to \infty$ and

$$|A - A_n| \leq \mathcal{O}(\lambda_n). \tag{31}$$

Then,

$$\begin{aligned} R^n f &= A_n f_0^* + g_n \\ &= A f_0^* + (A_n - A) f_0^* + g_n \end{aligned} \tag{32}$$

and

$$\|R_L^n f - A f_0^*\| \leq L^{-(1-\delta)n} + \mathcal{O}(\lambda_n) \tag{33}$$

which goes to zero as $n \to \infty$. Using (5,8,17), this proves the claim, at least for a special sequence of times; the extension to all times is easy [12].

Remark 1. From (33,26), one gets an estimate $\mathcal{O}(t^{-(\frac{1-\delta}{2})})$ on the rate of convergence.
Remark 2. The sign of λ is not important here. It matters only for $p \leq 3$, see below.
Remark 3. R_0 is the linearization of the component of the operator R that acts on initial data, around the fixed point f_0^*. All functions of the form $k^n e^{-k^2}, n \in N$, are (Hermite) eigenvectors:

$$R_0 k^n e^{-k^2} = L^{-n} k^n e^{-k^2} \tag{34}$$

The fact that the scaling transforms the linear evolution semigroup, $e^{t\partial^2}$, which has continuous spectrum, into an operator with discrete spectrum, is one of the interesting features of this method. Using it, one can extend the above analysis and obtain higher order asymptotics of the solution in inverse powers of t, see subsection 2.5 below, and [56, 66].

2.3 The RG flow in spaces of equations and data.

Of course, Theorem 1 can be extended to a much more general class of nonlinearities. Consider an equation of the form

$$u_t = u_{xx} + F(u, u_x, u_{xx}) \tag{35}$$

Then, the map R_L will be defined in general by

$$R_L = (L^\alpha u_L(\cdot, 1), F_L) \tag{36}$$

where

$$F_L(a,b,c) = L^{2+\alpha} F(L^{-\alpha} a, L^{-1-\alpha} b, L^{-2-\alpha} c) \tag{37}$$

If we consider a monomial $F(a,b,c) = a^{n_1} b^{n_2} c^{n_3}$, and the previous scaling $\alpha = 1$, we have $F_L = L^{-d_F} F$ with

$$d_F = n_1 + 2n_2 + 3n_3 - 3 \tag{38}$$

and using standard RG terminology, we call the monomial *relevant* if $d_F < 0$, *marginal* if $d_F = 0$, and *irrelevant* if $d_F > 0$.

Considering $n_i \geq 1$ or $n_i = 0$, the nonlinearities are easily classified: u^p is relevant for $p < 3$; u^3 and uu_x are marginal, and everything else is irrelevant. Then, Theorem 1 extends to the irrelevant case, provided we take $q > 3$ in (12) and $F : \mathbf{C}^3 \to \mathbf{C}$ analytic in a neighbourhood of zero [12].

Remark. More general irrelevant nonlinearities (nonlocal, integro-differential terms) were considered by Taskinen [63]. Besides, as mentioned in [12], we may replace the second derivative in (15) by $-(-\Delta)^{\beta/2}$ and obtain similar results.

A nice extension to the study of waves satisfying a generalized KdV-Burgers equation was made in [5]. There, the highest order (third) derivative turns out to be irrelevant.

The two marginal terms exhibit different behaviours. For u^3, it is essential to replace the plus sign by a minus one (with $\lambda > 0$) in equation (15) (with a plus sign, and $p \leq 3$, any positive initial data leads to a solution that belows up in a finite time [34, 55]). The nonlinearity turns out to be irrelevant in the next order. The coefficient A_n satisfies a recursion of the form $A_{n+1} = A_n - \lambda\beta A_n^3 +$ higher order terms, where $\beta > 0$ is an explicit constant. This implies that $A_n \sim n^{-\frac{1}{2}}$ (here we use λ positive) and since $t = L^{2n}$, this translates into the following logarithmic correction (see [12]):

$$u(\xi\sqrt{t}, t) \simeq (t \log t)^{-\frac{1}{2}} f_0^*(\xi) \tag{39}$$

On the other hand, for uu_x, we have Burgers' equation and, via the Cole-Hopf transformation, we get a new line of fixed points:

$$f^*(x) = \frac{d}{dx} \log(1 + Ae(x)) \tag{40}$$

where $e(x) = \int_{-\infty}^{x} e^{-\frac{y^2}{4}} dy$ is the error function and A is a parameter. A theorem analogous to Theorem 1 can be proven for these new fixed points [12]. Of course we can also add to these marginal terms a general irrelevant F and obtain a similar result.

2.4 Non-Gaussian fixed points.

Let us now turn to the relevant case, $u^p, 1 < p < 3$. Actually, since we will not restrict ourselves to integer p or to everywhere positive u's, we shall consider the equation:

$$u_t = u_{xx} - u|u|^{p-1}, \tag{41}$$

but the minus sign will again be essential. Here, the parameter λ of (15) has been eliminated by a rescaling of u.

Since the nonlinear term is relevant, the Gaussian fixed point will be of no use. In order to find another fixed point, consider the transformation (36) with $\alpha = \frac{2}{p-1}$. The reason for this choice is that equation (41) is then scale invariant: $F_L = F$. It is easy to see that we shall have a fixed point of the transformation if we have a solution of the form:

$$u(x,t) = t^{-\frac{1}{p-1}} f(xt^{-\frac{1}{2}}) \tag{42}$$

where $f(\xi)$ is a function of one variable which solves:

$$f'' + \frac{1}{2}\xi f' + \frac{f}{p-1} - f^p = 0. \tag{43}$$

These solutions are called self-similar. The theory of positive solutions of (43) has been developed in [8, 35, 48]. The main result is that, for any $p > 1$, there exist smooth, everywhere positive solutions, f_γ, of (43) with $f_\gamma'(0) = 0$ and $f_\gamma(0) = \gamma$ for γ larger than a certain critical value γ_p (but not too large). The decay at infinity of these solutions is given by

$$f_\gamma(x) \sim |x|^{-\frac{2}{p-1}} \tag{44}$$

as $|x| \to \infty$ if $\gamma > \gamma_p$, while, for $\gamma = \gamma_p$, it decays at infinity as

$$f_{\gamma_p}(x) \sim |x|^{\frac{2}{p-1}-1} e^{-\frac{x^2}{4}}. \tag{45}$$

Remark. The existence of a critical γ_p can be understood intuitively by viewing (43) as Newton's equation for a particle of mass one, whose "position" as a function of "time" is $f(\xi)$. The potential is then $U(f) = \frac{f^2}{2(p-1)} - \frac{f^{p+1}}{p+1}$ and the "friction term" $\frac{\xi}{2}f'$ depends on the "time" ξ. Hence, if $f_\gamma'(0) = 0$ and $f_\gamma(0) = \gamma$ is large enough, the time it takes to approach zero is long and, by then, the friction term has become sufficiently strong to prevent "overshooting". However, as p increases, the potential becomes flatter and one therefore expects γ_p to decrease with p.

We shall now explain why these self-similar solutions are stable. Consider the initial data (taken as before at time 1 for convenience)

$$u(x,1) = f_\gamma(x) + h(x) \tag{46}$$

with $\gamma \geq \gamma_p$ and $h \in B$, where B is the Banach space of L^∞ functions equiped with the norm (with some abuse of notation!)

$$\|h\|_\infty = \operatorname*{ess\,sup}_\xi |h(\xi)(1 + |\xi|^q)|. \tag{47}$$

One can show the following [16]:

Theorem 2 *Let $1 < p < 3$. There exist $\varepsilon > 0, C < \infty$ and $\mu > 0$ such that, if the initial data $u(x,1)$ of (41) is given by (46) with $h \in B$ and satisfies*

$$\|h\|_\infty \leq \varepsilon, \ \textit{or}\ h \geq 0 (a.e.)$$

then, (41) *has a unique classical solution and, for all* t,

$$\|t^{\frac{1}{p-1}}u(\cdot t^{\frac{1}{2}},t)-f_\gamma(\cdot)\|_\infty \le Ct^{-\mu}\|h\|_\infty$$

Remark. For related results on the stability of self-similar solutions, see [29, 30, 31, 35, 49].

Let us explain the main ideas of the proof. Given the initial data (46), it is convenient to rewrite (41) in terms of the variables $\xi = xt^{-\frac{1}{2}}$ and $\tau = \log t$; so, define $v(\xi,\tau)$ by:

$$u(x,t)=t^{-\frac{1}{p-1}}(f_\gamma(xt^{-\frac{1}{2}})+v(xt^{-\frac{1}{2}},\log t)) \tag{48}$$

where now

$$v(\xi,0)=h(\xi). \tag{49}$$

Then, (41) is equivalent to the equation

$$v_\tau = \mathcal{L}v - \left(|f_\gamma+v|^{p-1}(f_\gamma+v)-f_\gamma^p-pf_\gamma^{p-1}v\right)\equiv \mathcal{L}v+N(v) \tag{50}$$

where we used the fact that f_γ solves (43) and we gathered the linear terms in

$$\mathcal{L}=\mathcal{L}_0+V_\gamma,$$

with

$$\mathcal{L}_0=\frac{d^2}{d\xi^2}+\frac{\xi}{2}\frac{d}{d\xi}+\frac{1}{p-1}, \tag{51}$$

and

$$V_\gamma(\xi)=-pf_\gamma^{p-1}(\xi). \tag{52}$$

To prove the theorem, it suffices to show that the corresponding solution of (50) goes to zero as $\tau\to\infty$. For that, the main estimate will be that the semigroup $e^{\tau\mathcal{L}}$ contracts, at least for τ large:

$$\|e^{\tau\mathcal{L}}\|\le Ce^{-\mu\tau} \tag{53}$$

for some $\mu>0$, $C<\infty$. Given (53), the control over the nonlinear term in (50) is standard.

There are two important ingredients in the proof of (53). The first is the fact that $e^{\tau\mathcal{L}}$ is a contraction in a suitable Hilbert space of rapidly decreasing functions. To see this, note first that $\mathcal{L}_0$ can be conjugated to the following Schrödinger operator:

$$e^{\frac{\xi^2}{8}}\mathcal{L}_0e^{-\frac{\xi^2}{8}}=\frac{d^2}{d\xi^2}-\frac{\xi^2}{16}-\frac{1}{4}+\frac{1}{p-1}, \tag{54}$$

i.e. to minus the Hamiltonian of the harmonic oscillator. Thus, $\mathcal{L}_0$ is self-adjoint on its domain $\mathcal{D}(\mathcal{L}_0) \subset L^2(\mathbf{R}, d\mu)$, where

$$d\mu(\xi) = e^{\frac{\xi^2}{4}} d\xi.$$

$\mathcal{L}_0$ has a pure point spectrum $\{\frac{1}{p-1} - \frac{1}{2} - \frac{m}{2} \mid m = 0, 1, \ldots\}$ and the largest eigenvalue $\frac{1}{p-1} - \frac{1}{2}$ is *positive* if $1 < p < 3$. Thus $e^{\tau\mathcal{L}_0}$ is *not* contractive and, for $e^{\tau(\mathcal{L}_0 + V_\gamma)}$ to contract, we need to use the potential in a non-trivial way (this is the reason why $1 < p < 3$ is harder than the $p > 3$ case).

Remarkably, it is possible to prove that $\mathcal{L} < -E < 0$ without a detailed study of the function f_γ, but only using equation (43). Indeed, one first shows that $\mathcal{L}$ is self-adjoint and that $-E_\gamma$, its largest eigenvalue, satisfies $-E_\gamma \leq -E_{\gamma_p}$. So, writing $E \equiv E_{\gamma_p}$, it is enough to show that $-E < 0$. Next, it is easy to see, using the Feynman-Kac formula [61] and the the Perron-Frobenius theorem [41] that $\mathcal{L}$ has a unique eigenvector Ω with eigenvalue $-E$ and Ω can be chosen to be strictly positive. Write (43) as

$$\mathcal{L}f = -(p-1)f^p.$$

So,

$$(\Omega, \mathcal{L}f) = -(p-1)(\Omega, f^p)$$

where $(\cdot, \cdot)$ denotes the scalar product in $L^2(\mathbf{R}, d\mu)$. By the self-adjointness of $\mathcal{L}$ and the definition of Ω, i.e. $\mathcal{L}\Omega = -E\Omega$, we have

$$-E = -(p-1)\frac{(\Omega, f^p)}{(\Omega, f)} < 0$$

since Ω and f are strictly positive. This proves our claim.

Notice that functions in $L^2(\mathbf{R}, d\mu)$ have essentially a Gaussian decay at infinity, which is much faster than what is allowed in our Banach space B, see (47). An extra work is needed, using the Mehler's formula for the kernel of $e^{\tau\mathcal{L}_0}$ (see [16] for details).

Finally, let us express this result in the RG language. The RG map has a fixed point $R_L(f_\gamma, F^*) = (f_\gamma, F^*)$ for $F^*(u) = -u|u|^{p-1}$. There are different fixed points for different values of γ, but since the decay of the functions in B is faster than the one of any of these fixed points, each of them is unique in the corresponding set $\{f_\gamma + h\}$ for h as in Theorem 2. In other words, unlike the situation for $p > 3$, we do not have to deal with a line of fixed points and here there is no constant like A in Theorem 1. This is reflected in the fact, (53), that the linear semigroup $e^{\tau\mathcal{L}}$ contracts.

Remark. The reader who is familiar with the theory of critical phenomena will recognize an analogy between the behaviour of an Ising model at its critical point, when the dimension of the lattice changes and the asymptotic behaviour of the solution of (41) when p varies. This is further discussed in [12]. Here, our results for $p < 3$ are non-perturbative, i.e. we do not use any "ε-expansion" (see [66] for an analysis of the bifurcation at $p = 3$).

Also, the reader may notice that our approach to the renormalization group is close to the "Wilsonian" one in the theory of critical phenomena, while the method developed in

[43] is analogous to the renormalized perturbation of quantum field theory. For a rigorous treatment of Barenblatt's equation, which was one of the first example analyzed in [43], see [50].

2.5 Higher-order asymptotics.

It is interesting to reexamine the $p \geq 3$ case, using what we learned from the analysis of $p < 3$. This has been done by Wayne [66]. Namely, consider equation (15), and make the change of variable (48), but replacing f_γ by 0, i.e.

$$u(x,t) = t^{-\frac{1}{p-1}} v(xt^{-1/2}, \log t) \tag{55}$$

Then $v(\xi, \tau)$ satisfies

$$v_\tau = \mathcal{L}_0 v + \lambda v^p \tag{56}$$

with $\mathcal{L}_0$ as in (51); we have computed the spectrum of $\mathcal{L}_0$ which is $\{\frac{1}{p-1} - \frac{1}{2} - \frac{m}{2} | m = 0, 1, \cdots\}$. Now, notice that, for $p > 3$, this spectrum is entirely negative. Thus $e^{\tau \mathcal{L}_0}$ contracts exponentially and it is not hard to show that, for λ small, and $v(\xi, 0)$ bounded in a suitable norm, the same contraction holds for the solution of (56). Actually, since the largest eigenvalue of $\mathcal{L}_0$ corresponds to $m = 0$, we expect a decay like $e^{\tau(\frac{1}{p-1} - \frac{1}{2})}$, which, using $\tau = \log t$ and combining it with the $t^{-\frac{1}{p-1}}$ prefactor in (55), leads to a $t^{-1/2}$ decay of $u(x,t)$, for all $p \geq 3$. This of course has to be the case, in view of Theorem 1.

Actually, much more can be said. Take the n largest eigenvalues $\{\lambda_1, \cdots, \lambda_n\}$ of $\mathcal{L}_0$ and the corresponding subspaces. Then, using a theorem of Gallay [37], Wayne constructs an invariant manifold (in the v variable) such that any solution (for $p > 3$ and small initial data) will converge to that invariant manifold at a rate at least of order $t^{-(\lambda_{n+1} - \delta)}$ (with, as before, δ arbitrarily small for sufficiently small initial conditions).

Since, for $n = 1$, the eigenvector of $\mathcal{L}_0$ is $e^{-\xi^2/4}$, one easily recovers Theorem 1. But all higher order asymptotics in time of $v(\xi, \tau)$, hence of $u(x,t)$, can also be obtained in terms of Hermite functions (such higher order corrections were also derived in [56]).

For $p < 3$, an invariant manifold can still be constructed but it becomes unstable ($\frac{1}{p-1} - \frac{1}{2} > 0$) and thus, it does not, in general, give information on the long-time asymptotics. Of course, we know from Theorem 2 what the situation is: instead of 0, we have to consider the solution f_γ and, presumably, a similar picture (with invariant manifolds and higher order asymptotics) holds there, since the spectrum of the operators $\mathcal{L}$ for $p < 3$ is qualitatively similar to the one of $\mathcal{L}_0$ for $p > 3$.

2.6 An application to reaction-diffusion equations.

In [4], Berlyand and Xin consider the following model which leads to a nice RG analysis and which exhibits the novel feature of anomalous exponents depending on the initial data:

$$u_t = \Delta u + v u^{p-1} \tag{57}$$

$$v_t = \Lambda^{-1} \Delta v - v u^{p-1} \tag{58}$$

where $u = u(x,t), v = v(x,t), x \in \mathbf{R}^n$ (we take $n = 1$ for simplicity) and $\Lambda > 0$ is the Lewis number. Equations (57,58) model a chemical reaction $A \to B$ where v is the mass fraction of reactant A and u the one of reactant B.

First, observe that for $p > 3$, the analysis done for (15) applies and the asymptotics of u, v is Gaussian. So let $p = 3$, so that the nonlinear terms are marginal, and let us see heuristically what happens. If u and v are positive (which is physically necessary and is preserved by (57,58) due to the maximum principle) then, again by the maximum principle, u is larger than the solution of the heat equation with the same initial data:

$$u(x,t) \geq \frac{A}{t^{1/2}} e^{-\frac{x^2}{4t}}. \tag{59}$$

where A depends on $u(x,0)$.

Then, v is less than the solution of (58) with $u^{p-1} = u^2$ replaced by its lower bound (59). Calling $\bar{v}$ this upper bound on v, $\bar{v}$ solves a linear equation

$$\bar{v}_t = \Lambda^{-1} \Delta \bar{v} - A^2 t^{-1} e^{-\frac{x^2}{2t}} \bar{v}. \tag{60}$$

It is not hard to see, by direct substitution, that (60) admits a self-similar solution of the form

$$\bar{v}(x,t) = t^{-\frac{\alpha}{2}} f^*_\alpha\left(\frac{x}{\sqrt{t}}\right). \tag{61}$$

Indeed, writing $f^*_\alpha(\xi) = e^{-\frac{\Lambda \xi^2}{8}} \Psi(\xi)$, Ψ solves

$$-\Lambda^{-1} \Psi'' + \left(\frac{\Lambda}{16}\xi^2 + \frac{A^2}{4\pi} e^{-\xi^2/2} + \frac{1}{4}\right)\Psi = \frac{\alpha}{2}\Psi \tag{62}$$

We see that, requiring f^*_α to be positive means that Ψ is the ground state of the operator in the LHS of (62) i.e. of an harmonic oscillator perturbed by a positive, rapidly decaying, potential. The exponent $\frac{\alpha}{2}$ is the corresponding ground state energy. For the harmonic oscillator, $\frac{\alpha}{2} = \frac{1}{2}$ and, perturbation theory tells us that, for A small,

$$\frac{\alpha}{2} = \frac{1}{2} + \frac{A^2}{4\pi\sqrt{2\Lambda^{-1} + 1}} + h.o.t. \text{ in } A \; > \; \frac{1}{2}. \tag{63}$$

Thus, α depends on A, i.e. on $u(x,0)$. Since $\alpha > 1$, $\bar{v}$, and hence v, decay strictly faster than the solution of the heat equation. Inserting this in (57) gives an upper solution for u, which solves an equation where the nonlinear term is now *irrelevant*.

These considerations lead us to expect u to have the heat equation decay:

$$u(x,t) \simeq \frac{A}{\sqrt{t}} e^{-x^2/4t} \tag{64}$$

and v to have the anomalous decay:

$$v(x,t) = \frac{B}{t^{\alpha/2}} f^*_\alpha\left(\frac{x}{\sqrt{t}}\right) \tag{65}$$

where A, B and α depend on the initial data. Upper and lower bounds of the form (64,65) are proven in [4]. From the RG point of view, one can define a map

$$R_{L,\alpha}(u,v) = (u_L(\cdot,1), v_L(\cdot,1))$$

where

$$\begin{aligned} u_L(x,t) &= Lu(Lx, L^2t), && (66)\\ v_L(x,t) &= L^\alpha v(Lx, L^2t). && (67) \end{aligned}$$

We have thus a two-parameter family of fixed points $(Af_0^*, Bf_{\alpha(A)}^*)$, and the scaling exponent varies continuously with A.

3 Patterns and fronts.

3.1 The Ginzburg-Landau equation.

In the previous section, we have seen that nonlinear equations with initial data decaying at infinity produce universal, Gaussian or non-Gaussian, diffusive profiles. Here we shall show that other types of asymptotic behaviour can also exhibit such universality. To discuss a concrete example, consider the Ginzburg-Landau equation

$$u_t = u_{xx} + u - |u|^2 u \tag{1}$$

where $u : \mathbf{R}\times\mathbf{R} \to \mathbf{C}$ is complex. This equation has a two parameter family of stationary solutions

$$u_{q\theta}(x) = \sqrt{1-q^2}e^{i(qx+\theta)} \tag{2}$$

and a natural question is to inquire about the time development of initial data $u(x)$ which approach two such solutions at $\pm\infty$:

$$\lim_{x\to\pm\infty} |u(x) - u_{q_\pm\theta_\pm}(x)| = 0. \tag{3}$$

This problem has been extensively studied for equation (1) with u *real* and $u(-\infty) = 1$, $u(\infty) = 0$ (i.e. $q_- = 0$, $q_+ = 0$) [1, 6]. Then the solution takes the form of a propagating front. This occurs because $u(x) = 1$ is a stable stationary solution, while $u(x) = 0$ is an unstable one. For complex u, the solutions (2) are stable, under small perturbations, for $q^2 < \frac{1}{3}$ (the Eckhaus stable domain) [23].

We shall consider two questions. The first one was suggested in [24], namely we take $q_\pm$ in (3) small, belonging to the Eckhaus stable domain, but not necessarily equal. What is then the long-time asymptotics of the solution? The second question concerns the stability of the real front solutions of (1) for real or complex u. For reviews on these questions, we refer the reader to [22, 25].

3.2 Non-Gaussian patterns.

For the first question, we considered in [13] a class of initial data satisfying (3) and we showed that, for any interval I,

$$\sup_{x\in I}|u(x,t)-e^{i\sqrt{t}\phi^*}u_{q^*\theta^*}(x)|\le\frac{C_I}{\sqrt{t}} \tag{4}$$

where the constants q^*, ϕ^* and θ^* depend only on the boundary conditions (3). For a more detailed bound, see Section 3 of [13]. Graphics of the solution can be found in [25].

In order to see where (4) comes from, we write, following [24],

$$u=(1-s)e^{i\phi}. \tag{5}$$

Equation (1) becomes, in these variables,

$$\begin{aligned} s_t &= s_{xx}-2s+3s^2-s^3+\phi_x^2-s\phi_x^2\equiv s_{xx}-2s+F(s,\phi_x) \\ \phi_t &= \phi_{xx}-\frac{2\phi_x s_x}{1-s}\equiv\phi_{xx}+G(s,s_x,\phi_x) \end{aligned} \tag{6}$$

with the initial data (taking again $t=1$ as initial time)

$$\lim_{x\to\pm\infty}s(x,1)=s_\pm\ ,\quad \lim_{x\to\pm\infty}|\phi(x,1)-\phi_\pm x-\theta_\pm|=0. \tag{7}$$

where $2s_\pm=F(s_\pm,\phi_\pm)$. We proved in [13] the following asymptotics, as $t\to\infty$:

$$\phi(x,t)=\sqrt{t}\phi^*(\frac{x}{\sqrt{t}})+\theta^*(\frac{x}{\sqrt{t}})+\mathcal{O}(\frac{1}{\sqrt{t}}) \tag{8}$$

$$s(x,t)=s^*(\frac{x}{\sqrt{t}})+\frac{1}{\sqrt{t}}r^*(\frac{x}{\sqrt{t}})+\mathcal{O}(\frac{1}{t}) \tag{9}$$

for a set of initial data satisfying (7), with $\phi_\pm$, and $\theta_\pm$ small enough (see again [13] for the precise definition of the norms in which these limits take place). The functions ϕ^*, θ^*, s^*, r^* are *universal*, depending on the initial data only through the boundary conditions (7). They are smooth and therefore (expanding ϕ^* to first order, and using (2)) the phase of u will have the asymptotics (4), with

$$\phi^*=\phi^*(0),\quad q^*=\phi^{*'}(0),\quad \theta^*=\theta^*(0).$$

The behaviour of s will be discussed below.

The peculiar scaling behaviour exhibited by (8) can already be understood through the linear heat equation. Previously, we considered the Gaussian fixed point of the RG in a space of integrable initial data. Since the heat equation has stationary solutions of the form $a+bx$, we may also consider the solution of $\phi_t=\phi_{xx}$ with initial data $\phi(x,1)=f(x)$ satisfying (7). The solution can be computed:

$$\begin{aligned} t^{-\frac12}\phi(\sqrt{t}x,t+1) &= \frac{1}{\sqrt{4\pi t}}\int e^{-\frac14(y-x)^2}f(\sqrt{t}y)dy, \\ &\to_{t\to\infty}\phi_-x+(\phi_+-\phi_-)(x+2\frac{d}{dx})e(x)\equiv\phi_0^*(x) \end{aligned} \tag{10}$$

where $e(x) = \int_{-\infty}^{x} e^{-\frac{y^2}{4}} dy$ is the error function. This gives the first term in (8), with ϕ^* replaced by ϕ_0^*. If we want to obtain the next correction, we write $\phi^* = \phi_0^* + \theta$, where $\theta(x,t)$ solves $\theta_t = \theta_{xx}$ with $\theta(x,1) \to \theta_\pm$ as $x \to \pm\infty$. One computes that

$$\theta(x,t) \to \theta_0^* \left(\frac{x}{\sqrt{t}}\right) = \theta_- + (\theta_+ - \theta_-) e \left(\frac{x}{\sqrt{t}}\right) \tag{11}$$

and this yields the second term in (8). It is trivial to check that ϕ_0^* (resp. θ_0^*) is a fixed point of the RG map (2.36) with $\alpha = -1$ (resp. $\alpha = 0$), and $F = 0$.

We shall explain below that ϕ^*, θ^* are small perturbations of ϕ_0^*, θ_0^*. However, as we saw, the scaling (8,4) can be understood qualitatively on the basis of the heat equation, and the boundary conditions (7). We still need to understand (9).

The latter comes from the "slaving" of s to ϕ, due to the linear $-2s$ term in (6). This will imply that s will have the right form so that (4) holds. Concretely, the slaving means that, to leading order, we may solve the algebraic equation $-2s + F(s, \phi_x) = 0$, which is equivalent to $1 - s = \sqrt{1 - (\phi_x)^2}$, and substitute the result in the equation (6) for ϕ. Since the derivative of s is proportional to the second derivative of ϕ, we obtain an equation for ϕ only, of the form:

$$\phi_t = (1 + a(\phi_x, \phi_{xx}))\phi_{xx}\ , \quad \phi(x,1) = f(x) \tag{12}$$

with the boundary conditions (7) for the initial data f. The function a is analytic around the origin.

The RG map acts again on pairs of initial data and equations and can be defined (with $\alpha = -1$) as

$$R_L(\phi, a) = (\phi_L, a_L) \tag{13}$$

where

$$\phi_L(x,t) = L^{-1}\phi(Lx, L^2 t)$$

and

$$a_L(u,v) = a(u, L^{-1}v). \tag{14}$$

The semigroup property follows by observing that ϕ_L satisfies

$$\phi_{Lt} = (1 + a_L(\phi_{Lx}, \phi_{Lxx}))\phi_{Lxx}. \tag{15}$$

We may now iterate R as before, i.e. solve a finite time problem, to study the asymptotics of (12). One wants to show that there exist functions ϕ^*, a^* such that

$$R_L^n(\phi, a) \to (\phi^*, a^*) \tag{16}$$

where

$$R_L(\phi^*, a^*) = (\phi^*, a^*) \tag{17}$$

is the fixed point of the RG, corresponding to the scale-invariant equation $\phi_t = (1 + a^*)\phi_{xx}$. Then, replacing $\frac{x}{\sqrt{t}}$ by ξ, the asymptotics of the original problem is given by

$$\phi(\xi t^{1/2}, t) \sim t^{\frac{1}{2}}\phi^*(\xi). \tag{18}$$

Because of the factor L^{-1} in the second argument of a in (14), the fixed point equation is

$$\phi_t = (1 + a^*(\phi_x))\phi_{xx} \tag{19}$$

where $a^*(\cdot) = a(\cdot, 0)$. The fixed point is the scale invariant solution $\phi(x,t) = \sqrt{t}\phi^*(\frac{x}{\sqrt{t}})$. We get for ϕ^* the equation

$$(1 + a^*)\frac{d^2}{d\xi^2}\phi^* + \frac{1}{2}\xi\frac{d}{d\xi}\phi^* - \frac{1}{2}\phi^* = 0 \tag{20}$$

with $a^* = a^*(\frac{d}{d\xi}\phi^*)$ and, for small $\phi_\pm$, we look for a solution

$$\phi^* = \phi_0^* + \rho \tag{21}$$

where $\rho(\pm\infty) = 0$ and ϕ_0^* is the "Gaussian" solution (10), which solves (20) with $a^* = 0$. This is easy to solve by a fixed point argument(see Proposition 1 in [13] or the Proposition in Section 4 of [12]).

This gives us the first term in (8). Turning to θ^*, we write

$$\phi(x,t) = \phi^*(x,t) + \theta(x,t) \tag{22}$$

where, with an abuse of notation, $\phi^*(x,t) = \sqrt{t}\phi^*(\frac{x}{\sqrt{t}})$ and ϕ^* is given by (21), while ϕ solves (12). Then,

$$\theta_t = \theta_{xx} + (a\phi_{xx} - a^*\phi^*_{xx}) \tag{23}$$

with $\theta(\pm\infty) = \theta_\pm$. Now we set

$$\theta_L(x', t') = \theta(Lx', L^2t'). \tag{24}$$

Putting $x = Lx', t = L^2t'$, we have $\frac{d^l}{dx^l}\phi^*(Lx', L^2t') = L^{1-l}\frac{d^l}{dx'^l}\phi^*(x', t')$ and therefore θ_L, satisfies the equation (replacing (x', t') by (x,t)):

$$\begin{aligned} \theta_{Lt} \;&=\; \theta_{Lxx} + L(a(\phi^*_x + L^{-1}\theta_{Lx}, L^{-1}\phi^*_{xx} + L^{-2}\theta_{Lxx})(\phi^*_{xx} + L^{-1}\theta_{Lxx}) \\ &\quad - a^*(\phi^*_x)\phi^*_{xx}). \end{aligned} \tag{25}$$

Thus, reasoning as above, we expect

$$\theta_{L^n} \to \theta^* \tag{26}$$

where $\theta^*(x,t) = \theta^*(\frac{x}{\sqrt{t}})$ satisfies the $L \to \infty$ form of (25):

$$\theta^*_t = \theta^*_{xx} + a^*\theta^*_{xx} + (a_u(\phi^*_x, 0)\theta^*_x + a_v(\phi^*_x, 0)\phi^*_{xx})\phi^*_{xx}. \tag{27}$$

This is a linear equation, easy to solve, whose solution is, for $\theta_\pm$ small, a small perturbation of the "Gaussian" solution (11) (which solves (27) with $a = 0$). Finally, one sets (with the same abuse of notation)

$$\phi(x,t) = \phi^*(x,t) + \theta^*(x,t) + \psi(x,t) = \sqrt{t}\phi^*(\frac{x}{\sqrt{t}}) + \theta^*(\frac{x}{\sqrt{t}}) + \psi(x,t) \tag{28}$$

As for the s variable, one gets

$$s(x,t) = s^*(x,t) + r^*(x,t) + v(x,t) \tag{29}$$

where

$$s^*(x,t) = s^*(\frac{x}{\sqrt{t}}), \quad r^*(x,t) = \frac{1}{\sqrt{t}} r^*(\frac{x}{\sqrt{t}}) \tag{30}$$

are fixed points "slaved" respectively to ϕ^* and θ^*. They satisfy the boundary conditions

$$\lim_{x\to\pm\infty} |s^*(x) - s_\pm| = 0 \ , \ \lim_{x\to\pm\infty} r^*(x) = 0 \ . \tag{31}$$

The equations for ψ and v are rather complicated, but are essentially of the form heat equation plus irrelevant terms, in the sense of the previous section. Since now ψ and v decay to zero at infinity, we are in the situation of that section and, taking small initial data (in a suitable norm), one shows that the corresponding solution diffuses to zero. This, then, allows us to prove equation (4).

3.3 Stability of Fronts in the Ginzburg-Landau Equation.

Let us write the Ginzburg-Landau equation (1) in radial and angle variables, $u = re^{i\phi}$:

$$r_t = r_{xx} + r(1 - \phi_x^2) - r^3 \tag{32}$$

$$\phi_t = \phi_{xx} + 2r^{-1}\phi_x r_x \tag{33}$$

and let us discuss the stability of the front solutions of these equations. It is well known that these equations have real, positive, front solutions, i.e. solutions of the form

$$\phi = 0, r = r_c(x - ct) \geq 0, \tag{34}$$

such that r_c interpolates between a stable and an unstable solution of (32), i.e. $r_c \to +1$ for $x \to -\infty, r_c \to 0$, for $x \to +\infty$. Indeed, from (32,34), we see that r_c satisfies

$$r_c'' + cr_c' + r_c - r_c^3 = 0 \tag{35}$$

which, if we reinterpret the variable as "time", can be seen as Newton's equation of motion of a particle of mass one subjected to a friction term cr_c' and to a force deriving from the potential $\frac{r^2}{2} - \frac{r^4}{4}$, which is an inverted double-well. It is intuitively clear and easily proved that, for c not too small, solutions exist that satisfy the required conditions, i.e. such that r_c tends, as "time" goes to $+\infty$, to zero, the stable critical point of the potential, and to one as "time" goes to $-\infty$. For large "time" $u = x - ct, r_c(u)$ will decay exponentially, as is seen from the linearization of (35) at $r = 0$. One gets

$$r_c(u) \leq (C_1 + C_2 u)e^{-\gamma u} \tag{36}$$

where γ is given by $\gamma^2 - c\gamma + 1 = 0$ i.e.

$$\gamma_c = \frac{1}{2}(c - \sqrt{c^2 - 4}) \tag{37}$$

which is real for $c \geq 2$, in which case $\gamma_c \leq 1$ (actually, one can take $C_2 = 0$ in (36), if $\gamma < 1$). Thus, the larger the friction, the slower the decay. For $c < 2$, the solution "overshoots" the minimum at zero, i.e. r_c is no longer positive. Each of the solutions r_c with $c \geq 2$ is stable under real perturbations ($\phi = 0$) : if we start with initial data $r(x, 0)$, with $r = r_c + s$ with $0 \leq r \leq 1$, s decaying faster than $e^{-\gamma_c x}$ for $x \to +\infty$, $r(x, t)$ will converge, as $t \to +\infty$, to $r_c(x - ct)$, see [1, 6, 22].

However the solution with $c = 2, \gamma_c = 1$ is more stable than the others in the sense that any initial data $r(x, 0)$ with $0 \leq r \leq 1$ which decays faster than e^{-x} as $x \to +\infty$ (in particular, if r is of compact support) will converge, as $t \to +\infty$, to $r_2(x - 2t)$ [1, 6]. From now on, we shall concentrate on the most interesting front, namely the one with $c = 2, \gamma = 1$ and we write r for r_2 .

We consider a complex perturbation of r : $r(x, 0) = r + s$ with $\phi(x, 0) \neq 0$ and $\phi(x, 0), s(x, 0)$ are small in a suitable sense. The equations satisfied by ϕ and s are :

$$\phi_t = \phi_{xx} + 2(r_x\phi_x + s_x\phi_x)(r + s)^{-1} \tag{38}$$

$$s_t = s_{xx} + s(1 - 3r^2) - r\phi_x^2 - s\phi_x^2 - 3rs^2 - s^3 \tag{39}$$

Since r is a function of $x-2t$ it is convenient to consider also the equation in the frame of reference of the front : let $u = x - 2t$ and $\phi_f(u, t) = \phi(u+2t, t)$,$s_f(u, t) = s(u+2t, t)$; then ϕ_f and s_f satisfy equations like (38,39), with $2\phi_{fu}$ added to the RHS of (38) and $2s_{fu}$ added to the one of (39). Now $r = r(u)$ is time-independent.

To understand the expected behaviour of $\phi_f(u, t)$, let us consider the linearized equation around the zero solution :

$$\phi_{ft} = \phi_{fuu} + 2r_u\phi_{fu}r^{-1} + 2\phi_{fu}. \tag{40}$$

It is convenient to rewrite this equation as a heat equation with a potential: Let

$$\phi_f(u, t) = e^{-u}r(u)^{-1}\psi(u, t). \tag{41}$$

Then, ψ satisfies

$$\psi_t = \psi_{uu} - V\psi \tag{42}$$

with

$$V = 1 + \frac{r''}{r} + 2\frac{r'}{r} = r^2. \tag{43}$$

To derive the last equality, we used eq.(35), satisfied by r. Since $r \simeq 1$ for $u \to -\infty$, $r \simeq e^{-u}$ for $u \to +\infty$, we have $V \simeq 1$ for $u \to -\infty, V \simeq 0$ for $u \to +\infty$.

So, starting with $\psi(u,0)$ localised around $u = 0$, we expect that

$$\psi(u,t) \sim \begin{cases} \frac{e^{-t-\frac{u^2}{4t}}}{\sqrt{t}} & u \to -\infty \\ \frac{e^{-\frac{u^2}{4t}}}{\sqrt{t}} & u \to +\infty. \end{cases} \tag{44}$$

Hence, using (41) and the asymptotic behaviour of $r(u)$,

$$\phi_f(u,t) \sim \begin{cases} \frac{e^{-t-\frac{u^2}{4t}-u}}{\sqrt{t}} & u \to -\infty \\ \frac{e^{-\frac{u^2}{4t}}}{\sqrt{t}} & u \to +\infty \end{cases} \tag{45}$$

which can be written as

$$\phi_f(u,t) \sim \begin{cases} \frac{e^{-\frac{(u+2t)^2}{4t}}}{\sqrt{t}} & u \to -\infty \\ \frac{e^{-\frac{u^2}{4t}}}{\sqrt{t}} & u \to +\infty. \end{cases} \tag{46}$$

Since $u+2t = x$, the first part of (46) is a diffusive wave stationary in the fixed frame. The second part represents a diffusive wave which is "carried along" by the front. This is a rough, but basically correct picture.

Let us consider the linear equation for s_f, in the front frame :

$$s_{ft} = s_{fuu} + s_{fu} + s_f(1 - 3r^2) \tag{47}$$

Writing $s_f = e^{-u}\sigma$, we get

$$\sigma_t = \sigma_{uu} - \tilde{V}\sigma \tag{48}$$

with $\tilde{V} = 3r^2$. Following the analysis leading to (44), we get

$$s(u,t) \sim \begin{cases} \frac{e^{-3t-\frac{u^2}{4t}-u}}{\sqrt{t}} = \frac{e^{-2t}}{\sqrt{t}} e^{-\frac{(u+2t)^2}{4t}} & \text{as} \quad u \to -\infty \\ \frac{e^{-\frac{u^2}{4t}}}{\sqrt{t}} e^{-u} & \text{as} \quad u \to +\infty \end{cases} \tag{49}$$

There is a "wave" which is stationary in the front frame, but exponentially decreasing in u , while the wave which stays in the fixed frame is suppressed by the factor e^{-2t}.

The rigorous results that justify this picture are of two types: Using the RG method, Gallay [35] was able to obtain very precise asymptotics on how a small perturbation of *real fronts* diffuses to zero (this improves previous results of [53, 60]). So Gallay considers equation (39) with $\phi = 0$, in the front frame, whose linearization is given by (47). He writes

$$s(u,t) = r'(u)w(u,t) \tag{50}$$

and studies the behaviour of $w(u,t)$ as $t \to \infty$. Since $r(u)$ goes exponentially to 1 or 0 as u goes to $-\infty$ or $+\infty$, $r'(u)$ will be localized around $u = 0$. The main result is that w has the following universal asymptotics [36]: let $u = \xi\sqrt{t}$, then

$$w(\xi\sqrt{t}, t) \simeq At^{-3/2} f^*(\xi) \tag{51}$$

where

$$f^*(\xi) = \begin{cases} 1 & \text{if } \xi \leq 0 \\ e^{-\xi^2/4} & \text{if } \xi > 0 \end{cases}$$

and A depends again on the initial conditions. The limit (51) holds in a weighted $L^1 \cap L^\infty$ norm. Going back to (47-49), the power $t^{-3/2}$ is easy to grasp intuitively: the potential $\tilde{V}$ in (48) plays the role of a barrier around $u = 0$ ($\tilde{V}$ goes exponentially to 3 for $u \to -\infty$ and exponentially to 0 far $u \to +\infty$). The simplest approximation is to replace the RHS of (48) by a Laplacian on $\mathbf{R}^+$ with Dirichlet boundary conditions at $u = 0$. And that operator leads to a $t^{-3/2}$ decay of the solution. This effect was not taken into account in (49).

The second type of results deals with the complex perturbations. In [14], we consider initial data in the Banach space of C^1-functions ϕ, s with the norm

$$\|(\phi, s)\| = \sup_x (1+ \mid x \mid)^{3+\delta} (\mid \phi(x) \mid + \mid \phi'(x) \mid + (1+e^x)(\mid s(x) \mid + \mid s'(x) \mid)) \tag{52}$$

and prove

Theorem 3. *For any $\delta > 0$ there exists an $\epsilon > 0$ such that equations (38, 39) with initial data $\phi(x,1) = \phi(x), s(x,1) = s(x)$, and $\|(\phi, s)\| < \epsilon$, have a unique classical solution $\phi(x,t), s(x,t)$, for all $t \geq 1$, such that*

$$|\phi(x,t)| \leq t^{-\frac{1}{2}+\delta} \tag{53}$$

$$(1+e^u)|s(x,t)| \leq t^{-1+\delta} \tag{54}$$

Remark 1. The power laws of the decay in time are presumably optimal (except for the δ) and are different from those of Gallay, because the diffusive wave that is stationary in the fixed frame, for the ϕ variable (see (45)), goes only diffusively to zero. This in turn slows downs the decay of the s variable, due to the nonlinear term $s\phi_x^2$ in (39).
Remark 2. The nonlinear terms in (38,39) turn out to be irrelevant, in the RG sense. However, this is not a simple affair: to show it, one has to take into account the precise decay of ϕ and s both in the fixed and the front frames. This makes the proofs rather complicated.
Remark 3. Finally, let us mention that Eckmann and Wayne [28], using a completely different (and simpler) method, namely coercive functionals, have proven similar results: they can consider a larger space of perturbations (s, ϕ) than the one defined by (52), but they do not obtain explicit upper bounds on the decay in time.

4 Universality in Blow-Up for Nonlinear Heat Equations

4.1 Statement of the problem.

Let us now consider equations for which global existence results do not hold: the solutions of the initial value problem

$$u_t = u_{xx} + u^p \tag{1}$$

where $p > 1, u = u(x,t), x \in \mathbf{R}$, and $u(\cdot, 0) = u_0 \in C^0(\mathbf{R})$, will, for a large class of initial data u_0, diverge in a finite time at a single point (for reviews on this problem, see [47, 55]). Again, we limit ourselves to one space dimension, but the generalization is straightforward.

The RG ideas can be applied to the analysis of the profile of the solution at the time of blow-up. To explain what this means, let us fix the blow-up point to be 0 and the blow-up time to be T. Then, we ask whether it is possible to find a function $f^*(x)$ and a rescaling $g(t,T)$ so that

$$\lim_{t \uparrow T} (T-t)^{\frac{1}{p-1}} \, u(g(t,T)x, t) = f^*(x) \tag{2}$$

Moreover, we want to see how g or f^* depend on the initial data.

The prefactor $(T-t)^{\frac{1}{p-1}}$ in (2) can be understood easily: for initial data $u_0(x)$ constant in x, $u(t)$ solves the ODE $u_t = u^p$, i.e. $u(t) = ((p-1)(T-t))^{\frac{1}{1-p}}$ for $T = (p-1)^{-1} u_0^{1-p}$. In [44, 45, 46, 64] (see also [32, 33]) several possible f^*'s are discussed, and the set of initial data that will lead to a given f^* is partially characterized. In [15], we showed that there exists, in the space of initial data $C^0(\mathbf{R})$, sets $\mathcal{M}_k$ of codimension $2k$, such that, for $u_0 \in \mathcal{M}_k$, the limiting behaviour (2) is obtained, in the case $k=1$, for

$$g(t,T) = ((T-t)|\log(T-t)|)^{\frac{1}{2}} \tag{3}$$

$$f^*(x) = \left(p - 1 + b^* x^2\right)^{\frac{1}{1-p}} \tag{4}$$

where $b^* = \frac{(p-1)^2}{4p}$, and in the case $k > 1$ for

$$g(t,T) = (T-t)^{\frac{1}{2k}} \tag{5}$$

$$f_b^*(x) = \left(p - 1 + b x^{2k}\right)^{\frac{1}{1-p}}. \tag{6}$$

where now b is an arbitary positive number. As shown in [15], one can also add suitable (irrelevant) terms to (1) without affecting the result. It was shown in [44, 46, 64, 33] that, under quite general hypotheses, (3,4) or (5,6) are the only possibilities (see also [65] for a formal analysis). Moreover, solutions that behave as in (5,6) for $k=2$ were constructed in [44].

The codimension of $\mathcal{M}_k$ for $k=1$ is easy to understand: since we have fixed the blow-up point (to zero) and the blow-up time (to T), we have to fix two parameters in the initial data. To reach the other profiles, $2k-2$ additional parameters need to be fixed in the initial data. The $k=1$ situation is therefore the most generic one.

In the RG language, f^* and f_b^* can be viewed as fixed points of a renormalization group transformation having $2k$ unstable ("relevant", in renormalization group terminology) directions. Thus, to converge towards the fixed point, one has to fine-tune $2k$ parameters (one for each unstable direction) and this explains why $\mathcal{M}_k$ is of codimension $2k$, and in what sense f^*, f_b^* are "universal". In addition, we encounter also one neutral ("marginal") mode, which, for $k=1$, turns out to be stable when nonlinear effects are taken into account and for $k>1$ parametrizes a curve of fixed points.

Our results are perturbative, i.e. the sets $\mathcal{M}_k$ consist of initial data that are close to the corresponding fixed point. Therefore, our results are similar to those of Bressan [6] who considers a nonlinearity e^u instead of u^p and obtains the universal profile analogous to our $k=1$ case. The connection of blow-up and center manifold theory was used earlier in the work of Filippas, Kohn and Liu [32, 33] and of Herrero, Velazquez [44, 45, 46, 47, 64] and Galaktionov [65]. Futhermore, the scaling and the dynamical systems aspect of our work goes back to Giga and Kohn [38, 39, 40]. Rescalings as in (7) below were used as a technique for numerical computation in [3].

Let us first describe a change of variables that transforms the problem (1) into a problem of long time asymptotics: we write (1) in the "blow–up–variables": given a $u:\mathbf{R}\times[0,T)\to\mathbf{R}$, define $\phi:\mathbf{R}\times[-\log\ T,\infty)\to\mathbf{R}$ by

$$u(x,t)=(T-t)^{-\frac{1}{p-1}}\phi(\frac{x}{(T-t)^{1/2k}},-\log\ (T-t)). \tag{7}$$

Then u is a classical solution of (1) if and only if $\phi(\xi,\tau)$ is a classical solution of

$$\begin{aligned}\phi_\tau &= L_\tau^{-2}\phi_{\xi\xi}-\frac{1}{2k}\xi\phi_\xi-\frac{1}{p-1}\phi+\phi^p && (8)\\ \phi(\xi,\tau_0) &= T^{\frac{1}{p-1}}u_0(T^{\frac{1}{2k}}\xi) && (9)\end{aligned}$$

where $\tau_0=-\log T$, and

$$L_\tau=e^{\frac{1}{2}\tau(1-1/k)}. \tag{10}$$

We construct in [15] global solutions of (8), with suitable initial data, thereby establishing blow–up for (1). Note that, for $k=1$, the scaling in (7) differs from the one used in (3) by a factor $|\log(T-t)|^{1/2}=\tau^{1/2}$. Actually, the situation where $k>1$ is easier to understand heuristically, so let us start by discussing the latter.

4.2 Analysis of $k>1$.

For $k>1$, as $\tau\to\infty$, the factor L_τ^{-2} in front of the second derivative in (8) leads us to consider the solutions of

$$\phi_\tau=-\frac{1}{2k}\xi\phi_\xi-\frac{1}{p-1}\phi+\phi^p \tag{11}$$

Observe that the "fixed points" $f_b^*(x) = \left(p-1+bx^{2k}\right)^{\frac{1}{1-p}}$ of (6) are stationary solutions of that equation. The latter can of course be integrated in closed form, but before doing that, let us first look at its linearization around the constant solution $\phi = (p-1)^{\frac{1}{1-p}}$. The linear problem is $\phi_\tau = \mathcal{L}_\infty \phi$, where

$$\mathcal{L}_\infty = -\frac{1}{2k}\xi\frac{d}{d\xi} + 1 \ . \tag{12}$$

and so, in the space of polynomials, we have now $2k$ expanding directions corresponding to ξ^n, for $n < 2k$.

Equation (11) is solved by putting $\phi(\xi,\tau) = e^{-\frac{\tau}{p-1}} h(e^{-\tau/2k}\xi, \tau)$ whereby $\frac{d}{d\tau}h(y,\tau) = e^{-\tau}h(y,\tau)^p$ and so, for $\rho = \tau - \tau_0$

$$\phi(\xi,\tau) = \frac{e^{-\frac{\rho}{p-1}} f(e^{-\rho/2k}\xi)}{[1-(p-1)f(e^{-\rho/2k}\xi)^{p-1}(1-e^{-\rho})]^{1/p-1}} \tag{13}$$

where $\phi(\xi,\tau_0) = f(\xi)$. The stationary solutions f_b^* are stable in a suitable codimension $2k$ space: let us consider f smooth, with

$$f(0) = (p-1)^{-\frac{1}{p-1}}, \ f^{(\ell)}(0) = 0 \quad \ell < 2k, \ f^{(2k)}(0) = \beta < 0 \tag{14}$$

and

$$0 \leq f(\xi) < (p-1)^{-\frac{1}{p-1}} \quad \xi \neq 0. \tag{15}$$

Then, for all $\xi \in \mathbf{R}$,

$$|\phi(\xi,\tau) - f_b^*(\xi)| \underset{\tau\to\infty}{\longrightarrow} 0 \tag{16}$$

where

$$f_b^*(\xi) = (p-1+b\xi^{2k})^{-\frac{1}{p-1}} \tag{17}$$

for some b depending on β,k, p.

These considerations thus lead us to expect (8) to have global solutions with initial data in a suitable codimension $2k$ set in a ball around (17) in a suitable Banach space. Of course, the perturbation $L_\tau^{-2}\phi_{\xi\xi}$ in (8) is very singular, but, basically, the picture is not much modified: the unstable modes turn out to be τ-dependent Hermite functions instead of the monomials ξ^n, and one has to fine-tune the projection of the initial data on these Hermite functions instead of imposing the vanishing of the derivatives as in (14). See [15] for details.

4.3 A non-conventional center manifold problem: $k = 1$.

Consider now the case $k = 1$. There are several differences with respect to the previous one: there is no damping factor in front of the second derivative in (8), and the asymptotics is given by (4), with a "universal" b^*, and an extra logarithmic factor in the definition (3) of $g(t,T)$. To understand the dynamics of (8) in that case, let us start by considering again its linearization around the constant solution $\phi = (p-1)^{\frac{1}{1-p}}$. The linear problem is $\phi_\tau = \mathcal{L}\phi$, where now

$$\mathcal{L} = \frac{d^2}{d\xi^2} - \frac{1}{2}\xi\frac{d}{d\xi} + 1 \; . \tag{18}$$

Hence, the first thing we have to do, in order to understand the stability of the constant solution, is to study the spectrum of the linear operator $\mathcal{L}$. $\mathcal{L}$ is self–adjoint on $\mathcal{D}(\mathcal{L}) \subset L^2(\mathbf{R}, d\mu)$ with

$$d\mu(\xi) = \frac{e^{-\xi^2/4}d\xi}{\sqrt{4\pi}} \tag{19}$$

The spectrum of $\mathcal{L}$ is

$$spec(\mathcal{L}) = \{1 - \frac{n}{2} \mid n \in \mathbf{N}\} \tag{20}$$

and we take as eigenfunctions multiples of Hermite polynomials

$$h_n(\xi) = \sum_{m=0}^{[\frac{n}{2}]} \frac{n!}{m!(n-2m)!}(-1)^m \xi^{n-2m} \tag{21}$$

that satisfy

$$\int h_n h_m d\mu = 2^n n! \delta_{nm} \tag{22}$$

and

$$\mathcal{L}h_n = (1 - \frac{n}{2})h_n. \tag{23}$$

Thus the linearization of (8) at the constant solution, for $k = 1$, has two expanding ("relevant") modes, h_0 and h_1, and one neutral ("marginal") one, $h_2 = \xi^2 - 2$.

Our goal is therefore to construct a center manifold for the flow of (8), in a neighbourhood of the fixed point. Formally, we would expand,

$$\phi(\xi,\tau) = (p-1)^{\frac{1}{p-1}} + \psi(\xi,\tau) \;\text{ as }\; \psi = \sum_{n=0}^{\infty} \psi_n(\tau)h_n(\xi), \tag{24}$$

and rewrite (8) for the $\psi_n(\tau)$ as an infinite set of ODE's:

$$\frac{d}{d\tau}\psi_n(\tau) = (1 - \frac{n}{2})\psi_n(\tau) + \text{ nonlinear terms}$$

A formal solution of this flow yields (see below for the calculation): $\psi_2(\tau) \simeq C_p(\log\tau)\tau^{-1}$ with $C_p = \frac{-(p-1)^{\frac{1}{1-p}}}{4p}$ as in (4).

However, there are severe problems with this approach. Since the eigenfunction h_n of the linearization $\mathcal{L}$ *increase* at infinity, the expansion (24) is not useful for ξ large, and, in particular, we cannot use any standard infinite dimensional center manifold theorem. The key to the solution to this problem comes again from a scaling argument, which will explain the emergence of the fixed point f^*. Let

$$\phi_L(\xi,\tau) = \phi(L\xi, L^2\tau) \tag{25}$$

Then, ϕ_L satisfies the equation

$$H(\phi_L) = L^{-2}(-\phi_{L\tau} + \phi_{L\xi\xi}),$$

where we defined

$$H(\phi) = \frac{1}{2}\xi\phi_\xi + \frac{1}{p-1}\phi - \phi^p \; . \tag{26}$$

Hence, as $L \to \infty$, we expect the solutions of $H(\phi) = 0$ to be relevant. These are (like the stationary solutions of (11)) given by the one-parameter family f_b^*, given by (6), with $k = 1$. Therefore, we have the following picture: instead of perturbing around the constant solution, introduce $\phi_b(\xi,\tau) = (p-1+b\xi^2/\tau)^{\frac{1}{p-1}}$, and write

$$\phi(\xi,\tau) = \phi_b(\xi,\tau) + \eta(\xi,\tau). \tag{27}$$

A local (i.e. for ξ small) center manifold analysis then fixes $b = b^*$, as in (4). For large ξ, namely $\frac{\xi^2}{\tau} > \mathcal{O}(1)$, the linearization $\tilde{\mathcal{L}}$ of (8) around ϕ_b differs from $\mathcal{L}$, and, actually, in that region (see below) $\tilde{\mathcal{L}} \simeq \mathcal{L} - 2$. By (20), the spectrum of $\mathcal{L} - 2$ is entirely negative. Hence, the dynamics tends to *contract* η.

We will now explain the calculation that yields b^* in (4). For simplicity of notations, we shall consider only $p = 2$, i.e.

$$\phi_b(\xi,\tau) = (1 + b\xi^2/\tau)^{-1}. \tag{28}$$

We get, using $H(\phi_b) = 0$,

$$\begin{aligned} \eta_\tau &= \eta_{\xi\xi} - H(\phi_b+\eta) + H(\phi_b) + \phi_{b\xi\xi} - \phi_{b\tau} \\ &= (\mathcal{L} + W)\eta + M(\eta) + \phi_{b\xi\xi} - \phi_{b\tau} \end{aligned} \tag{29}$$

where we introduce

$$W = 2(\phi_b - 1) \tag{30}$$
$$M(\eta) = (\phi_b+\eta)^2 - \phi_b^2 - 2\phi_b\eta. \tag{31}$$

The operator $\mathcal{L}$, given by (18), has two unstable modes. Note that, formally, (i.e., for ξ of order one) W is $\mathcal{O}(\tau^{-1})$, M is nonlinear in η and $\phi_{b\xi\xi} - \phi_{b\tau}$ is $\mathcal{O}(\tau^{-1})$. We want to construct a center manifold for (29), i.e. to see how to fix two parameters in the initial data of η, such that the flow of (29) drives η to zero. A simple calculation will show that this can only be achieved through a suitable choice of b in (28).

To see this, let us simplify further and consider η even in ξ, which will imply that we need to fix only one parameter. This example contains all the relevant features of the general case. It is convenient to write

$$\eta_0(\tau) = \frac{a}{\tau} \tag{32}$$

and define ψ by

$$\eta = \eta_0 + \psi.$$

Then ψ satisfies the equation

$$\psi_\tau = \tilde{\mathcal{L}}\psi + N(\psi) + \alpha \tag{33}$$

with

$$\begin{aligned}
\tilde{\mathcal{L}} &= \mathcal{L} + V \\
V &= 2(\phi_b + \eta_0 - 1) && (34)\\
N(\psi) &= (\phi_b + \eta_0 + \psi)^2 - (\phi_b + \eta_0)^2 - 2(\phi_b + \eta_0)\psi && (35)\\
\alpha &= \phi_{b\xi\xi} - \phi_{b\tau} + (\mathcal{L} + W)\eta_0 - \eta_{0\tau} + M(\eta_0) \\
&= \phi_{b\xi\xi} - \phi_{b\tau} + \eta_0 + W\eta_0 - \eta_{0\tau} + M(\eta_0). && (36)
\end{aligned}$$

Let us decompose ψ (which is even in η) as

$$\psi = \psi_0(\tau) + \psi_2(\tau)h_2 + \psi^\perp \tag{37}$$

where $\psi^\perp$ is orthogonal to $h_n,\ n \leq 2$. Next we expand V and α (for $\xi = \mathcal{O}(1)$):

$$\begin{aligned}
V &= -\frac{2b\xi^2}{\tau} + \frac{2a}{\tau} + \mathcal{O}(\frac{1}{\tau^2}) && (38)\\
\alpha &= (a - 2b)\tau^{-1} + (a + a^2 + (12b^2 - b - 2ab)\xi^2))\tau^{-2} + \mathcal{O}(\tau^{-3}). && (39)
\end{aligned}$$

Inserting (32), (37) in (33) and retaining only the leading terms in $1/\tau$ and $\psi_i,\ i = 0, 2$, we get from $\psi_{i\tau} = (2^i i!)^{-1}(h_i, \psi_\tau)$ ($(\cdot, \cdot)$ is the scalar product of $L^2(\mathbf{R}, d\mu)$ and we use (22)):

$$\begin{aligned}
\psi_{0\tau} &= \psi_0 + (a - 2b)\tau^{-1} + R_0 && (40)\\
\psi_{2\tau} &= \beta\tau^{-1}\psi_2 + (12b^2 - b - 2ab)\tau^{-2} + R_2 && (41)
\end{aligned}$$

where $R_0 = \mathcal{O}(\tau^{-2} + \tau^{-1}|\psi| + |\psi|^2), R_2 = \mathcal{O}(\tau^{-3} + \tau^{-1}|\psi_0| + \tau^{-2}|\psi_2| + |\psi|^2)$,and $\beta = 2a - \frac{1}{4}b(\xi^2 h_2, h_2) = 2a - 20b$ (coming from the $V\psi$ term in (33)). We choose now a so that the $\mathcal{O}(\tau^{-1})$ term in $\psi_{0\tau}$ vanishes i.e.

$$a = 2b \tag{42}$$

and b such that the $\mathcal{O}(\tau^{-2})$ term in $\psi_{2\tau}$ is zero:

$$b = b^* = 1/8. \tag{43}$$

Note that this choice correspond to $b = b^*$ in (4) for $p = 2$. Then $\beta = -2$ and our equations read

$$\psi_{0\tau} = \psi_0 + R_0, \quad \psi_{2\tau} = -\frac{2}{\tau}\psi_2 + R_2. \tag{44}$$

Now, keeping in mind the presence of the R_0, R_2 terms,

$$\psi_0 = \mathcal{O}(\tau^{-2}), \quad \psi_2 = \mathcal{O}((\log\ \tau)\tau^{-2}) \tag{45}$$

would be consistent solutions. Of course, we need to show that the expanding variable ψ_0 will satisfy (45) by a suitable choice of $\psi_0(\tau_0)$. This is rather easy to do, using the fact that ψ_0 is expanding; in the general case (with more than one parameter to fix), we used a topological argument.

In the rigorous proof, we set up a suitable Banach space for the function η, parametrized by the ψ_i's for $\xi^2/\tau < \mathcal{O}(1)$, and a function η_l for $\xi^2/\tau > \mathcal{O}(1)$. The function η_l will

contract under the action of $\tilde{\mathcal{L}}$: indeed, from (34,28,32), we see that the potential V tends to -2 as ξ^2/τ (and τ) $\to \infty$.

5 Open Problems

There are several patterns and fronts in dissipative equations, and their stability can probably be studied using RG methods. For example, the Cahn-Hillard equation [21]:

$$u_t = \Delta(-\Delta u - u - u^3) \tag{1}$$

is often used to study the phase separation in alloys and fluids. One would like to study the stability (and possibly the dynamics), in infinite volume, of interface solutions of that equation.

Another major open problem consists in the extension of the RG method to hyperbolic equations. The latter have their own scaling laws, see [55, 62]. A lot is known on the stability of soliton solutions of (generalized) KdV equations [58], but the stability of localized solutions of other nonparabolic equations is quite open. Also, there are open problem concerning the blow-up of solutions, most notably in the nonlinear Schrödinger equation [54].

So far, we have only considered equations whose solutions have a rather simple asymptotic behaviour. Of course, it is well known that finite dimensional dynamical systems described by differential equations can have a chaotic asymptotic behaviour, i.e. depend sensitively on the initial data but have also some statistical regularity in the sense that the long time average along the orbits is described by an invariant (SRB) measure [27]. For certain classes of F in (1.1), one would like to find natural invariant measures for the flow.

A class of dynamical systems, modeling PDE's, is obtained by discretizing space and time and considering a recursion

$$u(x, t+1) = F(x, u(\cdot, t)) \tag{2}$$

i.e. $u(x, t+1)$, with x being a site of a lattice, is determined by the values taken by u at time t (usually on the sites in a neighbourhood of x). For a suitable class of F's such dynamical systems are called Coupled Map Lattices (CML) [51, 52]. The study of the invariant measures, and their properties, even for these "toy models" is essentially a terra incognita. In [20, 59, 17] various "high-temperatures" (weak coupling) results are obtained. Almost nothing is known about the "low temperatures" (strong coupling) side (see [18, 19, 57]). One would like to find models for which the set of invariant measures changes as the coupling is varied; this phenomenon would then be interpreted as a kind of nonequilibrium phase transition.

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