Topics on chaotic dynamics.

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Abstract: Various kinematical quantities associated with the statistical properties of dynamical systems are examined: statistics of the motion, dynamical bases and Lyapunov exponents. Markov partitons for chaotic systems, without any attempt at describing "optimal results". The Ruelle principle is illustrated via its relation with the theory of gases. An example of an application predicts the results of an experiment along the lines of Evans, Cohen, Morriss' work on viscosity fluctuations. A sequence of mathematically oriented problems discusses the details of the main abstract ergodic theorems guiding to a proof of Oseledec's theorem for the Lyapunov exponents and products of random matrices.

Keywords: chaos, nonequilibrium ensembles, Markov partitions, Ruelle principle, Lyapunov exponents, random matrices, gaussian thermostats, ergodic theory, billiards, conductivity, gas.

§*1 Dynamical systems and their statistics.*

A dynamical system $(\mathcal{C},S)$ will consist of a piecewise smooth compact manifold and of a piecewise diffeomorphic map S of $\mathcal{C}$ into itself with a piecewise diffeomorphic inverse S^{-1}.[1] The set $\mathcal{N}$ of the singularities of either S or of the manifold $\mathcal{C}$ will be called the set of the singularities of $(\mathcal{C},S)$.[2]

The reader disturbed by the above generality can simply think that all I am discussing is the case of a C^∞ diffeomorphism of a C^∞ compact manifold without boundary. The concession to the generality is due to the fact that some of the most important dynamical systems really show physically significant singularities, like the billiards or the hard sphere gases.

Example 1: A Hamiltonian system with l degrees of freedom is observed at the instants in which a certain event, *the timing event*, happens. This is the event in which the point representing the system passes through a predefined surface in phase space (*e.g.* one particle of the system passes through an ideal wall in physical space or the distance between two particles takes some prefixed value r). If $\mathcal{C}$ is the family of timing events with a prefixed energy H_0, and S is the transformation mapping one event in $\mathcal{C}$ into the following, then $(\mathcal{C},S)$ is a dynamical system.

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1 By piecewise smooth manifold I mean a manifold that can be regarded as the union of a finite number of C^∞ compact manifolds, with any two of them having in common only boundary points: the union of the boundaries of such manifolds will be called the set of singularities of $\mathcal{C}$. Similarly a map of $\mathcal{C}$ into itself is said to be piecewise smooth if $\mathcal{C}$ can be regarded as the union of finitely many compact C^∞ manifolds and S is a diffeomorphism as a map of the interior of each such manifold and its image. A compact C^∞ manifold, of dimension d, is a C^∞ manifold with boundary consisting of finitely many compact C^∞ manifolds of dimension $< d$ with only boundary points in common; the inductive definition is started by declaring that a point is a C^∞ manifold of dimension 0.

2 Sometimes one may wish to consider as singular points some points where S is in fact regular: in this case such points will also be supposed to lye on piecewise smoooth submanifolds of $\mathcal{C}$ (of lower dimension) and will be included in the set of singularities. Furthermore, since the action of S on the singular points will not be studied, one may wish to require that S^{-1} be undefined or arbitrarily defined on the points $x \ni S(\mathcal{C}/\mathcal{N})$ so that in such points the property $S(S^{-1}x) = x$ may even fail. And likewise one may avoid defining S on $\mathcal{N}$. I will not try to be so general: see [P].

The probability distribution μ_L defined by assigning to a (measurable) set of events $E \subset \mathcal{C}$ the Liouville measure of the set of phase space points on the energy surface experiencing the first timing event in E, is an invariant measure (*i.e.* $\mu_L(E) = \mu_L(S^{-1}E)$). The μ_L will still be called the Liouville measure.

Example 2: consider a box, $[-\frac{1}{2}L, \frac{1}{2}L]^2$, with side L and periodic boundary conditions (*i.e.* opposite sides identified) and with a few circular regions C_i, "obstacle of radius" r_i, in it. A particle moves freely among them and collides elastically, at velocity v. The space $\mathcal{C}$ consists of the *collisions*, parameterized by the point α on the obstacle where the collision takes place and by the angle φ formed between the incoming velocity and the outer normal to the collision point: hence $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. We see that $\mathcal{C}$ consists in the union of $\partial C_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. A point in $\mathcal{C}$ is determined by 2 coordinates (α, φ), with $\alpha \in [0, 2\pi]$ being the angular position of the collision point and a label i denoting the obstacle on which the collision takes place. The obstacles will be supposed so distributed so that no collisionless trajectory is possible. The map S is smooth everywhere except on the boundary of $\mathcal{C}$ (*i.e.* $\varphi = \pm\pi$) and on the collisions $x = (i, \alpha, \varphi)$ such that Sx is a tangent collision. This is the *billiard system.* A natural metric on $\mathcal{C}$ is $d\alpha^2 + d\varphi^2$. The system being hamiltonian it is easy to find the Liouville distribution: in the coordinates (α, φ) it is given by $\Gamma r_i d\alpha \cos\varphi\, d\varphi$ where (α, φ) represent collisions with the obstacle i and Γ is defined so that $\mu_L(\mathcal{C}) = 1$.

Example 3: many particles, say N, in a periodic box as in the example 2. The particles interact with a radial pair potential v (with range smaller than L) and with an external potential represented by the hard core due to the obstacles. The phase space is $\mathcal{C}^N$ and the equations of motion are:

$$\underline{\dot{q}}_j = \frac{1}{m}\underline{p}_j, \quad \underline{\dot{p}}_j = \underline{F}_j + E\underline{i} - \alpha\underline{p}_j \tag{1.1}$$

where $\underline{i}$ is the 1–axis unit vector, E is an external constant field, $\underline{F}_j$ is the force generated by the total potential Φ on the j–th particle; the obstacles are taken into account by the elastic reflection rule and α is so defined that the energy $H = \sum_j \frac{1}{2m}\underline{p}_j^2 + \Phi$ is a constant of motion (*gaussian thermostat*): *i.e.* $\alpha = E\underline{i} \cdot \sum_j \underline{p}_j / \sum_j \underline{p}_j^2/2m$, as a simple calculation proves; m is the particles mass. The phase space $\mathcal{C}$ consists of the energy H_0 configurations in which one particle (any one) is colliding with some obstacle. The time evolution S maps one such configuration into the following configuration of the same type ("next collision configuration"). The phase space thus defined is a $4N - 2$ dimensional subspace of the total $4N$ dimensional phase space; and a simple calculation of the divergence of the r.h.s. of (1.1) shows us that the phase space volume changes at a rate $(2N - 1)\alpha$ in the total $4N$ dimensional phase space $\mathcal{F}$. This example has a rather involved set of singularities: nevertheless it is piecewise smooth and it defines a dynamical system in the above sense. Physically it is a model for electric conductivity (*Lorentz gas conductor*).

Example 4: The map of the torus $\mathcal{C} = [0, 2\pi]^2$ defined by: $S(\varphi_1, \varphi_2) = (\varphi_1 + \varphi_2, \varphi_1 + 2\varphi_2) \bmod 2\pi$. One also writes:

$$S\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} \bmod 2\pi \tag{1.2}$$

Then S preserves the distribution $\mu_0(d\underline{\varphi}) = d\underline{\varphi}/(2\pi)^2$. This system has *no singularities.*

Given a dynamical system $(\mathcal{C}, S)$ we shall consider initial data chosen randomly with respect to a given probability distribution μ_0. It is customary to consider the "Liouville distribution": the latter means a probability distribution with a positive density with respect to the volume on the phase space, as one often calls the space $\mathcal{C}$. Sometimes it could literally coincide with the Liouville measure of analytical mechanics; *e.g.* when the system $(\mathcal{C}, S)$ is obtained by timing the evolution of a hamiltonian system: this is the case in examples 1,2.

The *qualitative theory* of $(\mathcal{C}, S)$ studies the asymptotic properties of the motions following initial data chosen randomly with distribution μ_0, which we call "μ_0–random data".

More generally one could investigate the properties enjoyed by motions with initial data chosen randomly with respect to other probability distributions μ not absolutely continuous with respect to the volume measure. Sometimes this is regarded as a less interesting question on the grounds that the Liouville distribution is the *natural one* to use in selecting initial data. This is a preconception; in fact it is well known that it is equally easy (or, rather, difficult) to produce random initial data with a distribution singular with respect to the uniform distribution: therefore such data are equally interesting; or at least one has to find better reasons to regard them less interesting. This being not the place to undertake a learned philosophical discussion on the *preminence* of the Liouville measure μ_0, I shall concentrate on the most studied question of which is the asymptotic behaviour of motions with initial data chosen randomly with distribution μ_0. Many of the properties that will emerge will be relevant also for the other random choices with respect to less appreciated distributions.

The key notion is that of *statistics*:

1 Definition: *Given a dynamical system $(\mathcal{C}, S)$ and a probability distribution μ attributing probability 1 to the set of points x which never in their evolution fall on the singularity set $\mathcal{N}$ (i.e. $\mu(\cap_{k=0}^{\infty} S^{-k}(\mathcal{C}/\mathcal{N})) = 1$), we say that μ_0 has a statistics with respect to the evolution S if for any continuous function F on $\mathcal{C}$, called* observable, *the time average of F on the motion generated by μ–almost all points $x \in \mathcal{C}$ exists, and has the form:*

$$\lim_{T\to\infty} \frac{1}{T} \sum_{k=0}^{T-1} F(S^k x) = \int_{\mathcal{C}} F(y)\overline{\mu}(dy) \tag{1.3}$$

where $\overline{\mu}$ is a suitable probability distribution on $\mathcal{C}$, called the statistics of μ in the dynamical system $(\mathcal{C}, S)$. If $\mu = \mu_0$ is the Liouville distribution then $\overline{\mu}$ will be called simply the statistics of $(\mathcal{C}, S)$ and we shall say that "$(\mathcal{C}, S)$ has a statistics" without reference to μ_0.[3]

Remarks:

(i) Usually the map S is such that $S^{-k}\mathcal{N}$ is contained in a lower dimensional compact submanifold of $\mathcal{C}$: therefore almost all points (randomly chosen with respect to the Liouville measure) will never fall on $\mathcal{N}$ and it will make sense to ask if the Liouville measure has a statistics.

(ii) Note that $\overline{\mu}$ *does depend on* μ_0. It makes no sense to talk about the statistics of a dynamical system without specifying the distribution with which the initial data are randomly selected.

(iii) If $(\mathcal{C}, S)$ has a statistics (*i.e.* if the Liouville distribution has a statistics) then one says also that $(\mathcal{C}, S)$ has a *unique attractor*. It might happen, and in some interesting cases it does happen, that the space $\mathcal{C}$ can be represented as a union of several open sets U_i and of some 0 μ_0–measure sets N_k: for almost all points in U_i the average of F is given by a formula like (1.3) but with some $\overline{\mu}_i$ replacing $\overline{\mu}$. In this case one would say that $(\mathcal{C}, S)$ has several attractors each with its own statistics. In the present analysis I shall not consider such cases: often they can be reduced to cases with unique statistics simply by redefining the phase space to be $\overline{U}_i$ rather than $\mathcal{C}$; although this is not the most general case.

A further general notion is that of *attractor*:

2 Definition: *The set $A \subset \mathcal{C}$ is an "attractor" for μ–random data if μ has a statistics in the dynamical system $(\mathcal{C}, S)$ and (1): $SA = A$ ("A is invariant"), (2): $\overline{\mu}(A) = 1$ (A has "probability 1 with respect to the statistics), (3): the fractal (Hausdorff) dimension of A is minimal among*

[3] Adhering, only for lack of time, to the prejudice that the volume measure is somewhat privileged.

the A's with the properties (1),(2).

Remarks:

It is not convenient to require that the attractor be a closed set (as sometimes done). In many cases the attractors A in the above sense are dense in $\mathcal{C}$ but have a fractal dimension strictly less than that of $\mathcal{C}$. So the above notion is subtler than the ones that imply or require that an attractor be closed. It also stresses that an attractor will usually have a non trivial fractal dimension. On the other hand the attractor in this sense is not unique (*e.g.* one can usually remove from it countably many points, except in special cases like when A is a fixed point or a periodic orbit, or even larger sets with zero Hausdorff dimension and zero $\overline{\mu}$ measure).

With the above notions in mind we can proceed to define some of the main qualitative features of the motions following data chosen randomly with respect to a distribution μ_0.

§2 Dynamical bases and Lyapunov exponents.

A given motion analysis can starts with the attempt at understanding the behaviour of nearby trajectories: this means understanding the linearization of the motions taking place near it.
For this purpose the notion of regular point is necessary:

3 Definition: *A point x is regular if the map S is regular in the vicinity of $S^k x$ for all $k \ge 0$. Recalling that S is always supposed to be piecewise regular this simply means that $S^k x$ does not fall into the set $\mathcal{N}$ of singularities of S: i.e. $x \notin \cup_{k=0}^{\infty} S^{-k}\mathcal{N}$ or $x \in \cap_{k=0}^{\infty} S^{-k}(\mathcal{C}/\mathcal{N})$.*

Let $S^k x$ be a trajectory starting at a regular point x, $k = 0, 1, \ldots$: the trajectory $S^k(x + dx)$ with dx infinitesimal will stay close forever to that of x departing from it by an infinitesimal amount simply given by $\partial S^k(x)dx$, where $\partial S^k(x)$ is the jacobian matrix of S^k evaluated at the point x. It is a $d \times d$ matrix if d is the dimension of $\mathcal{C}$; by the chain rule $\partial S^k(x)$ is :

$$T_k(x) \equiv \partial S^k(x) = \partial S(S^{k-1}x) \cdot \ldots \cdot \partial S(x) \equiv \prod_{j=1}^{k} \partial S(S^{k-j}x) \tag{2.1}$$

where the factors appear ordered from left to right as j increases.

I shall assume that for some $\bar{C}, \bar{\varepsilon} > 0$ it is $|\det \partial S(x)| \ge \bar{\varepsilon} > 0$ and $|\partial S(x)| < \bar{C}$, for all regular points x. This assumption could be greatly weakened, see the appended problems and [P], but it is convenient continuing the discussion without having to worry about such matter.
Therefore the square of the *stretching* of dx will be described by the ratio:

$$\frac{(\partial S^k dx, \partial S^k dx)}{(dx, dx)} \tag{2.2}$$

where the scalar products are evaluated in the metric defined on $\mathcal{C}$. If $G(x)$ is the metric tensor in x and if the *dilatation* matrix $(M_k)_{ij}$ is defined by setting: $M_k = (\partial S^k(x))^* \partial S^k(x)$, we see that in the local coordinates the matrix is expressed by: $(M_k)_{ij} = (\partial S^k(x))^T G(S^k x)(\partial S^k(x))_{ij}$. So that (2.2) can be explicitly written as:

$$\frac{\sum_{ij}(M_k)_{ij} dx_i\, dx_j}{\sum_{ij} G(x)_{ij} dx_i dx_j} \tag{2.3}$$

Suppose that the symmetric matrix $M_k^{1/2k}$ has a limit $D_+(x)$, as $k \to \infty$. Then tere is $\tilde{c}$ such that $\tilde{c} > |\det D_+(x)| \ge \bar{\varepsilon}$ by the assumption, and we can define $l_1(x) \ge l_2(x) \ge \ldots l_d(x) > 0$ to be its ordered eigenvalues, counted according to multiplicity. Let $\underline{v}_1(x), \ldots, \underline{v}_d(x)$ be the corresponding eigenvectors, which can be taken orthonormal with respect to the "naive" scalar product $(a,b)=\sum_i a_i b_i$ (which is the scalar product in which M_k and D_+ are symmetric). Then $M_k \underline{v}_j(x)$ grows as $l_j(x)^k$, in the sense that $\frac{1}{k}\log|M_k\underline{v}_j(x)| \xrightarrow[k\to\infty]{} l_j(x)$, as an elementary estimate shows.

We also call $\overline{l}_1(x) > \overline{l}_2(x) > \ldots > \overline{l}_{s(x)}(x)$, respectively $\overline{\lambda}_1(x) > \overline{\lambda}_2(x) > \ldots > \overline{\lambda}_{s(x)}(x)$, the *distinct eigenvalues* (respectively their logarithms) of $D_+(x)$ and $n_1(x),\ldots,n_{s(x)}(x)$ their multiplicities, *i.e.* the dimensions of the eigenspaces $U_1(x),\ldots,U_{s(x)}(x)$ spanned by the eigenvectors with the same eigenvalue. The $l_j(x), \lambda_j(x)$ can be called the spectral or scaling coefficients at x and, respectively, the spectral or scaling exponents at x.

Then it is clear that the vector space R^d can be regarded as containing the planes $V_1(x) \equiv R^d \supset V_2(x) \supset V_3(x) \supset V_{s(x)}(x)$ where $V_j(x)$ is the plane spanned by the eigenvectors of $D_+(x)$ with eigenvalue $\le \overline{l}_j(x)$: the dimensions of $V_j(x)$ are $d_j = n_j(x)+\ldots+n_{s(x)}(x)$ and the planes $V_j(x)$ are $V_j(x) = U_j(x)\oplus\ldots\oplus U_{s(x)}(x)$.

It is very important to realize that the vectors $\underline{v}_j$ *do depend on the choice of coordinates (and of the metric G)*: by changing the metric or the coordinates the vectors $\underline{v}_j$ may, in general, change. However the eigenvalues $\overline{l}_j(x)$, and their multiplicities $n_j(x)$ cannot change, and *also the sequence of decreasing subspaces $V_j(x)$ cannot change*, as one can see by noting that the scaling propeties of the vectors in such spaces are not metric poperties, but have an intrinsic geometric meaning. They can be *characterized* by the property that $\frac{1}{k}\log|\partial S^k(x)u| \xrightarrow[k\to+\infty]{} \overline{\lambda}_j(x)$ if $u \in V_j(x)/V_{j-1}(x)$.

One could try to define the planes with given contraction rate $\overline{\lambda}_j$ to be just $U_j(x)$. But such spaces would not have an intrinsic dynamical meaning: because one could "tilt" $U_j(x)$ sligthly and still keep the property that all the vectors contract at the rate $\overline{\lambda}_j$ (provided the tilting does not generate components along the $U_{j+p}(x)$ with $p>0$, of course).

Likewise one cannot assign a special meaning to the planes $U_1(x)\oplus\ldots\oplus U_j(x)$: this would in fact generate a system of planes such that if u is a vector in the j-th plane and not in the $(j-1)$-th then the exponent of dilatation will be $\overline{\lambda}_j$; *but such system is not uniquely determined*, for the same reasons discussed in the previous paragraph. On the other hand the system $V_j(x)$ has the property of being uniquely determined by the action of S, and it defines a system of planes along which the scaling size becomes weaker.

4 Definition: *Given the dynamical system $(\mathcal{C},S)$ a point $x\in\mathcal{C}$ admits a "system of scaling (or contracting) planes" for the forward motion, or for S, if:*
(1) x is a regular point.
(2) there exist numbers $\overline{\lambda}_j(x)$ and positive integers $s(x)$ and $n_j(x)$, with $j=1,\ldots,s(x)$, with the properties:
(3) the space R^d of the infinitesimal vectors out of x can be regarded as containing a sequence of $s(x)$ subspaces $V_1(x)\equiv R^d \supset V_2(x)\supset\ldots\supset V_{s(x)}(x)\supset V_{s(x)+1}\equiv \underline{0}$ with $V_j(x)$ having dimension $n_{s(x)}+\ldots+n_j(x)$ and
(4) the following limits hold:

$$\lim_{k\to+\infty}\frac{1}{k}\log|\partial S^k(x)v| = \overline{\lambda}_j(x) \qquad \text{if } v\in V_j(x)/V_{j+1}(x) \tag{2.4}$$

Then one says that the numbers $\overline{\lambda}_j(x)$ are the scaling exponents of S at x in the forward direction, their exponentials $l_j(x) = e^{\overline{\lambda}_j(x)}$ are the scaling coefficients, the number $n_j(x)$ is the multiplicity of the j-th coefficient. Sometimes the coefficients are repeated according to the multiplicity: in this case their number is, of course, exactly the phase space dimension. The scaling exponents (coefficients) for S are often called the forward Lyapunov exponents *(coefficients).*

Remarks:

(i) There is no reason why a point should admit a system of scaling planes.
(ii) A sufficient condition for the existence of contracting planes is that the limit of the sequence of matrices $(\partial S^k(x)^* \partial S^k(x))^{1/2k}$ exists and is a positive matrix D, see the discussion preceding the definition 4,
(iii) If $\overline{\lambda}_1(x) > 0$, $\overline{\lambda}_{s(x)} < 0$ and $\overline{\lambda}_j(x) \neq 0$ for all j's one says that x is a *hyperbolic point*. In this case if r_- is such that $\overline{\lambda}_j > 0$ for $j < r_-$ and $\overline{\lambda}_j < 0$ for $j \geq r_-$ the plane $V^s = V_{r_-}$ will be called the *contraction*, or *stable, plane*. The motion of a hyperbolic point is very unstable in, essentially, all possible senses.
(iv) If a point x admits a system of scaling planes so do all the points on the trajectory generated by x. Such points do have the same coefficients (multiplicities included) and the spaces $V_j(x)$ and $V_j(Sx)$ are related by: $V_j(Sx) = \partial S(x) V_j(x)$, for all j's. The spaces $U_j(x)$ instead *do not have, in general, any covariance property* because they are not intrinsically defined by the dynamics (see above).
(v) If x admits a system of scaling planes then it may not admit such a system in the dynamical system $(\mathcal{C}, S^{-1})$; and even if it does there is no reason why there should be any relation between the two systems or the relative exponents, c.f.r. problem (32).
(vi) Note, once more, that the subspaces in which the contraction exponent is $< \lambda$ for some real λ, intermediate between the exponents values, are well defined (and coincide with $V_j(x)$ if $\overline{\lambda}_{j-1} > \lambda > \overline{\lambda}_j$). But the ones in which the rate is $> \lambda$ are *not* well defined and this explains the impossibility of associating expanding planes to the forward motion.
(vii) Note that the scaling planes for S at a point, when existent, concern the *forward* motion: *i.e.* they are properties of the trajectory $S^k x$, $k \geq 0$. Likewise the scaling planes for S^{-1} concern only the *backward* motion.

In view of the above remarks it is important to try to establish some general results about the existence of systems of scaling planes.

A simple case concerns the regular hyperbolic fixed points x: the latter always admit a system of scaling planes for the forward and backward motion, *i.e.* for S and for S^{-1}. The linearization of the map S around x, *i.e.* the matrix $T = \partial S(x)$ defines the V_x^s plane spanned by the spectral planes of the eigenvalues of T with modulus < 1 and the subspace V_x^u spanned by the planes relative to the eigenvalues of T with modulus > 1. The two subspaces V_x^s, V_x^u can be continued into two small regular connected manifolds tangent to them in x, Δ_x^s, Δ_x^u, such that if $y \in \Delta_x^s$ then $|S^k y - S^k x| \xrightarrow[k\to\infty]{} 0$ bounded proportionally to l_-^k, if l_- denote the absolute values of the eigenvalue of T closest to the unit circle and < 1. Or, similarly, if $y \in \Delta_x^u$ or > 1 then $|S^{-k} y - S^{-k} x| \xrightarrow[k\to\infty]{} 0$ bounded proportionally to l_+^{-k}, if l_+ denotes the absolute value of the eigenvalue of T closest to the unit circle and > 1.

In this case the forward scaling coefficients coincide with the absolute values of the eigenvalues of D, called in stability theory the *Lyapunov coefficients* of the fixed point. They can also be defined as the eigenvalues of the matrix D (whose existence is not completely trivial, see problems): $\lim_{n\to\infty} ((T^*)^n T^n)^{1/2n} = D$. The backward scaling coefficients can likewise be identified with the eigenvalues of the matrix $\lim_{n\to\infty} ((T^{-k})^* T^{-k})^{1/2k} = D_-$ and, therefore, they are the reciprocals of the forward coefficients. The contracting plane V_x^u for S^{-1} is, also, spanned by the eigenplanes of D_- corresponding to the eigenvalues < 1.

It is convenient to set up the following definition for future use:

5 Definition: *If x is a fixed point the set $\cup_{k=0}^{\infty} S^k \Delta_x^u \equiv W_x^u$ will be called the "global unstable* *manifold", while the set $\cup_{k=0}^{\infty} S^{-k}\Delta_x^s \equiv W_x^s$ will be called the global stable manifolds of x.*

The two manifolds are locally regular around all their points that do not fall on the singularities of S^k or S^{-k}, respectively, for some $k \ge 0$. But they may be even disconnected, in general.

The subspace tangent to Δ_x^s at x coincides with the plane generated by the vectors associated with the Lyapunov numbers less than 1 which, as said above, is intrinsically associated with S. It consists of the infinitesimal vectors that contract at exponential rate under the action of S.

One cannot identify "similarly" the subspace tangent to Δ_x^u, at x, as the plane generated by the infinitesimal vectors expanding at exponential rate under S because, as already pointed out, such vectors are not unambiguously defined (given a vector expanding at exponential rate one can add to it a vector that expands at a lower rate, getting a vector expanding at the same rate). However one can identify Δ_x^u in the same way as Δ_x^s by replacing S with S^{-1}.

The extension of the analysis to the case of a periodic point is very easy and well known and it will be skipped. In the case x is not fixed, nor periodic, the discussion above leads to a natural extension expressed by the definition:

6 Definition: *A point is said to admit a dynamical base $(Z_1, \ldots, Z_s)$, consisting of s mutually transversal planes, if:*
(1) it is regular for S and S^{-1} and it admits, both in the forward and in the backward motions, scaling systems of planes with opposite Lyapunov exponents.
(2) no Lyapunov exponent vanishes.
(3) the following limit relation holds as $k \to +\infty$ as well as $k \to -\infty$:

$$\lim_{k\to\pm\infty} \frac{1}{k} \log |\partial S^k u| = \overline{\lambda}_j(x) \qquad \text{for } u \in Z_j \tag{2.5}$$

In this case the planes $V_j = Z_s \oplus \ldots \oplus Z_j$ will be called the system of contraction planes for S and the $\tilde{V}_j = Z_1 \oplus \ldots \oplus Z_j$ will be the system of expanding planes for S. If s_- is such that $\overline{\lambda}_j < 0$ for $j \ge s_-$ and $\overline{\lambda}_j > 0$ for $j < s_-$ the planes $V^s = V_{s_-}$ and $V^u = \tilde{V}_{s_- -1}$ will be called the stable and unstable planes of S at x.

Remarks:

(a) The corresponding notions for the backward motion (*i.e.* for S^{-1}) are trivially related to those for S.
(b) If $u \in V_j(x)/V_{j+1}(x)$ then (2.4) holds.
(c) The points $S^k x$ also admit a dynamical base and $Z_j(S^k x) = \partial S^k(x) Z_j(x)$ for all (signed) integers k.

Finally one more definition is useful to simplify the language:

7 Definition: *A point x is called normal if it admits a dynamical base.*

Hence a normal point is a generalization of a periodic point. And it is remarkable that such points do exist and in fact abund, in some sense. The first classical result concerns invariant ergodic distributions μ on $\mathcal{C}$: here invariance means that $\mu(\cap_{-\infty}^{\infty} S^k \mathcal{N})) = 0$ and $\mu(S^{-1}E) = \mu(E)$ for all (Borel)* sets E and ergodicity means that there are no non trivial constants of motion which are not μ–almost surely constant.

I Theorem: *Let μ be an invariant ergodic distribution for $(\mathcal{C}, S)$: then μ almost all points are normal for S. Furthermore the Lyapunov exponents and their multiplicities are almost*

* This is in parenthesis because all the sets I will mention are Borel sets or differ from a Borel set by a set of measure 0 with respect to the measure that is being considered and which I also call for brevity Borel sets if no ambiguity arises. In fact, I cannot even conceive the other sets.

everywhere constant.
If no contraction or expansion coefficient has value 1 *and if* V^s_x *and* V^u_x *are the contracting and expanding planes, then under further "weak" regularity assumptions, there exist (integrability property) manifolds* W^u_x *and* W^s_x *tangent in* x *to* V^u_x *and* V^s_x*, smooth near* x*, such that:*

$$
\begin{aligned}
|S^k x - S^k y| \xrightarrow[k\to\infty]{} 0, &\quad \textit{for } y \in W^s_x \\
|S^{-k} x - S^{-k} y| \xrightarrow[k\to\infty]{} 0, &\quad \textit{for } y \in W^u_x
\end{aligned}
\tag{2.6}
$$

and the approach to 0 *takes place exponentially fast bounded above proportionally to* $e^{-\lambda k}$*, where* $\lambda > 0$ *is a suitable constant (such that no Lyapunov exponent is in the interval* $[-\lambda, \lambda]$*).*

A discussion of the "further regularity assumptions" would lead us away from the themes chosen here for discussion: I just mention that they are assumptions on the speed at which a regular point x can approach, in its evolution, the singularities, [P]. The idea is that such speed should be slower than any exponential (or at least slower than the speed corresponding to a bound λ on the contraction and expansion rate).

The above first statement is the *Oseledec theorem*: a "guided" proof to it is described in the problems, the statements concerning the integrability property are part of Pesin's theory, [P]. The theorem shows that if the data are picked up randomly with respect to an ergodic distribution (any one!) then they provide (non constructive) examples of points with dynamical bases. It also shows that the scaling properties of S and those of S^{-1} are intimately related and the two maps show the "same" properties (or, rather, properties that are trivially related), provided the data are randomly distributed with an *invariant* ergodic distribution giving zero measure to data visiting, in their evolutions, the singularities. "Nothing about data randomly chosen with an egodic invariant distribution can be learnt by running the motion backwards, that cannot be learnt by running it forward". In other words one can say that the motion is "reversible" *on such data.*[4]

The case in which there are 0 contraction or expansion exponents is more involved and it will not be discussed. It certainly arises when the system has non trivial smooth constants of motion, each of which generates a vanishing Lyapunov exponent. But such cases can be easily eliminated because one usually can restrict the phase space to the data for which the constants of motion have a fixed value. When zero Lyapunov exponents are not related to smooth constants of motion, however, a genuinely more complicated situation arises: I shall not deal with it here, besides saying that some of the above kinematical properties *do not depend on the assumption of the absence of* 0 *Lyapunov exponents.* In particular the first sentence in the theorem I above *does not require* that the exponents be non zero.

Also the ergodicity, in some sense, is not really necessary and it has been introduced only to simplify the exposition. The case in which ergodicity is missing can be often reduced to the ergodic case by using the "ergodic decomposition theorem": a theorem and a discussion that I avoid here.

A more interesting question is what can be said when the data are picked up with a distribution that is *not* invariant. The case of non invariant distribution of the initial data will be analyzed, under suitable further assumptions, in the next section. Here I add only a few remarks on a frequently misrepresented procedure.

Note that the Lyapunov exponents for S and those of S^{-1} are opposite *if the data are chosen randomly with respect to an invariant distribution.* However usually one selects the data with respect to a *non invariant distribution*: in this case the above theorem does not say much. *In particular one should expect that the asymptotic properties of the motions in the future and in the past are different and described by different statistics.* For instance in cases in which there is a *time reversal symmetry* one will find that the Lyapunov exponents for the motions towards

[4] Remember, however, that such data are very special.

the past and the future are identical (rather than opposite). This will not contradict the fact that if the data are chosen randomly with respect to the *future* statistics then the Lyapunov exponents for the motion towards the past would be the *opposite* of the ones for the motion towards the future!

How comes that often a method to measure the minimum Lyapunov exponent is *said* to be: "just measure the maximum Lyapunov exponent for the backward motion and change sign"? In most cases this would seem wrong: for instance when the system is time reversible *because* one would, instead, get again the maximum Lyapunov exponent.

The "paradox" is understood if one examines what is really meant by the above statement, *i.e.* what is the actual measurement performed. One finds that the measurement consists (schematically) in picking at random with a distribution μ_0 (non invariant) a point x, following its trajectory for a long time T, togheter with the trajectory of a nearby point y until it reaches a point $S^T x$. The rate of separation of the two points gives[5] a measurement of the maximum Lyapunov coefficient. The trajectory of x is memorized and that of y "thrown away". One then starts from $S^T x$ and from a nearby point y', and runs the two motion backwards: of course the motion of $S^T x$ is already known and one computes that of y'. The motions of both points y, y' is computed as a perturbation of the motion of x or of $S^T x$ (forward and backwards respectively) thus it is very convenient to use twice the same trajectory as this saves considerable time.

But this is not merely a matter of convenience: we see that $S^T x$ is *no longer a random point with the initial distribution.* It is, rather, approximately a random point with respect to the statistics $\overline{\mu}$ of the distribution μ_0. Therefore it will show in the backward motion Lyapunov exponents opposite to the ones exhibited in the forward motion (which have the *same* value if the initial data are chosen with respect to μ_0 or $\overline{\mu}$), by the above theorem. Hence one gets the wanted result: in the *improper sense just discussed* the maximum exponent in the backward motion is the opposite of the minimum in the forward motion. This property is very reminiscent of the build up of correlations that appears in the theory of the molecular chaos in the Boltzmann equation, see [CB].

§3 Chaotic motions.

The definition that I shall use for a chaotic system is:

8 Definition: *A dynamical system $(\mathcal{C}, S)$ will be called "chaotic" if:*
(1) There exists a periodic hyperbolic point O such that the global stable and unstable manifolds of O consist of regular points for $S^{\pm 1}$, and are smooth, connected and dense on $\mathcal{C}$, (instability axiom).
(2) The restrictions of S^k to W^u_O, and of S^{-k} to W^s_O, are uniformly expansive for $k \ge 0$ and uniformly contractive for $k \le 0$, (expansivity axiom).
(3) If x_n, x'_n are points on W^u_O and $|x_n - x'_n| \xrightarrow[n\to\infty]{} 0$ then the tangent planes to W^u_O at such points become parallel uniformly at speed $|x_n - x'_n|^\beta$ for some $\beta > 0$, togheter with the matrix that linearizes the evolution S on W^u_O: a similar property holds for W^s_O; (continuity axiom).
(4) The manifolds W^s_O, W^u_O form, everywhere they cross, an angle uniformly bounded away from 0, (transversality axiom).
(5) Given any open sets $G, \Gamma_1, \ldots, \Gamma_n$ there is a k such that $S^h G$ has intersection with all the Γ_j for $h \ge l$ (topological mixing axiom).

Remarks:

(i) To understand the role of the various conditions in the above definition note that from the hyperbolicity of O, and from the definitions of the previous section, it follows that the tangent

[5] Unless the two points are very special, *e.g.* if y lies on the lower dimensional contracting manifold W^s_x.

planes to the manifolds W^u_O and W^s_O are the expansive and contractive scaling manifolds of O. However it does not follow from this only fact that the action of S on W^u_O, nor that the action of S^{-1} on W^s_O, are expansive. The latter property, as well as any statement on the Lyapunov exponents on W^u_O, concern properties of the *future* evolution on W^u_O, while the knowledge that W^u_O is the unstable manifold of the fixed point gives us only information about the motion towards the past. Hence it cannot follow that $(\partial S^k(x)^*_u \partial S^k(x)_u)^{1/2k}$, where $\partial S(x)_u$ denotes the jacobian of the transformation S regarded as a map of W^u_O, has a limit as $k \to \infty$. The role of (2) is to require such properties, or at least the part of them that we need, explicitly (as well as the corresponding ones for W^c_O). Its quantitative meaning is expressed as: there exist $D_0, \lambda > 0$ such that for all $x, y \in W^u_O$ and $k \ge 0$ the distance $d_u(S^{-k}x, S^{-k}y)$ between $S^{-k}x, S^{-k}y$, *measured along* W^u_O, is:

$$d_u(S^{-k}x, S^{-k}y) \le D_0 e^{-\lambda k} d_u(x, y) \tag{3.1}$$

and the corresponding properties are required to hold for W^s_O for $k \le 0$.
(ii) Property (3) on the jacobians means that there exists $D_1, \beta > 0$ such that the "unit vector orthogonal to W^u_O at x", which is in fact a tensor that we can denote $\partial W^u_O(x)$ verifies:

$$|\partial W^u_O(x) - \partial W^u_O(y)| \le D_1 d(x, y)^\beta \tag{3.2}$$

for all $x, y \in W^u_O$ and that there exists $D_2, \beta > 0$ such that the jacobian $\partial S(x)_u$ of S as a map of W^u_O into itself verify, for all $x, y \in W^u_0$:

$$|\partial S(x)_u (\partial S(y)_u)^{-1} - 1| \le D_2 d(x, y)^\beta \tag{3.3}$$

The analogous conditions are also imposed on W^s_O and on S^{-1}.
(iii) the property (4) can be put in a quantitative form as follows: if dw^u and dw^s are two surface elements tangent to W^u_O and W^s_O, respectively, at a common point x then the Liouville volume of the parallelogram generated by them is $\mu_0(dw^u dw^s) = b(x) dw^u dw^s$ and $e^{-B_0} < b(x) < e^{B_0}$ holds for some constant B_0.
(iv) The continuity above is a "Hölder continuity"; it is convenient here being generous on the weakness of this assumption and *not* requiring Lipshitz continuity or higher smoothness. Even very smooth dynamical systems may have just Hölder regularity of the foliation of phase space into stable or unstable manifolds.
(v) The definition essentially yields what is usually called an Anosov system. A more general notion (of axiom A-system) could be envisaged. However in the cases considered here the attractors will always be dense in the full phase space and the above generality will be sufficient. All the items in the definition are essential; but in a sense the axiom (1) is the major one among them.

In this section I suppose the system to be chaotic in the above sense. The following analysis (essentially due to Sinai, Bowen and Ruelle, [S2], [Bo], [R2]), will show that *there is a well defined statistics for the Liouville measure* and the attractor will be dense on $\mathcal{C}$. The statistics on it can be determined quite explicitly and it can be shown to have strong ergodic properties. The existence of a well defined statistics plays the role of the ergodic hypothesis and it will therefore be called *ergodicity property*: it will appear that it has been introduced here under the disguise of the chaoticity assumptions (essentially the smoothness and density of the stable and unstable manifolds of O).

We first describe how the measures $\overline{\mu}^{\pm}$, the past and future statistics of the Liouville measure, can be characterized. An apparently involved geometric construction is necessary: it is very simple in the case in which W^s_O and W^u_O are 1*–dimensional*, hence the phase space $\mathcal{C}$ has dimension $d = 2$. I shall, however, provide a sketch of the construction in the general case: but one should first understand it (via drawings) in the trivial $d = 2$ case, to realize later that the general case is its natural generalization. The geometrical construction leads to the definition 9 below of *Markov pavement*: the construction that follows is somewhat different from the classical constructions of [S2], [Bo1] and, admittedly sketchy (except in the $d = 2$ case, in which it becomes a well known construction whose generalization is attempted here): its details will not be really used in the following and one could invoke here the theorem of Sinai stating the existence of a Markov pavement, *i.e.* proceed to the paragraph preceding definition 9 below.

Given a small ball of radius δ (small compared to the curvature radii of the manifolds W^u_O, W^s_O) centered at $x \in W^u_O$ the manifold W^u_O will intersect the ball on a (dense) family of connected surfaces: only one of them will contain x and it will be called "the disk on W^u with center x and radius δ". The disks with radius δ on W^s are likewise defined.

The assumed density and continuity of W^s_O and W^u_O allows us to define "disks on W^u or W^s" centered at any point x: they are the surfaces obtained as limits of a sequence of disks of radius δ on W^s_O (or W^u_O) centered at the points $x_n \in W^s_O$ (or $x_n \in W^u_O$) of a sequence of converging to x as $n \to \infty$. And the union of the disks that match smoothly with a disk on W^s (or W^u) centered at x will define the manifold W^s_x (or W^u_x).

Let Δ, Δ' be two small disks on W^u_O and, respectively, W^s_O centered at O. Let ϑ_0 be so large that the web generated by $S^{\vartheta_0}\Delta$ and $S^{-\vartheta_0}\Delta'$, fills the phase space $\mathcal{C}$ so densely that there is no point further away from the web than a prefixed $\delta > 0$, and so that the disk of radius δ on W^s centered at any $x \in S^{\vartheta_0}\Delta$ has at least another intersection with $S^{\vartheta_0}\Delta$ (and "viceversa", exchanging Δ, Δ' and S, S^{-1}).

On $S^{\vartheta_0}\Delta$ we consider all the points $x_1, x_2, \ldots$ common to $S^{-\vartheta_0}\Delta'$: they form a rather dense set of points on $S^{\vartheta_0}\Delta$, and by taking ϑ_0 large we can suppose that there is a triangulation $\mathcal{T}$ of $S^{\vartheta_0}\Delta$ with base on such points[6] consisting of smooth triangles T^u_j which have vertices at distances $\le \delta$. By slightly deforming Δ we can also suppose that the boundary $\partial(S^{\vartheta_0}\Delta)$ consists of sides of some of the triangles of $\mathcal{T}$.[7] Let a be the *smallest distance between distinct triangles vertices* and let ε be the smallest opening angle at the vertices.

Let ∂^s_δ be the surface formed by drawing through *each* x on the *boundary* of *each* triangle a

[6] A smooth triangle (one should call it pyramid but I use the more reassuring word triangle to make the object look simpler) T is d–dimensional smooth manifold W consisting in a compact set with connected interior containing $d+1$ points, the *vertices*, and a boundary formed by joining them with d smooth surfaces of dimension $d-1$ each, *faces*, containing a different d–ple of the $d+1$ vertices and with the further property that the set is *conical* (see below) around each vertex and with the intersections of any pair of them forming a smooth $d-1$–dimensional triangle, *face*. The definition is recursive if one declares that a 0 dimensional triangle is a point and a 1 dimensional triangle is a smooth arc. Here conical means that at every vertex one can draw an open cone: this also requires a definition, as the surface may be not flat; for instance one can consider a family of smooth curves emerging from the vertex and tangent at the vertex to a true open right cone with apex at the vertex and opening angle $\varepsilon > 0$, and with all the points close enough to the vertex entirely contained inside the interior of the triangle. If the boundaries of the various triangles are required to be only Hölder continuous rather than smooth one obtains the more general notion of triangle: the key transversality property remains, however, unchanged for such more general triangles and we call $\varepsilon(T), \eta(T)$ the minimum of the opening angles at the vertices of the various faces of the triangle and, respectively, the minimum Hölder continuity exponents of the various faces. Note that all the d dimensional triangles have boundaries with 0 d–dimensional volume. A triangulation of a compact manifold with base on a given family of points is a covering by a family of triangles with the vertices on the base, no interior points in common and no base point in their interior. All this is quite trivial in the case the triangle is 1 dimensional. Consider a family of points, distributed on a compact manifold with smooth boundary, dense enough so that every point has another one closer than δ to it. Then it is possible to build a triangulation based on the given points by trimming the manifold boundary by an amount not exceeding, say, twice δ if δ is small enough. I take this as obvious, although a proof is likely to be somewhat verbose.

[7] The deformation is at most of the order of $\delta e^{-\lambda\vartheta_0}$.

disk on W^s centered at x and of radius δ. This is a codimension 1 surface. It is not difficult to see that if the expansion rate λ of S is large enough (so that $\delta D_0 e^{-\lambda} \ll a$) then we can slightly modify the sides of the triangles (hence Δ as well) so that the image of ∂^s_δ is contained in itself. The expansion rate will, however, be in general much weaker than needed. In that case we imagine, for the time being, replacing S with a high iterate of S so that the new transformation meets the expansivity requirement.

The construction can be done by successive approximations, recursively, as every change of a given triangle boundary performed to impose the wanted condition, will ruin the validity of the condition for some other triangle (but by a smaller amount). One will only be able to infer that the final triangles have Hölder continuous boundaries, even when the W^u_O and W^s_O are (locally) flat (it is not difficult to see that the exponent of Hölder continuity should be bounded below by $\eta = e^{-(\lambda_+ - \lambda_-)}$ if λ_+ is the maximal expansion rate and $\lambda_- \ge \lambda$ is the minimal expansion rate on W^u, which will be in general different: just think at what happens to an angle when the abscissa and the ordinate are contracted by a different scaling factor). In the case of one dimensional Δ, however, the construction is clearly completed "in one stroke" and the triangles are just smooth segments on W^u_O (and their boundaries are isolated points, hence necessarily smooth (0-dimensional manifolds). The reason why in general the construction leads to triangles with mildly regular boundaries can be understood through the celebrated elementary Bowen's example, [Bo2], ultimately relating it to the direction dependence of the expansion rate on the unstable manifold, when it has dimension > 1 and at least two distinct positive Lyapunov exponents (so that $\lambda_+ > \lambda_-$ as expected from the above bound on the Hölder continuity exponent). The deformation of the triangles, during the construction, is small (at most of the order of $\delta e^{-\lambda\vartheta_0}$) but the regularity of the deformation is quite out of control (because of the mechanism in [Bo2]).

An identical construction can be performed on the stable manifold leading to a triangulation $\mathcal{T}^s$ of $S^{-\vartheta_0}\Delta'$, with Hölder cotinuous triangles.

If $\mathcal{T}^u$ and $\mathcal{T}^s$ are the above triangulations of W^u_O and W^s_O we can consider the pairs of triangles T^u_h, T^s_k with one point in common and consider the sets $E_j = T^u_h \times T^s_k$ built with the points obtained by fixing $\xi \in T^u_h$ and $\eta \in T^s_k$ and by drawing the disk with radii 2δ on W^u_η centered at η and on W^u_ξ centered at ξ and by considering their intersection $x = x(\xi, \eta)$ as ξ, η vary in T^u_h and T^s_k, (here δ may have to be required to be small enough, always with respect to the curvature radii of the manifolds, so that the intersection defining x is certainly non empty).

Such sets will be called *prisms*: the name "parallelogram" or "rectangle" is used in the literature: but the construction here is somewhat different from the original one of [S2],[Bo] (which is not based on triangulations). Furthermore even in the original construction the sets called parallelograms look to me, no matter how I attempt to draw them, prisms except of course in the two dimensional case, when they really look like parallelograms. The sets T^u_h and T^s_k will be called, respectively, the *horizontal* and the *vertical axis* of the prism $E_j = T^u_h \times T^s_k$. Therefore a natural name for the sets $\partial^u E_j \equiv T^u_h \times \partial T^s_k$ and $\partial^s E_j \equiv \partial T^u_h \times T^s_k$ will be *horizontal* boundary of the prism E_j and, respectively, *vertical boundary* of E_j (or unstable and stable boundary of E_j). The set $\partial^s = \cup_j \partial^s E_j$ $\partial^u = \cup_j \partial^u E_j$ define the *total vertical* or the *total horizontal boundary* of the family of prisms E_j which themseves constitute a *pavement* of the phase space with small *prisms*.

The above notion of prism as a set having the form $E = T^u \times T^s$, see the above two last papragraphs, where T^u and T^s are small connected surfaces, with Hölder continuous boundaries, on the stable and unstable manifolds of a common point x is more general; so that one can consider a pavement $\mathcal{E}$ of the phase space $\mathcal{C}$ with prisms, $\mathcal{E} = (E_1, \ldots)$ and define, likewise, the stable and unstable boundaries $\partial^u_j, \partial^s_j$ of the prisms E_j and the total horizontal and the total vertical boundaries ∂^u and ∂^s: note that ∂^s, ∂^u have zero volume (by the Hölder continuity of the boundaries). Given such concepts the following definition is useful:

9 Definition: *A pavement $\mathcal{E} = (E_1, E_2, \ldots)$ by prisms is called a* Markov pavement *(or a* Markov partition*) if the total stable boundary is mapped into itself under the action of S and the total unstable boundary is mapped into itself under the action of S^{-1} and they have both* 0 *volume.*

The key property of the special pavement $\mathcal{E}$ constructed above, and motivating the last definition, is that by construction the total vertical boundary of the prisms is mapped into (a small portion of) itself under the action of S and likewise the total horizontal boundary ∂^u is mapped into (a small portion of) itself under the action of S^{-1}. Hence we have just seen that special Markov partitions, associated with the stable and unstable manifolds of a hyperbolic fixed point, exist for a chaotic system at least if S is replaced by a high enough power S^K of S. Such special Markov partitions are particularly suited for numerical applications, see [FZ].

But it follows immediately from the definition 9 just given (and the covariance of W^s_x, W^u_x: $SW^s_x = W^s_{Sx}, SW^u_x = W^u_{Sx}$) that if $\mathcal{E}$ is a Markov partition for S^K then the partition $S^p\mathcal{E}$ (whose prisms are obtained by transforming with the map S^q those of $\mathcal{E}$) is *also* a Markov partition for the iterate S^K of S. Furthermore the partition obtained by intersecting the prisms of the partitions $S^p\mathcal{E}$ with p taking K consecutive values is a Markov partition for S. Hence any dynamical system which is chaotic in the sense of definition 8 admits a Markov partition $\mathcal{E}$, [S2].

We can form, given any integer ϑ, and a Markov partition $\mathcal{E}$ a more general and finer Markov partition $\mathcal{E}_\vartheta$ simply by "intersecting" the partitions $S^{-\vartheta}\mathcal{E}, S^{-\vartheta+1}\mathcal{E}, \ldots, S^{\vartheta}\mathcal{E}$. We shall only consider sequences of Markov partitions of the above form $\mathcal{E}_\vartheta$.

Given $\mathcal{E}$ we consider the map $\Sigma:\ x \to \underline{\sigma}(x)$ mapping x to the sequence σ_j defined by $S^jx \in E_{\sigma_j}$, the *history* of x on the partition $\mathcal{E}$.

This map is unambiguosly defined for all x such that S^kx does not fall on the boundary of any one of the parellelograms E_σ of $\mathcal{E}$. If we define the *compatibility matrix* element $M_{\sigma\sigma'}$ to be 1 if $SE^0_\sigma \cap E^0_{\sigma'} \neq \emptyset$, and 0 otherwise, where E^0_σ is the set of interior points of E_σ, then it is clear that the history $\underline{\sigma}(x)$ of a point x, for which it is unambiguosly defined, is an *allowed sequence*: in the sense that $\prod_{j=-\infty}^{\infty} M_{\sigma_i,\sigma_{i+1}} = 1$. Viceversa, and this follows directly from the mappimng properties of the boundaries of the prisms in definition 9 (attempt a proof only after finding via a drawing in the trivial $d = 2$ case), if $\underline{\sigma}$ is an allowed sequence there is always a point x producing $\underline{\sigma}$ as history, provided one chooses conveniently the definition of $\sigma_j(x)$ in the cases in which this quantity is ambigously defined (because some iterate of x falls on the boundary of some E_σ). Item (5) of the chaoticity assumption, in definition 8, implies that the matrix M is "transitive", *i.e.* there is an integer p such that M^p has all entries non vanishing, [S2]. This important property may be implied by other more topological properties, see the discussion of the axiom–A in [Sm], but here we do not discuss this point (in the same spirit animating what precedes and follows, of not trying to get "optimal theorems" operating with "minimal assumptions").

If $\underline{\sigma}$ is the history of x: $\underline{\sigma} = \Sigma x$, then the history of Sx is $\varphi\underline{\sigma}$ where φ is the *shift* to the left of $\underline{\sigma}$. This means: $\Sigma(Sx)_j = (\Sigma x)_{j+1}$.

Even when the history of a point is ambiguous it can be seen that the number of possible *allowed* sequences is never more than a number that can be explicitly bounded in terms of the maximum number of triangles with one vertex in common and the dimensions of W^s, W^u. But we do not have to worry about this, since the boundaries of the prisms of $\mathcal{E}_\vartheta$ have zero volume measure and we are interested only on properties valid for the data x which have probability 1 with respect to the Liouville measure.

Therefore, for our purpose, we can think that the points of $\mathcal{C}$ can be described by allowed (in the above sense) sequences of symbols $\underline{\sigma}$: *we shall denote K the space of such sequences,* defining on it a metric that sets to 2^{-q} the distance between two sequences which agree on the sites $-q,\ldots,q$ but not on any larger symmetric interval. Here 2 is arbitrarily chosen and any number >1 could replace it, generating the same topology on the space of sequences. In other words we can use the sequences $\underline{\sigma}$ as a system of coordinates (a slight variation on the usual representation of the cartesian coordinates in some digital representation of the reals, *e.g.* base 10; but the present representation is intrinsically tied to the dynamics).

It is convenient to think of a history sequence $\underline{\sigma}=(\sigma_j)_{j=-\infty}^{\infty}$ as a *configuration of a one dimensional spin system*, to make more striking the analogy (that will emerge below) with the theory of one dimensional statistical mechanics : thus we call the labels j, labeling the times marking the history (in units of t_0), with the name of *sites.* Hence the value of $\underline{\sigma}$ at the site j will be σ_j.

The Liouville measure μ_0 becomes a probability distribution μ_0' on the space of the allowed sequences. In fact consider the set of sequences denoted

$$G'=\begin{pmatrix} -q & . & . & . & q\\ \sigma^0_{-q} & . & . & . & \sigma^0_q\end{pmatrix},$$

consisting in the sequences $\underline{\sigma}$ whose values σ_j with $j=-q,\ldots,q$ coincide with the given σ^0_j. Such set G' will be naturally given a probability equal to the μ_0–measure of the set: $G=\Sigma^{-1}G'$, if Σ is the above history map.

Thus setting $\mu_0'(G')\equiv\mu_0(G)$, with $G=\Sigma^{-1}G'$, for any (Borel) set G' in the space K allows us to think of the dynamical system $(\mathcal{C},S)$ as (K,φ) and the probability measure μ_0 becomes μ_0'.

The main point is that there is a simple formula expressing the probability $\mu_0'(G')$, (Sinai).

To find it just remark that the set G is (clearly) a prism for the Markov partition $\mathcal{E}_q$ (because $G\equiv\cap_{r=-q}^{q}S^{-r}E_{\sigma_r^0}$). If β^s_σ denotes the surface of the vertical axis T^s_σ of $E_\sigma=T^u_\sigma\times T^s_\sigma$ lying on W^s_O then the area of the vertical axis of G is essentially given by: $\beta^s_{\sigma_{-q}}\prod_{j=-q}^{-1}\Lambda_s(S^jx)$, where x is a point in G and $\Lambda_c(x)$ is the absolute value of the jacobian determinant of S, at x, as a map from W^s_x to W^s_{Sx}.

Likewise if β^u_σ is the surface of the horizontal axis T^u_σ of E_σ on W^u_O then the area of the horizontal axis of G is, essentially: $\beta^u_{\sigma_q}\prod_{j=0}^{q-1}\Lambda_u^{-1}(S^jx)$ where $\Lambda_u(x)$ is the absolute value of the jacobian determinant of S, at x, as a map from W^u_x to W^u_{Sx}.

Therefore the μ_0 measure of G is, essentially:

$$\beta^s_{\sigma_{-q}}\Big(\prod_{j=-q}^{-1}\Lambda_s(S^jx)\Big)\cdot\Big(\prod_{j=0}^{q-1}\Lambda_u^{-1}(S^jx)\Big)\beta^u_{\sigma_q}\cdot b(x) \tag{3.4}$$

where $b(x)dwdw'$ is the volume element (*i.e.* the $\mu_0(dwdw')$ corresponding to two surface elements dw,dw' tangent to, respectively, W^u_x,W^s_x (see remark (iv) above). The "essentially" means that there is an error in (3.4) due to the finite size of the sides of G: see below for its treatment.

To connect (3.4) with something more familiar we can go through the definitions and remarks that led Ruelle, [R1],[R3], to call what follows the *thermodynamic formalism* for strange attractors. Define:

$$\begin{aligned} h_+(\underline{\sigma})&=\log\Lambda_u(x(\underline{\sigma})),\qquad h_-(\underline{\sigma})=-\log\Lambda_s(x(\underline{\sigma}))\\ h_s(\underline{\sigma})&=\log\beta^s(\sigma_0),\qquad h_u(\underline{\sigma})=\log\beta_u(\underline{\sigma}_0),\qquad h_0(x)=-\log b(x(\underline{\sigma}))\end{aligned} \tag{3.5}$$

and remark that h_0,h_+,h_- as functions of σ_j depend very little on the σ_j's with j large in the sense that ther exist constants $H_\gamma,\kappa>0$ such that for $\gamma=u,s,+,-,0$:

$$|h_\gamma(\underline{\sigma})-h_\gamma(\underline{\sigma}')|<H_\gamma e^{-\kappa q} \tag{3.6}$$

if $\sigma_i = \sigma'_i$ for $|i| < q$. This follows again from the the continuity and hyperbolicity assumption, because varying $\underline{\sigma}$ on the sites $j > k$-th means varying $x(\underline{\sigma})$ in

$$\Sigma^{-1}\begin{pmatrix} -k & . & . & . & k \\ \sigma_{-k} & . & . & . & \sigma_k \end{pmatrix}$$

which is a prism whose diameter has size of order $O(\delta e^{-\lambda k})$, see (3.1). Therefore, by the continuity assumption in definition 8, we see that the variation of h_j, $j = 0, \pm, u, s$, is bounded by $O(e^{-\lambda k\beta}\delta^\beta)$. This is usually quoted by saying that dependence of h_j on σ_k "vanishes exponentially" or h_j has "memory vanishing exponentially".

In terms of the functions $h_\gamma, \gamma = s, u, +, -, 0$ we can rewrite (3.4) as:

$$e^{-h_s(\varphi^{-q}\underline{\sigma}) - \sum_{j=-q}^{-1} h_-(\varphi^j\underline{\sigma}) - h_0(\underline{\sigma}) - \sum_{j=0}^{q-1} h_+(\varphi^j\underline{\sigma}) - h_u(\varphi^q\underline{\sigma})} \tag{3.7}$$

This shows, unless the approximation involved in the discussion of the word "essentially" above spoils everything, that the probability distribution μ'_0 on K coincides with a Gibbs' state on K for the short range *non translation invariant* formal hamiltonian:

$$\sum_{j=-\infty}^{-1} h_-(\varphi^j\underline{\sigma}) + h_0(\underline{\sigma}) + \sum_{j=0}^{\infty} h_-(\varphi^j\underline{\sigma}) \tag{3.8}$$

Such a state, to the far right (*i.e.* in the far future) corresponds to the Gibbs state $\overline{\mu}'_+$ with hamiltonian $\sum_{-\infty}^{\infty} h_+(\varphi^j\underline{\sigma})$ and to the far left (*i.e.* in the far past) corresponds to the Gibbs state $\overline{\mu}'_-$ with hamiltonian $\sum_{-\infty}^{\infty} h_-(\varphi^j\underline{\sigma})$.[8]

The last statement defines unambiguously $\overline{\mu}'_\pm$ as probability measures on K, via the theory of one dimensional Gibbs states ([R5],[Bo],[Ga3]). This is so because the Gibbs states potentials have short range (by the above remark on the shortness of the memory, the potential decreases exponentially, see the paragraph following (3.5)), and no phase transitions are possible, the system being one dimensional. By the map Σ the distributions $\overline{\mu}'_\pm$ are transformed into the natural candidates for the forward and backward statistics $\overline{\mu}_\pm$ for the distribution μ_0.

Hence the only problem is the check that μ'_0 is really the Gibbs state on the allowed sequences with formal energy (3.8): this means discussing quantitatively the error mentioned in connection with (3.4). All the properties of $\overline{\mu}^\pm$ would then follow from the well known theory of the one dimensional short range Gibbs states for spin systems, [R5],[Bo1], including the property that the $\overline{\mu}'_\pm$ are the forward and the backward statistics of μ'_0. The latter property holds simply because the Gibbs state with hamiltonian (3.8) really converges to $\overline{\mu}'_+$ or to $\overline{\mu}_-$ if it is observed to the far right or to the far left. Thus the characterization of $\overline{\mu}^\pm$ would have received a complete description.

To correct the error involved in the above use of the word "essentially" one can use the deeper properties of Gibbs states; namely their characterization in terms of the DLR equations. To use the DLR theory one needs an expression of the ratio of the probabilities that two sequences $\underline{\sigma}^1$ and $\underline{\sigma}^2$ have the value σ or σ' at site 0, conditioned to the two sequences having the same value in all the other sites. This is the ratio of the probabilities for the events in which the spin at site 0 is σ or σ', *having fixed* the same configuration for the spins at *all* the other sites.

This means considering the ratio of the probabilities:

$$\lim_{q\to\infty} \frac{\mu'_0(\begin{pmatrix} -q & . & . & -1 & 0 & 1 & . & . & q \\ \sigma_{-q} & . & . & \sigma_{-1} & \sigma & \sigma_1 & . & . & \sigma_q \end{pmatrix})}{\mu'_0(\begin{pmatrix} -q & . & . & -1 & 0 & 1 & . & . & q \\ \sigma_{-q} & . & . & \sigma_{-1} & \sigma' & \sigma_1 & . & . & \sigma_q \end{pmatrix})} \tag{3.9}$$

[8] One should not forget that (3.8) is defined for the allowed sequences. Hence one should more precisely say that the hamiltonian of our spin system is (3.8) if the configuration is allowed and it is $+\infty$ otherwise: *i.e.* the spin system has a (short range) *hard core* coresponding to the compatibility condition.

where $\underline{\sigma} = (\sigma_j)_{-\infty}^{\infty}$ is an arbitrary (allowed) sequence. Then clearly the error involved in (3.4) can be regarded as a multiplicative factor correction to the ratio in (3.9) approaching 1 as $q \to \infty$. If $\underline{\sigma}^1$ and $\underline{\sigma}^2$ denote two allowed sequences differing only in the entry with label 0, one gets that the value of the limit in (3.9) is, rigorously:

$$e^{-\sum_{-\infty}^{-1}(h_-(\varphi^j\underline{\sigma}^1)-h_-(\varphi^j\underline{\sigma}^2))-(h_0(\underline{\sigma}^1)-h_0(\underline{\sigma}^2))-\sum_0^\infty(h_+(\varphi^j\underline{\sigma}^1)-h_+(\varphi^j\underline{\sigma}^2))} \tag{3.10}$$

which is the DLR characterization of μ_0' as the Gibbs state with formal hamiltonian (3.8): proving, by the uniqueness theory of the short range one dimensional Gibbs states and the DLR theory of the Gibbs states, that indeed μ_0' *is the Gibbs state with energy (3.8)*, [Bo1],[Ga3].

Hence the forward and backward statistics $\overline{\mu}^{\pm}$ are precisely identified and described, because the theory of one dimensional short range Gibbs states is very well developed. *It also follows that $\mu_\pm$ are ergodic (in fact isomorphic to a Bernoulli shift) measures*, (see [Ga2]).

We proceed to derive some properties of $\overline{\mu}^+$ by making full use of the short range (3.6) of the potential h_+ generating $\overline{\mu}^+$ as a Gibbs state: they will be used in the application in §4.

Let $\Gamma_T, T > 0$ denote the set of the allowed strings $\sigma_j^T, |j| < T$: these are the strings that can becontinued to infinite allowed strings $\underline{\hat{\sigma}}^T \in K$: we in fact imagine to continue each $\underline{\sigma}^T$ to some $\underline{\hat{\sigma}}^T$ (arbitrarily).

Then $\overline{\mu}^+$ can be constructed as a limit of the distributions μ_T defined by the average they assign to an arbitrary smooth function F:

$$\int \mu_T(dx)F(x) = \frac{\sum_{\underline{\sigma}^T\in\Gamma_T} e^{-\sum_{-T}^T h_+(\varphi^j\underline{\hat{\sigma}}^T)}F(x(\underline{\hat{\sigma}}))}{Z_{[-T,T]}} \tag{3.11}$$

where $Z_{[-T,T]}$ is the normalization factor,that should be called the "partition function" of the energy h_+ relative to the interval $[-T,T]$.

Since $\overline{\mu}^+$ is the Gibbs state with potential h_+ it follows that $\overline{\mu}^+$ is the limit of μ_T as $T \to \infty$, [S2]. We now investigate the probability distribution of a random variable $w = \sum_{j=-t}^t f(S^j x)$ with f smooth and $t > 0$. This is done by considering the probability distribution $\overline{\mu}_{T,t}$ defined, on the smooth functions F, by:

$$\int \mu_{T,t}(dx)F(x) = \frac{\sum_{\underline{\sigma}^T\in\Gamma_T} e^{-\sum_{-t}^t h_+(\varphi^j\underline{\hat{\sigma}}^T)}F(x(\underline{\hat{\sigma}}))}{\text{normalization}} \tag{3.12}$$

We want to show:

Theorem II: *If f is smooth the probability distribution of $F = \sum_{-t}^t f(S^j x) = w$ computed by using the distribution $\overline{\mu}_{T,t}$, $T > t$ is different from that obtained by using $\overline{\mu}^+$ by a factor bounded, for each value of w, between e^{-B_f} and e^{B_f} for some B_f which is T, t independent.*

Remark: this is a simple consequence of the short range ((3.6) of h_+ and the genral theory of Gibbs states on one dimensional lattices. Note that the set of strings in K (the space of the compatible strings) which agree with a given $\underline{\sigma}^T \in \Gamma_T$ is a prism of the Markov partition $\mathcal{E}_T = \cap_{k=-T}^T S^k\mathcal{E}$; so that the sum in (3.12) can be regarded as a sum on the prisms of $\mathcal{E}_T$

partial proof: since f is smooth the function $f'(\underline{\sigma}) = f(x(\underline{\sigma})$ also has exponentially vanishimng memory, as the h_γ in (3.6): hence there exist C, ν such that $f'(\underline{\sigma}) - f'(\underline{\sigma}')| < Ce^{-\nu q}$ if $\underline{\sigma}, \underline{\sigma}'$ agree on the sites between $-q$ and q. Therefore we suppose first that f' depends *only* on the sites $j \in [-r, r]$, ("finite range assumption"). The the value of $w = \sum_{-t}^t f(S^j x(\underline{\sigma}))$ is entirely determined by $\underline{\sigma}_j^T$, $|j| < t + r$, so that the probability of w can be computed with the distribution:

$$e^{-\sum_{j=-t}^{t} h_+(\varphi^j\underline{\sigma}^T)}.\frac{\sum_{\substack{\underline{\sigma}^T\in\Gamma_T\\ \underline{\sigma}^{t+r}\ \mathrm{fixed}}} e^{-\sum_{j=-T}^{-t-1} h_+(\varphi^j\underline{\sigma}^T)}e^{-\sum_{j=t+1}^{T} h_+(\varphi^j\underline{\sigma}^T)}}{Z_{[-T,T]}} \tag{3.13}$$

but since h_+ has short range the above numerator is essentially the product of the partition functions relative to the intervals $[-T,-t-1]$ and $[t+1,T]$, with some $\underline{\sigma}^t$ dependent "boundary condition" (in the sense of the Gibbs states terminology). Therefore the ratio is bounded *between*:

$$e^{-\sum_{j=-t}^{t} h_+(\varphi^j\underline{\sigma}^T)}.\frac{Z_{[-T,-t-1]}Z_{[t+1,T]}}{Z_{[-T,T]}}.e^{\pm B_r} \tag{3.14}$$

where B_r is a T,f–independent constant: note that the ratio of the partition functions is large or small (exponentially in T,T) but *it is not random* (*i.e.* it is independent of $\underline{\sigma}^{t+r}$).

This proves the theorem if f has finite memory r: if f has exponentially vanishing memory the result has to be proven by approximating it with a finite memory function and passing to the limit; I do not discuss this problem further.

§*4 Time reversible systems. Ruelle's principle.*

As an application I consider the system in example 3, §1, to show that the analysis of §3 can have some far reaching consequences and relevance for a fundamental theory of non equilibrium ensembles. Consider a 2 dimensional periodic box containing a few circular obstacles disposed so that there are no collisionless trajectories (this is the case called $0H$ in [GG], "no horizon"). The box contains N particles and the equations of motion are the (1.1), *i.e.* with the usual symbols:

$$\dot{\underline{q}}_j=\frac{1}{m}\underline{p}_j,\quad \dot{\underline{p}}_j=-\underline{\partial}_{\underline{q}_j}V_{ext}(\underline{q}_j)-\underline{\partial}_{\underline{q}_j}V(\underline{q}_1,\ldots,\underline{q}_N)+E\underline{i}-\alpha(\underline{p})\underline{p}_j \tag{4.1}$$

where V_{ext} is the hard core potential (*i.e.* it is really a boundary condition imposing elastic reflections on the obstacles) and V is a short range pair potential energy. The α represents a friction mechanism to keep the total energy $H=\sum_i \underline{p}_i^2/2m+V(\underline{q}_1,\ldots,\underline{q}_N)$ bounded. We choose: $\alpha(\underline{p})=E\underline{i}\cdot\sum_j\underline{p}_j/\sum_j\underline{p}_j^2$ which keeps H constant, exactly. This is called a "gaussian thermostat" from the meaning it has when $V=0$. The case $V=0,N=1$ has been "completely" studied in [S1],[CELS].

The dynamical system generated by (4.1) can be regarded as defining a dynamics on the $4N$ dimensional full phase space $\mathcal{F}$, or on the $4N-1$ dimensional surface of constant energy $\mathcal{E}$, or on the $4N-2$ dimensional manifold $\mathcal{C}$ consisting in the phase space points of $\mathcal{E}$ in which one particle is exactly colliding with one of the hard obstacles. We denote S_t^0 the dynamics on $\mathcal{F}$, S_t that on $\mathcal{E}$ and S will denote the map defined, on $\mathcal{C}$, by mapping one collision to the next.

The system properties are:

(A) Dissipativity: the phase spaces $\mathcal{F},\mathcal{E},\mathcal{C}$ contract at an average rate $(2N-1)\langle\alpha\rangle$, where $\langle\alpha\rangle$ is the time average of α, at least if we suppose that the forward infinite time average $\langle\alpha\rangle$ is positive, as it seems intuitively correct and as some numerical experiments seem to support.

(B) Reversibility: the map $i:(\underline{q},\underline{p})\to(\underline{q},-\underline{p})$ is such that, if $t\to x(t)$ is a solution of (4.1), then $t\to ix(-t)\equiv(ix)(t)$ is also a solution.

I will furthermore suppose that the system verifies the following property:

(C) Chaoticity: in the sense that "things go as if the system was chaotic".

The strict validity of this assumption in the sense of definition 8, §3, is *not true* for the model (4.1) already for $N=1$, simply because the system is *not* smooth. For large N testing its *approximate* validity is very hard, but hopefully something will be done in the future. The best one can hope is that the failure to be verified is not "relevant" for the discussion of the thermodynamics of the system, in the same sense that the failure of ergodicity is believed to be, in many cases, irrelevant for the equilibrium thermodynamics (*i.e.* in the limit as $L \to \infty$ with fixed particle density and periodically repeated obstacles).

This is essentially Ruelle's principle: its operational meaning is to proceed as if the system was chaotic in the sense of the definition 8 of §3, or some similar definition, and then to suppose that the deductions are correct even though we have no way to check the chaoticity assumption or even though it does, strictly speaking, fail (as in the case at hand, (1.1)). One can hope that the assumptions should become more and more "true" as N grows. They fail, as just mentioned, for instance if $N = 1$, see [CELS]: but, in this particular case, some conclusions of §3 would nevertheless be correct *even if* $N = 1$, as it seems likely to be implied by combining the present analysis with that of [CELS].

For instance an example of a property that is expected to hold, if the chaoticity assumption holds, is a strong, exponential, decay of the correlations between smooth observables, [S2], [R2], [Bo1]. The above assumption is therefore expected to correspond to a strong exponential decay of the correlations, at least over a time scale that is reachable by the experiments that can be conceived in order to check the picture that we are developing.

The latter decay has not been experimentally investigated (because it is really difficult). But after a decade of uncertainty it is becoming increasingly acceptable that indeed the correlations, in the $N = 1$, $E = 0$ case, may decay exponentially: in the paper [GG] the "simple" cases $V = 0, N = 1$ and several choices for V_u are studied. No evidence for a non exponential decay could be detected, at least on time scales reliably attainable by numerical experiments (and, of course, for the simple observables considered). Essentially all the data confirm, instead, an exponential decay, both for the observables of the collision system $(\mathcal{C}, S)$ and for the continuous system $(\mathcal{E}, S_t)$. In this respect "things go as if the system was chaotic".[11] In spite of the evidence for the failure of the assumption, we have therefore an example, in the case least favourable for thermodynamic interpretations (as $N = 1$), strengthening the view that "things may go as if the system was hyperbolic" expressed here.

The existence of time reversal invariant periodic orbits can be deduced for many arrangements of the obstacles: hence we shall also suppose, for simplicity that the periodic point discussed in the definition of chaoticity is just a time reversal invariant periodic point. And to simplify further the analysis we assume that in fact O is a fixed point (it should be clear that the analysis that follows would not really change if O was just a periodic motion).

Other properties can be expected to hold for our model, on the basis of analogies with similar models in which they have been shown to hold. I make here a list, hoping that someone will find their experimental check worth of some effort. They *will not bee used in the following* but they make the general picture brighter.

(D) Density: following [LPR] we label the non negative Lyapunov exponents of S_t (on $\mathcal{E}$) as λ_j^+, $j = 2N-1, 2N-2, \ldots, n_+$, in decreasing order, and the negative ones as λ_j^-, $j = 2N-1, \ldots, n_-$ in increasing order, discarding from the enumeration the trivially zero exponent associated with the flow direction. Thus the maximum Lyapunov exponent is $\lambda_{\max} \equiv \lambda_{2N-1}^+$. Then in various models it has been experimentally[12] found, starting with the work [LPR] which has been followed by other interesting cases,(*e.g.* [ECM1], [SEM]), that the graphs of $x = j/(2N-1) \to \lambda_j^\pm$ seem to have a smooth limit $f_\pm(x)$ as $N \to \infty$ and $x > x^\pm$. No

[11] In our paper there is only one experimental result that can rise, in our opinion, doubts about the decay law. It is the result corresponding to the extremely long run leading to fig. 6 of [GG]: see the discussion there. More recent work, [ACC], provides much stronger further evidence on the exponential decay.

[12] By *experimental measurement* I mean here also a computer experiment: as I think it does not conceptually differ from the older notions of experimental measurement.

experimental result seems to be available in the case (4.1): I shall assume this property to hold for it. The Lyapunov exponents of S (on $\mathcal{C}$) will be $t_0\lambda_j^{\pm}$, where t_0 is the average time between any two collisions, which is the time scale associated with the map S).

(E) Pairing: it has been noted that in various cases, starting with [Dr],[ECM1], that:

$$\lambda_j^+ + \lambda_j^- = j-\text{independent} = -\langle\alpha\rangle \qquad j = 2N-1,\dots \tag{4.2}$$

Where $\langle.\rangle$ denotes (forward) time average (over ∞ time).

The relation (4.2), that will be called *strong pairing property*, has been experimentally found to hold for many other models, as it emerges from the subsequent analysis in [SEM]. It is not yet clear whether it holds strictly or up to corrections of $O(N^{-1})$.

Remarks:

(1) The pairing property appears to be related to several features of the equations of motion among which the reversibility. As essentially suggested in [Dr] one may think that the property holds, for instance, for locally hamiltonian equations (as (4.1) with $\alpha = 0$) with a constraint imposed via a variational principle (like the Gauss' least constraint principle or some generalizations of it). But no proofs are available and, furthermore, the case [ECM1] and others in [SEM] do not fall into this cathegory, although (4.2) holds for them.

(2) In other models considered in [SEM] a weaker property seems to hold:

$$-C\langle\alpha\rangle \le \lambda_j^+ + \lambda_j^- \le 0 \tag{4.3}$$

for some $C > 0$, that I will call the *weak pairing property*, whose essential feature is the $N-$independence of C.[13] Neither (4.2) nor (4.3) have been tested numerically for the present model. I consider quite likely that the strong pairing rule holds in this case.

(3) The pairing property and the density property imply that the Kaplan Yorke fractal dimension of the attractor (also called the Lyapunov dimension), see [ER], is *macroscopically smaller* than the full dimension $4N-4$ of the phase space, by an amount $O(\lambda_{\max}^{-1}\langle\alpha\rangle N)$: the proportionality to N of the loss in dimension is sometimes called "dimensional reduction", [PH], [CELS].

Going back to the system (4.1), after the assumption (C) above, I want to discuss its consequences, with the main objective of finding some that could be tested and thus provide a test for Ruelle's principle for the statistics $\overline{\mu}$, and to see if any experimental consequences can be drawn from it (accessible by using the present day technical capabilities).

Under the assumptions (C) the "principle" is in fact a *theorem* (see theorem II,§3): the reason it is called a principle is that, as already mentioned, the hypotheses of definition 8 are not to

[13] In [SEM] the Lyapunov exponents are considered in the full phase space $\mathcal{F}$. As discussed in [SEM] the rules (4.2) or (4.3) may fail to be obeyed in a few cases: but this seems often related to the way the exponents are counted and computed. For instance in some systems there may exist observables called in [SEM] "constants of motion" or "conservation laws" which, if initially having some special value, conserve it in the evolution, (note that this is a somewhat unusual notion of conservation law, as it holds only for special initial values), or have slow oscillations around it . For instance H or the flow direction in (4.1), or the total momentum $\underline{P}$ in the case $V_{ext} = 0$ (in the latter case this is conserved only if $\underline{P} = \underline{0}$). And one is naturally led to fix the values of such observables or to perform the measurements in some special way (*e.g.* by timing them appropriately when there is a periodic forcing). This is done to simplify the calculations, eliminating trivially behaving coordinates: but the end result might be in a odd number of exponents and in a consequent apparent "failure" of the pairing rule.

be expected to hold rigorously for our model: they must be regarded to hold approximately, and this is the reason we say that to use them is a "principle", which we regard as correct for all the consequences it may have about thermodynamic quantities.

Note first that the time reversal symmetry implies that W^u_O and W^s_O have the same dimension, equal to half that of the phase space, *i.e.* $2N-1$.

A key idea behind all what is said in this paper, and on which intuition can be built, [Ga1], is that of regarding A, rather than as a fractal set of $\sim 4N$ dimensions, as a *smooth surface* of dimension $2N-1$ consisting in the unstable manifold W^u_O of some periodic point O. Note that, in this way, there is unification between the equilibrium ($E=0$) and the non equilibrium cases (in both cases W^u_O is smooth and it has the same dimension).

One should avoid thinking of A as a nasty fractal: following [Ga1] one must think of it as an infinite uncut *folio* (*i.e.* W^u_O) confined into a finite region (the phase space $\mathcal{C}$) and folded over and over again to fit into it, thus forming the (uncut) "book" A. Forcing simply introduces some (small if E is small) wrinkles on the folio accounting for the (mild) fractal nature of A. In particular the dimension of W^u_O stays $2N-1$ and does not change because of the introduction of the forcing (while the dimension of A changes by $O(N)$).

The unification between the equilibrium theory and the non equilibrium one is made possible by regarding the attractor in the phase space $\mathcal{C}$ as $2N-1$ dimensional (rather than $\sim 4N-2$ dimensional as one would be normally tempted to do).

The real question is whether, if $E\neq 0$, we can predict anything that could be experimentally checked as a test of Ruelle's principle.

The work [ECM2] provides evidence in this direction. I apply here the ideas of [ECM2], [CG], to show their relevance to the present case (as an example of the claim made in [CG] about the generality of the ideas).

We can study the fluctuations, in the stationary state $\overline{\mu}$, of the average $(2N-1)\langle\alpha\rangle_t$ of the phase space contraction rate $(2N-1)\alpha$ over time stretches $t\,\tau_0$:

$$\langle\alpha\rangle_t(x)=\frac{1}{t\tau_0}\int_{-t\,\tau_0/2}^{t\tau_0/2}\alpha(S_\vartheta x)d\vartheta=\frac{1}{t\,\tau_0}\sum_{j=-t/2}^{t/2-1}\int_0^{\tau_0(S^jx)}\alpha(S_\vartheta S^jx)\,d\vartheta \tag{4.4}$$

where τ_0 is the average time elapsing between timing events, *i.e.* the average (over infinite time) of the time interval $\tau_0(x)$ between the event x and the successive Sx; the S_ϑ is the continuoum time evolution on the surface of constant energy.

The quantity $(2N-1)\langle\alpha\rangle_t$ is also called the *entropy production rate* on the trajectory stretch between $S^{-t/2}x$ and $S^{t/2}x$: it is related to a transport coefficient (the conductivity, *i.e.* to the ratio between the particle current and the field E, see [CELS]), at least in the present model (because the relation between entropy production and a suitable transport coefficient might only hold for a special class of examples, see [CL]).

If the quantity t is small compared to the duration of the experiment the quantity $\langle\alpha\rangle_t$ *fluctuates* around the mean value $\langle\alpha\rangle$ (defined as the (forward) average of α over infinite time). If we write:

$$\langle\alpha\rangle_t(x)=\langle\alpha\rangle\,a_t(x) \tag{4.5}$$

we define a dimensionless random variable $a_t(x)$ with (forward) average 1. We can divide the axis a_t into intervals $I_{j\delta}$, $j=0,\pm1,\ldots$, with $I_{j\delta}=[j\delta,(j+1)\delta]$ for some small δ and measure the probability distribution of a_t by counting how many times a_t takes a value in each $I_{j\delta}$ when measured at the phase space points into which x evolves at times multiples of $t\,\tau_0$,

The measurement of $\langle\alpha\rangle_t(x)$ requires measuring the value of α points $S^{-t/2}x,\ldots,S^{t/2-1}x$.

The probability $\pi(p)dp$ that $a_t\in(p,p+dp)$ can be computed from (3.13) with $T\gg t$ by using the Markov partition $\mathcal{E}_T$ constructed in §3. The prisms of the partition $\mathcal{E}_T$ are naturally labeled by the allowed strings $\underline{\sigma}\in\Gamma_T$ (see the paragraph preceding (3.11)); for brevity we label them with a label denoted j. If E_j is a generic prism of this partition with axes intersecting at x_j, consider the measure $\mu_{T,t}$ defined by setting, for every smooth F:

$$\int_{\mathcal{C}} F(x)\,\mu_{T,t}(dx) = \frac{\sum_j \overline{\Lambda}^{-1}_{u,t}(x_j)F(x_j)}{\sum_j \overline{\Lambda}^{-1}_{u,t}(x_j)} \tag{4.6}$$

where $\overline{\Lambda}_{u,t}(x) \equiv \prod_{j=-t/2}^{t/2-1} \Lambda_u(S^j x)$ is defined before (3.4) and x_j is a point in E_j. In the limit $T\to\infty$ the distribution μ gives the correct probability to the values of an observable F which is smooth and has a t–dependence like the one in (4.4), (*i.e.* like the one considered in theorem II, with $f(x) \equiv \tau_0^{-1}\int_0^{t_0(x)} \alpha(S_\vartheta x)d\vartheta$) up to an error which is a *factor bounded above and below uniformly in* t, by theorem II.

The partition $\mathcal{E}$ constructed in §3 turns out to be time reversal invariant, if the triangulation on W^s is taken to be the i image of the triangulation on W^u: this means that if $E_j\in\mathcal{E}$ then $iE_j = E_{j'}\in\mathcal{E}$ for some j'. Note that if we used another Markov partition $\mathcal{E}'$, constructed for instance via the classical proofs of existence of Markov partitions of [S2],[Bo1], then the time reversal symmetry of our system would still imply that $i\mathcal{E}'$ is also a Markov partition (as $W^u_x = iW^s_{ix}$) so that the partition obtained by intersecting the two would still be a Markov partition, with the extra property of being time reversal invariant; hence by using $\mathcal{E}'\cap i\mathcal{E}'$ instead of $\mathcal{E}$ we could still carry the argument that follows which only depends on the time reversal invariance of the Markov partition $\mathcal{E}$ used for the construction of $\overline{\mu}_+$ and not on the particular Markov partition used. Note that if $\mathcal{E} = i\mathcal{E}$ then also the partition $\mathcal{E}_T$ in theorem II, $\mathcal{E}_T = \cap_{q=-T}^{T} s^q\mathcal{E}$, has the property of time reversal invariance $\mathcal{E}_T = i\mathcal{E}_T$.

Thus, for instance, up to the mentioned error of a factor (T,t)–independent:

$$\frac{\pi(p)}{\pi(-p)} = \frac{\sum_{j,a_t(x_j)=p}\overline{\Lambda}^{-1}_{u,t}(x_j)}{\sum_{j,a_t(x_j)=-p}\overline{\Lambda}^{-1}_{u,t}(x_j)} \tag{4.7}$$

where the sums run over the labels j of the prisms $E_j\in\mathcal{E}_T$ (with axes intersecting at x_j) verifying $a_t(x_j)\in(p,p+\delta)$.

By using the time reversal symmetry the above sums can be seen to consist of sums with the same number of addends. This can be done by pairing the contribution from $E_j\in\mathcal{E}_T$ with that of $iE_j\equiv E_{j'}$ that can be easily seen to give, if T is large (so that the prisms are really small), a value $-p$ to $a_t(x_{j'})$ if $a_t(x_j)=p$ and at the same time$\overline{\Lambda}_{u,t}(x_{j'}) = \overline{\Lambda}^{-1}_{s,t}(x_j)$. [14] Hence the addends can be paired so that the ratios of corresponding addends is:

$$\frac{\overline{\Lambda}^{-1}_{u,t}(x_j)}{\overline{\Lambda}_{s,t}(x_j)} = e^{(2N-1)\langle\alpha\rangle pt\,\tau_0}\frac{b(S^{t/2}x_j)}{b(S^{-t/2}x_j)} \tag{4.8}$$

where $\overline{\Lambda}_{s,t}(x)$ is defined as $\overline{\Lambda}_{u,t}(x) \equiv \prod_{j=-t/2\tau_0}^{t/2\tau_0-1}\Lambda_u(S^j x)$, with $\Lambda_c(x)$ being the absolute value of the determinant of the jacobian matrix of S as a map of W^s_x into W^s_{Sx}. Eq. (4.8) simply follows from the fact that a volume element around $S^{-t/2}x$ varies under the action of S^t by a factor

[14] For instance, [ECM2], from the two identities $S^{-\tau}(S^\tau x) = x$ and $S^{-\tau}(iS^{-\tau}x) = ix$ we deduce:

$$\begin{aligned}&\partial S^\tau(S^{-\tau}x)\,\partial S^{-\tau}(x) = 1 = \partial S^{-\tau}(S^\tau x)\,\partial S^\tau(x)\\ &\partial S^{-\tau}(iS^{-\tau}x)\,i\,\partial S^{-\tau}(x) = i\end{aligned} \tag{$*$}$$

We see that, by applying the first relation with x replaced by $\tilde{x}'\equiv iS^{-\tau}x$ and by applying successively the third (with $x\to\tilde{x}'$) and the second relations (with $x\to S^{-\tau}x$, using also $x = iS^{-\tau}\tilde{x}'$) in (*) one deduces:

$$\partial S^\tau(S^{-\tau}\tilde{x}')i\partial S^\tau(S^{-\tau}x)i = 1 \tag{$**$}$$

Therefore: $\big(\det\partial S^\tau(S^{-\tau}\tilde{x}')\big)^{-1} = \det\partial S^\tau(S^{-\tau}x) = e^{-(2N-1)\tau\tau_0\langle\alpha\rangle a_\tau(x)}$, where τ_0 is the value of the timing interval in real units (for us τ is an integer); *i.e.* , replacing x with $S^{-\tau/2}x$: $a_\tau(x) = -a_\tau(x')$.

$\overline{\Lambda}_{u,t}(x)\overline{\Lambda}_{s,t}(x)b(S^{t/2}x)/b(S^{-t/2}x)$ where $b(x)$ describes the transversality of the intersection of the stable and unstable manifolds at x, as defined in remark (iii) to definition 8: so that this equals $\exp -(2N-1)t\,\tau_0\langle\alpha\rangle_t(x)$, (here τ_0 is considered extremely small, for simplicity).

As repeatedly mentioned above (4.6),(4.7) require a correction: but it has been remarked after (4.6) that the correction is a factor bounded by $e^{\pm B}$ for some B which is p–independent because of theorem II; and we can include in B also the correction due to the last ratio in (4.8) (bounded uniformly in t, by transversality property in the chaoticity assumption, see definition 8 in §3).

This means that $\pi(p)$ verifies (N large):

$$\begin{aligned}&\frac{1}{2Nt\,\tau_0 p}\log\frac{\pi(p)}{\pi(-p)}\xrightarrow[N\to\infty,\,t\to\infty]{}\langle\alpha\rangle = p-\text{independent}\\ &\pi(p)=e^{-\xi(p)+p\langle\alpha\rangle Nt\pm B}\end{aligned}\tag{4.9}$$

where $\xi(p)$ is an even function of p (over which we have no control) and B does not depend on p nor on t (but it may depend on N,E). The (4.9) can be considered as a *large deviation* result (both in N and t): its peculiarity is the p independence of the coefficient of p in the odd part othe argument of the exponential, to leading order in t.

The p,t,N independence of the first relation in (4.9), and the equality to $\langle\alpha\rangle$ can be quite easily tested even if N is moderately large (but not too large: as in such case the measurements simply could not be carried out): $N=64$ seems feasible.

The idea of the above test, and its realization in a model different from (4.1), is in the basic paper [ECM2] and it has been further developed in [CG]. The test *does not* require the measurement of the individual Lyapunov exponents: therefore it should be (relatively) easy to carry out. Particularly after its feasibility has been proved in similar models in [ECM2].

Finally it is remarkable that one can, at all, find tests of the principle: in the case $E\neq 0$ there is no really well established non equilibrium thermodynamics with which one could compare the results of the principle. Such general results would constitute the real test of the principle from the point of view of Physics. A non controversial theory of non equilibrium thermodynamics could, actually, follow from the principle. For this, however, we must learn how to extract other consequences, just as we learnt to extract consequences from the Boltzmann Gibbs principle. And it is important to stress that whatever predictions it gives they should be as true as the statements derived in equilibrium are: *i.e.* essentially *exactly true.* Therefore the check above (and others that might be devised) will be satisfactory only if it gives exactly the expected results, within the experimental errors. This seems to be the case in [ECM2] for the model considered there, see also [CG].

§5 *Summary and outlook*

Here I regarded the Ruelle's principle as valid: to clarify its status I considered of some interest setting up a list of assumptions that would imply it rigorously. This essentially amounted to saying that the system behaves as an Anosov system (or more generally I could have supposed something like an axiom-A behaviour) and then use the theory of Sinai (or, respectively, of Bowen and Ruelle) to "prove" the principle, see §3. The interest of course is not in the mathematical theorem as I just made enough assumptions to make it valid and checked its validity by adapting the ideas of the key papers [S2],[Bo1],[R2], (while attempting, to avoid repetitions, at describing a more intuitive construction of Markov partitions), but rather the interest lies in the clarification of the meaning of the property (C) in §3. One cannot be too demanding on the matter of mathematical rigour: it should not be forgotten that even the ergodic hypothesis of Boltzmann is far from being proved, particularly in the generality one would want.

The "application" deviced in §4 is not very satisfactory as a test of validity of the principle because it can be performed only if N,τ are not too large. It would be, of course, nice to find a true thermodynamic property that could be computed and tested via the principle.

1) One should remark that the above analysis should improve with N, as the non chaotic phenomena should become less important: they are excluded by the strict chaoticity assumption, see §2, but they are possibly present if the assumption is made in the loose form of §3, (C). They would be certainly present if the hard core potential in (4.1) was replaced by a smooth finite range potential steeply diverging at zero distance (by general results from KAM theory).
2) Predictions analogous to the ones at the end of §4 could be made for other models, see [CG]: but except in the cases in [ECM2] the numerical experiments do not seem to exist, yet.
3) The new principle appears to play the role that the ergodic hypothesis plays in equilibrium statistical mechanics: therefore one may be led to think that the time of approach to equilibrium should be of the order of the recurrence time on the attractor. Having set up a unified point of view for the equilibrium and the non equilibrium cases, we can adopt the classical explanation of Boltzmann, [B96], [B02], apparently relying also on an earlier suggestion by Thomson, [T], intended to contradict such a hasty conclusion. The rate of approach to equilibrium is very short (essentially determined by the Boltzmann equation in the case of rarefied gases, as exemplified, for instance, by the Lorentz theory of conductivity in metals, [Be], closely related to model 1) at least if one looks at the very few macroscopic observables relevant for equilibrium thermodynamics and for the transport coefficients: the reason being that such observables have the same value on most of the surface W_O^u, see [Ga1].
6) Concerning the particularity of the gaussian thermostat, which could be called an "unphysical fiction", I think, see [Ga1], that there should be, also in non equilibrium, several equivalent ways of describing the same stationary distribution corresponding to different $\overline{\mu}$ and to different physical ways of reaching the stationary state. And it might well be that the gaussian thermostat turned out to be equivalent to other models of thermostats, which could be described by rather different attractors. For instance a stochastic thermostat, in which a particle colliding with the wall comes out with a maxwellian distribution at given temperature, will certainly be described by a statistics $\overline{\mu}$ which is absolutely continuous with respect to the Liouville measure.[15] In the thermodynamic limit this might just be the same as the result obtained with a statistics which, for finite N, is on a strange attractor. This mechanism is like the one realized by the microcanonical and the canonical ensembles (the first is concentrated on a set of configurations which has zero probability with respect to the second, as long as $N < \infty$). This is clearly a question that requires further investigations.

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[15] Note that a stochastic model of thermostat is described by a stochastic differential equation and therefore our discussion does not apply without some major modification.

Appendix: Philosophical questions; and a few concrete ones

The following is a guided series of problems to the general theory of the Lyapunov exponents: it contains the ergodic commutative, subadditive and non commutatitive theorems and the Oseledec theorem. The theorems are mostly philosophical, i.e. they hold with essentially no assumptions. The problems are taken out of the preprint of the book Meccanica dei Fluidi, in italian, circulating in the form of a draft: the Oseledec theorem exposition is essentially taken from [R4].

(1) If μ is a probability measure on the Borel sets of R^n and if Δ_n is a sequence of measurable sets such that $\sum_n \mu(\Delta_n) < +\infty$ then almost all points are contained in at most a finite number of sets in the sequence (*Borel Cantelli theorem*). (Idea: the set of the points in an infinite number of Δ_n's is $N = \cap_{k=1}^{\infty}(\cup_{h=k}^{\infty}\Delta_h)$, of course. Therefore $\mu(N) \le \sum_{h=k}^{\infty}\mu(\Delta_h)$ for all k's, hence $\mu(N) = 0$ because the series converges.)

(2) Let $(\mathcal{C}, S, \mu)$ be a triple formed by an invertible dynamical system and an invariant probability measure on (the Borel sets of) $\mathcal{C}$ (*i.e.* there is a zero measure set N such that S is invertible outside N and $\mu(E) = \mu(SE) = \mu(S^{-1}E)$ for all Borel sets $E \subset \mathcal{C}/N$). Let f be a (measurable) function bounded by a constant K almost everywhere (with respect to μ). Let D_n be the set of points x such that some average of f over a time $\le n$ is *non negative*: *i.e.* $m^{-1}\sum_{j=0}^{m-1} f(S^j x) \ge 0$ for some $m \le n$. Then $\int_{D_n} f(x)\mu(dx) \ge 0$. (*Garsia maximal averages theorem*). (Idea: if $n = 1$ the condition defining D_1 is simply $f(x) \ge 0$ and nothing has to be proved, besides the obvious. If $n = 2$ the condition defining D_2 is either $f(x) \ge 0$ or $f(x)+f(Sx) \ge 0$. Therefore the *new* points, *i.e.* those in D_2 but not in D_1, are points x where $f(x) < 0$ which can be paired with point $Sx \in D_1$ so that $f(x)+f(Sx) \ge 0$. Hence we can subdivide D_2 in the disjoint union of $D_2/D_1 \cup S(D_2/D_1)$ and of D_1/SD_2. On the last set it is $f(x) \ge 0$ while the integral over the first union can be written as $\int_{D_2/D_1}(f(x)+f(Sx))\mu(dx)$ because, by the invariance of μ the integral $\int_{S(D_2/D_1)} f(x)\mu(dx) \equiv \int_{D_2/D_1} f(Sx)\mu(dx)$. The case $n = 3$ is only slightly more involved and it is left to the reader (proceed in the "same way"), and, once understood, the general case becomes crystal clear.)

(3) The invertibility assumption is not necessary in (2): the invariance in the ordinary sense $\mu(E) = \mu(S^{-1}E)$ for all Borel sets E is sufficient. Prove this statement. (Idea: one has just to try to formulate what said by always using S^{-1}: for instance D_2 will consist of the points in D_1 and the inverse images of those in D_1 such that $f(x) + f(S^{-1}x) \ge 0$, etc.)

(4) Show that (2) implies the almost everywhere existence of the limit $\lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k-1} f(S^j x)$. (Idea: let $f_{sup}(x) = \limsup_{k\to\infty}\frac{1}{k}\sum_{j=0}^{k-1} f(S^j x)$ and $f_{inf}(x) = \liminf_{k\to\infty}\frac{1}{k}\sum_{j=0}^{k-1} f(S^j x)$. The denial of the statement is that $f_{sup}(x) > \alpha > \beta > f_{inf}(x)$ on a set D which has *non zero measure* for a suitable pair $\alpha > \beta$. Quite absurd because on the (obviously) invariant set D the functions $f(x) - \alpha$ and $\beta - f(x)$ would have some non negative average. Hence by the previous theorem their integrals over D would have to be ≥ 0, but their sum would therefore be $(\beta - \alpha)\mu(D)$ which would be < 0, because we are (foolishly) thinking that $\mu(D) > 0$!)

(5) The boundedness assunption in (4) can be replaced by the summability assumption $f \in L_1(\mu)$. Furthermore if $\overline{f}(x)$ denotes the average of f, defined μ almost everywhere, show that:

$$|\overline{f}|_{L_1} \equiv \int_{\mathcal{C}} \mu(dx)\,|\overline{f}(x)| \le \int_{\mathcal{C}} \mu(dx)|f(x)| \equiv |f|_{L_1}$$

$$\int_{\mathcal{C}} \overline{f}(x)\mu(dx) = \int_{\mathcal{C}} f(x)\mu(dx)$$

If there are no non trivial functions $f(x)$ which are constant of motion μ almost everywhere, *i.e.* if the system is *ergodic*, then the function $\overline{f}(x)$ is a constant (almost everywhere) and $\overline{f} = \int_{\mathcal{C}} \mu(dy)f(y)$ for all $f \in L_1(\mu)$.

(6) Let $(\mathcal{C}, S, \mu)$ be a dynamical system as in problem (4). And let $f_n(x)$ be a sequence of measurable functions such that:

$$|f_1(x)| < K, \qquad f_{n+m}(x) \le f_n(x) + f_m(S^n x) \qquad \mu \text{ almost everywhere}$$

Suppose μ ergodic; by applying the ergodic theorem (3),(4) above show the existence, μ almost everywhere, of the limit $\lim_{n\to\infty}\frac{1}{n}f_n(x) = \overline{f}(x)$, (*Kingman's subadditive ergodic theorem*). (Idea: Remark that the functions:

$$f_{sup}(x) = \limsup_{n\to\infty}\frac{1}{n}f_n(x), \qquad f_{inf}(x) = \liminf_{n\to\infty}\frac{1}{n}f_n(x)$$

verify $f_{sup}(Sx) \ge f_{sup}(x)$ and $f_{inf}(Sx) \ge f_{inf}(x)$ (because $f_n(x) \le f_1(x) + f_{n-1}(Sx)$, then divide by n), so that the μ invariance of μ implies that they are constants of motion (because $\int(f_{sup}(Sx) - f(x))d\mu = 0$).

Therefore, μ almost everywhere, $f_{sup}(x) = \beta$ and $f_{inf}(x) = \alpha$ where $\alpha < \beta$ are suitable constants. Let us consider a number $\eta > 0$ such that $\alpha + \eta < \beta$.
Let Δ_n be the set of points where $f_n(x) \le \alpha + \eta$ for at least one value of $m \le n$: hence $\lim_{n\to\infty} \mu(\Delta_n) = 1$. Given $\varepsilon > 0$ there is, therefore, a n_ε such that $\mu(\Delta^c_{n_\varepsilon}) < \varepsilon$ if ^{c} denotes the complementation operation on sets. If $x \in \mathcal{C}$ one can suppose that the frequency of visit of x to $\Delta^c_{n_\varepsilon}$ is $< \varepsilon$, because such frequency is the average value of $\chi_{\Delta^c_{n_\varepsilon}}(S^j x)$ (over j) if χ_Δ denotes the characteristic function of the set Δ (by the ergodicity).
Consider the sequence of times $j_1 < j_2 < \ldots$ when, instead, $S^{j_k} x \in \Delta^c_{n_\varepsilon}$: it follows that the number p of such j's with $j_k \le T$ is such that $p/T < \varepsilon$ for T large enough (as p/T tends to the frequency of visit to $\Delta^c_{n_\varepsilon}$).
Let k_0 be the first time $\le T$ in which $S^{k_0} x \in \Delta_{n_\varepsilon}$ and let k'_0 be the largest integer $\le T$ such that $f_{k'_0 - k_0}(S^{k_0} x) \le (k'_0 - k_0)(\alpha + \eta)$. The point $k'_0 + 1$ must be one of the j's, otherwise it could not be the largest integer k' such that $\frac{1}{k' - k_0} f_{k'-k_0}(S^{k_0} x) \le \alpha + \eta$ by the subadditivity of f, unless of course $k'_0 = T$.
Let $k_1 > k'_0$ be the first value, with $k_1 \le T$, not among the j's, *i.e.* such that $S^{k_1} x \in \Delta_{n_\varepsilon}$ and let $k'_1 > k_1$ be the largest value such that $f_{k'_1 - k_1}(S^{k_1} x) \le (k'_1 - k_1)(\alpha + \eta)$, and so on.
In this way a sequence $[k_0, k'_0], \ldots, [k_s, k'_s]$ is constructed in the interval $[0, T]$. All the values $k < k'_s$ outside the intervals must be among the j's (hence their number is $\le p$). The last value k'_s will be, possibly, followed by a string of values among the j's but the first value k_{s+1} that does not have this property, if existing at all, ***must*** be within n_ε of the value T, otherwise we could form the interval $[k_{s+1}, k'_{s+1}]$. Therefore the number of values outside the intervals is bounded above by $p + n_\varepsilon$. The subadditivity then implies:

$$\frac{1}{T} f_T(x) \le \frac{1}{T}\Big(K(p + n_\varepsilon) + \sum_{i=1}^{s} f_{k'_i - k_i}(S^{k_i} x)\Big) \le$$
$$\le \frac{1}{T}\Big(K(p + n_\varepsilon) + T(\alpha + \eta)\Big) \xrightarrow[T\to\infty]{} \alpha + \eta$$

which shows the one ***cannot*** contemplate the case $\alpha < \beta$, by the contradiction it does provoke: hence $\alpha = \beta$.)

(7) Show that (6) implies that $\lim_{n\to\infty} \frac{1}{n}(x) = \overline{f}(x)$ is a μ-almost everywhere constant, and:

$$\lim_{n\to\infty} \frac{1}{n} f_n(x) \equiv \overline{f} = \lim_{n\to\infty} \frac{1}{n} \int_{\mathcal{C}} f_n(x)\mu(dx) = \inf_n \frac{1}{n} \int_{\mathcal{C}} f_n(x)\mu(dx)$$

(Idea: subaddititivity implies $f_n(x) \le f_1(x) + f_{n-1}(Sx)$; hence $\overline{f}(x) \le \overline{f}(Sx)$ and $0 \le \int_{\mathcal{C}} (\overline{f}(Sx) - \overline{f}(x)) d\mu = 0$ imply $\overline{f}(Sx) = \overline{f}(x)$ μ–almost everywhere. Then ergodicity yields that $\overline{f}$ is constant. By dominated convergence the first limit relation follows. The sequence $n \to \langle f_n \rangle \equiv \int_{\mathcal{C}} f_n(x)\mu(dx)$ is subadditive and bounded by K; hence $\frac{1}{n}\langle f_n \rangle \xrightarrow[n\to\infty]{} \inf_n \frac{1}{n}\langle f_n \rangle$, by an elementary argument.)

(8) Show that the assumption $|f_1(x)| < K$ in (6) can be replaced by the summability of f^+, $f^+(x) \in L_1(\mu)$, if $f^+(x) = \max(0, f_1(x))$. (Idea: no idea is necessary; just a careful examination of the proofs in (6),(7).)

(9) Show that the ergodic theorem implies that if N has zero measure and $S^j N$ is measurable then $\mu(SN) = 0$. (Idea the frequency of visit of the motion starting at x to the set SN, $\varphi_x(SN)$, is equal to that to N itself: $\varphi_x(N) = \varphi_x(SN)$; hence by the ergodic theorem $\mu(SN) = \int \varphi_x(N) d\mu \equiv \int \varphi_x(SN) d\mu = \mu(N) = 0$.)

(10) Define, via the ergodic theorem, the "future" and "past" averages of $f \in L_1(\mu)$ as the limits $f^\pm(x) = \lim \frac{1}{n} \sum_{j=0}^{n-1} f(S^{\pm j} x)$. Show that $f^+(x) = f^-(x)$ μ–almost everywhere. (Idea: Let D, α, β be such that $\mu(D) > 0$ and $f^+(x) > \beta > \alpha > f^-(x)$ for $x \in D$. Let D^+_n be the set of points $x \in D$ such that $\frac{1}{m} \sum_{j=0}^{n-1} f(S^j x) \equiv \langle f \rangle^+_m(x) > \beta$ for all $m \ge n$. Let D^-_n be the correspoonding set where $\langle f \rangle^-_m(x) < \alpha$. Then, if n is large enough $S^{-(n-1)} D^-_n \cap D^+_n \ne \emptyset$, because the quantities $\mu(D^-_n) \equiv \mu(S^{-(n-1)} D^-_n)$ and $\mu(D^+_n)$ are both very close to $\mu(D)$ for n large. Hence for x in the latter intersection it is $\alpha > \langle f \rangle^-_n(x) \equiv \langle f \rangle^+_n(S^{-(n-1)} x) > \beta$ by the properties defining D, D^+_n, D^-_n. This is impossible.)

The guided sequence of problems will now come "closer to Earth" by studying some finite matrix theorems.

(11) Let L be a real $d \times d$ non singular matrix. Consider the matrix $M_n \equiv ((L^*)^n L^n)^{1/2}$ and call $t^{(n)}_j$ the eigenvalues of M_n ordered by decreasing size. Show that the largest eigenvalue of M_n is such that the limit $\frac{1}{n} \log t^{(n)}_1 \xrightarrow[n\to\infty]{} \lambda_1$ exists. Here * means transposition with respect to the scalar product $(u, v) = \sum_i u_i v_i$. (Idea: first show that the eigenvalues $t^{(n)}_j$ of $((L^T)^n L^n)^{1/2n}$, ordered according to decreasing size, do have a limit as $n \to \infty$. Note that $f_n = \log |L^n|$ verifies subadditivity $f_{n+m} \le f_n + f_m$ and $f_n \le n \log |L|$, if $|L|$ is the norm of the matrix L (*i.e.* $|L| = \max |Lv|/|v|$, with $|v| = (v, v)^{1/2}$). Hence the limit $n^{-1}\log|L^n| \xrightarrow[n\to\infty]{} t_1 = \inf_n \frac{1}{n}\log|L^n|$ exists, and $\frac{1}{n}f_n \ge \min\log|\mu_j|$ if μ_j are the eigenvalues of L. But $|L^n v| = (L^n v, L^n v)^{1/2} = (v, (L^*)^n L^n v)^{1/2} \le \max_{1\le j\le d}(t_j^{(n)})^{1/n}$, if $|v| = 1$. Hence the largest eigenvalues have the appropriate convergence property.)

(12) Think of the vectors in R^d as functions $i \to u_i$ on the finite space $F \equiv (1, 2, \dots, d)$. Let $(R^d)^{\wedge q}$ be the space of the functions on F^q which are *antisymmetric*: these are the functions $u_{i-1\dots i_q}$ which are antisymmetric. Define a scalar product on such functions by setting $(U, v) \equiv \sum u_{i_1\dots i_q} v_{i_1\dots i_q}$. Define the matrix $L^{\wedge q}$ by setting:

$$(L^{\wedge q}u)_{i_1\dots i_q} = \sum_{j_1\dots j_q}^{1,d} L_{i-1j_1} L_{i_2j_2}\dots L_{i_qj_q} u_{j_1\dots j_q}$$

Show that the result of (12) implies that the matrix $(((L^{\wedge q})^*)^n (L^{\wedge q})^n)^{1/2n}$, where the * denotes the adjunction operation with respect to the scalar product (u, v), is such that its largest eigenvalue logarithm divided by n has a limit as $n \to \infty$.

(13) Let $(t_j^{(n)})^2$ be the eigenvalues of $(L^*)^n L^n$ in decreasing order, repeated according to multiplicity. Prove that the eigenvalues of $((L^{\wedge q})^n)^*(L^{\wedge q})^n$ are just the products of the q-ples of eigenvalues $(t_{j_1}^{(n)})^2 \dots (t_{j_q}^{(n)})^2$ with $j_i \ne j_j$ (*i.e.* products of q–ples of pairwise distinct eigenvalues). (Idea: no idea is necessary.)

(14) Combine (12) and (13) to infer that the limits $\frac{1}{n}\log t_j^{(n)} = \lambda_j$ exist for all $j = 1, \dots, d$. Let $\overline{\lambda}_1, \dots, \overline{\lambda}_s$ be the *distinct* limits λ_j and let $m_1, \dots, m_s$ be their multiplicities (*i.e.* the number of λ_i which are equal to $\overline{\lambda}_j$). Verify that $\sum_{i=1}^s m_i = d$ and define $r(i) = j$ if $\lambda_i = \overline{\lambda}_j$.

(15) Let $U_1^{(n)}$ be the linear space spanned by the first m_1 eigenvectors of $\Lambda_n = ((L^*)^n L^n)^{1/2}$; likewise $U_2^{(n)}$ will be the space spanned by the next m_2, and so on until $U_s^{(n)}$ is defined. Show that the notion of multiplicity introduced in (14) is even more justified by proving the following "orthogonality" property between unit vectors $u \in U_r^{(n)}$ and $u' \in U_{r'}^{(n+k)}$, $k \ge 0$. Given $\delta > 0$ there exists $C > 0$ such that:

$$|(u, u')| \le Ce^{-(|\overline{\lambda}_r - \overline{\lambda}_{r'}|-\delta)n}$$

(Idea: nothing to prove if $r = r'$, of course. The "easy case" is $r' > r$. Let $\Lambda_{n+1} = \sum_{i=1}^d t_i^{(n+1)} P_i$ be a spectral decompositon for the matrix Λ_{n+1}. Then supposing n so large that for $m \ge n$ it is $|\frac{1}{m}\log t_j^{(m)} - \overline{\lambda}_i| < \delta_1$ for all $r(j) = i$ and all i, where for a given $\delta_1 > 0$, $\delta_1 < \frac{1}{2}\min(|\lambda_r - \lambda_{r'}|)$:

$$\begin{aligned}
|(u, u')| \equiv& e^{-(n+1)\overline{\lambda}_{r'}} \max_{u'\in U_{r'}^{(n+1)}} |(u, \textstyle\sum_{r(i)=r'} e^{(n+1)\overline{\lambda}_{r'}} P_i u') \equiv \\
\equiv& e^{-(n+1)\overline{\lambda}_{r'}} \max_{u'\in U_{r'}^{(n+1)}} |(\textstyle\sum_{r(i)=r'} e^{(n+1)\overline{\lambda}_{r'}} P_i u, u') \le \\
\le& e^{-(n+1)\overline{\lambda}_{r'}} \Big|\textstyle\sum_{r(i)=r'} e^{(n+1)\overline{\lambda}_{r'}} P_i u\Big| \le \\
\le& e^{-(n+1)\overline{\lambda}_{r'}+(n+1)\delta_1} \Big|\textstyle\sum_{r(i)=r'} t_i^{(n+1)} P_i u\Big| \le \\
\le& e^{-(n+1)\overline{\lambda}_{r'}+(n+1)\delta_1} \Big|((L^*)^{n+1}L^{n+1})^{1/2}u\Big| \\
=& e^{-(n+1)\overline{\lambda}_{r'}+(n+1)\delta_1} (u, (L^{n+1})^* L^{n+1}u)^{1/2}
\end{aligned}$$

Thus, again by the spectral theorem and by $|TT'| \le |T|\,|T'|$ we deduce:

$$\begin{aligned}
|(u, u')| \le& e^{-(n+1)(\overline{\lambda}_{r'}-\delta_1)} \Big|L^{n+1}u\Big| \le \\
&\le e^{-(n+1)(\overline{\lambda}_{r'}-\delta_1)} |L| \Big|L^n u\Big| \le e^{-(n+1)(\overline{\lambda}_{r'}-\overline{\lambda}_r-2\delta_1)} |L|
\end{aligned}$$

because $u \in U_r^{(n)}$. This completes the proof in the case $r' > r$ and $k = 1$. The case $k > 1$ is simply obtained from the case $k = 1$ by applying the latter inequality k times. One finds, since the series converges, the result with a C that can be taken $C_1 = |L|(1 - e^{-x})^{-1}$ if $0 < x < \min(|\mu_r - \mu_{r'}|) - 2\delta_1$ for all n large enough and all $k \ge 0$.

Consider the case $r > r'$. Let u_α be an orthonormal base with the first m_1 vectors spanning $U_1^{(n)}$, the next m_2 spanning $U_2^{(n)}$ and so on; let u'_α be the corresponding base for $n + k$. The orthogonal matrix $W_{\alpha\alpha'} = (u_\alpha, u'_{\alpha'})$ verifies the inequalities:

$$\begin{cases} |W_{\alpha,\alpha'}| \le 1 & \text{for all } \alpha, \alpha' \\ |W_{\alpha,\alpha'}| \le C_1 e^{-|\overline{\mu}_{r(\alpha)}\overline{\mu}_{r(\alpha')}-2\delta_1|n} & \text{if } r(\alpha') > r(\alpha) \end{cases}$$

If $r(\alpha') < r(\alpha)$, *however*, the orthogonality implies that $W_{\alpha\alpha'} \equiv (W^{-1})_{\alpha'\alpha}$. The latter quantity is "just" the determinant obtained by deleting the row α and the column α' from the matrix W. Such determinant consists in a sum of $(d-1)!$ products of $d-1$ matrix elements W_{ij} picked up in pairwise distinct rows and columns (by Cramers' rule). Since we are interested in the non diagonal elements of $W_{\alpha\alpha'}$ we see that in each product there must be at least enough factors $W_{\beta\beta'}$ with $r(\beta') > r(\beta)$ so that $\sum \mu_{r(\beta')} - \mu_{r(\beta)} \geq \mu_{r(\alpha)} - \mu_{r(\alpha')}$ (hint: check this first when there are no degenracies, *i.e.* $r(\alpha) \equiv \alpha$). Hence we use for such factors the second inequality and just bound the others by 1. We finally get the result with $C = C_1(d-1)!$ if $2\delta_1(d-1) < \delta$.)

(16) Show that (15) implies that the planes $U_j^{(n)} \subset R^d$ have a limit U_j as $n \to \infty$ and the planes U_j are pairwise orthogonal. (Idea the planes $U_1^{(n+h)}$ and $U_1^{(n+k)}$ must form with the planes $\bigoplus_{r>1} U_r^{(n)}$ an angle closer to 90^o by an amonut prefixed arbitrarily if n is large enough. Hence they form a Cauchy sequence of planes converging to some U_1. The other planes are treated analogously.)

(17) Check that the previous problems (11)÷(16) imply that if L is a real $d \times d$ non singular matrix the limit $\lim_{n\to\infty}((L^*)^n L^n)^{1/2n} = D$ exists and it is a non singular positive definite matrix, with eigenvalues equal to the absolute values of those of L, counted according to multiplicity. The eigenspaces $U_j^{(n)}$ of $D_n \equiv ((L^*)^n L^n)^{1/2n}$ spanned by the $r(j)$ eigenvalues of D_n that converge to the j-th distinct limit value are planes that converge to limit planes U_j which are the eigenplanes of D which correspond to distinct eigenvalues. (Idea the only thing still to check is the relation between the eigenvalues of D and those of L. Suppose for simplicity that the eigenvalues of L have pairwise distinct absolute values (hence they must be all real) and order them by decreasing absolute values $\mu_1, \ldots, \mu_d$. Let $L = \sum_j \mu_j v_j \times v_j^*$ be the spectral resolution of L where $Lv_j = \mu_j v_j$, $L^* v_j^* = \mu_j v_j^*$ and $(v^*, v) = 1$. Clearly $|L^n v_j| = |\mu_j|^n |v_j|$. Hence $|\mu_1| = \lambda_1$. The other equalities can be deduced for instance by the "trick" of considering the matrices $L^{\wedge q}$, as in (13).)

Going back to the realm of abstract thinking the following guided problems, combining the concrete theory (11)÷(17) and the philosophical results (1)÷(10), lead to the Oseledec theorem.

(18) Given an ergodic dynamical system $(\mathcal{C}, S, \mu)$ with an invariant distribution μ such that there is a zero μ measure invariant set N outside which the transformation S is invertible and non singular, consider the matrix $\partial S^n(x) \equiv T^n(x)$. Suppose that $|\det T(x)| \geq \varepsilon > 0$ and that $|T(x)| < E$ in $\mathcal{C}/N$. Check that $T_n(x) = T(S^{n-1}x) \cdot \ldots \cdot T(Sx) \cdot T(x)$, and that $f_n(x) = \log |T_n(x)|$ verifies the subadditivity property of the subadditive ergodic theorem of (6). Therefore: $\lim_{n\to\infty} \frac{1}{n} \log |T_n(x)| = \lambda_1(x)$ exists μ almost everywhere.

((**19**) Define the matrices $T_n^{\wedge q}(x)$ on $R^{\wedge q}$ as in (12),(13) above, and by repeating the argument there (with (18) replacing (11)) prove that the limits $\lim_{n\to\infty} \frac{1}{n} \log t_j^{(n)}(x) = \lambda_j(x)$ exist almost everywhere for μ almost all x.

(20) Show that the analysis in (15) can be repeated word by word even when the matrices L^n are replaced by $T^n(x)$, in the points where the limits in (19) exist, *i.e.* almost everywhere. The matrix $D = \lim_{k\to\infty}((T_k^*(x)T_k(x))^{1/2k}$ has eigenplanes $U_1(x), \ldots, U_{s(x)}(x)$. Show that the spaces $V_j(x) = U_j(x) \oplus \ldots \oplus U_{s(x)}$ can be identified with the system of scaling planes for S of definition 6, §2.

(21) The contraction exponents $\lambda_j(x)$ are defined almost everywhere and they are constants of motion togheter with their multiplicities $m_j(x)$. Therefore they are μ almost everywhere constants. (Idea: By (20) and an argument like that in (17).)

(22) Check that all the above results can be derived by just requiring that $T(x)$ is such that $\log^+ |T(x)|$, where $\log^+$ denotes the positive part of the logarithm function, is μ–summable. (Idea: just careful examination of the proofs.)

(23) If $T(x)$ is replaced by any $d' \times d'$–matrix valued function $x \to O(x)$ with $\log^+ |O(x)|$ μ–summable the "same results" can be proved. For instance if $O_n(x) = O(S^{n-1}x) \cdot \ldots \cdot O(x)$ then $D_n = (O_n^*(x)O_n(x))^{1/2n}$ has a limit D as $n \to \infty$ which is almost surely independent of x, and the eigenspaces, spanned by the eigenvalues of D_n whose eigenvalues converge to the same limit $\overline{\lambda}_j$, converge to the eigenspace of D with eigenvalue λ_j. In other words the above theory extends trivially to a theory of products of random matrices $O(x)$ generated by selecting randomly by a distribution μ on $\mathcal{C}$ a point x and the corresponding random matrix $O(x)$ and by multipliyng $O(S^{n-1}(x) \ldots O(x)$.

(24) Let $\tilde{T}(x) = \partial S^{-1}(x)$, $\partial S(x) = T(x)$. Then $\tilde{T}(x) = T(S^{-1}x)^{-1}$ and $\tilde{T}_k(S^k x) = \partial S^{-k}(S^k x)$ can be written either $\tilde{T}(Sx) \ldots \tilde{T}(S^k x)$ or also as $T^{-1}(x) \ldots T^{-1}(S^{k-1}x)$. Show that if x admits a system of contracting planes, *i.e.* μ–almost everywhere, the eigenvalues $\tilde{t}_j^{(k)}$ (ordered by decreasing size) of $(\tilde{T}_k^*(S^k x)\tilde{T}_k(S^k x))^{1/2} \equiv \tilde{D}_k$ have the limit: $\lim \frac{1}{k} \log \tilde{t}_j^{(k)} = \tilde{\lambda}_j = -\lambda_{d-j}$, μ–almost everywhere. (Idea: note the following expression for $\tilde{T}_k$: $\tilde{T}_k^*(S^k x) \equiv \tilde{T}^*(S^k x) \ldots \tilde{T}(Sx)$ so that the matrices are multiplied in the correct order for the application of (23): hence $(\tilde{T}_k(S^k x)\tilde{T}_k^*(S^k x))^{1/2k}$ (identical to $(T_k^*(x)T_k(x))^{-1}$) converges to a matrix $\tilde{D} \equiv D^{-1}$, μ–almost surely (or if x admits a contracting system of planes). Thus the limits exist, because the spectrum of T^*T

and that of TT^* coincide in general, if T is non singular; their relation with the opposites of the contraction exponents λ_j in the forward direction is derived from $\tilde{T}_k(S^kx)\tilde{T}_k^*(S^kx) = (T_k^*(Sx)T_k(Sx))^{-1}$.)

(25) In the context of the problems (20),(24) consider the system of planes $\hat{V}_j(x)$, at x, defined by the eigenplanes $U_j(x)$ of the matrix D of problem (20) by $\hat{V}_j = U_{s-j+1} \oplus \ldots \oplus U_1(x)$, and existing μ–almost everywhere. Show that if $w_k \in U_j(S^kx)/U_{j+1}(S^kx)$ and $w_k = \partial S^k(x)u$ (hence $u \in \hat{U}_j(x)/\hat{U}_{j+1}(x)$) then $\lim_{k\to\infty} \frac{1}{k}\log|\tilde{T}(S^k)w_k|/|w_k| = -\overline{\lambda}_{s-j}$. Why one should not call $\hat{V}_j(x)$ an expanding plane when j is such that $\overline{\lambda}_{s-j'} < 0$ for all $j' < j$ and positive otherwise?

(26) Let us call the spectrum of a sequence of matrices O_n the d numbers $\omega_1, \ldots, \omega_d$ obtained by considering the logarithms of the eigenvalues of $(O_k^*O_k)^{1/2k}$, $o_1^{(k)} \geq \ldots o_d^{(k)}$, and setting, when they exist: $\lim_{k\to\infty} o_j^{(k)} = \omega_j$. Show that the spectrum of the sequence $O_k(x) = T_k(S^{-k}x)$ coincides with that of $T_k(x)$ μ–almost surely. (Idea: if the functions $n \to f_n(x)$ and $n \to f_n(S^nx)$ are subadditive, in the sense of problem (6), the functions $n \to f_n(S^nx)$ and $n \to f_n(x)$ are such that $F = \lim_{n\to\infty}\frac{1}{n}f_n(S^nx)$ and $G = \lim_{n\to\infty}\frac{1}{n}f_n(x)$ exist and are constant μ–almost everywhere, by (6) above. Suppose that $F > G$. Consider the sets D_n and D'_n consisting in the points where $\frac{1}{m}f_m(S^mx) > F - \varepsilon > G$ for all $m \geq n$, or respectively where $\frac{1}{m}f_m(x) < F - \varepsilon$ for *all* $m \geq n$. Then $\mu(D_n), \mu(D'_n) \to 1$. Hence for n large enough $S^{-n}D'_n \cap D_n \neq \emptyset$: if x is in this set then $\frac{1}{n}f(S^nx) < F - \varepsilon$ because $S^nx \in D'_n$ and $\frac{1}{n}f_n(S^nx) > F - \varepsilon$ because $x \in D_n$. Hence this contradiction shows that it is sufficient to check that the two functions are subadditive: this comes from the inequality $|AB| \leq |A||B|$.)

(27) The spectrum, see (26), of $\tilde{T}_{-k}(x) \equiv \tilde{T}(S^{-(k-1)}x)\ldots\tilde{T}(x)$ is the opposite of that of $T_k(x)$: $\omega_j = -\overline{\lambda}_{d-j+1}$. (Idea: $\tilde{T}_{-k}(x) = (T_k(S^{-k}x))^{-1}$; hence $\tilde{T}_{-k}(x)^*\tilde{T}_{-k}(x) = ((T_k(S^{-k}x)(T_k(S^{-k}x))^*)^{-1}$; but the spectrum of the last matrix is the same as that of the matrix $(T_k(S^{-k}x)^*T_k(S^{-k}x))^{-1}$ which by (26) leads to the result.)

(28) The forward and backward systems of scaling planes exist μ–almost everywhere and have opposite corresponding exponents: $\lambda_j = -\omega_{d-j+1}$. (Idea: this is a corollary of what already discussed in the few preceding problems.)

(29) Show that if $V_{-j}(x)$ are the contracting planes for S^{-1}, defined μ–almost everywhere, then $V_r(x) \cap V_{-(s-r+2)}(x) = 0$ and $V_r(x) \oplus V_{-(s-r+2)} = R^d$, μ–almost everywhere. (Idea: since, by definition, the dimensions of V_r and of $V_{-(s-r+2)}$ are complementary to d one only has to check the first statement. Note that if $u \in V_r(x)$ it is $|T_k(x)u| \leq e^{k(\lambda_r+\delta)}|u|$ for any prefixed $\delta > 0$ and for k large enough and all $x \in D_k$ where D_k is a set such that $\mu(D_k) \to 1$. For the same reason $|T_{-k}(S^kx)v| \leq e^{-(\lambda_{r-1}+\delta)k}|v|$ for $v \in V_{-r}(S^kx)$ provided $S^kx \in D_k$. Since $T_{-k}(S^kx)V_{-(s-r+2)}(S^kx) = V_{-(s-r+2)}(x)$ we see that all the $u \in V_{-(s-r+2)}(x)$ have the form $u = T_{-k}(S^kx)v$ with $v \in V_{-r}(S^kx)$. So that using $T_k(x)T_{-k}(S^kx) = 1$ we get:

$$\begin{aligned}&|T_{-k}(S^kx)v| \leq e^{-(\lambda_{r-1}-\delta)k}|T_k(x)T_{-k}(S^kx)v| \to \\ &\to |T_k(x)u| \geq e^{(\lambda_{r-1}-\delta)k}|u| \qquad \text{for } x \in S^{-k}D_k,\ u \in V_{-(s-r+2)}(x)\end{aligned}$$

and if $x \in S^{-k}D_k \cap D_k$ this is impossible because $\lambda_{r-1} > \lambda_r$ if also $u \in V_r(x)$ (so that $|T_k(x)u| \leq e^{(\lambda_r-\delta)k}|u|$) and δ is smaller than the twice the minimal difference between the scaling exponents.)

(30) The planes $W_j(x) = V_j(x) \cap V_{-(s-j+1)}(x)$ are such that

$$\frac{1}{k}\log|T_{\pm k}(x)u| \xrightarrow[k\to\infty]{} \lambda_j \qquad \text{for } u \in W_j(x)$$

and $V_j = W_j \oplus \ldots \oplus W_s$, $V_{-(s-j+2)} = W_1 \oplus \ldots \oplus W_{-(j-1)}$. Also $T(x)W_j(x) = W_j(Sx)$. This also implies the relation between the dynamical bases claimed in the theorem II. (Idea: this is just a corollary, or a summary, of the previous problems.)

(31) Show that if μ is non invariant the foward and backward Lyapunov exponents (and the relative systems of planes) may exist and be different. (Idea: consider the example on $\mathcal{C} = [-1,1] \times T^2$, where T^2 is the two dimensional torus. Let $x = (z, \varphi_1, \varphi_2)$ be a point in $\mathcal{C}$. Let $z \to f(z)$ be a smooth map such that $f^n(z) \xrightarrow[n\to\pm\infty]{} \pm 1$, and let $\nu(z) = 2$ if $z > 0$ and $\nu(z) = 1$ if $z < 0$; define:

$$x' = (z', \varphi'_1\varphi'_2) = \begin{cases} f(z) \\ \varphi_1 + \nu(z)\varphi_2 \mod 2\pi \\ \nu(z)\varphi_1 + (\nu(z)^2+1)\varphi_2 \mod 2\pi \end{cases}$$

and check the statements using $\mu = dz\,d\varphi_1\,d\varphi_2/2(2\pi)^2$.)

(32) Find the positive and negative Lyapunov exponents in the example suggested in problem (31); find also the dynamical bases (of the points which admit them).

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