ON THE PROBLEM OF UNIVERSALITY OF QUANTUM THEORY†)

PAVEL BÓNA

Department of Theoretical Physics, Comenius University
842 15 Bratislava, Slovakia
and
Department of Mathematics, Silesian University
746 01 Opava, Czech Republic

ABSTRACT

Two aspects of the problem of "objectification" in quantum mechanics (QM) are considered: (1) The question of description of the usually requested nonexistence of interference of macroscopically different states, and (2) the possibility of distinction of Genuine mixture (or "Gemenge") from the Elementary mixture (or "improper mixture").

Some possibilities for a necessary modification of QM are considered. Especially modifications of the set of "observables" allowing, e.g., description of quantal measurement process in framework of systems with finite number of degrees of freedom are discussed. An idealised solvable model of measurement process in quantum theory is presented. A nonlinear version of quantum mechanics (NLQM – a consistent generalisation of the Weinberg's one) describing different evolutions for Genuine and Elementary mixtures is presented. "Relative isolation" of quantal systems is defined in this connection. NLQM contains QM, classical mechanics, and some "quasiclassical" theories as its exact subtheories.

1. Introduction

Quantum theory is formulated in terms of statistical distributions of values of observed quantities. The probability distributions of QT cannot be endowed with a "simultaneous ignorance interpretation": It cannot be reformulated in terms of Kolmogorovian model of probability theory describing statistical distributions on a *firmly given* set of elementary events (representing in applications a "background reality" – some sort of "Universe of all possible events"); Kolmogorovian model works well, however, in classical theory (CT) of physical phenomena, where an "elementary event" determines sharp values of all possible quantities. On the other hand, classical description of a "background" (resp. a "pretheory", cf. [11]) for QT seems to be a necessary prerequisite for introduction into QT such basic concepts as reference frames, basic observables, preparations, and measurements; hence, mere definition of "a microsystem" depends crucially on a use of classical concepts. No clear boundary is known in physics between phenomena described by CT, resp. QT. Moreover, interests of physicists are attracted presently by "mesoscopic phenomena" for description of which both CT and QT are useful. Hence, the requirement of *universality* of an adequate physical theory containing both quantum and classical concepts is quite actual. We expect that QT can be a basis for such a universal theory.

CT cannot be included into the traditional quantum mechanics (QM) of systems with finite number of degrees of freedom, cf. [2]: Even classical "observables" are not present in QM. Hence, QM needs some modifications (resp. extensions) to become a universal QT. We intend to consider here some questionable features of QM which could be, perhaps, modified without changing any

†) Supported from Grant No. 201/93/2245 of the Czech Grant Agency.

essential and successful feature of standard QT, with the aim to reach universality of (such a minimal modification of) QT, cf. also [3].

One of those essential features of QT which should be kept in its universal modification (let us call UQT the intended universal theory) is, according to present author's opinion, the irreducibly probabilistic character of QT. Hence, also description of classical phenomena in framework of UQT should be probabilistic. This might seem to contradict to everyday experience with "individual objects" which can be observed repeatedly as "the same object" with unchanged (sharp) values of all their relevant quantities. The mentioned "relevant quantities" seem to be stable also with respect to measurements of any other quantities of the observed empirical object. (Warning: It should be distinguished between theoretically defined "physical system" specified by a given set of "observables", "states", etc., and an "empirical object" specified by some empirical procedures, some features of which – e.g. some specific observable quantities – serve as an empirical realization of the theoretical system; that "empirical object" can be used for an empirical realization of also other theoretically defined "systems" which, e.g., are described by larger sets of "observables".) It seems to be no serious problem to reformulate CT in terms of very narrow (Kolmogorovian) statistical distributions, to match just described intuition. More problems with intuition arise in trials of description of "macroscopically unstable" situations, e.g., of the process of quantal measurement. In such situations, QT leads (e.g., via a version of the "reduction postulate") to statistical distributions of values of macroscopic (classical) quantities with large dispersions, and, in the same time, QT doesn't contain any (although *intuitively desirable*) possibility of dealing with individual events (e.g., with the process of acquirement of a specific pointer position). We shall accept here such a state of affairs as granted: We believe that there are physical situations (as in the case of measurement of some quantities of a microscopic system) in which there is impossible to forecast deterministically a motion of otherwise classical objects (e.g., the pointer). The instability of situations allows us to foresee only a time evolution of statistical ensembles (in the sense of Genuine mixtures) of states of macroscopic (classical) systems, hence we cannot theoretically describe observed individual results of a measurement. This could be considered as a source of effective irreversibility (and of a loss of information contained in theoretical description of actual states) in formally time-reversible theory. (Let us mention here, however, that also in pure CT there are unstable situations, e.g. systems with unstable flows – see Anosov diffeomorphisms, etc., where the formally deterministic description is practically as stochastic, as the above mentioned quantum processes; this "practical stochasticity" is due to our restricted ability of distinction between points in a phase space corresponding to different initial conditions of mutually quickly diverging trajectories.)

Let us mention that a sort of the desired UQT can be obtained in framework of C^*-algebraic theory of infinite quantal systems (cf. [12], [13]): CT can be included there together with QT, but the unstable processes of the type of quantal measurement are lasting there for infinite time interval, hence no evolution of irreversible situations is described by those theories.

Another generalization of QM is the form of nonlinear quantum mechanics (NLQM) described in the last section of this paper: It also includes QM together with CM ($\equiv$ classical mechanics), but the problem of description of measurement is not solved yet in that framework.

2. Controversial "Observables"

In standard formulations of QM, any selfadjoint operator on separable Hilbert state-space $\mathcal{H}$ of the considered system represents an "observable". No differences of observability of different observables are formulated on this general level. Hence, the set of all bounded "observables" consists of all symmetric bounded operators $\mathcal{L}(\mathcal{H})_s$. The set $\mathcal{L}(\mathcal{H})_s$ is the same for all finite-particle systems (with some modifications for systems of identical paricles, or systems with absolutely conserved charges): A structurless point particle is described by the same set of abstract "observables" as a collection of 10^{25} atoms. In practice, however, some "basic observables" of a given system are picked out, e.g. generators of $2f+1$ – dimensional Weyl-Heisenberg group and a few other operators (e.g. generators of Galilean group representation) for a system with f degrees of freedom. This shows that in practice some "hierarchy of observability" of formally given operators is respected in QM.

A possible restriction can be anticipated for measurability of "observables" of multiparticle systems (let us keep in mind – for simplicity – mutually distinguishable particles only, e.g. particles placed in sites of a crystalline solid): To measure interference of two distinct permutationally invariant multiparticle states it would be necessary to measure very–high–order correlation functions, those of the order of number of particles present in the system, cf. also [3]. To illustrate this idea, let us take a large system to be described by tensor-product Hilbert space $\mathcal{H}^N := \otimes_{p=1}^N \mathcal{H}_p \ni \Phi, \Psi$; $\Phi := \otimes_{p=1}^N \varphi_p$, $\Psi := \otimes_{p=1}^N \psi_p$, with φ_p and $\psi_p \in \mathcal{H}_p$. Assume that $\|\varphi_p\| = \|\psi_p\| = 1$, and $|\langle\varphi_p|\psi_p\rangle| \leq \lambda < 1$ for all p. Let us consider the observables of the form

$$A := \sum_{p_1,p_2,\ldots p_k} X^1_{p_1} \otimes X^2_{p_2} \otimes \ldots \otimes X^k_{p_k} \otimes I_{\{N\setminus\{p_1,p_2,\ldots p_k\}\}} \tag{1}$$

with X^j_p being selfadjoint in $\mathcal{L}(\mathcal{H}_p)$ of the norm ≤ 1. Let $k(\ll N)$ be fixed. Then, for $N \to \infty$, one has

$$|\langle\Psi|A\Phi\rangle| < N^k \lambda^{N-k} \approx O(e^{-\alpha N}) \tag{2}$$

for some $\alpha > 0$. If there would not be any other observables of the considered system, then (2) would present a kind of "effective disjoitness" of the states described by Ψ and Φ with respect to the chosen set of observables of the form (1). Conventional QM admits, however, observables with k in (1) of the order of N: $k \approx N$. Such "observables" (i.e. products of large number of one particle operators) seem to be practically unobservable, [4]. This might be a reason of "practical classicality" of some subsystems (determined by their observables, etc.) of large quantal systems.

The following note (although related to previous ones) could further support the requirement for extension of QT by some criteria for "observability of observables": Projection operators are used to be considered to represent "questions" in QT with "yes"–"no" answers (cf. [5], [6], [8]). Their interpretation is, however, determined by interpretation of vectors in corresponding subspaces of states in $\mathcal{H}$. But interpretation of vectors in $\mathcal{H}$ is equally problematic as the interpretation of operators: vectors correspond to "preparation procedures", whereas operators are ascribed to "registration procedures", cf. [1], [10], [11]. Hence, all systems with any finite number of degrees of freedom are described formall by "the same number of questions", what seems to be absurd from the point of wiev of physical intuition. (The situation is here analogous as that one occuring at "counting points" on the real line, resp. on its intervals of nonzero lenght: mathematically there is no difference, but endowing the reals by a metric a natural difference is created.)

Our last argument for necessity of distinction between "degrees of observability" of different formally defined "observables" is, perhaps, the most "physical" one but, simultaneously, it seems to be most difficult to be taken seriously in building of a general theory. We propose here a natural *principle of measurability* according to which the set of (really observable) "observables" is determined as the set of those selfadjoint operators which can be measured by a "realistic" apparatus.

I.e., to any "observable", an apparatus can be (in principle, at least theoretically) constructed by a use of observed interactions only (electroweak, gravitational, etc.). One could pose the problem alternatively: Let us look for observables which can be measured by a given set of apparatuses (trying to construct the "realistic" ones). An (unrealistic) example of such a construction is given in the next section.

It might be superfluous to mention here, that a consistent restriction of the set of observables in QT could be a difficult task, since there are strong and naturally-looking requirements on that set, e.g. its Jordan–algebraic, or quantum logic structures, cf. [5], [16]. The NLQM of the last section represents an extension of the sets of observables, states, etc. which introduces, however, a richer structures into these sets, hence offers some possibilities to attempt to solutions to our present problem.

3. An Observable Determined by Quantum Measurements

We shall present here a model of an infinite size apparatus (formulated rigorously in C^*-algebraic terms, [7]) detecting a scalar particle moving on a line. This has to illustrate how a measured quantity can be defined from the interaction of the apparatus with the measured "microsystem" (this scalar nonrelativistic particle interacts with *first two* spins in the chain). The state space of a particle (modelled also as a quantum of scalar Fermi field) is simply $\mathcal{H} \equiv L^2(\mathbf{R}, dx)$. A suitably chosen element $\sigma \in L^2(\mathbf{R}, dx)$ describes also interaction with the apparatus "placed" on the line within the set $supp(\sigma) \subset \mathbf{R}$. The apparatus is modelled by a semiinfinite spin chain described by the spin creation - annihilation operators $a_n^{(*)}$ $(n \in \mathbf{Z}_+)$, and by the Hamiltonian (cf.[17])

$$H_\infty := \sum_{n\in\mathbf{Z}_+} a_n^* a_n (a_{n+1} + a_{n+1}^*) a_{n+2} a_{n+2}^*. \tag{3}$$

Let us introduce now the model which represents a nonideal measurement. The microsystem (particle) is described by the Fermi field described by annihilation operators $b(\varphi)$, with $\varphi \in L^2(\mathbf{R}, dx)$. The total Hamiltonian is now

$$H := H_\infty + d\Gamma(\hat{p}^2) + W, \tag{4}$$

where the interaction W is defined by

$$W := \gamma(a_1^* + a_1) a_2 a_2^* \otimes b^*(\sigma) b(\sigma), \tag{5}$$

with $\gamma \in \mathbf{R}$ and $\sigma \in L^2(\mathbf{R}, dx)$, $\|\sigma\| = 1$. The symbol $d\Gamma(\hat{p}^2)$ denotes the second quantized, [14], one particle Hamiltonian $\hat{p}^2$, with $\hat{p} := -i\partial_x$. Let the chain be at $t = 0$ in its "vacuum", i.e. all spins are pointig down. Let the initial state of the particle be described by an arbitrary normalized wave function $\varphi_0 \in L^2(\mathbf{R}, dx)$. (We do not use here the "incoming" asymptotics for $t \to -\infty$ of the scattering theory.) Denoting by ω_t the state (i.e. the expectation functional) of the total system in time t, and by A an arbitrary observable of that composed system, one can obtain that

$$\bar{\omega}(A) := \lim_{t\to\infty} \omega_t(A) \tag{6}$$

exists, and one arrives for suitable parameters of W at:

$$\bar{\omega} = (w\omega_1^{chain} + (1-w)\omega_0^{chain}) \otimes \omega_0^{field},\ w \not\equiv 0, \tag{7}$$

where ω_j^{chain} $(j = 0, 1)$ denotes the state of the chain with all spins pointing down, resp. pointing up. The field state ω_0^{field} is the Fermi vacuum. We obtained here a mixture of two disjoint states.

The probability w of switching the apparatus can be expressed explicitly in terms of dynamics of the chain, as well as of the particle-chain interaction parameters, cf. [7]. It can be written in the form

$$w = \langle\varphi_0|\ \Lambda\varphi_0\rangle. \tag{8}$$

Here Λ is a positive operator of the norm $\|\Lambda\| \leq 1$ acting on the Hilbert space $L^2(\mathbf{R}, dx)$. This operator represents the measured (unsharp) observable of the particle. Its explicit expression can be written in the form

$$\Lambda = \gamma^2 (F_\sigma^+)^* \hat{f} F_\sigma^+. \tag{9}$$

Here $F_\sigma^+ \in \mathcal{L}(\mathcal{H}, L^2(\mathbf{R}, dt))$, with

$$F_\sigma^+(\sigma') := \vartheta(t)\langle\beta_0 \otimes \Omega(\sigma)|\ U_t\beta_0 \otimes \Omega(\sigma')\rangle, \tag{10}$$

where $\vartheta(t) = 1$ for $t > 0$, $\vartheta(t) = 0$ for $t < 0$; $U_t := exp(-itH)$ is the total system's evolution operator, β_0 is the spin–chain "vacuum", and $\Omega(\sigma')$ is the one–particle state with the particle wave–function σ'; $\hat{f} \in \mathcal{L}(L^2(\mathbf{R}, dt))$ is the convolution operator with the function $f : t \mapsto f(t)$ determined as a diagonal matrix element of the free evolution $U_t^0 := exp(-itH_0)$ of the total system with $H_0 := H - W$:

$$f(t) := \langle\beta_1 \otimes \Omega(\sigma)|\ U_t^0\beta_1 \otimes \Omega(\sigma)\rangle,$$

with $\beta_1 := a_1^*\beta_0,\ \ a_m\beta_0 := 0$ for all $m \in \mathbf{Z}_+$, $\Omega(\sigma) := b^*(\sigma)\Omega_0$, and $b(\sigma')\Omega_0 := 0$ for all $\sigma' \in \mathcal{H}$. The operator F_σ^+ can be found explicitly in terms of the free evolution and of the interaction parameters, [7], hence Λ is a known "effect" defining an "unsharp observable". Physical interpretation of this observable is "in principle" given by the measurement process just described.

4. On Different Interpretations of Mixed States

Let us consider a microsystem in (conventional formulation of) QM: It is characterized by irreducibility of its set $\mathcal{L}(\mathcal{H})$ of "observables". Hence, any vector $\varphi \in \mathcal{H}$ represents a pure state. Let ϱ be any density matrix on $\mathcal{H}$ different from one–dimensional projections. Then ϱ can be nontrivially decomposed in many different ways into a convex combination

$$\varrho := \sum_j \lambda_j \varrho_j \tag{11}$$

of density matrices ϱ_j. Different decompositions (11) can correspond to decompositions with respect to different, mutually incompatible (i.e. mutually noncommuting) commutative subsets of observables. Such "incompatible decompositions" cannot be both interpreted as Genuine mixtures although they are in the framework of QM *of the considered microsystem* mutually equivalent. This indicates that for a separately taken isolated microsystems the *concept of Genuine mixture (i.e. "Gemenge") is meaningless* in the framework of conventional QM.

On the other side, for classical systems one has instead of ϱ a probability measure on its phase space, and this is uniquelly decomposable into its extremal (i.e. pure) components – into an integral of Dirac measures concentrated in points of the phase space.

In the case of quantal systems with (some kind of) superselection rules, e.g. in the C^*-algebraic description of infinite systems, there are distinguished unique decompositions of states (e.g. the central decomposition, cf. [14]). In the both last mentioned situations the components of the decompositions are mutually "noninterfering", i.e. the considered systems (also in any of their empirical realizations) cannot be coupled to any other system in such a way where a quantummechanical interference would be observed between states of the composed system corresponding to different components of the decomposition.

To be more specific, let us try to express some ideas in terms of QM. Let us consider one–dimensional projections P_{φ_j} on $\mathcal{H}$ corresponding to an orthonormal set $\{\varphi_j, j = 1, 2, \ldots\}$ of vectors in $\mathcal{H}$. In the case if a "classical" subsystem is present in the considered system, the vectors φ_j for different values of j can correspond to different superselection sectors (e.g. to different values of some classical quantity) and, in that case, there is no "observable" $B \in \mathcal{L}(\mathcal{H})$ such that

$$\langle\varphi_j|\, B\varphi_k\rangle \neq 0, \quad \text{for } \textit{some} \;\; j \neq k. \tag{12}$$

For a microscopic system there are always $B \in \mathcal{L}(\mathcal{H})$ such that (12) is valid. Let now $\varrho_j := P_{\varphi_j}$ in (11). Then there is always possible to find (in mathematical theory of QM) a system with its Hilbert space $\mathcal{K}$ and an orthonormal basis $\{\psi_j, j = 1, 2, \ldots\} \subset \mathcal{K}$ such that ϱ is obtained as the partial trace (cf. [8]) with respect to $\mathcal{K}$ from the vector–state Ψ,

$$\Psi := \sum_j \sqrt{\lambda_j}\; \varphi_j \otimes \psi_j \;\in\; \mathcal{H} \otimes \mathcal{K}. \tag{13}$$

If, for any choice of "the system $\mathcal{K}$", and for any "observable R" of the "combined system $\mathcal{H} \otimes \mathcal{K}$", one has

$$\langle\Psi|\, R\Psi\rangle = \sum_j \lambda_j \;\langle\varphi_j \otimes \psi_j|\, R\varphi_j \otimes \psi_j\rangle, \tag{14}$$

we say that the states P_{φ_j} of "the system $\mathcal{H}$" are mutually noninterfering. This is the case of the mentioned belonging of different φ_j into different superselection sectors in otherwise standard QM, [18].

Hence, for macroscopic (classical) systems the concept of Genuine mixture (cf. also its rough description in [3]) makes sense: The distinguished unique decomposition of the state (let us denote it ϱ also in general situation) into mutually noninterfering components ϱ_j can be interpreted in terms of a statistical ensemble of states ϱ_j describing a collection of systems (also in their empirical meaning, if interpreted) each of which "really occurs" in one of the states ϱ_j. Let us note here that concepts like "existence", "real occurence", or "being" of some specific "objects" seem to be of profound macroscopic (and, conceptually: of classical) character: Their empirical definition rests on repeatability of empirical measurement procedures and on reproducibility of their (individual – sharp) results with respect to all their (macroscopic) observables. If these "classical features" of quantummechanically desribed macrosystems would be only "effective" in the sense of considerations of preceding sections, then also such concepts as "existence" could be only effective.

On the other hand, we would like to have a concept of Genuine mixture also for microsystems, because there are situations which intuitively fit in those described above for macrosystems with classical behaviour. E.g., we can prepare a collection of microsystems by choosing during the preparation a procedure preparing the state P_{φ_j} with probability λ_j, for all j's. Let us see, however, that now we have not an ensemble of separated microsystems, but, at the best, an ensemble of couples *microsystem & preparation device*, and the "Genuine interpretation" is possible due to the presence of macroscopic devices in macroscopically distinct states correlated with the states of the microsystem in the sense of eq. (13). We can ask, however, whether there is some possibility of distinction of the last mentioned situation (in which the density matrix of the micro – ensemble was ϱ from (11)) from that one described by (13) with both subsystems being microscopic, and the partial density matrix being again the same ϱ. Since microsystems can be coupled both with microsystems, as well as with macrosystems, this question addressed to the state ϱ of a separately considered system in an instant of time has (at least in the framework of QM) negative answer. Another possibility arises if we assume that there is present an interaction of the considered microsystem with the correlated macrosystem where interaction Hamiltonian depends on values of the macroscopic quantity distinguishing different ϱ_j in (11). Then we could ask for future time evolution of the state ϱ, and we could look for some characteristic differences between evolutions in the cases when the considered microsystem is correlated with macrosystems (with different values of the macroscopic quantity) and in the cases in which such a correlation is missing (or the microsystem's states ϱ_j are all correlated with a unique value of the macroscopic quantity determining the interaction). Since general evolution of the state (13) of the coupled system might lead to an arbitrary change of its partial trace ϱ_t, we have to restrict our interest to some class of evolutions where the posed question could obtain a reasonable meaning.

Let us choose a class of such (sub–)systems in QM (or also in UQT) which satisfy the following two conditions.
(a) Time evolution of the chosen systems (in considered global situations including also description of other relevant systems) can be described by unitary operators, i.e.

$$\varrho_t = U(t, t_o; \varrho_{t_o})\varrho_{t_o} U(t, t_o; \varrho_{t_o})^{-1}, \quad t \in \mathbf{R}, \ \varrho_{t_o} \in \mathcal{T}_{+1}(\mathcal{H}) \equiv \mathcal{S}_*. \tag{15a}$$

The possible dependence of U on ϱ_{t_o} leads, however, to a problem: The ambiguity of the expression of state ϱ in various convex combinations could lead to ambiguity in time evolutions, if calculated as the evolution of the components of the convex decompositions. The possibility of the ϱ_{t_o} – dependence of U will be interpreted here as a possible presence of the above described correlations of microscopic states with (at least weakly interacting) macrosystem's states, cf. [21]. We shall assume further:
(b) The function $\varrho \mapsto U(t_1, t_2; \varrho)$ is defined on $\mathcal{S}_*$ for all $t_j \in \mathbf{R}$ $(j = 1, 2)$ and satisfies a 'cocycle property' (up to possible phase factors):

$$U(t_2, t_1; \varrho_{t_1})U(t_1, t_0; \varrho_{t_0}) \equiv U(t_2, t_0; \varrho_{t_0}), \ U(t, t; \varrho) \equiv I. \tag{15b}$$

Now it is natural to define two kinds of (mixed) states of the considered (sub–)systems:
(i) Density matrices describe indecomposable quantal states evolving according to (15), and they should be dealt within the theory as a whole (their components in formal decompositions (11) have no interpretation, hence for this states eq. (11) is physically irrelevant).
(ii) States corresponding to Genuine mixtures are described by *specific decompositions* of the form (11) (here different components ϱ_j correspond to different sharp values of classical quantities of a correlated macrosystem); in these states each component of the (specific!) decomposition (11) evolves separately according to (15).

The systems satisfying the conditions (a) and (b) could be imagined intuitively as microsystems moving in an external (classical) field which can depend moreover on the microscopic state of the system. Some correspondence between actual states of microsystems and their macroscopic environment can be understood due to necessity of preparation of specified microstates by macrodevices. Since there is no evolution of quantum correlations changing spectral characteristics of density matrices of states of systems fulfilling (a) and (b), we shall call such systems to be *relatively isolated*. Genuine mixtures correspond to convex combinations (11) of microstates correlated with macroscopically different states of the "relevant macrosystems".

Systems satisfying nontrivially the above conditions are, e.g., individual quantal systems of an infinite collection of equal quantal systems interacting mutually by specific very weak, and of very long range ("mean–field", cf.[19], [20]) interactions. This describes in fact nonlinear situations in QM, and a theory of such situations is sketched in the next section. We assume that such nonlinear situation could occur in measurement – like situations, as it was mentioned also in [3].

5. On Nonlinear Extension of QM

Let us consider a complex Hilbert space $\mathcal{H}$ and the set $\mathcal{S}_*$ of all density matrices on it. The set $\mathcal{S}_*$ is embedded into the Banach space $\mathcal{T}(\mathcal{H})$ of all trace–class operators on $\mathcal{H}$. Let $\mathcal{U}(\mathcal{H})$ be the unitary group of $\mathcal{L}(\mathcal{H})$. $\mathcal{S}_*$ can be decomposed into orbits of $\mathcal{U}(\mathcal{H})$ with respect to the coadjoint representation:

$$Ad^*(u)\varrho := u\varrho u^*$$

for all $u \in \mathcal{U}(\mathcal{H})$, and all $\varrho \in \mathcal{S}_*$. Each orbit consists of all density matrices with the same spectral characteristics (i.e. spectrum and spectral multiplicities). Orbits $\mathcal{O}_\varrho$ corresponding to ϱ with zero eigenvalue subspaces in $\mathcal{H}$ of finite codimension form embedded submanifolds of $\mathcal{T}(\mathcal{H})$, other orbits are not submanifolds of $\mathcal{T}(\mathcal{H})$, cf. [21]. Hence, the projective Hilbert space $P(\mathcal{H})$ elements of which are one-dimensional projections in $\mathcal{H}$ (e.g., $\mathcal{H} := L^2(\mathbf{R}, dx)$) forms an orbit of the first mentioned kind. The set $\mathcal{S}_*$ will be considered here as the quantal analogy of the phase space of Hamiltonian CM. Let us consider the set $\mathcal{F}$ of real valued differentiable (for simplicity; for more general – also unbounded – cases see [21]) functions on $\mathcal{T}(\mathcal{H})$ representing *generators* of transformations of the generalized phase space $\mathcal{S}_*$. We shall define Poisson brackets on $\mathcal{S}_*$. Let for all "sufficiently nice" real-valued functions $f,\ h, \dots$ the symbols $d_\varrho f,\ d_\nu h \dots$ denote the (Frèchet) differentials at the points $\varrho,\ \nu, \dots$ of $\mathcal{S}_*$. We can consider these differentials as unambigously defined bouned operators in $\mathcal{L}(\mathcal{H})$. Then we define the Poisson bracket of f and h by, [21],

$$\{f, h\}(\nu) := i\,Tr(\nu[d_\nu f, d_\nu h]), \tag{16}$$

where $[\cdot,\cdot]$ means the commutator of bounded operators in $\mathcal{L}(\mathcal{H})$, and Tr is the trace in the space $\mathcal{T}(\mathcal{H})$.

Time evolution given by the Hamiltonian function h is determined now in the usual way from Eq. (16). It can be shown, [21], that the evolution of density matrices can be described by the 'cocycle' of unitary operators $u_h(t,\nu)$ (with $t \in \mathbf{R},\ \nu \in \mathcal{S}_*$) satisfying Schrödinger-like (in general nonlinear) equation with the 'Hamilton operator' $d_\varrho h$:

$$i\frac{d}{dt}u_h(t,\nu) = d_{\nu(t)}h\ u_h(t,\nu). \tag{17}$$

Here $\nu(t) := u_h(t,\nu)\,\nu\,u_h^*(t,\nu)$, if $\nu(0) := \nu$; this determines time evolution of any $\nu \in \mathcal{S}_*$ under the Hamiltonian dynamics corresponding to Poisson brackets (16) (there is an ambiguity in formulation of eq. (17) for $u_h(t,\nu)$, cf.[21]). We see that the evolution in this NLQM satisfies conditions (a) and (b) of the preceding section, with $U(t_1,t_2;\varrho) \equiv u_h(t_1 - t_2, \varrho)$. If $h(\nu) \equiv Tr(\nu H)$ for some ν-independent operator H, then the Eq. (17) converts into the usual Schrödinger equation with the Hamiltonian H; the dynamics is linear in the usual sense exactly in this last mentioned case. A nonllinear Schrödinger equation of the form (17) (for vector states $\rho^2 \equiv \rho$ only) was published in [22]; the corresponding NLQM contained, however, interpretation, work with mixtures, and other aspects different from those of the present one.

Let us now define (formal) *observables* as differentiable functions **f, h** of two variables $(\nu;\rho)(\in \mathcal{S}_* \times \mathcal{S}_*) \mapsto \mathbf{R}$ depending on the first variable ν affinely:

$$\mathbf{f}(\lambda_1\nu_1 + \lambda_2\nu_2;\rho) = \lambda_1\mathbf{f}(\nu_1;\rho) + \lambda_2\mathbf{f}(\nu_2;\rho). \tag{18}$$

Such functions can be represented by operator–valued functions $\mathbf{F} : \rho(\in \mathcal{S}_*) \mapsto \mathbf{F}(\rho) \in \mathcal{L}(\mathcal{H})$ in the following way:

$$\mathbf{f}(\nu;\rho) = Tr(\nu\mathbf{F}(\rho)). \tag{19}$$

The proposed interpretation of these "observables" is a natural extension of the one from QM: $Tr(\rho\mathbf{F}(\rho)^n)$ equals to n–th moment of the probability distribution of the observable **f** (resp. equivalently, of **F**) in the state ρ (corresponding modifications for POV – measures, [8], can be formulated easily). Classical observables are described by $\mathbf{R}$ – valued functions $f(\rho)$ on $\mathcal{S}_*$ representing the operator – valued function

$$\mathbf{F}(\rho) \equiv f(\rho)\mathbf{I} \tag{20}$$

with **I** denoting the identity operator on $\mathcal{H}$.

Elements of $\mathcal{S}_*$ can be, in accordance with the preceding section, interpreted as Elementary (hence "improper") mixtures. The Genuine mixtures (i.e. Gemenge) are proposed here to be described by probability measures on $\mathcal{S}_*$.

We claim, [21], that the presented scheme is a mathematically consistent model containing conventional QM as its subtheory: If we choose only those "generators", resp. "observables" which are of the form $f(\rho) \equiv Tr(\rho F)$, $(F \in \mathcal{L}(\mathcal{H}))$, resp. $\mathbf{f}(\nu;\rho) \equiv Tr(\nu\mathbf{F})$, $\mathbf{F} \in \mathcal{L}(\mathcal{H})$, we obtain QM (with restriction to bounded operators, cf., however, [21]). CT can be formulated in this framework with a help of the "observables" (19) with **F** from (20). The Hamilton functions h in (17) can be chosen such that the corresponding time evolution leaves some arbitrarily chosen manifolds of generalized coherent states (in the sense of Perelomov, [23]) invariant. In this way, approximative theories to QT, like the time dependent Hartree–Fock theory (cf. [24]), together with finite dimensional Hamiltonian CT are included in our NLQM as exact (i.e. obtained without any approximations) subtheories.

6. References

1. P. Busch, P. J. Lahti, and P. Mittelstaedt (Eds.), *Symposium on the Foundations of Modern Physics 1993 – Quantum Measurement, Irreversibility and The Physics of Information*, Cologne, Germany 1 – 5 June 1993 (World Scientific, London, 1993).
2. P. Mittelstaedt in [1].

3. P. Bóna in [1].
4. According to some estimates the size of the corresponding apparatus would be of the order (or even larger) of the size of our Universe.
5. V. S. Varadarajan, *Geometry of Quantum Theory*, Vols. I and II, (Van Nostrand Reinhold Co., New York, 1968, 1970).
6. G. W. Mackey, *The Mathematical Foundations of Quantum Mechanics*, (W. A. Benjamin, New York, 1963).
7. P. Bóna, *ACTA F.R.N. Univ. Comen. PHYSICA* **XX** (1980) 65.
8. E. B. Davies, *Quantum Theory of Open Systems*, (Academic Press, London, 1976).
9. P. Busch, in *Classical and Quantum Systems – Foundations and Symmetries*, (Eds. H. D. Doebner, W. Scherer and F. Schroeck, Jr.), (World Scientific, Singapore, 1993).
10. P. Busch, P. J. Lahti and P. Mittelstaedt, *The Quantum Theory of Measurement*, (Springer, Berlin, 1992).
11. G. Ludwig, *An Axiomatic Basis for Quantum Mechanics*, Vols. I and II (Springer, Berlin, 1985 and 1987).
12. G. Ludwig in [1].
13. K. Hepp, *Helv. Phys. Acta* **45** (1972) 237.
14. O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics* (Springer, New York - Berlin, 1979 and 1981), Vols. I and II.
    R. Haag and D. Kastler, *J. Math. Phys.* **5** (1964) 848.
15. P. Bóna: *On Nonlinear Quantum Mechanics*, in Differential Geometry and Applications, *Proceedings of 5 – th International Conference on Differential Geometry and Its Applications, August 24 – 28, 1992, Opava, Czechoslovakia*, Eds. O. Kowalski, and D. Krupka (Silesian University Press, Opava, 1993).
16. P. Pták, S. Pulmannová, *Orthomodular Structures as Quantum Logics*, (Kluwer, Dordrecht, 1991).
17. P. Bóna, *acta phys. slov.* **27** (1977) 101.
18. The present author apologizes to readers for this way of expression where from time to time into the formalism of QM are incorporated intrinsically foreign elements as "superselection rules" in a trial to express classical features of observed empirical systems in a modified version of QT.
19. K. Hepp and E. H. Lieb, *Helv. Phys. Acta* **46** (1973) 573.
20. P. Bóna, *Czech. J. Phys.* **B37** (1987) 482; *J. Math. Phys.* **29** (1988) 2223; *ibid* **30** (1989) 2994.
21. P. Bóna, *Quantum Mechanics With Mean-Field Background*, (Comenius University, Bratislava 1991), preprint Ph10-91.
22. S. Weinberg, *Ann. Phys.*, **194** (1989), 336.
23. A. M. Perelomov, *Commun. Math. Phys.*, **26** (1972) 222.
24. D. J. Rowe, A. Ryman, and G. Rosensteel, *Phys. Rev.* **A 22** (1980) 2362.