PERCOLATION TECHNIQUES IN DISORDERED SPIN SYSTEMS: THE UNIQUENESS REGIME
I. STATICS
II. DYNAMICS

G. Gielis[1] and C. Maes[2]

Instituut voor Theoretische Fysica
K.U. Leuven
Celestijnenlaan 200D
B–3001 Leuven, Belgium

Abstract

We consider lattice spin systems with short range but random and unbounded interactions.
Statics : We give an elementary proof of uniqueness of Gibbs measures at high temperature or strong magnetic fields, and of the exponential decay of the corresponding quenched correlation functions. The analysis is based on the study of disagreement percolation (as initiated in van den Berg–Maes (1994)).
Dynamics : We give criteria for ergodicity of spin flip dynamics and estimate the speed of convergence to the unique invariant measure. We find for this convergence a stretched exponential in time for a class of "directed" dynamics (such as in the disordered Toom or Stavskaya model). For the general case, we show that the relaxation is faster than any power in time. No assumptions of reversibility are made.
The methods are based on relating the problem to an oriented percolation

[1]I.I.K.W Onderzoeker Belgium. e–mail: Guy.Gielis@fys.kuleuven.ac.be

[2]Onderzoeksleider N.F.W.O. Belgium. e–mail: Christian.Maes@fys.kuleuven.ac.be

problem (contact process) and (for the general case) using a slightly modified version of the multiscale analysis of e.g. Klein (1993).
Keywords : quenched disorder, spin glasses, disagreement percolation, spin flip dynamics, Gibbs measures, Griffiths'singularities, multiscale analysis, ergodicity.

1 Introduction

Adding disorder to a system of many interacting particles may in general be a highly non–trivial perturbation. The study of spin glasses and spin flip systems with quenched disorder is certainly not so well developed as their corresponding versions without disorder. In this paper we show how percolation techniques can be useful for investigating that part of the phase diagram in which the disordered system typically forgets about initial data or about boundary conditions. This behavior appears in the high noise, high temperature or low density limit in a variety of interacting particle systems.

Our analysis concerns two types of classical problems in the study of disordered systems (topics (B) and (C) in the review of Fröhlich (1986)): one is in the context of equilibrium and the other in the theory of stochastic spin flip dynamics. Our analysis is however restricted to the uniqueness regime (high temperature, strong magnetic field, high noise, strong bias,...) and the main problem consists therefore in circumventing the static and dynamical consequences of the so–called Griffiths' singularities, see Griffiths (1969).
For the equilibrium part we give conditions assuring the uniqueness of Gibbs fields with a random potential and the exponential decay of the corresponding quenched correlation functions.
For the non–equilibrium part we investigate the relaxation properties of spin flip dynamics with random transition rates.

Both questions have of course been considered before. For the case of random Gibbs fields, there are the standard works of Olivieri *et al* (1983), Berretti (1985), Fröhlich–Imbrie (1984) and Bassalygo–Dobrushin (1986).
Moreover, more recently there came out a series of papers containing a simple and detailed analysis of the uniqueness regime of short range spin glasses in equilibrium, cf. Perez (1993), Klein (1994) and von Dreifus *et al* (1994).
In this paper we continue along the lines of Bassalygo–Dobrushin (1986).

Their formulation is closely related to a bond percolation problem. We use disagreement site percolation as in van den Berg–Maes (1994).
For Glauber dynamics, Zegarlinski (1994) recently studied the problem in one dimension. We study more general models, making the correspondence with (temporal-) oriented percolation. For an important subclass of models the problem is further reduced to the study of space–time oriented percolation. No assumptions of reversibility are made. An important ingredient already appeared in the work of Campanino–Klein (1991), Klein (1993) and Klein (1994).

Our results can be summarized as follows : the influence of the boundary conditions on Gibbs states with a random potential can be analyzed via a corresponding percolation problem with random densities. The uniqueness conditions for Gibbs states are stated in terms of absence of percolation. Truncated correlation functions are bounded by connectivity functions for independent site percolation. The arguments are direct applications of the van den Berg–Maes (1994) results. For the non–equilibrium case we describe the influence of the initial data on the asymptotic state via oriented percolation (contact process) in a random environment. We give explicit criteria under which ergodic behavior typically results and obtain bounds for the rate of convergence to the unique stationary measures. For the subclass of dynamics in which the spatial dependence of the transition rates is "directed", we get a stretched exponential for the decay in time. For the more general case, we only get a decay faster than any power of time.

The outline of this paper is as follows : Section 2 contains general definitions. In Section 3 the equilibrium problem is investigated. Section 4 is devoted to the anologous problem for both discrete time and continuous time spin flip dynamics. In the Appendix we collect the more technical arguments and modifications with respect to the multiscale analysis of Klein (1993).

2 Definitions

For convenience we consider spin systems defined on the regular d-dimensional lattice $\mathbb{Z}^d$. As will become clear, the arguments that follow are valid on a more general periodic lattice (e.g. the triangular or the FCC lattice) . $\mathbb{Z}^d$ comes equipped with the usual structure of nearest neighbor sites x, y connected by bonds $\langle x, y\rangle$. If two sites x and y are nearest neighbors (or adjacent) we will write $x \sim y$.

A configuration σ puts a spin value $\sigma(x) = 1$ or $\sigma(x) = -1$ on every site $x \in \mathbb{Z}^d$. $\Omega = \{-1, +1\}^{\mathbb{Z}^d}$ is the set of all configurations. Our results can easily be extended to other finite single site state spaces.

One typically considers two types of quenched disorder. One is realized by the nearest neighbor couplings $\{J_{xy}\}_{x\sim y}$ and the other by a set of single site parameters $\{h_x\}$ (also denoted below by $\{\gamma_x\}$ if not referring to a random magnetic field). We assume that the $\{J_{xy}\}$ are real (possibly infinite) valued mutually independent and identically distributed random variables. Similarly for the $\{h_x\}(\{\gamma_x\})$. Examples will follow where these parameters enter explicitly. Sometimes it is however more convenient to speak about "realizations" in general without specifying exactly how the disorder is frozen in in the interactions or transition rates. Indeed, the relevant objects for our analysis are the (random) specifications and/or transition kernels (see below) and we do not need to refer to specific forms of the interaction. We therefore write π to denote such a general (random) realization (of the disorder).
Π is the set of all these realizations. $\mathbf{Q}$ is the probability law on the realizations. $\mathbf{E}$ the expectation value with respect to the distribution $\mathbf{Q}$.

Independent percolation plays a crucial role in our construction. We consider site percolation on $\mathbb{Z}^d$. One independently assigns to every site a value 0 or 1 with a certain density. A site with value one is called open. An open path is a sequence of neighboring open sites. We say that site percolation occurs if there is a positive probability for the event that there exists an infinite open path.

In (time) oriented percolation, only paths with decreasing time coordinate on the space–time lattice are considered. For more details and definitions we refer to e.g. Grimmett (1989).

In all cases, the densities at each site may (and frequently will) differ, depending on the (random) realization.

3 Equilibrium States with Random Potentials

A probability measure ν on Ω is a Markov field if for every finite $A \subset \mathbb{Z}^d, \eta \in \{-1,1\}^A$,

$$\nu[\sigma = \eta \text{ on } A|\sigma(i), i \in A^c] = \nu[\sigma = \eta \text{ on } A|\sigma(i), i \in \partial A], \tag{1}$$

where $A^c = \mathbb{Z}^d \setminus A$ and ∂A is the set of all sites in A^c that are adjacent to A. A major problem in statistical mechanics is to determine the Markov fields ν which satisfy for all finite $A \subset \mathbb{Z}^d$, all $\eta \in \{-1,1\}^A$, $\eta' \in \{-1,1\}^{\partial A}$

$$\nu[\sigma = \eta|\sigma = \eta' \text{on } \partial A] = Y_A(\eta, \eta'), \tag{2}$$

with

$$\{Y_A(\cdot, \eta'), A \subset \mathbb{Z}^d, \text{ finite}, \eta' \in \{-1,1\}^{\partial A}\} \tag{3}$$

a given set of self–consistent conditional probabilities (a specification) possibly parametrized (among other things) by the inverse temperature $\beta \geq 0$, external fields, etc. (see Georgii (1988)). In that case we say that ν is a Gibbs measure with respect to the specification $\{Y_A\}$. We look for conditions on the set $\{Y_A(\cdot, \eta)\}$ such that there exists just one associated Gibbs measure. What is specific to our study here is that the specification is *random*, i.e. the

$Y_A = Y_A^\pi$ depend not only on the values of certain fixed parameters but also on the realization π of the randomness.

We do not wish to formalize here the notion of specification–valued random field. The reader who is bothered by this is referred to the examples.

An important example is the following random field short range spin glass with formal Hamiltonian:

$$H = -\sum_{\langle x,y\rangle} J_{xy}\sigma(x)\sigma(y) - b\sum_x h_x\sigma(x) - h\sum_x \sigma(x), \tag{4}$$

determined by a realization of one–(h_x) and two–point interactions (J_{xy}). The specification $\{Y_A\}$ is obtained by taking the finite volume Gibbs measures (fixed boundary conditions outside A) with respect to the Hamiltonian H at inverse temperature β. For $A = \{x\}$ we then have, (with some abuse of notations)

$$Y_x(\sigma(x),\sigma) = \frac{\exp[\beta\sum_{y\sim x} J_{xy}\sigma(x)\sigma(y) + (\beta b h_x + \beta h)\sigma(x)]}{Z_x^\beta(\{J_{xy},\sigma(y)\}_{y\sim x}, h, h_x, b)}. \tag{5}$$

The construction of a Gibbs measure ν_π with respect to the specification $\{Y_A = Y_A^\pi\}$ will obviously depend on the realization π. The uniqueness of the Gibbs measure should be understood in the sense that with $\mathbf{Q}$-probability one there is just one such Gibbs measure. For these equilibrium measures, we define the truncated correlation function of the local functions f and g on Ω as

$$\langle f;g\rangle_\pi = \nu_\pi(fg) - \nu_\pi(f)\nu_\pi(g). \tag{6}$$

Bassalygo–Dobrushin (1986) give a "simple" proof for the uniqueness of the Gibbs state for the specification (5) with random unbounded interactions at sufficiently high temperatures. A key idea is to reduce the problem to that of the uniqueness of a Gibbs state with uniformly small pair interactions on an associated "arbitrary" locally finite graph. We do not make

this correspondence here but investigate how far we can take quite similar stochastic–geometric arguments to include random interactions.
von Dreifus *et al* (1994) carry out an analysis which is also inspired by Bassalygo–Dobrushin (1986) and combine it with expansions on duplicated systems (similar to the objects considered in Fröhlich-Imbrie (1984)). These expansions very much resemble Hammersley sums for independent percolation but the relation between the two problems stops there because the "densities" in these sums can take values larger than one.
In what follows we get a direct and explicit relation with an independent percolation process, called disagreement percolation in van den Berg–Maes (1994), from which all the results follow almost for free.
Let

$$q_x = \max_{\eta,\eta'} \mathrm{var}(Y_x(\cdot,\eta), Y_x(\cdot,\eta')), \tag{7}$$

where $\mathrm{var}(\cdot,\cdot)$ $(\in [0,1])$ is the variational distance.

Everything that follows is expressed in terms of the distribution of the field $\{q_x\}$. Remember that the q_x depend on the realization π and on extra parameters (such as the temperature and external fields) as inherited from the specification. So instead of referring to the high temperature or strong external field regimes separately, the single phase regime of our disordered system will be obtained if "typically" the q_x are "small" for all x in a sufficiently "big" set. Using Definition (7) one can then for every specific model get explicit conditions on the realizations and the external parameters.
Important to observe is that q_x and q_y may be correlated for $x \neq y$. However, in all relevant examples the randomness in the specification enters locally and has a high degree of independence. While the arguments that follow essentially go through unchanged under the assumption that there is a finite "distance" R for which q_x and q_y are independent whenever the "distance"

between x and y exceeds R, for simplicitly we require that this already happens for $R = 1$, i.e. q_x and q_y may be correlated for $x \neq y$ only if $x \sim y$, otherwise they are independent; $\{q_x\}$ is a one–dependent random field. This is verified in all examples discussed here.

Another feature present in all our examples of interest is that, with **Q**-probability one, there are finite regions of all sizes on which q_x is large. These regions are responsible for the so called Griffiths' singularities, Griffiths (1969).

Example 1 (spin glass)

In a spin glass, the realization π consists of a collection of real independent and identically distributed variables J_{xy} associated to the bonds $\langle x, y\rangle$.

$$Y_x(\sigma(x), \sigma) = \frac{\exp[\beta \sum_{y\sim x} J_{xy}\sigma(x)\sigma(y) + \beta h\sigma(x)]}{Z_\pi^\beta(\{J_{xy}\sigma(y)\}_{y\sim x}, h)}, \tag{8}$$

for $h > 0$ an externally applied magnetic field. Then

$$q_x = 1/2[\tanh(\beta \sum_{y\sim x} |J_{xy}| + \beta h) + \tanh(\beta \sum_{y\sim x} |J_{xy}| - \beta h)]. \tag{9}$$

Note that q_x can be made "small" both by taking $\beta > 0$ small or by taking h large. Note also that a realization where $J_{xy} = \infty$ for some bond $\langle x, y\rangle$ causes $q_x = q_y = 1$ for all values of $\beta, h > 0$. Moreover, with **Q**-probability one, $\sup_x q_x = 1$ because the interaction couplings are unbounded.

Example 2 (random field Ising model)

For the Hamiltonian (4) with $0 \leq J_{xy} = J < \infty$ fixed and $h = 0$ we have

$$q_x = \frac{1}{2}[\tanh\beta(2dJ + bh_x) + \tanh\beta(2dJ - bh_x)]. \tag{10}$$

q_x can now also be made small by letting b grow, except when the realization puts $h_x = 0$ because then $q_x = \tanh(2d\beta J)$, independent of b.

Example 3 (hard core lattice gas)
Here a realization is a random choice of activities $a_x = e^{\lambda\gamma_x} - 1$ with $\gamma_x \geq 0$ random and $\lambda \geq 0$ an extra parameter.

$$Y_x(1,\eta) = \begin{cases} 1 - e^{-\lambda\gamma_x} & \text{if} \quad \text{for all } y \sim x \ \eta(y) = -1 \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Then,

$$q_x = 1 - e^{-\lambda\gamma_x}. \tag{12}$$

will be "small" for λ large except in a realization where $\gamma_x = \infty$.

The idea is now to use the $\{q_x\}$ as densities for independent site percolation on $\mathbb{Z}^d$ (independently a site x is open with probability q_x, and is closed with probability $1 - q_x$). Denote by $G_\pi(x, y)$ the probability to find an open path from x to y if the realization is π. Next Proposition 1 is a direct consequence of van den Berg–Maes (1994) and it is the first key-ingredient for the main result of this section.
Following notations are used. The distance $\text{dist}(f, g)$ between the local functions f and g on Ω is

$$\text{dist}(f, g) = \min_{\substack{x \in \text{supp} f \\ x \in \text{supp} g}} |x - y|, \tag{13}$$

with $\text{supp} f$ the support of f and

$$|x - y| = \inf\{|\omega|, \omega \text{ a path connecting } x \text{ and } y\}. \tag{14}$$

Definition (14) has a somewhat more general flavor than needed on $\mathbb{Z}^d$, but is useful when considering other lattices. Further we denote by $||f||$ the usual supremum norm of f and

$$\delta_x = \sup_\eta |f(\eta^x) - f(\eta)| \tag{15}$$

the oscillation of f at x. The total oscillation is then

$$|||f||| = \sum_{x\in \mathbb{Z}^d} \delta_x f. \tag{16}$$

Proposition 1 *Absense of independent site percolation with densities* $\{q_x\}_{x\in \mathbb{Z}^d}$ *implies the uniqueness of the Gibbs state for the specifications used in the definition of the* $\{q_x\}_{x\in \mathbb{Z}^d}$.
Moreover, the truncated correlation function in that Gibbs state obey the bound

$$|\langle f, g\rangle_\pi| \leq ||f||\ |||g|||\max_{x\in \mathrm{supp} g} \sum_{y\in \mathrm{supp} f} G_\pi(x,y) \tag{17}$$

for f *and* g *local functions.*

Proof The uniqueness of the Gibbs state follows directly from Corollary 2 in van den Berg–Maes (1994). Let ν_π be the unique Gibbs measure. The truncated correlation function can be estimated as follows.

$$\begin{aligned} |\langle f; g\rangle_\pi| &= |\textstyle\int \nu_\pi(d\sigma)(\nu_\pi(fg|\sigma \text{ on supp} f)) - \nu_\pi(f)\nu_\pi(g)| \\ &= |\textstyle\int \nu_\pi(d\sigma) f(\sigma)(\nu_\pi(g|\sigma \text{ on supp} f) - \nu_\pi(g))| \\ &\leq ||f|| \max_{\eta,\eta'} |\nu_\pi(g|\sigma = \eta \text{ on supp} f) \\ &\qquad\qquad -\nu_\pi(g|\sigma = \eta' \text{ on supp} f)|. \end{aligned} \tag{18}$$

Using Corollary 1 in van den Berg–Maes (1994) we get that

$$\begin{aligned} |\langle f; g\rangle_\pi| &\leq ||f|| \sum_{\substack{x \\ y\in \mathrm{supp} f}} \delta_x g\ G_\pi(x,y) \\ &\leq ||f||\ |||g||| \max_{x\in \mathrm{supp} g} \sum_{y\in \mathrm{supp} f} G_\pi(x,y). \end{aligned} \tag{19}$$

This proves the Proposition.

∎

¿From Proposition 1 we learn that the structure of the unique Gibbs measure will inherit all the nice proporties of the connectivity function of the associated percolation process. We must therefore study the latter more closely.

Theorem 1 *If* $\{q_x\}_{x\in \mathbb{Z}^d}$, *as defined by (7), is a stationary one–dependent random field verifying*

$$\mathbf{E}(q_x) < \frac{1}{(2d-1)^2} \tag{20}$$

then—with $\mathbf{Q}$*-probability one—there is a unique Gibbs measure* ν_π.
Moreover,

$$\mathbf{E}(|\langle f; g\rangle_\pi|) \leq C(f,g)\mathrm{e}^{-m \ \mathrm{dist}(f,g)} \tag{21}$$

for all local functions f *and* g, *with* $m > 0$ *and* $C(f,g) = C\|f\| \ |||g||| < \infty$.

Proof For any path ω, we denote with ω_i the i-th site in the path.
The Theorem follows from Proposition 1 and the bound

$$G_\pi(x,y) \leq \sum_{n\geq|x-y|} \sum_{|\omega|=n} \prod_{i \ \mathrm{even}} q_{\omega_i}, \tag{22}$$

with the second sum running over all the self–avoiding walks $\omega, |\omega| = n$ from x to y. Taking the expectation of both sides of (22) and using the independence of $\{q_x\}_{\{x \ \mathrm{even}\}}$ completes the proof. ∎

Remarks

1. From the examples it is clear that it is not always possible to tune an external parameter to make q_x (pointwise) arbitrarily small on any site x. For instance in Example 1 (3) always $q_x = 1$ if $J_{xy} = \infty(\gamma_x = \infty)$ or in Example 2 $q_x = \tanh 2d\beta J$ when $h_x = 0$, independent of b. When this happens we call a site x "bad". Therefore, a necessary condition for the assumption of Theorem 1 to be verified is that the $\mathbf{Q}$–probability of a site to be "bad" is itself small enough. In a way this condition is also sufficient: see e.g. Example 1, if $\mathbf{Q}\{J_{xy} = \infty\} < 1/(2d-1)^2$, then for $\beta > 0$ sufficiently small condition (20) is satisfied.

2. Theorem 1 does not give the best possible bound on the smallness of $\mathbf{E}(q_x)$. Depending on specific models and using the main underlying idea (as from Proposition 1) we can improve substantially on this bound. E.g. for example 2 with $\{h_x\}$ an independent identically distributed field, when $\mathbf{E}(q_x) < p_c(\mathbb{Z}^d)$ the threshold for independent site percolation on $\mathbb{Z}^d$, then the same conclusions as in Theorem 1 hold.

3. We did not exhaust all possible conclusions that can drawn under the assumptions of Theorem 1. Every property of the connectivity function $G_\pi(x, y)$ could in principle be carried over to the level of the unique Gibbs measure. von Dreifus *et al* (1994) show that for the case of the Hamiltonian (4), the exponential decay of the truncated correlation functions (as in (21)) implies the existence of thermodynamical limits and the infinite differentiability of the correlation functions with respect to the external magnetic field h.

4. Finally, we want to mention that Proposition 1 and 2 can also be proven for a more general lattice Γ. Also lattices without a bipartite structure, but with a bounded number (N) of nearest neighbors can be considered. Then in every set of M sites we can find at least $M/(N+1)$ non–neighboring sites, where the q_x are independent.

4 Dynamics with random transition rates

4.1 Introduction

In this Section we study the influence of disorder on the ergodic regime of spin flip dynamics. More specifically, we derive conditions implying the uniqueness of the invariant measure and we estimate the speed at which the system reaches this state. Because this invariant measure depends on the realization, we are interested in statements that are true with probability one (with respect to the distribution of these realizations).

As far as we know, there exist very few rigorous results concerning the ergodicity of dynamics governed by random parameters. Mostly they concern Glauber (or heat–bath) dynamics for spin glasses with nearest neighbor interactions. For one–dimensional spin glasses with unbounded pair interactions, Zegarlinski (1994) recently found a convergence to equilibrium faster than $c \exp -t^\delta$ with $\delta \in (0,1)$ and c dependent on the realization. Moreover he argues that for any dimension this is the best possible result : the dynamics has no spectral gap. This assertion is confirmed by the extensive computer simulations of spin glasses by Ogielski (1985). He observed the time dependent correlation function $q(t) = \mathbf{E}(\langle \sigma_x(0), \sigma_x(t) \rangle)$ for large systems during long observation times and also found a stretched exponential decay. There, $q(t)$ is calculated in thermal equilibrium and is an average over all realizations. Randeria *et al.* (1985) argued for a lower bound for this same correlation function, by considering the contribution to the relaxation process of independent tightly coupled and unfrustrated clusters. The energy needed to flip a cluster with diameter L typically goes as $\sim \sigma L^{d-1}$, with σ the surface tension. The relaxation time thus goes as $\exp(-c\sigma L^{d-1})$. Using this and the fact that the probability to find such a cluster is $\sim \exp(-\lambda L^d)$ they get $q(t) > \exp -c(\log t)^{\frac{d}{d-1}}$. This does not agree with the simulations of Ogielski (1985). Palmer *et al.* (1984) introduced different degrees of freedom to elucidate the relaxation in strongly interacting glassy materials. They propose a serial relaxation process, slower modes are constrained by the faster ones : the former can't decay before the latter are finished. The formula they obtain for $q(t)$ contains two parameters that can be chosen so that a stretched exponential decay appears.

Here we do not restrict ourselves to *reversible* continuous time spin glasses. Also other continuous and discrete time spin flip dynamics are regarded. The common characteristic is a short range interaction governed by random and possibly unbounded parameters. A consequence of such disorder is that there will typically be large regions on which the spins are strongly interacting. This is completely analogous to the situation in equilibrium. What is far worse here however, is that these "bad" regions should be thought of as infinitely extending in the time direction, i.e., spins there will relax only very slowly in the course of time. Depending on the size and the "badness" of these regions, the relaxation time may become arbitrarily large. Therefore we cannot expect to see *typically* an exponential decay to the invariant measure. One could view this as a dynamical consequence of the Griffiths' singularities. Our main goal for the rest of the paper is to see what to do about this.

4.2 Disordered Probabilistic Cellular Automata

A Probabilistic Cellular Automaton (PCA) is a parallel updating discrete time evolution $\sigma_n, n = 0, 1, \ldots$ on Ω. σ_n is a Markov process defined by transition probabilities $p_x(\pm 1, \eta)$ and for every $\Lambda \in \mathbb{Z}^d$

$$\text{Prob}[\sigma_n(x) = \xi(x), \forall x \in \Lambda | \sigma_{n-1} = \eta] = \prod_{x \in \Lambda} p_x(\xi(x)|\eta) \tag{23}$$

for $\xi, \eta \in \Omega$.

For simplicity we assume that the $p_x(\cdot|\eta)$ depend only on the nearest neighbor spins $\eta(y), y \sim x$ and on $\eta(x)$. While the dynamics is time homogeneous we add quenched disorder by letting the transition probabilities $\{p_x\}$ depend on the realization π having the same nature as in the previous Section. It may for example enter in the coupling between $x \sim y$ or in the bias (see the examples below). As before there will typically be extra parameters (such as the noise level or temperature) available to modify the behavior of our PCA for the same distribution of realizations.

We use P_π to denote the transfer operator of the PCA and $P_\pi^N = P_\pi^{N-1} P_\pi$. A probability measure ν_π on Ω is invariant if $\nu_\pi P_\pi = \nu_\pi$. The PCA is ergodic if there exists a unique invariant measure ν_π, such that for all probability measures μ on Ω

$$\mu P_\pi^N \longrightarrow \nu_\pi. \tag{24}$$

See Lebowitz *et al* (1990) for more details.

The PCA evolution is of course not deterministic because P_π applied to a configuration $\eta \in \Omega$ (the delta measure concentrated at η) gives the product measure with densities $\{p_x(.|\eta)\}$. Our object of study is therefore a stochastic dynamics in which the degree and/or nature of the stochasticity (noise,...) itself is randomly determined (via the quenched disorder realized by π).

Define

$$k_x \equiv \max_{\eta,\eta'} \text{var}(p_x(.|\eta), p_x(.|\eta')). \tag{25}$$

k_x is a function of the realization π but we assume that π only enters locally: $\{k_x\}$ is a one–dependent random field. In particular, the k_x, x in the even (odd) sublattice of $\mathbb{Z}^d$, are jointly independent. At the same time there typically will be large regions on which the k_x are large (close to one). This is similar to what happens to the $\{q_x\}$ in the equilibrium case (see (5)) but, as remarked before, these regions are copied at every time step and therefore give rise to infinite "cylinders" on the space–time lattice. This is the reason why, in contrary to the equilibrium case, we cannot allow a k_x to be one with positive probability.

Example 4 (Discrete time spin glass)
This is the discrete time (non–equilibrium) analogue of Example 1 ($h = 0$). With $J_{xy} = J_{yx}$ for $x \sim y$ an independent family of random couplings, the transition probability is

$$p_x(\sigma(x)|\eta) = \frac{1}{2}[1 + \sigma(x)\tanh(\beta \sum_{y \sim x} J_{xy}\eta(y)]. \tag{26}$$

Here,

$$k_x = \tanh\beta \sum_{y \sim x} |J_{xy}|. \tag{27}$$

Example 5 (A random version of Stavskaya's PCA)
The $\{\gamma_x\}$ are independent and identically distributed non–negative random variables. The transition probability is

$$p_x(+1|\eta) = \begin{cases} 1 & \text{if } \eta(x) = \eta(x+1) = +1 \\ \mathrm{e}^{-\lambda\gamma_x} & \text{otherwise} \end{cases} \tag{28}$$

In the case $\gamma_x = \gamma$ with γ large, the PCA has more than one invariant measure. Here,

$$k_x = 1 - \mathrm{e}^{-\lambda\gamma_x}. \tag{29}$$

The basic coupling of a PCA with itself is a new Markov process (a new PCA) (σ_N, σ'_N) on the product space $\Omega \times \Omega$, whose transition probabilities satisfy

$$\mathrm{Prob}_\pi[\sigma_N(x) \neq \sigma'_N(x) | \sigma_{N-1} = \eta, \sigma'_{N-1} = \eta'] = \mathrm{var}(p_x(.|\eta), p_x(.|\eta')) \leq k_x \tag{30}$$

The basic coupling has (as every other coupling) the property that

$$\sup_{\sigma,\sigma'} |P_\pi^N f(\sigma) - P_\pi^N f(\sigma')| \leq \sum_x \delta_x f \mathrm{Prob}_\pi[\sigma_N(x) \neq \sigma'{}_N(x) | \sigma, \sigma'] \tag{31}$$

What is however typical about the basic coupling is that with probability one, if $\sigma_N(x) \neq \sigma'_N(x)$ then there must be some $y \sim x$ or $y = x$ for which $\sigma_{N-1}(y) \neq \sigma'_{N-1}(y)$ as well, etc. Or, disagreement can only be inherited. Hence it is natural to associate the behavior of forgetting about initial conditions with the absence of oriented percolation of disagreements on the space–time graph. Before we come to that we must specify the percolation part more explicitly.

The space–time graph of a PCA is obtained as follows. The set of vertices are the space–time points $(x, n), x \in \mathbb{Z}^d, n = 0, 1, ...$ in the time–ordered stacking of the spatial lattice $\mathbb{Z}^d$; n is the time–coordinate. The graph is completed by drawing arrows (oriented edges) from each space–time point (x, n) to another point $(y, n-1)$ ($y = x$ or $y \sim x$ in so far that there is a non–trivial dependence of $p_x(1|\eta)$ on $\eta(y)$ for some realization π).
A path ω on the space–time graph (starting at (x_0, N)) is an (oriented) sequence of space–time points $\omega : (x_0, N), (x_1, N-1), \ldots, (x_k, N-k), x_l \in \mathbb{Z}^d, l = 0, \ldots, k < N$ in which at each step (x, n) to $(y, n-1)$ there is an arrow in the graph from the point (x, n) to the point $(y, n-1)$.

Consider now the independent oriented site percolation problem on the space–time graph with (random but highly correlated) densities $\{k_{(x,N)}\} =$

$\{k_x\}$; each point (x,N) is open (closed) with probability $k_x(1-k_x)$. Let $G_\pi^N(x,y)$ denote the probability to have an open path from (x,N) to $(y,0)$.

$$G_\pi^N(x,y) \equiv \mathrm{Prob}_\pi \left[\text{There is a path of disagreement from } (x,N) \text{ to } (y,0)\right]. \tag{32}$$

It is immediate from the previous discussion and the inequality in (30) that

$$\mathrm{Prob}_\pi[\sigma_N(x) \neq {\sigma'}_N(x)|\sigma_0, {\sigma'}_0] \leq \sum_{y\in \mathbb{Z}^d} G_\pi^N(x,y). \tag{33}$$

Combining this with (31) and using that there is at least one invariant measure for the dynamics we get Proposition 2 and see that all depends on how well we are able to control the connectivity function $G_\pi(\cdot,\cdot)$.

Proposition 2 *For every local function f*

$$\|P_\pi^N f - \nu_\pi(f)\| \leq |||f||| \sup_{x\in \mathrm{supp} f} \sum_{y\in \mathbb{Z}^d} G_\pi^N(x,y), \tag{34}$$

with ν_π an invariant measure for the dynamics.

4.2.1 Ergodicity for a "directed" PCA

We consider here PCA with transition probabilities of the form

$$p_x(\cdot|\eta) = p_x(\cdot|\eta_x, \eta(x+e_\alpha), \alpha = 1,\ldots,d) \tag{35}$$

where the $\{e_\alpha\}$ are the unit vectors on $\mathbb{Z}^d$.
The nice thing about "directed" PCA is that its space–time graph is not only timelike oriented but it is also spatially directed : there is no path connecting (x,N) with (x,M) passing by another space–time point (y,n) with $y\neq x$. In particular, when we project this graph on the spatial lattice (by identifying all points $(x,n), n = 0,1,\ldots$ with $(x,0)$) we get an oriented spatial lattice.

Every path $\omega : (x, N), (x_1, N-1), \ldots, (x_l, N-l), \ldots, (y, 0)$ from the point (x, N) to the point $(y, 0)$ on the space–time graph gives rise to a directed path on this spatial lattice. The opposite relation can also be uniquely defined if in addition we specify the times $n_l = 1, 2, \ldots$ the path stays on the site x_l visited by the spatial path.
Here we restrict ourselves to PCA's for which $\{k_x\}$ is a set of independent identically distributed random variables. Stavskaya's PCA (Example 5) is a well–known example. Another one (in two dimensions) is Toom's model.

Example 6 (Toom's model)
The transition probability is derived from a majority rule. With $\{\gamma_x\}$ as in Example 5:

$$p(\sigma(x)|\eta) = \frac{1}{2}[1 + (1 - \mathrm{e}^{-\lambda\gamma_x})\mathrm{sgn}(\eta(x) + \eta(x + e_1) + \eta(x + e_2))]. \tag{36}$$

Then,

$$k_x = 1 - \mathrm{e}^{-\lambda\gamma_x}. \tag{37}$$

In Propositions 3 and 4 we estimate the connectivity functions $G_\pi^N(x, y)$ of a directed PCA. First we derive an upperbound uniform in N.

Proposition 3 *Suppose*

$$\mathbf{E}(\frac{k_x}{1 - k_x}) < \frac{1}{d}, \tag{38}$$

then

$$\mathbf{E}[G_\pi^N(x, y)] \le C_1 e^{-\lambda'|x-y|} \tag{39}$$

for constants $C_1 < \infty$, $\lambda' > 0$.

Proof Let $|x - y| = m$. Every path $\omega : (x, N) \rightsquigarrow (y, 0)$ on the directed space–time graph gives rise to a spatial path $\omega' : x \rightsquigarrow y$ on $\mathbb{Z}^d$ of lenght $|\omega'| = m$. Following the constuction explained above we may thus write that

$$
\begin{aligned}
G_\pi^N(x,y) &\le \sum_{\substack{\omega':x\rightsquigarrow y\\ |\omega'|=m}} \sum_{\substack{l_1,\dots,l_m\ge 1\\ l_1+\dots+l_m=N+1}} \prod_{i=1}^m k_{\omega'_i}^{l_i} \\
&\le \sum_{\substack{\omega':x\rightsquigarrow y\\ |\omega'|=m}} \prod_{i=1}^m \frac{k_{\omega'_i}}{1-k_{\omega'_i}}
\end{aligned}
\tag{40}
$$

from which the conclusion readily follows. ■

Remarks

1. From the Borel–Cantelli lemma we have from Proposition 3 that $\mathbf{Q}$-a.s. there is a $N_{x,1} = N_{x,1}(\pi, \mathbf{E}(\frac{k_x}{1-k_x}), d) < \infty$ such that if $|x-y| > N_{x,1}$, then

$$
G_\pi^N(x,y) \le \mathrm{e}^{-\lambda|x-y|}, \tag{41}
$$

with $\lambda < \lambda'$.

2. The same conclusions remain valid in $d = 1$ under the assumption that

$$
\mathbf{E}(\log \frac{k_x}{1-k_x}) < 0. \tag{42}
$$

This is clear from (40) (but now there is only one path $\omega' : x \rightsquigarrow y$) by applying the strong law of large numbers.

Define l_x^* such that for all $l > l_x^*$, $k_x^l < \exp(-\lambda' l^\delta)$, with $0 \le \delta < 1$, i.e.

$$
l_x^* = (\frac{-\lambda'}{\log k_x})^{(\frac{1}{1-\delta})}. \tag{43}
$$

Lemma 1 *Take* $0 \le \beta \le 1$ *and* $v > 0$ *such that* $(1-\beta)(1-\delta)v > \beta d + 1$. *If*

$$
\mathbf{Q}(k_x \ge 1-\tau) < \tau^v, \tag{44}
$$

then —with $\mathbf{Q}$*-probability one— there exists a* $L_0 = L_0(\pi)$ *such that for* $L > L_0$

$$
l_x^* \le L^{1-\beta} \text{ for all } x \in [-L^\beta, L^\beta]^d \cap \mathbb{Z}^d. \tag{45}
$$

Proof A straightforward calculation shows that

$$\mathbf{Q}\{l_x^* \leq L^{1-\beta} : x \in [L^\beta, L^\beta]^d\} \geq \Big(1 - \Big(1 - \exp(-\lambda' L^{(\delta-1)(1-\beta)v})\Big)\Big)^{(2L^\beta+1)^d}. \tag{46}$$

Remember that $\mathrm{e}^{-x} \geq 1 - x$ for $0 \leq x \leq 1$ and that $\lambda' L^{(1-\beta)(\delta-1)} < 1$ for L large enough. So,

$$\mathbf{Q}\{\exists x \in [L^\beta, L^\beta]^d \cap \mathbb{Z}^d : l_x^* > L^{1-\beta}\} \leq 1 - (1 - \lambda' L^{(1-\beta)(\delta-1)v})^{(2L^\beta+1)^d} \tag{47}$$

This is summable if $(1-\beta)(1-\delta)v > \beta d + 1$. Hence we can use the Borel-Cantelli lemma to conclude. ∎

The second step in the study of $G_\pi^N(x, y)$ concerns deriving an upperbound uniformly in $|x - y|$.

Proposition 4 *If*

$$\mathbf{E}(\frac{k_x}{1-k_x}) < \frac{1}{d}, \tag{48}$$

then for every $0 < \epsilon < 1/2$ we have that if

$$\mathbf{Q}\{k_x > 1 - \tau\} < \tau^v, \tag{49}$$

with $v = v(\epsilon)$ high enough, there exist a time $N_{x,2} = N_{x,2}(\pi, d, \mathbf{E}(\frac{k_x}{1-k_x})) < \infty$ and a constant λ, independent of x and π such that for $N > N_{x,2}$

$$G_\pi^N(x, y) \leq \exp(-\lambda N^\epsilon). \tag{50}$$

Proof For $0 < \beta < 1$ (will be specified later on) take ${N'}_{x,2}$ such that $({N'}_{x,2})^\beta > N_{x,1}$ (see Proposition 3) and $\Big(({N'}_{x,2})^{1-\beta} - 1\Big)^{\frac{1}{1-\beta}} > L_0$ (See Lemma 1).

Let $N > N'_{x,2}$.
First we assume that $m = |x-y| < N^\beta$ and start again from

$$G_\pi^N(x,y) \le \sum_{\omega':x\rightsquigarrow y} \sum_{\substack{l_1,\ldots,l_m\ge 1\\ l_1+\ldots+l_m=N+1}} \prod_{i=1}^m k_{\omega'_i}^{l_i}. \tag{51}$$

In every path ω there is at least one site ω_i for which l_i exceeds $(N+1)/N^\beta > N^{1-\beta}$. So,

$$(51) \quad \le \sum_{\omega':x\rightsquigarrow y} \sum_{i=1}^m k_{\omega'_i}^{N^{1-\beta}-1} \prod_{i=1}^m (\tfrac{k_{\omega'_i}}{1-k_{\omega'_i}}). \tag{52}$$

Using Lemma 1 with $\lambda < \lambda'$ and δ as specified later on, and the summing over y gives

$$(52) \le \exp -(\lambda(N^{1-\beta}-1)^\delta) \sum_{y\in \mathbb{Z}^d} \sum_{\omega':x\rightsquigarrow y} \sum_{i=1}^m \prod_{i=1}^m \frac{k_{\omega_i}}{1-k_{\omega_i}}. \tag{53}$$

Condition (48) assures that the sum is finite with $\mathbf{Q}$-probability one. Hence for $\beta' > \beta$ and N large enough.

$$G_\pi^N(x,y) \le \exp -(\lambda N^{(1-\beta')\delta}) \tag{54}$$

In the case $|y-x| \ge N^\beta$ we can use Proposition 3 to see that

$$G_\pi((x,0),(y,N)) \le \exp -(\lambda N^\beta). \tag{55}$$

Now for $0 < \epsilon < 1/2$ we can choose δ and $\beta' > \beta$ such that $\beta > \epsilon$ and $(1-\beta')\delta > \epsilon$. This proves the proposition for $N_{2,x} > {N'}_{2,x}$ large enough. ∎

Theorem 2 *If*

$$\mathbf{E}(\frac{k_x}{1-k_x}) < \frac{1}{d}, \tag{56}$$

then the PCA has a unique invariant measure ν_π.
Moreover for every $0 < \delta < 1/2$ we have that if

$$\mathbf{Q}\{k_x > 1-\tau\} < \tau^v, \tag{57}$$

with $v = v(\delta)$ high enough, then for every local function f there exist a time $N_0 = N_0(\pi, \mathrm{supp} f, d, \mathbf{E}(\frac{k_x}{1-k_x})) < \infty$ and a constant $m > 0$ such that for $N > N_0$

$$\|P_\pi^N f - \nu_\pi(f)\| \leq \exp(-mN^\delta). \tag{58}$$

Proof Combining Proposition 2 with Proposition 3 we see that with **Q**-probability one there is a unique invariant measure. Furthermore,

$$\begin{aligned} \|P_\pi^N f - \nu_\pi(f)\| \leq \quad & |||f||| \sup_{z \in \mathrm{supp} f} \sum_{|y-z| \leq N^\epsilon} G_\pi^N(z,y) \\ & + \sup_{z \in \mathrm{supp} f} \sum_{|y-z| > N^\epsilon} G_\pi^N(z,y)\} \end{aligned} \tag{59}$$

Take $N_0 > \max\{\sup_{x \in \mathrm{supp} f} N_{x,1}, \sup_{x \in \mathrm{supp} f} N_{x,2}\}$, then for $N^\epsilon > N_0$ and $\delta < \epsilon$,

$$(59) \leq \exp -(mN^\delta) \tag{60}$$

with $0 < m < \lambda$ and N_0 large enough. ■

4.2.2 Ergodicity for a general PCA

The paths considered in the associated percolation problem are not spatially directed and k_x and k_y can be mutually dependent for $x \sim y$.

Proposition 5 *Take*

$$K > 2d^2(1 + \sqrt{1 + \frac{1}{d}} + \frac{1}{2d}).$$

If

$$\mathbf{E}[\{\log(1 - \log(1 - k_x))\}^K] < \infty, \tag{61}$$

then — for $\mathbf{Q}$*-almost every realization — there exists a* $v(K,d) > 1$ *such that for every* $1 < v < v(K,d)$ *and* $m > 0$ *we can find constants* $0 < C_1, C_2 < 1$ *such that when*

$$\mathbf{Q}\{k_x > C_1\} < C_2, \tag{62}$$

there exists a $N_{0,x} = N_{0,x}(\pi)$ *for which*

$$G_\pi^N(x,y) \leq \exp(-m \max\{|x-y|, (\log(1+N))^v\}) \tag{63}$$

whenever $|N - M| > N_{0,x}$.

Campanino–Klein (1991) proved an analogous result for independent bond percolation and Klein (1993) for (continuous time) percolation on $\mathbb{Z}^d \times \mathbb{R}$. In their problems the set corresponding to $\{k_x\}$ contains *independent* random variables. Although we deal with directed site percolation and k_x and k_y are *dependent* if $x \sim y$, we can almost copy their proof. The modifications are given in Appendix A.

Theorem 3 *Take*

$$K > 2d^2(1 + \sqrt{1 + \frac{1}{d}} + \frac{1}{2d}).$$

If

$$\mathbf{E}[\{\log(1 - \log(1 - k_x))\}^K] < \infty, \tag{64}$$

then— for $\mathbf{Q}$*-almost every realization — there exists a* $v(K,d) > 1$ *such that for every* $1 < v < v(K,d)$ *and* $c > 0$ *we can find constants* $0 < C_1, C_2 < 1$ *such that when*

$$\mathbf{Q}\{k_x > C_1\} < C_2 \tag{65}$$

the PCA has a unique invariant measure ν_π*. Moreover, for every local function* f *we can find a time* $N_0 = N_0(\pi, f)$ *and a constant* $B(d,f) < \infty$ *so that for all* $N > N_c$

$$\|P_\pi^N f - \nu_\pi(f)\| \leq B(d,f) \exp(-c(\log(1+N))^v) \tag{66}$$

Proof

Write (34) as

$$\begin{aligned}\|P_\pi^N f - \nu_\pi(f)\| \leq \quad & |||f|||\{ \sup_{z\in \mathrm{supp} f} \sum_{|y-z|\leq (\log(1+N))^v} G_\pi^N(z,y) \\ & + \sup_{z\in \mathrm{supp} f} \sum_{|y-z|>(\log(1+N))^v} G_\pi^N(z,y)).\}\end{aligned} \tag{67}$$

Using (63) and take $N_0 > \sup_{x\in \mathrm{supp} f} N_{0,x}$, then for $N > N_0$

$$\|P_\pi^N f - \nu_\pi(f)\| \leq B(d,f)\exp -c(\log(1+N))^v. \tag{68}$$

with $0 < c < m$ and N_0 large enough. ■

4.3 Disordered Interacting Particle Systems

The continuous time version of the previous dynamics is defined in terms of spin flip rates $c(x,\eta)$. They give the probability per unit time of flipping the value of the spin at site x if the configuration is η.

$$\mathrm{Prob}_\pi[\sigma_t(x) = \eta(x)|\sigma_0 = \eta] = c(x,\eta)t + o(t). \tag{69}$$

See Liggett (1985) for more details on these spinflip dynamics as a class of interacting particle systems (IPS). Also here we have that $c(x,\eta)$ only depends on η via $\eta(y)$, $y = x$ or $y \sim x$, via a random interaction.

Define

$$\begin{cases} \lambda_x & \equiv \sup_{\eta,\eta'}\{|c(x,\eta) - c(x,\eta')|, \eta(x) = \eta'(x)\} \\ \delta_x & \equiv \inf_{\eta,\eta'}\{c(x,\eta) + c(x,\eta'), \eta(x) \neq \eta'(x)\} \end{cases} \tag{70}$$

Example 7 (Continuous time spin glass)
The spin flip rate is of Glauber type :

$$c(x,\eta) = \exp -(\frac{\beta}{2}[H_\pi(\eta^x) - H_\pi(\eta)]) \tag{71}$$

with η^x, the configuration that differs from η in x and

$$H_\pi(\eta) = -\sum_{y\sim x} J_{xy}\eta(x)\eta(y). \tag{72}$$

The $\{J_{xy}\}$ is a family of real valued independent and identically distributed random variables.
Here,

$$\begin{aligned} \delta_x &= 2\exp(-\beta \textstyle\sum_{y\sim x} |J_{xy}|), \\ \lambda_x &= \exp(\beta \textstyle\sum_{y\sim x} |J_{xy}|) - \exp(-\beta \textstyle\sum_{y\sim x} |J_{xy}|). \end{aligned} \tag{73}$$

Example 8 (Random version of a majority vote IPS)
Let $\{\gamma_x\}_{x\in \mathbb{Z}^d}$ be independent identically distributed positive random variables. The spin flip rate is

$$c(x,\eta) = \begin{cases} \frac{2-\mathrm{e}^{-\lambda\gamma_x}}{2} & \text{if } \eta_x \neq \text{sign } (\eta(x) + \sum_{y\sim x} \eta(y)) \\ \frac{\mathrm{e}^{-\lambda\gamma_x}}{2} & \text{if } \eta_x = \text{sign } (\eta(x) + \sum_{y\sim x} \eta(y)). \end{cases} \tag{74}$$

Here,

$$\begin{aligned} \delta_x &= \mathrm{e}^{-\lambda\gamma_x}, \\ \lambda_x &= 1 - \mathrm{e}^{-\lambda\gamma_x}. \end{aligned} \tag{75}$$

To show the ergodicity for the continuous time dynamics, we connect to the IPS a Contact Process on the configuration space $\{0,1\}^{\mathbb{Z}^d}$. (See Liggett (1985).) The extinction of the population of 1's in this process, or absence of percolation in its graphical representation, implies ergodicity of the original IPS. The Contact Process is defined by the following rates :

$$\overline{c}(x,\xi) = \begin{cases} \delta_x & \text{if } \xi(x) = 1 \\ \lambda_x \sum\limits_{y \in A_x} \xi(y) & \text{if } \xi(x) = 0 \end{cases} \tag{76}$$

with λ_x and δ_x as defined in (70) and A_x the set of all neighbors y of x for which $c(x,\eta)$ depends on $\eta(x)$.

The graphical representation of the process on the continuous time lattice $\mathbb{Z}^d \times \mathbb{R}$ is obtained by putting cuts on every line $\{x\} \times \mathbb{R}$ according to a Poisson process with rate δ_x and placing arrows from the line $y \times \mathbb{R}$, $y \in A_x$ to the line $x \times \mathbb{R}$ with intensity λ_x.(See e.g. Liggett(1985), Klein (1993)). $G^t_\pi(x,y)$ is the probability that (x,t) and $(y,0)$ are connected in this construction. Absence of percolation (no infinite clusters) implies extinction of the Contact Process with the same parameters.

Proposition 6 *For every local function f*

$$\|P^t_\pi f - \nu_\pi(f)\| \leq |||f||| \sup_{z \in \mathrm{supp} f} \sum_{y \in \mathbb{Z}^d} G^t_\pi(z,y) \tag{77}$$

with ν_π an invariant measure for the dynamics.

Proof We couple three IPS: $\eta_t, {\eta'}_t$ and ξ_t with rates $c(x,\eta), c(x,\eta')$ and $\overline{c}(x,\xi)$ in such a way that if $|\eta_0(x) - {\eta'}_0(x)| \leq \xi_t(x)$ for all $x \in \mathbb{Z}^d$, then $|\eta_t(x) - {\eta'}_t(x)| \leq 2\xi_t(x)$ with probability one for all $t \geq 0$. This can be done by using the coupling described in Liggett (1985) (p. 130).

Then for every local function

$$
\begin{aligned}
|P^t_\pi f(\sigma) \quad & - P^t_\pi f(\sigma')| \\
& \leq \sum_x \delta_x f \mathrm{Prob}_\pi[\sigma_t(x) \neq {\sigma'}_t(x) | \sigma_0 = \sigma, {\sigma_0}' = \sigma'] \\
& \leq \sum_x \delta_x f \mathrm{Prob}_\pi[\xi_t(x) | \xi(0)] \\
& \leq |||f||| \sup_{x \in \mathrm{supp} f} \mathrm{Prob}_\pi[\xi_t(x) = 1] \\
& \leq |||f||| \sup_{x \in \mathrm{supp} f} \sum_{y \in \mathbb{Z}^d} G^t_\pi(x, y)
\end{aligned}
\tag{78}
$$

Which is stronger then (77).

■

4.3.1 Ergodicity in a "directed" IPS

Just as for the PCA, it is worthwhile considering the class of IPS with transition rates for which

$$
A_x = \{x + e_\alpha, \alpha = 1, \ldots, d\} \tag{79}
$$

Then, in the graphical representation, there are only arrows pointing in the opposite direction of one of the unit vectors and there are no paths that reach one spatial point twice. Further, we suppose that $\{(\delta_x, \lambda_x)\}$ is a set of independent couples of random variables. (Of coarse, δ_x and λ_x can be mutually dependent.)

We discretize the corresponding Contact Process to apply the arguments used for a directed PCA. The space time lattice becomes $\mathbb{Z}^d \times a\mathbb{Z}$. To each space–time point $(x, t) \in \mathbb{Z}^d \times a\mathbb{Z}$ we assign the occupation value 1(0) with probability $\mathrm{e}^{-a\delta_x}(1 - \mathrm{e}^{-a\delta_x})$. Between each two time layers $\mathbb{Z}^d \times \{na\}$ and $\mathbb{Z}^d \times \{(n+1)a\}$ we draw arrows from $x + e_\alpha$ to x with probability $1 - \mathrm{e}^{-a\lambda_x}$. Two sites (x, t_1) and (y, t_2) are connected if there exists a time– and space–directed path composed of arrows and open sites. The probability that these two points are linked is denoted by $G^{t_2-t_1}_{a,\pi}((x,y))$. Finally, for $\Lambda = \mathbb{Z}^d$, one can show that $G^t_{a,\pi}(x,y) \Rightarrow G^t_\pi(x,y)$ when $a \downarrow 0$. (For more details, see Bezuidenhout–Grimmett (1991).)

Proposition 7 *Suppose*

$$\mathbf{E}(\frac{\lambda_x}{\delta_x}) < \frac{1}{d}, \tag{80}$$

then

$$\mathbf{E}(G^t_\pi(x,y)) \le C_1 \exp(-\lambda'|x-y|) \tag{81}$$

for constants $C_2 < \infty$ and $\lambda' > 0$.

Proof Let $m = |x-y|$ and ω' the projection of the space–time path ω on $\mathbb{Z}^d$.

$$\begin{aligned} G^t_{a,\pi}(x,y) &\le \sum_{\omega':x\rightsquigarrow y} \sum_{\substack{l_1,\ldots,l_m\ge 1\\ l_1+\ldots+l_m=\frac{t}{a}+1}} \prod_{i=1}^m 1-\mathrm{e}^{-a\lambda_{\omega'_i}} \prod_{j=1}^m \mathrm{e}^{-a\delta_{\omega'_j} l_{\omega'_j}} \\ &\le \sum_{\omega':x\rightsquigarrow y} \prod_{i=1}^m \frac{1-\mathrm{e}^{-a\lambda_{\omega'_i}}}{1-\mathrm{e}^{-a\delta_{\omega'_i}}} \prod_{j=1}^m \mathrm{e}^{-a\delta_{\omega'_j}} \end{aligned} \tag{82}$$

Letting $a \downarrow 0$ the Proposition easily follows. ∎

Remark

¿From Proposition 7 we can deduce that there exists a $T_{x,1} = T_{x,1}(\pi, \mathbf{E}(\frac{k_x}{1-k_x}), d)$ such that for $|x-y| > T_{x,y}$

$$G^t_\pi(x,y) \le \exp\lambda|x-y| \tag{83}$$

with $\lambda < \lambda'$.

Proposition 8 *If*

$$\mathbf{E}(\frac{\lambda_x}{\delta_x}) < \frac{1}{d}, \tag{84}$$

then for every $0 < \epsilon < 1/2$ we have that if

$$\mathbf{Q}\{e^{-\delta_x} > 1 - \tau\} < \tau^v, \tag{85}$$

with $v = v(\epsilon)$ high enough, there exist a time $T_{x,2} = T_{x,2}(\pi, d, \mathbf{E}(\frac{\lambda_x}{\delta_x})) < \infty$ and a constant λ , independent of x and π such that for $t > T_{x,2}$

$$G^t_\pi(x, y) \leq \exp(-\lambda t^\epsilon). \tag{86}$$

Proof Take m and ω' as before. In the same way as we did in the proof of Proposition 4 for a PCA, we can show that for T high enough and $|x-y| < T^\beta$

$$G^t_\pi(x, y) \leq \sum_{\omega' : x \leadsto y} \sum_{i=1}^{m} \left(\mathrm{e}^{-a\delta_{\omega'_i}}\right)^{\frac{T^{1-\beta}}{a}-1} \prod_{i=1}^{m} \frac{1 - \mathrm{e}^{-a\lambda_{\omega'_i}}}{1 - \mathrm{e}^{-a\delta_{\omega'_i}}}. \tag{87}$$

Let $a \downarrow 0$, sum over $y \in \mathbb{Z}^d$ and use Lemma 1, with $\lambda < \lambda'$, to see that

$$G^t_\pi(x, y) \leq \exp -\lambda t^{(1-\beta)\delta} \sum_{y \in \mathbb{Z}^d} \sum_{\omega' : x \leadsto y} \sum_{i=1}^{m} \prod_{i=1}^{m} \left(\frac{\lambda_{\omega'_i}}{\delta_{\omega'_i}}\right). \tag{88}$$

The sum is finite with $\mathbf{Q}$-probability one. Hence for $\beta' > \beta$

$$G^t_\pi(x, y) \leq \exp(-\lambda t^{(1-\beta')\delta}). \tag{89}$$

If $|x - y| \geq T^\beta$, Proposition 7 is saying that for T large enough

$$G^T_\pi(x, y) \leq \exp(-\lambda T^\beta). \tag{90}$$

Hence, for a given $0 < \epsilon < 1/2$ choose δ and $\beta' > \beta$ such that $\beta > \epsilon$ and $(1 - \beta')\delta > \epsilon$. ■

Proposition 7 and 8 allow us to prove the main result for a "directed" IPS.

Theorem 4 *If*

$$\mathbf{E}(\frac{\lambda_x}{\delta_x}) < \frac{1}{d}, \tag{91}$$

then the IPS has a unique invariant measure ν_π.
Moreover for every $0 < \delta < 1/2$ we have that if

$$\mathbf{Q}\{e^{-\delta_x} > 1 - \tau\} < \tau^v, \tag{92}$$

with $v = v(\delta)$ high enough, then for every local function f there exist a time $T_0 = T_0(\pi, \mathrm{supp} f, d, \mathbf{E}(\frac{\lambda_x}{\delta_x})) < \infty$ and a constant $m > 0$ such that for $N > N_0$

$$\|P_\pi^N f - \nu_\pi(f)\| \leq \exp(-mN^\delta). \tag{93}$$

The proof of Theorem 8 is similar to the one of Theorem 2.

4.3.2 Ergodicity in a general IPS

Define

$$\lambda_{\langle x,y \rangle} = \lambda_x + \lambda_y. \tag{94}$$

Note that $\lambda_{\langle xy \rangle}$ and δ_x can be *correlated* if x and y are (next–) nearest neighbors.

Proposition 9 *Let*

$$K > 2d^2(1 + \sqrt{1 + \frac{1}{d}} + \frac{1}{2d}).$$

Suppose

$$\Gamma \equiv \max\{\ \mathbf{E}\ [\{\log(1 + \lambda_{\langle x,y \rangle})\}^K], \mathbf{E}\ [\{\log(1 + 1/\delta_x)\}^K]\} < \infty, \tag{95}$$

Then there exists a constant C_1 such that if

$$\mathbf{E}\{(\log(1 + \frac{\lambda_{\langle x,y \rangle}}{\delta_x}))^K\} < C_1 \tag{96}$$

then — for $\mathbf{Q}$-almost every realization — there exists a $v(K, d) > 1$ so that for every $1 < v < v(K, d)$ and $m > 0$ we can find constants $0 < C_2, C_3 < 1$ such that when

$$\mathbf{Q}\{k_x > C_2\} < C_3 \tag{97}$$

there exists a $T_{c,x} = T_{c,x}(\pi)$ *for which*

$$G^t_\pi(x,y) \leq \exp(-m \max[|x-y|, (\log(1+t|))^v]) \tag{98}$$

whenever $|t-s| > T_{c,x}$.

Proposition 9 is formally the same as Theorem 3.2 in Klein (1993). However, we must take care of the consequences of the non-trivial correlations between δ_x and $\lambda_{\langle x,y\rangle}$. In Appendix B we briefly show the necessary modifications to the argument of Klein. (1993)

Theorem 5 *Let*

$$K > 2d^2(1+\sqrt{1+\frac{1}{d}}+\frac{1}{2d}).$$

Suppose

$$\Gamma \equiv \max\{ \ \mathbf{E}\ [\{log(1+\lambda_{\langle x,y\rangle})\}^K], \mathbf{E}\ [\{\log(1+1/\delta_x)\}^K]\} < \infty, \tag{99}$$

Then there exists a constant C_1 *such that if*

$$\mathbf{E}((\log(1+\frac{\lambda_{\langle x,y\rangle}}{\delta_x}))^K) < C_1, \tag{100}$$

then — for $\mathbf{Q}$*-almost every realization — there exists a* $v(K,d) > 1$ *so that for every* $1 < v < v(K,d)$ *and* $c > 0$ *we can find constants* $0 < C_2, C_3 < 1$ *such that if*

$$\mathbf{Q}\{k_x > C_2\} < C_3 \tag{101}$$

for every local function f *there is a time* $T_0 = T_0(\pi, f)$ *and a constant* $B(d,f) < \infty$ *so that*

$$||P^t_\pi f - \nu_\pi(f)|| \leq B(d,f)\exp(-c(\log(1+t))^v) \tag{102}$$

when $t > T_0$.

Proof Combining (77) and (98) in the same way as we did for Theorem 3 gives a proof of Theorem 5.

■

Acknowledgments: We are grateful to L. Gray, G. Grimmett and E. Speer for interesting discussions in an early stage of this work while C.M. was visiting the Newton Institute (Cambridge). We also thank A. van Enter and A. Klein for suggestions and careful reading of a first draft of this paper. This work is partially supported by EC grant CHRX–CT93–0411.

Appendix A

Remark In the Appendices A and B we give the necessary modifications to the argument of Klein (1993) and Campanino–Klein (1991).

As we already mentioned, the proof of Proposition 5 is similar to the one in Klein (1993) and Campanino–Klein (1991). In this paragraph the argument of (mainly) Klein (1993) is summarized. The most important lemma's are presented, with emphasis on the necessary modifications due to the mutual dependence of the k_x's.

Let $W, W' \subset \mathbb{Z}^{d+1}$. Define

$$\partial(W, W') = \{(y, M) \in W \cap W' | \exists (x, N) \in W' \setminus W, |(y, M) - (x, N)| = 1\} \tag{103}$$

$$\partial(W, \mathbb{Z}^{d+1}) \equiv \partial W \tag{104}$$

For $x \in \mathbb{Z}^d$ and $L > 0$ let

$$\Lambda_L(x) = \{y \in \mathbb{Z}^d; |y - x| \leq L\}. \tag{105}$$

For $(x, N), L > 0, T > 0$, we set

$$B_{L,T}((x,N)) = \Lambda_L(x) \times ([N-T, N+T] \cap \mathbb{Z}) \tag{106}$$

and

$$B_L((x,N)) = B_{L,e^{L^{1/v}}}((x,N)). \tag{107}$$

Define $G_{\pi,B_L((x,N))}((y_1,N_1),(y_2,N_2))$ the probability to find an open path from (y_1, N_1) to (y_2, N_2) in $B_L((x,N))$.

Definition 1 *Let $m, L > 0$. We say that $x \in \mathbb{Z}^d$ is (m,L)-regular if*

$$G_{\pi,B_L((x,0))}((x,0),(y,M)) \leq e^{-mL} \tag{108}$$

for all $(y, M) \in \partial B_L((x,0))$. Otherwise x is called (m,L)-singular. A space–time point $(x,t) \in \mathbb{Z}^{d+1}$ will be called (m,L)-regular if x is (m,L)-regular. $\Lambda \subset \mathbb{Z}^d$ will be called (m,L)-regular if every $x \in \Lambda$ is (m,L)-regular. Otherwise Λ is (m,L)-singular.

Klein (1993) uses a multiscale analysis. The proof is by induction. In Lemma 2 the induction hypothesis is proven: for every (m_0, L_0) we can find constants $0 < C_1, C_2 < 1$ such that if $\mathbf{Q}\{k_x > C_1\} < C_2$ then

$$\mathbf{Q}\{\text{The origin is } (m_0, L_0) - \text{regular}\} \geq 1 - \frac{1}{L_0^p}, \tag{109}$$

In the induction step (Lemma 3, Sublemma's 1 and 2) is proven that if

$$\mathbf{Q}\{\text{The origin is } (m_k, L_k) - \text{regular}\} \geq 1 - \frac{1}{L_k^p}. \tag{110}$$

then

$$\mathbf{Q}\{\text{The origin is } (m_{k+1}, L_{k+1}) - \text{regular}\} \geq 1 - \frac{1}{L_{k+1}^p}. \tag{111}$$

with $L_{k+1} = L_k^\alpha, \alpha > 1$ and $m_{k+1} < m_k$. One can show that for every choice of $0 < m_\infty < m_0$ we can choose L_0 such that $m_\infty < \lim_{k\uparrow\infty} m_k < m_0$. Then

$$\mathbf{Q}\{\text{The origin is } (m_\infty, L_k) - \text{singular}\} \leq \frac{1}{L_k^p}. \tag{112}$$

and using the Borel-Cantelli Lemma we know that —with $\mathbf{Q}$-probability one—there exists a K such that if $k > K$ the origin is (m_∞, L_k)-regular. Klein (1993) estimates $G_\pi^N(x, y)$ in such a (m_∞, L_k)-regular region.

Lemma 2 *For every $m, L > 0$ there exist constants $0 < C_1, C_2 < 1$ such that if*

$$\mathbf{Q}\{k_x > C_1\} < C_2 \tag{113}$$

then,

$$\mathbf{Q}\{\textit{The origin is } (m, L)\textit{-regular}\} \geq 1 - 1/L^p. \tag{114}$$

Proof For any $C > 0$ there exist constants $0 < C_1, C_2 < 1$ such that if $\mathbf{Q}\{k_x > C_1\} < C_2$ then,

$$\mathbf{Q}\{\sup_{x\in\Lambda_L(0)} k_x < C\} \leq \frac{1}{L^p}. \tag{115}$$

Because $G_{\pi,B_L(0)}$ becomes arbitrary small for decreasing C, the lemma is proven by taking C small enough. ■

Lemma 3 *Suppose that condition (61) holds.*
Let

$$\alpha = d + \sqrt{d^2 + d}. \tag{116}$$

and choose v and p such that

$$\frac{\alpha d(d + \beta + 1)}{\beta(\alpha - d + \alpha d)} < 1/v < 1, \tag{117}$$

$$\alpha d < p < \frac{\beta(1/v(\alpha - d + \alpha d) - \alpha) - \alpha d}{\alpha}. \tag{118}$$

For every m_0 *and* m_∞*, with* $m_\infty < m_0$*, there exists*
$\overline{L} = \overline{L}(d, \beta, \Gamma, 1/v, p, m_0, m_\infty) < \infty$ *such that if for some* $L_0 > \overline{L}$ *we have*

$$\mathbf{Q}\{0 \text{ is } (m_0, L_0) - regular\} \geq 1 - \frac{1}{L_0^p}. \tag{119}$$

Taking $L_{k+1} = L_k^\alpha, k = 1, 2 \ldots,$ *we also have*

$$\mathbf{Q}\{0 \text{ is } (m_\infty, L_k)\text{-regular } \} \geq 1 - \frac{1}{L_k^p}, \tag{120}$$

for all $k = 0, 1, 2, \ldots$ *.*

Proof Lemma 3 To prove Lemma 3 we need Sublemma's 1 and 2.

Sublemma 1 *Pick positive integers* R, κ *and* b *such that*

$$\alpha < \frac{(R+1)p}{p + (R+1)d}, \tag{121}$$

$$1 < \alpha(1 - 1/v) + 1/v < \kappa < \frac{b}{d} < \frac{\alpha}{vd}. \tag{122}$$

Let $m_0 \geq m_k \geq 1/L_k^\theta$*, with* $0 < \theta < \theta_0 = \min\{\alpha(1 - 1/v), 1\}$.

$$\text{Suppose there exist } x_1, \ldots, x_R \in \Lambda_{L_{k+1}}(0) \text{ such that } \Lambda_L \setminus \bigcup_{j=0}^{R} \Lambda_{2L_k+1}(x_j)$$

$$\text{is a } (m_k, L_k)\text{-regular region.} \tag{123}$$

Define

$$\tilde{\Lambda} = \bigcup_{j=1}^{R} \Lambda_{l^\kappa(x_j)} \cap \Lambda_L(0). \tag{124}$$

Suppose

$$\sum_{x \in \tilde{\Lambda}} \log (1 - k_x)^{-1} \leq L_k^b. \tag{125}$$

Then, if L_k *is large enough* $\Lambda_{L_{k+1}}(0)$ *is a* (m_{k+1}, L_{k+1})*-regular region with* $m_{k+1} \geq 1/L_{k+1}^{\theta}$.

Proof The proof is analogous to the proofs of sublemmas 3.6 and 3.7 in Campanino–Klein (1991) and the sublemmas 4.2 and 4.3 in Klein (1993). However, there is one necessary modification that should be mentioned.
Consider the sets $V_1, \ldots, V_k \subset V_0 \subset \mathbb{Z}^d$ with $V_i \cap V_j = \emptyset$.
For $i : 1, \ldots, k$ we define

$$B_i = V_i \times \{(-T, T) \cap \mathbb{Z}\}, \tag{126}$$

$$\partial B = \partial(B_i, B) \text{ and } B' = B \setminus \bigcup_{i=1}^{k} B_i. \tag{127}$$

Let $\{(x, N_1) \xrightarrow{B} (y, N_2)\}$ the event that there is an oriented open path in B from (x, N_1) to (y, N_2).
For $(y, t) \in \partial B$

$$\begin{aligned}
&\{(0,0) \xrightarrow{B} (y, N)\} \\
\subseteq \bigcup_{r=0}^{k} \bigcup_{\{i_1,\ldots,i_k\} \subset \{1,\ldots,k\}} &\{\text{There exist oriented open paths } (0,0) \xrightarrow{B'} \partial B_{i_1}, \\
&\partial B_{i_1} \xrightarrow{B'} \partial B_{i_2}, \ldots, \partial B_{i_r} \xrightarrow{B'} (y, N)\} \\
\subseteq \bigcup_{r=0}^{k} \bigcup_{\{i_1,\ldots,i_k\} \subset \{1,\ldots,k\}} &\{(0,0) \xrightarrow{B'} \partial B_{i_1}\} \circ \{\partial B_{i_1} \xrightarrow{B'} \partial B_{i_2}\} \circ \ldots \\
&\ldots \circ \{\partial B_{i_r} \xrightarrow{B'} (y, N)\}.
\end{aligned} \tag{128}$$

Note that we used the time–oriented character of the paths.
The van den Berg–Kesten inequality can be used to estimate the probability of the event $\{(0,0) \xrightarrow{B} (y, N)\}$ and we can complete the proof as in Klein

(1993) and Campanino–Klein (1991). ■

Sublemma 2 *If*

$$\mathbf{Q}\{0 \textit{ is } (m_k, L_k)\textit{-regular}\} \geq 1 - \frac{1}{L_k^p}, \tag{129}$$

then $\mathbf{Q}\{(123) \textit{ and } (125)\} \geq 1 - \frac{1}{L_k^p}$.

Proof We call two points x_1 and x_2 non l-touching if it is not possible to walk from the box $\Lambda_l(x_1)$ to $\Lambda_l(x_2)$ without passing a point that does not belong to any of the sets. If $R+1$ points $x_1, \ldots, x_{R+1}$ are non l-touching, the events $\{x_i$ is (m, l)-regular$\}$ i:1,2,... are independent.

$$\begin{aligned} \mathbf{Q}\{\exists x_1, x_2, \ldots, x_{R+1} \ & \in \Lambda_{L_{k+1}}(0) \text{ that are non } L_k\text{-touching and} \\ & (m_k, L_k) \text{ -singular } \} \\ & \leq \tfrac{(2L_{k+1}+1)^{d(R+1)}}{L_k^{p(R+1)}} < \tfrac{1}{L_k^p} \end{aligned} \tag{130}$$

for L_k sufficiently large by the choice of R. Hence

$$\begin{aligned} \mathbf{Q}\{\exists x_1, x_2, \ldots, x_R \ & \in \Lambda_{L_k}(0) \text{ such that } \Lambda_{L_{k+1}}(0) \setminus \bigcup_{j=0}^{R} \Lambda_{2L_k+1}(x_j) \\ & \text{is a } (m_k, L_k) - \text{regular region}\} \\ & \geq 1 - \tfrac{1}{2L_{k+1}^p} \end{aligned} \tag{131}$$

For the $\mathbf{Q}$-probability of (125), we have that

$$\begin{aligned} \mathbf{Q} \ & \{\sum_{x\in\tilde{\Lambda}} \log(1-k_x)^{-1} > L_k^b\} \\ & \leq \mathbf{Q}\{\exists x \in \tilde{\Lambda} : \log(1-k_x)^{-1} > \tfrac{L_k^b}{(2L_k^\kappa+1)^d}\} \\ & \leq (2L_k^\kappa+1)^d \mathbf{Q}\{\log(1-k_x)^{-1} > \tfrac{L_k^b}{(2L_k^\kappa+1)^d}\} \\ & \leq (2L_k^\kappa+1)^d \mathbf{Q}\{\log(1+\log(1-k_x)^{-1}) > \log(1+\tfrac{L_k^b}{(2L_k^\kappa+1)^d})\} \end{aligned} \tag{132}$$

$$
\begin{aligned}
\mathbf{Q} &\leq (2L_k^\kappa+1)^d \mathbf{Q}\{\log(1+\log(1-k_x)^{-1}) > \tfrac{L_k^b}{(2L_k^\kappa+1)^d}\} \\
&\leq \tfrac{(2L_k^\kappa+1)^d (2L_k^\kappa+1)^{Kd}}{L_k^{Kb}} \mathbf{E}([\log(1+\log(1-k_x)^{-1})]^K) \\
&\leq \tfrac{\mathbf{E}([\log(1+\log(1-k_x)^{-1})]^K)}{L_k^{K(b-\kappa d)-\kappa d}} \\
&\leq \tfrac{1}{2L_{k+1}^p}
\end{aligned} \tag{133}
$$

To prove the sublemma, note that both (123) and (125) are decreasing events.

■

Appendix B

The only thing to be done is to observe that the events (1.6), (4.1) and (4.2) in Klein (1993) have the same $\mathbf{Q}$ probability.

Choose κ, b, γ, τ such that

$$\alpha(1-1/v)+1/v < \kappa < \frac{b}{d} < \frac{\alpha}{vd}, 0 < \gamma < b-\kappa d, 1/v < \tau < \alpha(1-1/v). \tag{134}$$

Consider the following events:

$$8d\rho_{L_0} < e^{2m_0} \tag{135}$$

$$\delta_{L_0} > e^{-1/2L_0^{1/v}} \tag{136}$$

$$\lambda_{L_0} < e^{L_0^{1/v}}, \tag{137}$$

with

$$\rho_L = \sup_{x\in\Lambda_L(0)} \{\frac{1}{\delta_x} \max_{y\sim x} \lambda_{\langle xy\rangle}\} \tag{138}$$

$$\delta_L = \inf_{x\in\Lambda_L(0)} \delta_x \tag{139}$$

$$\lambda_L = \max_{(x,y)\in\partial(\Lambda_L(0),\mathbb{Z}^d)} \lambda_{\langle xy\rangle} \tag{140}$$

$$\partial(\Lambda_L(0), \mathbb{Z}^d) = \{(x,y) \in (\mathbb{Z}^d)^2 : x \in \Lambda_L(0), y \in \Lambda_L^c(0) : y \sim x\} \tag{141}$$

and the events

$$\lambda_{L_{k+1}} < e^{L_{k+1}^{1/v}} \tag{142}$$

$$\{\text{There exist } x_1, x_2, \ldots, x_R \in \Lambda_{L_{k+1}}(0) \text{ such that } \Lambda_{L_{k+1}}(0) \setminus \bigcup_{j=0}^{R} \Lambda_{2L_k+1}$$

$$\text{is a } (m_k, L_k)\text{-regular region}\}, \tag{143}$$

$$e^{L_k^\gamma} \sum_{\langle x,y\rangle \subset \tilde{\Lambda}} \lambda_{\langle x,y\rangle} - \sum_{x \in \tilde{\Lambda}} \log(1 - e^{-\delta_x \exp L_k^\gamma}) \leq L_k^b, \tag{144}$$

with

$$\tilde{\Lambda} = \bigcup_{j=1}^{R} \Lambda_{L_k}^{\kappa}(x_j) \cap \Lambda_{L_{k+1}}(0). \tag{145}$$

We only have to prove that there exists an L_0, large enough such that

$$\mathbf{Q}\{(135), (136), (137)\} \geq 1 - \frac{1}{L_0^p}. \tag{146}$$

and that if

$$\mathbf{Q}\{0 \text{ is } (m, L_k) - \text{regular}\} \geq 1 - \frac{1}{L_k^p}, \tag{147}$$

then

$$\mathbf{Q}\{(142), (143), (144)\} \geq 1 - \frac{1}{L_{k+1}^p}. \tag{148}$$

The lower bounds for $\mathbf{Q}\{(135)\}$, $\mathbf{Q}\{(136)\}$, $\mathbf{Q}\{(137)\}$, $\mathbf{Q}\{(142)\}$ and $\mathbf{Q}\{(144)\}$ are calculated in Klein (1993).

To estimate $\mathbf{Q}\{(143)\}$ we modify the definition of non l-touching points: Two points x_i, x_j are non l-touching if it is not possible to walk from the box $\Lambda_l(x_i)$

to the box $\Lambda_l(x_j)$ without passing two points that don't belong to any of the boxes.
Note that (135) to (144) are all decreasing events. Hence, we can use the FKG-inequality to prove (146) and (148).

References

1. Bassalygo, L.A. and Dobrushin, R.L.: Uniqueness of a Gibbs field with random potential – an elementary approach. Theory Prob. Appl. **31**, 572–589 (1986)

2. Beretti, A.: Some properties of random Ising models. J. Stat. Phys. **38**, 483–496 (1985)

3. Bezuidenhout, C. and Grimmett,G.: Exponential decay for subcritical contact and percolation processes. Ann. Prob. **3**, 984-1009 (1991)

4. van den Berg, J. and Maes, C.: Disagreement percolation in the study of Markov fields. Ann. Prob. **22**, 749–763 (1994)

5. Campanino, M. and Klein, A.: Decay of two–point functions for $(d+1)$–dimensional percolation, Ising and Potts model with d–dimensional disorder. Commun. Math. Phys. **135**, 499–515 (1991)

6. von Dreifus, H., Klein, A. and Perez, J.F.: Taming Griffith's singularities: infinite differentiability of quenched correlation functions. Preprint (1994)

7. Fröhlich, J.: Mathematical aspects of the physics of disordered systems. In: Ostwalder, K. and Stora, R. (eds.), Critical Phenomena Random Systems, Gauge Theories. Elsevier 1986

8. Fröhlich, J. and Imbrie, J.: Improved perturbation expansion for disordered systems : beating Griffiths' singularities. Commun. Math. Phys. **96**, 145–180 (1984)

9. Georgii, H.-O.: Gibbs measures and Phase Transitions. De Gruyter Studies in Mathematics, Berlin–New York 1988

10. Griffiths, R.: Non–analytic behaviour above the critical point in a random Ising ferromagnet. Phys. Rev. Lett. **23** 17–19 (1969)

11. Grimmett, G.: Percolation. New York, Berlin: Springer–Verlag 1989

12. Klein, A.: Disordered quantum spin systems, percolation and contact processes. In: Kotecký, R. (ed.), Phase Transitions: Mathematics, Physics, Biology, ... pp. 129–137, New Jersey, London: World Scientific 1992

13. Klein, A.: Extinction of contact and percolation processes in a random environment. To appear in Ann. of Prob. (1993)

14. Klein, A.: Multiscale analyses in disordered systems: percolation and contactpocesses in a random environment. In: Grimmett, G. (ed.), Probability and Phase Transitions. pp. 139–152, Dordrecht, Boston, London: Kluwer Academic Publishers 1993a

15. Klein, A.: Who is afraid of Griffith's singularities? In: Fannes, M., Maes, C. and Verbeure, A., (eds.), On Three Levels. Micro, Meso and Macroscopic Approaches in Physics. pp. 253–258, New York, London: Plenum Press 1994

16. Kesten, H.: Aspects of first passage percolation. In: Hennequin, P.L. (ed.), Ecole d'Eté de Probabilités de Saint–Flour XIV. Lecture Notes in Mathematics. **1180**, pp.125–264, Berlin: Springer 1996

17. Liggett, T.M.: Interacting Particle Systems. New york: Springer–Verlag 1985

18. Ogielski, A.T.: Dynamics of three–dimensional spin glasses in thermal equilibrium. Phys. Rev. B **32**, 7384–7398 (1985)

19. Olivieri, E., Perez, J.F. and Rosa Jr., S.G.: Some rigorous results on the phase diagram of dilute Ising systems. Phys. Lett. **94A**, 309 (1983)

20. Palmer, R.G., Stein, D.L., Abrahams, E. and Anderson, P.W.: Models of hierarchically constrained dynamics for glassy relaxation. Phys. Rev. Lett. **53**, 958–961 (1984)

21. Perez, J.F.: Controlling the effect of Griffith's singularities in random ferromagnets. Braz. J. Phys. **23**, 356-362 (1993)

22. Randeria, M., Sethna, J.P. and Palmer, R.G.: Low–Frequency relaxation in Ising spin glasses. Phys. Rev. Lett. **54**, 1321–1324 (1985)

23. Zegarlinski, B.: Strong decay to equilibrium in one–dimensional random spin systems. Preprint (1994)