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C^* Dynamical Systems that Asymptotically are Highly Anticommutative

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Abstract

It is shown that with probability 1 on Θ resp. on g_X the irrational rotation algebra $\mathcal{M}_\Theta$ with respect to the CAT map and the generalized Price–Powers shift $\mathcal{A}_X$ are asymptotically highly anticommutative.

1 Introduction

In [NT] the concept of an automorphism that is asymptotically highly anticommutative was introduced. In [AN] this property was shown to imply zero dynamical entropy. In [NST] the concept was slightly generalized and it was found that for some system with this property the dynamical entropy is not additive for the tensor product. In addition an explicit example in the framework of the Price–Powers shift was constructed. In this note we want to show that this example is not so exceptional but on the contrary for generalized Price–Powers shifts as well as for the irrational rotation algebra the shift resp. the CAT map are asymptotically highly anticommuative with probability 1.

2 The definition and its consequence

Definition 1: [NST] An automorphism α of a unital C^* algebra $\mathcal{A}$ is asymptotically highly anticommutative, if $\mathcal{A}$ contains a selfadjoint subset S of $\mathcal{A}$ such that $S \cup \{1\}$ is total in $\mathcal{A}$ and for which the following condition holds:

$$\forall\, w \in S,\ \forall\, \varepsilon > 0,\ \forall\, N \in \mathbf{N} \text{ there exists } k_1, \ldots, k_N \in \mathbf{N}$$

such that for $i \neq j$

$$\|[\alpha^{k_i} w, \alpha^{k_j} w]_+\| < \varepsilon. \tag{1}$$

Consequence: For such an automorphism there exists a unique invariant state [NT, NST]. The dynamical entropy of the automorphism in the sense of [CS, CNT] or [ST] is zero, since the stationary state in a coupling with an abelian system [ST] has to be of product form.

3 The irrational rotation algebra

Definition 2: The irrational rotation algebra $\mathcal{M}_\Theta$ is built by unitaries U, V which obey $UV = e^{2\pi i\Theta}VU$, $\Theta \in [0,1)$ is irrational. An automorphism $\alpha : \mathcal{M}_\Theta \to \mathcal{M}_\Theta$ is given by

$$\alpha(U^n V^m) = U^{an+bm} V^{cn+dm} \tag{2}$$

with $(a, b, c, d) \in \mathbf{Z}$, $ad - bc = 1$, $a + d > 2$. $(\mathcal{M}_\Theta, \alpha)$ is the C^* dynamical system we are considering here.

Remark: Θ was chosen to be irrational so that the center of the algebra is trivial. For Θ rational the center is the classical function algebra over $\mathcal{T}^2$ and determines the ergodic behaviour [BNS]. It was shown in [BNS] and [N] that α is for all Θ in the tracial state weakly asymptotic abelian but only for a countable set of irrational Θ's it is also strongly asymptotic abelian.

$\mathcal{M}_\Theta$ is linearly spanned by the unitaries

$$W(\vec{n}) = e^{-i\pi\Theta n_1 n_2} U^{n_1} V^{n_2}, \qquad \vec{n} = (n_1, n_2) \in \mathbf{Z}^2. \tag{3}$$

They obey the relations

$$W(\vec{n})W(\vec{m}) = e^{i\pi\Theta\sigma(n,m)} W(\vec{n}+\vec{m}), \qquad \sigma(\vec{n},\vec{m}) = n_1 m_2 - n_2 m_1. \tag{4}$$

In matrix notation (2) becomes

$$\alpha(W(\vec{n})) = W(T\vec{n}), \qquad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{5}$$

The eigenvalues $\lambda^{\pm 1}$ of T depend only on the trace $t = a+d > 2$ and obey $\lambda^2 - \lambda t + 1 = 0$. They are irrational and we take $\lambda > 1$. The eigenvectors of T

$$\vec{\mu}_\pm = \frac{1}{\sqrt{(a-\lambda^{\pm 1})^2 + b^2}} \left| \begin{matrix} b \\ \lambda^{\pm 1} - a \end{matrix} \right\rangle$$

are orthogonal iff $b = c$ but in any case we can expand $\vec{n} = c_+\vec{\mu}_+ + c_-\vec{\mu}_-$ such that

$$T^k \vec{n} = c_+ \lambda^k \vec{\mu}_+ + c_- \lambda^{-k} \mu_-. \tag{6}$$

Since

$$[W(\vec{n}), \alpha^k W(\vec{n})]_+ = 2\cos(\pi\Theta\sigma(\vec{n}, T^k\vec{n})) W(\vec{n} + T^k\vec{n}) \tag{7}$$

and $\|W(\vec{n})\| = 1$, anticommutativity depends on the closeness of $\Theta\sigma(\vec{n}, T^k\vec{n})$ to 1/2 mod $\mathbf{Z}$. The rest of this section is devoted to studying when this happens.

¿From (5) and (6) we deduce

$$\sigma(\vec{n}, T^k\vec{n}) = c(\lambda^k - \lambda^{-k}), \qquad c = c_+ c_- \sigma(\vec{\mu}_-, \vec{\mu}_+). \tag{8}$$

Since c depends only $\vec{n}$ and T but not on k we note

$$c = \frac{1}{\lambda - 1/\lambda}\, \sigma(\vec{n}, T\vec{n}). \tag{9}$$

With $\lambda^{\pm 2} - t\lambda^{\pm 1} + 1 = 0$ we can write higher powers of $\lambda^{\pm 1}$ as

$$\lambda^{\pm k} = \alpha_k \lambda^{\pm 1} + \beta_k \tag{10}$$

where $\vec{v}_k := (\alpha_k, \beta_k) \in \mathbf{Z}^2$ obey the recursion relation

$$\vec{v}_{k+1} = M\vec{v}_k, \qquad M = \begin{pmatrix} t & 1 \\ -1 & 0 \end{pmatrix}, \qquad \vec{v}_0 = (0,1). \tag{11}$$

M has also eigenvalues $\lambda^{\pm 1}$. With these notations and (8) and (9) we can write the quantity of interest as

$$\sigma(\vec{n}, T^k \vec{n}) = \frac{\lambda^k - \lambda^{-k}}{\lambda - \lambda^{-1}}\, \sigma(\vec{n}, T\vec{n}) = \alpha_k\, \sigma(\vec{n}, T\vec{n}) = \langle \vec{v} | M^k \vec{v}_0 \rangle\, \sigma(\vec{n}, T\vec{n}) \tag{12}$$

where $\vec{v} = (1, 0)$.

At some instance we shall need the dependence of $\sigma(\vec{n}, T^{k-\bar{k}}\vec{n})$ on k and $\bar{k}$ separately which we get by the observation

$$\begin{aligned} \alpha_{k-\bar{k}} &= (\lambda - \lambda^{-1})^{-1}(\lambda^k \lambda^{-\bar{k}} - \lambda^{-k}\lambda^{\bar{k}}) \\ &= (\lambda - \lambda^{-1})^{-1}[(\alpha_k \lambda + \beta_k)(\alpha_{\bar{k}}\lambda^{-1} + \beta_{\bar{k}}) - (\alpha_k \lambda^{-1} + \beta_k)(\alpha_{\bar{k}}\lambda + \beta_{\bar{k}})] \\ &= \alpha_k \beta_{\bar{k}} - \alpha_{\bar{k}}\beta_k = \left\langle \vec{v}_k \middle| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{v}_{\bar{k}} \right\rangle . \end{aligned} \tag{13}$$

After these elementary preparations we are ready for the

Lemma 1: $\forall\ \vec{v} = (v_1, v_2) \in \mathbf{Z}^2 \setminus \{0\}$ the numbers $\Theta\langle \vec{v} | \vec{v}_k \rangle \pmod{\mathbf{Z}}$ are in $k \in \mathbf{N}^*$ uniformly distributed over $\mathcal{T}^1$ with probability 1 in Θ.

Uniform distribution means [W, H] that $\forall\ f \in C(\mathcal{T}^1)$ we get

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} f(\Theta\langle \vec{v} | \vec{v}_k \rangle) = \int_0^1 dx\ f(x).$$

This is equivalent to require that it holds $\forall\ f(x) = e^{2\pi i h x}$, $h \in \mathbf{Z}$ or that

$$\bar{f}_{N,\Theta} := \frac{1}{N} \sum_{k=1}^{N} e^{2\pi i \Theta h \langle \vec{v} | \vec{v}_k \rangle} \to 0 \quad \forall\ h \in \mathbf{Z} \setminus \{0\}.$$

To see whether this holds for almost all Θ we consider

$$\int_0^1 d\Theta \left| \bar{f}_{N,\Theta} \right|^2 = \frac{1}{N^2} \sum_{k,\bar{k}=1}^{N} \int_0^1 d\Theta\ e^{2\pi i \Theta h \langle \vec{v} | \vec{v}_k - \vec{v}_{\bar{k}} \rangle}.$$

Now

$$\int_0^1 d\Theta\ e^{2\pi i \Theta h \langle \vec{v} | \vec{v}_k - \vec{v}_{\bar{k}} \rangle} = \begin{cases} 1 & \text{if } \langle \vec{v} | \vec{v}_k - \vec{v}_{\bar{k}} \rangle = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{since } \langle \vec{v} | \vec{v}_k - \vec{v}_{\bar{k}} \rangle \in \mathbf{Z}.$$

But

$$\langle \vec{v} | \vec{v}_k - \vec{v}_{\bar{k}} \rangle = \langle \vec{v} | (M^k - M^{\bar{k}})\vec{v}_0 \rangle = 0 \Longleftrightarrow |(M^k - M^{\bar{k}})\vec{v}_0\rangle = c|\vec{v}_\perp\rangle$$

with $|\vec{v}_\perp\rangle = \left| \begin{array}{c} v_2 \\ -v_1 \end{array} \right\rangle$, $c \in \mathbf{Q}$. This can happen for $k = \bar{k}$ or for fixed $k - \bar{k} = d$ only once. First note that $c \neq 0$ because $(M^{k+d} - M^k)|\vec{v}_0\rangle = 0 \Longleftrightarrow M^d|\vec{v}_0\rangle = |\vec{v}_0\rangle$ but the eigenvalues of M are also $\lambda^{\pm 1}$. If we also had for $\ell \in \mathbf{Z} \setminus \{0\}$

$$(M^{k+\ell} - M^{\bar{k}+\ell})|\vec{v}_0\rangle = \bar{c}|\vec{v}_\perp\rangle = \frac{\bar{c}}{c}(M^k - M^{\bar{k}})|\vec{v}_0\rangle$$

then M^ℓ were to have an eigenvalue $\bar{c}/c \in \mathbf{Q}$ but its eigenvalues $\lambda^{\pm\ell}$ are irrational. Thus $\langle \vec{v}|\vec{v}_k - \vec{v}_{\bar{k}}\rangle$ is zero for at most $N + N - 1$ values of $(k, \bar{k}) \in (1, \dots, N)^2$ and

$$\int_0^1 d\Theta \, \left|\bar{f}_{N,\Theta}\right|^2 \leq \frac{2N-1}{N^2} \to 0.$$

This means that the set of Θ's for which $\bar{f}_{N,\Theta} \to 0$ is of measure 1.

Lemma 2: For any sequence $\{k_1, \dots, k_r\} \in \mathbf{Z}^r$ and any $\vec{n} \in \mathbf{Z}^2 \setminus \{0\}$ the elements of $\mathcal{T}^r$, $\Theta\sigma(\vec{n}, T^{k-k_i}\vec{n})$ mod $\mathbf{Z}^r$, $i = 1, \dots, r$, are in $k \in \mathbf{N}^*$ uniformly distributed on the 2–dimensional submanifold

$$\mathcal{S}_2 = \{\vec{v} \in \mathcal{T}^r; v_i = x\alpha_{k_i}(\mathrm{mod} z) + \bar{x}\beta_{k_i}(\mathrm{mod} z); (x, \bar{x}) \in \mathbf{R}^2, i = 1, \dots, r\} \subset \mathcal{T}^r$$

with probability 1 in Θ.

Proof: From (12) and (13) we infer

$$\sigma(\vec{n}, T^{k-k_i}\vec{n}) = (\alpha_k\beta_{k_i} - \beta_k\alpha_{k_i})\sigma(\vec{n}, T\vec{n})$$

and the claim is that $\forall\ \{h_j\} \in \sigma(\vec{n}, T\vec{n}) \cdot \mathbf{Z}^r$ with $j = 1, \dots, r$ we have

$$\begin{aligned} \frac{1}{N}\sum_{k=1}^{N} \exp\left[2\pi i\Theta \sum_{j=1}^{r} h_j(\alpha_k\beta_{k_j} - \beta_k\alpha_{k_j})\right] &\to \\ \to \int_0^1 dx d\bar{x} \exp\left[2\pi i \sum_{j=1}^{r} h_j(\beta_{k_j}x - \alpha_{k_j}\bar{x})\right] & \\ = \begin{cases} 1 & \text{for } \sum_{j=1}^r h_j\beta_{k_j} = 0 = \sum_{j=1}^r h_j\alpha_{k_j} \\ 0 & \text{otherwise.} \end{cases} & \end{aligned}$$

With $v_1 = \sum_j h_j\beta_{k_j}$, $v_2 = -\sum_j h_j\alpha_{k_j}$ this means

$$\bar{f}_{\Theta,N} = \frac{1}{N}\sum_{k=1}^{N} e^{2\pi i\Theta(\alpha_k v_1 + \beta_k v_2)} \to 0 \quad \forall\ (v_1, v_2) \neq (0, 0)$$

with probability 1 in Θ. This is exactly the statement of Lemma 1.

Lemma 3: If $\alpha_j/\alpha_{j-1} > 2/\varepsilon$, $\alpha_j \in \mathbf{N}^*$, $j = 1, \ldots, r$ then $\mathcal{S}_1 := \{\vec{v} \in \mathcal{T}^r; v_j = x\alpha_j \mod \mathbf{Z}, x \in \mathbf{R}\} \subset \mathcal{S}_2 \subset \mathcal{T}^r$ meets the open set

$$\mathcal{O} = \{\vec{v} \in \mathcal{T}^r; 1/2 - \varepsilon < v_j < 1/2 + \varepsilon \;\forall\; j = 1, \ldots, r\} \subset \mathcal{T}^r.$$

Proof: We shall proceed inductively in j and use the quantities $\delta_j \in (-\varepsilon, \varepsilon)$, $z_j \in \mathbf{Z}$, $\gamma_j \in (-\varepsilon^j/2, \varepsilon^j/2)$. The first component in $\mathcal{T}^r$ is in $\mathcal{O}$ if $v_i = x\alpha_1 = \frac{1}{2} + \delta_i$ or $x = \frac{1}{\alpha_1}(\frac{1}{2} + \delta_1)$. The next component requires

$$v_2 = \alpha_2 x = \frac{\alpha_2}{\alpha_1}\left(\frac{1}{2} + \delta_1\right) = \frac{1}{2} + \delta_2 + z_2$$

or

$$\delta_1 = \frac{\alpha_1}{\alpha_2}\left(\frac{1}{2} + z_2 + \delta_2\right) - \frac{1}{2}.$$

Since by assumption $\alpha_1/\alpha_2 < \varepsilon/2$ we can choose z_2 such that

$$\delta_1 = \gamma_1 + \frac{\alpha_1}{\alpha_2}\delta_2 \in \left(\gamma_1 - \varepsilon\,\frac{\alpha_1}{\alpha_2}, \gamma_1 + \varepsilon\,\frac{\alpha_1}{\alpha_2}\right) \subset (-\varepsilon, \varepsilon).$$

The next step requires

$$\frac{\alpha_3}{\alpha_1}\left(\frac{1}{2} + \delta_1\right) = \frac{1}{2} + \delta_3 + z_3$$

or

$$\delta_1 = \frac{\alpha_1}{\alpha_3}\left(\frac{1}{2} + z_3 + \delta_3\right) - \frac{1}{2}.$$

Since $\dfrac{\alpha_1}{\alpha_3} < \dfrac{\varepsilon}{2}\,\dfrac{\alpha_1}{\alpha_2}$ we can choose z_3 such that

$$\begin{aligned}\delta_1 &= \gamma_1 + \gamma_2 + \frac{\alpha_1}{\alpha_3}\delta_3 \in \left(\gamma_1 + \gamma_2 - \varepsilon\,\frac{\alpha_1}{\alpha_3}, \gamma_1 + \gamma_2 + \varepsilon\,\frac{\alpha_1}{\alpha_3}\right)\\ &\subset \left(\gamma_1 - \varepsilon\,\frac{\alpha_1}{\alpha_2}, \gamma_1 + \varepsilon\,\frac{\alpha_1}{\alpha_2}\right).\end{aligned}$$

Proceeding in this way we see that the open subset of $\mathcal{S}_1$, where

$$x = \frac{1}{\alpha_1}\left(\frac{1}{2} + \delta\right), \qquad \delta \in \left(\gamma_1 + \gamma_2 + \ldots + \gamma_{r-1} - \varepsilon\,\frac{\alpha_1}{\alpha_r}, \gamma_1 + \gamma_2 + \ldots + \gamma_{r-1} + \varepsilon\,\frac{\alpha_1}{\alpha_r}\right)$$

is contained in Θ.

Theorem 1: With probability 1 in Θ one can find for any $\varepsilon > 0$, $\vec{n} \in \mathbf{Z}^2 \setminus \{0\}$ and $N \in \mathbf{N}^*$ a sequence k_i, $i = 1, 2, \ldots, N$ such that $|\cos \pi\Theta\sigma(\vec{n}, T^{k_i - k_j}\vec{n})| < \varepsilon \;\forall\; i \neq j$.

Proof: Having fixed ε and $\vec{n}$, we proceed by induction for k_i since Lemma 1 guarantees the existence of a k_1. Having found $(k_1, \ldots, k_r)$ Lemma 2 tells us that the k_{r+1} which qualify are uniformly distributed in $\mathcal{S}_2$ which by Lemma 3 meets the open set $\mathcal{O}$. The intersection is open in $\mathcal{S}_2$ and because of uniform distribution it contains infinitely many qualifying k's. Since α_k goes with k to infinity we can find a k_{r+1} which satisfies $\alpha_{k_{r+1}}/\alpha_{k_r} > 2/\varepsilon$. We denote the set of Θ's for which this holds $\Sigma_{\varepsilon,\vec{n},r} \subset [0,1)$ and observe $\mu(\Sigma_{\varepsilon,\vec{n},r}) = 1$. For $\Theta \in \bigcap_{r=1}^{N} \Sigma_{\varepsilon,\vec{n},r}$ we can find $(k_1, \ldots, k_N)$ and since finite intersections of sets with probability 1 still have probability 1 we have proved Theorem 1.

Conclusion: With probability 1 in Θ the C^*–dynamical system $(\mathcal{M}_\Theta, \alpha)$ is highly anticimmutative and therefore has only one invariant state, zero dynamical entropy, the latter not being additive for $(\mathcal{M}_\Theta \otimes \mathcal{M}_\Theta, \alpha \otimes \alpha)$.

Proof: The first statement follows from Theorem 1 from which the next two were deduced in [NT] and [AN, NST]. The last statement follows because $\mathcal{M}_\Theta \otimes \mathcal{M}_\Theta$ contains the subalgebra generated by $W(n_1, n_2) \otimes W(n_1, -n_2)$ which is for all Θ abelian and isomorphic to $\mathcal{M}_0 = (C(\mathcal{T}^2), \alpha)$. The latter hass dynamical entropy $\ln \lambda > 0$ and therefore only the general conclusion that the dynamical entropy is superadditive for the tensor product remains.

Remarks

1. Voiculecu [V] has recently defined another dynamical entropy for C^* systems which is subadditive for the tensor product. In the abelian case it also agrees with the KS entropy. Since for the dynamical entropy it is the large time behaviour which matters, one might think that also for asymptotic abelian system all definitions should agree. Since $(\mathcal{M}_\Theta, \alpha)$ is weakly asymptotic abelian we conclude that weak asymptotic abelianness is not enough. It is an open question which degree of asymptotic abelianness is necessary for the definitions to agree. When they do then dynamical entropy is necessarily additive for the tensor product.

2. The result shows how sensitive CS entropy is to the structure of the algebra. For rational $\Theta = p/q$, $p, q \in \mathbf{Z}$ it equals the KS entropy of the center $\{W(n)\}$, $n \in q\mathbf{Z}^2$ and therefore is $\ln \lambda$.

3. Our result implies that for most Θ's the tracial state is the only invariant state for $(\mathcal{M}_\Theta, \alpha)$. It exists for all Θ's and thus $\mathcal{M}_\Theta$ is of type II_1. Since Voiculescu's entropy is subadditive for the tensor product it is positive for $(\mathcal{M}_\Theta, \alpha)$ for all Θ's and thus different from the CS entropy and not only of its subsequent generalizations.

4 The Price–Powers Shift

In [NST] a special Proce–Powers shift was constructed as example of a system which satisfies (1). Several generalizations of the construction are possible. In this note we want to keep the probabilistic point of view as for the rotation algebra and show that it is also highly anticommutative with probability 1. We repeat the definition of the Price–Powers shift and keep the notation of [NST].

Definition 3: Let X be a subset of $\mathbf{N}^*$ and let g_X be its characteristic function. Let (s_i), $i \in \mathbf{Z}$ be a sequence of selfadjoint unitaries satisfying the commutation relations

$$s_i\, s_j = (-)^{g_X(|i-j|)} s_j\, s_i.$$

Denote by $\mathcal{A}_X$ the C^* algebra generated by the set of $s_i \in \mathbf{Z}$. It is linearly generated by the words

$$w_I = \prod_{i \in I \subset \mathbf{Z}} s_i, \qquad I = \{i_1 < i_2 < \ldots i_k\}, \qquad \text{card } I = k < \infty.$$

$\alpha \in \text{Aut } \mathcal{A}_X$ is defined by $\alpha(s_i) = s_{i+1}$.

It follows that a word w_I is either hermitian or antihermitian and two words either commute or anticommute. In fact

$$w_I w_J = w_J w_I (-)^{\sum_{i\in I}\sum_{j \in J} g_X(|i-j|)} = w_J w_I (-)^{\text{card}\,(|I-J|_o) \cap X} \tag{14}$$

where we define $|I-J|_o = \{|i-j|, i \in I, j \in J, \text{which occur as odd time}\} \subset \mathbf{N}^*$.

To assign a probability to the set of X's for which $(\mathcal{A}_X, \alpha)$ is highly anticommutative we map those elements of $P(P(\mathbf{N}^*))$, the second power set of $\mathbf{N}^*$, which are cylindrical subsets of $P(\mathbf{N}^*)$ onto the Ising algebra

$$\mathcal{B} := \prod_{i=1}^{\infty} D_i, \qquad = \text{diag } M_2$$

generated by

$$P_i = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad Q_i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The mapping γ associates bijectively to

$$\chi_{n,m} := \{X \subset \mathbf{N}^* : \{n_1, \ldots, n_k\} \subset X, \{m_i, \ldots, m_\ell\} \subset X^\ell\} \in P(P(\mathbf{N}^*)),$$

the element

$$\gamma(\chi_{n,m}) = \prod_{i=1}^{k} \prod_{j=1}^{\ell} P_{n_i} Q_{m_j}.$$

The probability μ for $\chi_{n,m}$ is given by the pull–back with γ of the standard product state ω over $\mathcal{B}$, $\omega(P_i) = 1/2$, $\mu = \omega \circ \gamma$ or

$$\mu(\chi_{n,m}) = \omega(\gamma(\chi_{n,m})) = \begin{cases} 2^{-k-\ell} & \text{if } n_i \neq m_j \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Lemma 4: For all $I \subset \mathbf{N}^*$

$$\mu\{X : \text{card } X \cap I = \text{odd}\} = \frac{1}{2}.$$

Proof: First consider the case where I has only one element $I = \{i\}$. Then $\gamma\{X : \text{card } X \cap I = \text{odd}\} = P_i$ and the result follows from $\omega(P_i) = 1/2$ (15). Next let I have ℓ elements, $I = \{i_1, i_2, \ldots, i_\ell\}$. Consider all possible partitions $I = I_+ \cup I_-$. Then

$$\mu\{X : \text{card } X \cap I = \text{odd}\} = \omega\left(\sum_{\text{card } I_+ = \text{ odd}} \prod_{i \in I_+} P_i \prod_{k \in I_-} Q_k\right) = 2^{\ell-1} 2^{-\ell}.$$

Corollary: The probability that two words w_I and w_J commute (or anticommute) is 1/2.

Proof:

$$\mu\{X : \text{card } X \cap |I - J|_o = \text{even (or odd}\} = \frac{1}{2}$$

according to Lemma 4.

Lemma 5: For a given set of words $w_{I_1}, \ldots, w_{I_\ell}$ the probability that a word w_J with $|J - I_i|_o \cap |J - I_\ell|_o = \emptyset$, $i \neq \ell$ anticommutes with all w_{I_i} is $2^{-\ell}$.

Proof: Denote $K_i = |J - I_i|_o = K_i^+ \cup K_i^-$,

$$\begin{aligned}
&\mu\{X : \text{card } X \cap K_r = \text{odd } \forall\, r \in (1, \ldots, \ell)\} = \\
&= \omega\left(\prod_{r=1}^{\ell} \sum_{\text{card } K_r^+ = \text{ odd}} \prod_{i_r \in K_r^+} \prod_{j_r \in K_r^-} P_{i_r} Q_{j_r}\right) \\
&= \prod_{r=1}^{\ell} \omega\left(\sum_{\text{card } K_r^+ = \text{ odd}} \prod_{i_r \in K_r^+} \prod_{j_r \in K_r^-} P_{i_r} Q_{j_r}\right) = 2^{-\ell}
\end{aligned}$$

because ω is a product state and we required $K_r \cap K_s = \emptyset\ \forall\, r \neq s$.

Theorem 2: Assume for a word w_I there exist numbers $n_1, \ldots, n_r$ such that

$$|I - I + n_s - n_k|_o \cap |I - I + n_s - n_j|_o = \emptyset\ \forall\, n_1 \leq n_k < n_j < n_s \leq n_r.$$

Then with probability 1 there exists an n_{r+1} such that

$$[\alpha^{n_s} w_I, \alpha^{n_{r+1}} w_I]_+ = 0 \quad \forall\, 1 \leq s \leq n_r$$

and

$$|I - I + n_{r+1} - n_k|_o \cap |I - I + n_{r+1} - n_j|_o = \emptyset\ \forall\, n_1 \leq n_k < n_j \leq n_r.$$

Proof: Consider the shifted word $\alpha^\ell w_I$ with ℓ such that $|I-I+\ell-n_k|_o \cap |I-I+\ell-n_j|_o = \emptyset \ \forall\ n_1 \leq n_k < n_\ell \leq n_r$. There are infinitely many such ℓ's, say, $\ell_1, \ell_2, \ldots, \ell_k, \ldots$. The probability that $w_{I+\ell}$ commutes with at least one of the $\alpha^{n_s} w_I$, $s = 1, \ldots, r$ equals $1 - 2^{-r} < 1$. Because of the disjointness assumption the shifted words $\alpha^{\ell_j} w_I$ correspond to independent variables P_i, Q_j. Because of the product structure of the state ω the probability that among k shifts $\ell_1, \ldots, \ell_k$ at least once the shifted word anticommutes with all given $\alpha^{n_s} w_I$ is therefore $1-(1-2^{-r})^k$. Since k can be arbitrarily big we can with certainty find an ℓ which qualifies for the n_{r+1} in Theorem 2.

Corollary: $(\mathcal{A}_X, \alpha)$ is with probability 1 in X highly anticommutative.

Proof: For each word w_I start with $n_1 = 0$ and follow the proof of Theorem 2 to find an n_2 such that $[w_I, \alpha^{n_2} w_I]_+ = 0$ and $|I-I|_o \cap |I-I+n_2|_0 = \emptyset$. Then proceed inductively in r with Theorem 2.

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