Trees, forests and jungles:

a botanical garden for cluster expansions

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ABSTRACT

Combinatoric formulas for cluster expansions have been improved many times over the years. Here we develop some new combinatoric proofs and extensions of the tree formulas of Brydges and Kennedy, and test them on a series of pedagogical examples.

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I. Introduction

Cluster expansions have a reputation of being hard to use; this is largely due to the difficulty to capture them in a single short formula. These expansions were introduced in constructive theory by Glimm, Jaffe and Spencer [GJS1-2] and they were improved or generalized over the years [BF][BaF][Bat][B1]. For many years the Ecole Polytechnique was happy using a cluster "tree formula" due to Brydges, Battle and Federbush (see [R1-2] and references therein). This formula expresses connected amplitudes more naturally as sums over "ordered trees" rather than regular trees; but a combinatoric lemma due to Battle and Federbush, shows that the sum over all ordered trees corresponding to a given ordinary Cayley tree has total weight 1. A slightly disturbing dissymmetry in this formula reveals itself in the need to order in an arbitrary way the elements on which the cluster expansion is performed, and in the particular rôle played by the first element or root of the tree.

Brydges kept digging for a truly satisfying, still more beautiful formula, and with Kennedy they obtained it in [BK] (see also [BY][B2]). This formula shows a clear conceptual progress; it does not require the use of arbitrary choices such as an arbitrary ordering of the objects to decouple, and the outcome can be written in a somewhat shorter form, involving directly standard Cayley trees. Both formulas share a positivity preserving property which is crucial in constructive theory: the corresponding interpolations of positive matrices remain positive. Both can be built by iterating inductively some perturbation step. But the first formula insists in completing the cluster containing the "first" cube, before building the next one. The second and better formula blindly derives connections, hence all clusters grow symmetrically at the same time. Therefore we prefer to call it a forest formula (see below).

However the original proof of this formula in [BK] relies on a differential equation (Hamilton-Jacobi) which is perhaps not totally transparent. The purpose of this paper is to derive a fully combinatoric or algebraic proof of this type of formula, and to show how to apply it to various examples chosen for their pedagogic value. Many situations in constructive theory in fact require several cluster expansions on top of each other, and for this case we derive a generalization of the tree or forest formula, which we call the jungle formula.

We also derive a generalization that applies not only to exponentials of two-body interaction potentials, or to perturbations of Gaussian measures. It is a general interpolation formula we call the Taylor forest formula.

Finally we propose a formula that performs, in a single move, the succession of a cluster and a Mayer expansion.

II. Two forest formulas and their combinatoric proofs

First let us recall the Brydges-Kennedy formula [BK] under the most convenient setting for the following discussion. Let $n \geq 1$, be an integer , $I_n = \{1, \ldots, n\}$, $\mathcal{P}_n = \big\{\{i,j\}/i,j \in I_n, i \neq j\big\}$ (the set of unordered pairs in I_n). Consider $n(n-1)/2$ elements $u_{\{ij\}}$ of a commutative Banach algebra $\mathcal{B}$, indexed by the elements $\{ij\}$ of $\mathcal{P}_n$. An element of $\mathcal{P}_n$ will be called a *link*, a subset of $\mathcal{P}_n$, a *graph*. A graph $\mathfrak{F} = \{l_1, \ldots, l_\tau\}$ containing no loops, i.e. no subset $\big\{\{i_1,i_2\},\{i_2,i_3\},\ldots,\{i_k,i_1\}\big\}$ with $k \geq 3$ elements, will be called a *u-forest* (unordered forest). A sequence $F = (l_1, \ldots, l_\tau)$ of links the range of which $\{l_1, \ldots, l_\tau\}$ is a u-forest, will be called an *o-forest* (ordered forest).

A u-forest is a union of disconnected trees, the supports of which are disjoint subsets of I_n called the *connected components* or *clusters* of $\mathfrak{F}$. Isolated points also form clusters reduced to singletons, hence the total number of clusters is $n - \tau$, where $\tau = |\mathfrak{F}|$. Two points in the same cluster are said *connected by* $\mathfrak{F}$.

Now the Brydges-Kennedy forest formula states:

Theorem II.1 [BK]

$$\exp\Big(\sum_{l \in \mathcal{P}_n} u_l\Big) = \sum_{\substack{\mathfrak{F}=\{l_1,\ldots,l_\tau\} \\ \mathrm{u-forest}}} \Big(\prod_{\nu=1}^{\tau} \int_0^1 dh_{l_\nu}\Big)\Big(\prod_{\nu=1}^{\tau} u_{l_\nu}\Big) \exp\big(\sum_{l \in \mathcal{P}_n} h_l^{\mathfrak{F}}(\mathbf{h}).u_l\big) \ , \tag{II.1}$$

where the summation extends over all possible lengths τ of $\mathfrak{F}$, including $\tau = 0$ hence the empty forest. To each link of $\mathfrak{F}$ is attached a variable of integration h_l; and $h^{\mathfrak{F}}_{\{ij\}}(\mathbf{h}) = \inf\big\{h_l, l \in L_{\mathfrak{F}}\{ij\}\big\}$ where $L_{\mathfrak{F}}\{ij\}$ is the unique path in the forest $\mathfrak{F}$ connecting i to j. If no such path exists, by convention $h^{\mathfrak{F}}_{\{ij\}}(\mathbf{h}) = 0$.

Our second forest formula writes exactly the same, except that the definition of the $h^{\mathfrak{F}}_{\{ij\}}(h)$ is different, and it involves for each cluster a particular choice of a root. For simplicity let us slightly restrict this arbitrary choice by imposing the following rule : for each non empty subset or cluster C of I_n, choose r_C, the least element in the natural ordering of $I_n = \{1, \ldots, n\}$, to be the root of all the trees with support C that appear in the following expansion. Now if i is in some tree $\mathfrak{T}$ with support C we call the *height* of i the number of links in the unique path of the tree $\mathfrak{T}$ that goes from i to the root r_C. We denote it by $l^{\mathfrak{T}}(i)$. The set of points i with a fixed height k is called the k-th *layer* of the tree.

We now have

Theorem II.2

$$\exp\Big(\sum_{l\in\mathcal{P}_n} u_l\Big) = \sum_{\substack{\mathfrak{F}=\{l_1,\ldots,l_\tau\}\\ \mathrm{u-forest}}} \Big(\prod_{\nu=1}^{\tau}\int_0^1 dw_{l_\nu}\Big)\Big(\prod_{\nu=1}^{\tau} u_{l_\nu}\Big)\exp\big(\sum_{l\in\mathcal{P}_n} w_l^{\mathfrak{F}}(\mathbf{w}).u_l\big)\ , \qquad \text{(II.2)}$$

where the summation extends over all possible lengths τ of $\mathfrak{F}$, including $\tau = 0$ hence the empty forest. To each link of $\mathfrak{F}$ is attached a variable of integration w_l. We define the $w_l^{\mathfrak{F}}$ as follows. $w^{\mathfrak{F}}_{\{ij\}}(\mathbf{w}) = 0$ if i and j are not connected by the $\mathfrak{F}$. If i and j fall in the support C of the same tree $\mathfrak{T}$ of $\mathfrak{F}$ then

$w^{\mathfrak{F}}_{\{ij\}}(\mathbf{w}) = 0$ if $|l^{\mathfrak{T}}(i) - l^{\mathfrak{T}}(j)| \geq 2$ (i and j in distant layers)

$w^{\mathfrak{F}}_{\{ij\}}(\mathbf{w}) = 1$ if $l^{\mathfrak{T}}(i) = l^{\mathfrak{T}}(j)$ (i and j in the same layer)

$w^{\mathfrak{F}}_{\{ij\}}(\mathbf{w}) = w_{\{ii'\}}$ if $l^{\mathfrak{T}}(i) - 1 = l^{\mathfrak{T}}(j) = l^{\mathfrak{T}}(i')$, and $\{ii'\} \in \mathfrak{T}$. (i and j in neighboring layers, i' is then unique). In particular, if $\{ij\} \in \mathfrak{F}$, then $w^{\mathfrak{F}}_{\{ij\}}(\mathbf{w}) = w_{\{ij\}}$.

The proof we give of theorem II.1 relies on two lemmas.

Lemma II.1 *Let $a_0, \ldots, a_p, (p \geq 1)$ be distinct complex numbers, then:*

$$\int_{\substack{t_0\geq 0,\ldots,t_p\geq 0\\ t_0+\ldots+t_p=1}} dt_0\ldots dt_p\ \exp(t_0a_0 + \ldots + t_pa_p) = \sum_{i=0}^{p} \frac{e^{a_i}}{\prod_{\substack{j=0\\ j\neq i}}^{p}(a_i - a_j)} \qquad \text{(II.3)}$$

Proof: By induction. $p = 1$ is an easy computation. We assume the result is true for $p \geq 1$ thus

$$\int_{\substack{t_0\geq 0,\ldots,t_{p+1}\geq 0\\ t_0+\ldots+t_{p+1}=1}} dt_0\ldots dt_{p+1}\ \exp(t_0a_0+\ldots+t_{p+1}a_{p+1}) =$$

$$\int_0^1 dt_{p+1}e^{t_{p+1}a_{p+1}}(1-t_{p+1})^p \int_{\substack{w_0\geq 0,\ldots,w_p\geq 0\\ w_0+\ldots+w_p=1}} dw_0\ldots dw_p\ \exp(w_0b_0+\ldots+w_pb_p) \tag{II.4}$$

where we performed first the integration on t_{p+1} and made the following change of variables: $t_i=(1-t_{p+1})w_i, b_i=a_i(1-t_{p+1})$ for $0\leq i\leq p$. By the induction hypothesis this becomes

$$\int_0^1 dt_{p+1}e^{t_{p+1}a_{p+1}}(1-t_{p+1})^p \sum_{i=0}^{p}\frac{e^{b_i}}{\prod_{\substack{j=0\\ j\neq i}}^{p}(b_i-b_j)} =$$

$$\sum_{i=0}^{p}\frac{e^{a_i}}{\prod_{\substack{j=0\\ j\neq i}}^{p+1}(a_i-a_j)} + e^{a_{p+1}}\sum_{i=0}^{p}\frac{1}{\left(\prod_{\substack{j=0\\ j\neq i}}^{p}(a_i-a_j)\right)(a_{p+1}-a_i)}\ . \tag{II.5}$$

This boils down to the wanted expression for $p+1$ after remarking that we have the following rational fraction decomposition with simple poles:

$$\frac{1}{(X-a_0)\ldots(X-a_p)} = \sum_{i=0}^{p}\frac{1}{(X-a_i)}\times\frac{1}{\prod_{\substack{j=0\\ j\neq i}}^{p}(a_i-a_j)}\ ,$$

and putting $X=a_{p+1}$. ■

Lemma II.2 *With the same notation as in the beginning of this section, assume that we are given two sets of* $n(n-1)/2$ *indeterminates* $u_{\{ij\}}$ *and* $v_{\{ij\}}$*; then the following algebraic identity is true in the field of rational fractions* $\mathbb{C}(u_{\{ij\}},v_{\{ij\}})$.

$$\prod_{l\in\mathcal{P}_n} v_l = \sum_{\substack{F=(l_1,\ldots,l_\tau)\\ \text{o-forest}}} u_{l_1}\ldots u_{l_\tau}\ .\left(\sum_{\nu=0}^{\tau}\frac{b_\nu^F}{\prod_{\substack{\mu=0\\ \mu\neq\nu}}^{\tau}(a_\nu^F-a_\mu^F)}\right) \tag{II.6}$$

where, for $0\leq\nu\leq\tau$, $a_\nu^F=\sum_{\{ij\}/\nu}u_{\{ij\}}$, $b_\nu^F=\prod_{\{ij\}/\nu}v_{\{ij\}}$, $\{ij\}/\nu$ *meaning that the points* i *and* j *are connected by the sub-o-forest* $(l_1,\ldots,l_\nu)$ *of* F. *In particular since no points are connected by the empty forest* $a_0^F=0, b_0^F=0$.

Proof: Both sides of this identity are polynomials in the v_l's, and we prove equality coefficient by coefficient. First consider the case of the constant monomial. We must show

$$\sum_{\substack{F=(l_1,\dots,l_\tau)\\ \text{o-forest}}} (-1)^\tau \frac{u_{l_1}\dots u_{l_\tau}}{a_1^F\dots a_\tau^F} = \begin{cases} 0 \; if \; n\geq 2 \\ 1 \; if \; n=1 \end{cases} \tag{II.7}$$

For n=1 it is trivial, so assume $n \geq 2$ and denote $\mathcal{A}_F$ the contribution of F to the left hand side. Given an o-forest $F=(l_1,\dots,l_\tau)$, an o-forest F' of the form $(l_1,\dots,l_\tau,l_{\tau+1},\dots,l_{\tau+\kappa})$ will be called a *κ-extension* of F. When we need not know κ we simply say an *extension* of F.

Let $\mathcal{R}_F = \mathcal{A}_F \, . \, \dfrac{a_\tau^F}{a_{\text{total}}}$ where $a_{\text{total}} = \sum_{l\in\mathcal{P}_n} u_l$ $(\neq 0$ since $n\geq 2)$.

Notice that

$$a_{\text{total}} = a_\tau^F + \sum_{F' \text{ 1-extension of } F} u_{l_{\tau+1}} \quad , \tag{II.8}$$

because summing over F' is the same as summing over all links $\{ij\}$ where i and j lie in different clusters of F, while a_τ^F handles summing over links where i and j are in the same cluster.

Hence

$$a_\tau^F = a_{\text{total}} + \sum_{F' \text{ 1-extension of } F} \left(-\frac{u_{l_{\tau+1}}}{a_{\tau+1}^{F'}}\right) a_{\tau+1}^{F'} \tag{II.9}$$

i.e multiplying by $\mathcal{A}_F/a_{\text{total}}$

$$\mathcal{R}_F = \mathcal{A}_F + \sum_{F' \text{ 1-extension of } F} \mathcal{R}_{F'} \quad . \tag{II.10}$$

Multiple (but finite because $\tau \leq n-1$) iteration of (II.10) yields

$$\mathcal{R}_F = \sum_{F' \text{ extension of } F} \mathcal{A}_F, \tag{II.11}$$

where the sum is over extensions of all possible lengths. In particular

$$\sum_F \mathcal{A}_F = \sum_{F \text{ extension of } \emptyset} \mathcal{A}_F = \mathcal{R}_\emptyset = \mathcal{A}_\emptyset . \frac{a_0^\emptyset}{a_{\text{total}}} = 0 \tag{II.12}$$

since $\mathcal{A}_\emptyset = 1,\; a_0^\emptyset = 0,\; a_{\text{total}} \neq 0$. (II.7) is now proven.

Let us now check the other monomials in the v's. If $F=(l_1,\ldots,l_\tau)$ is a forest we say that it *creates* the partition $\mathcal{D}$ of I_n if $\mathcal{D}$ is the set of clusters of F. We then write, with a slight abuse of notation $a_{\mathcal{D}}=\sum_{\{ij\}/\mathcal{D}} u_{\{ij\}}$, $b_{\mathcal{D}}=\prod_{\{ij\}/\mathcal{D}} v_{\{ij\}}$, where $\{ij\}/\mathcal{D}$ means that i and j are in the same element or *component* of $\mathcal{D}$.

Monomials generated by (II.5) are of the form $b_{\mathcal{D}}$, $\mathcal{D}$ being created by a sub-o-forest of F. Remark the if $\mathcal{D}$ is created by a forest of length τ then $|\mathcal{D}|=n-\tau$. There are two cases to be treated:

Case 1: $\mathcal{D}\neq\{I_n\}$

Here the coefficient of $b_{\mathcal{D}}$ is zero in the left hand side, we must show that also for the right hand side. Let $\nu=n-\tau$, then (st abbreviates "such that")

$$\sum_{\tau\geq\nu}\ \sum_{\substack{F=(l_1,\ldots,l_\tau)\\ \text{o-forest st}\\ (l_1,\ldots,l_\nu)\text{ creates }\mathcal{D}}} \frac{u_{l_1}\ldots u_{l_\tau}}{\prod_{\substack{\mu=0\\ \mu\neq\nu}}^{\tau}(a_\nu^F-a_\mu^F)}=$$

$$\sum_{\substack{F_1=(l_1,\ldots,l_\nu)\\ \text{o-forest}\\ \text{creating }\mathcal{D}}} \frac{u_{l_1}\ldots u_{l_\nu}}{\prod_{\mu=0}^{\nu-1}(a_{\mathcal{D}}-a_\mu^{F_1})} \sum_{\substack{F_2=(l_{\nu+1},\ldots,l_\tau)\\ \text{st }F=(F_1,F_2)\text{ is}\\ \text{an o-forest}}} (-1)^{\tau-\nu}\frac{u_{\nu+1}\ldots u_{l_\tau}}{\prod_{\mu=\nu+1}^{\tau}(a_\mu^F-a_{\mathcal{D}})}\ . \tag{II.13}$$

Again, summation is over all possible τ's. We now arrive at the heart of the inductive argument. We show using (II.7) that F_1 being fixed the sum on F_2 vanishes. In fact, this sum is the analog of (II.7) when instead of I_n we use $\mathcal{D}$ as a point set. If a and b are two elements of the partition $\mathcal{D}$, we let $\overline{u}_{\{ab\}}=\sum_{i\in a,j\in b} u_{\{ij\}}$. Given an o-forest $F=(F_1,F_2)$ on I_n such that F_1 creates $\mathcal{D}$, $F_2=(l_{\nu+1},\ldots,l_\tau)$ *induces* an o-forest $\overline{F}=(\bar{l}_1,\ldots,\bar{l}_{\tau-\nu})$ on $\mathcal{D}$ in the following way: if $l_\kappa=\{ij\}$, $\nu+1\leq\kappa\leq\mu$, with $i\in a$ and $j\in b$, a and b elements of $\mathcal{D}$, then set $\bar{l}_{\kappa-\nu}=\{ab\}$. $\overline{F}$ is simply obtained by forgetting the details of structure inside the components of $\mathcal{D}$, which are due to the forest F_1. Demanding that the whole of (F_1,F_2) be an o-forest insures that, in the process, $\overline{F}$ remains a forest. As a result, the analog of the a_ν^F for the forest $\overline{F}$ are $\overline{a}_\rho^{\overline{F}}=a_{\rho+\nu}^F-a_{\mathcal{D}}$, for $1\leq\rho\leq\mu-\nu$; and with $\overline{\tau}=\tau-\nu$ we have

$$\sum_{\substack{F_2=(l_{\nu+1},\ldots,l_\tau)\\ \text{st }F=(F_1,F_2)\text{ is}\\ \text{an o-forest}}} (-1)^{\tau-\nu}\frac{u_{\nu+1}\ldots u_{l_\tau}}{\prod_{\mu=\nu+1}^{\tau}(a_\mu^F-a_{\mathcal{D}})} = \sum_{\overline{F}=(\bar{l}_1,\ldots,\bar{l}_{\overline{\tau}})} (-1)^{\overline{\tau}}\frac{\overline{u}_{\bar{l}_1}\ldots\overline{u}_{\bar{l}_{\overline{\tau}}}}{\overline{a}_1^{\overline{F}}\ldots\overline{a}_{\overline{\tau}}^{\overline{F}}}\ . \tag{II.14}$$

The last sum is zero by (II.7) because $|\mathcal{D}|\geq 2$, since we are in the case $\mathcal{D}\neq\{I_n\}$.

Case 2: $\mathcal{D}=\{I_n\}$

Here $a_{\mathcal{D}} = a_{\text{total}}$. Equating the coefficients of $b_{\mathcal{D}} = \prod_{l\in\mathcal{P}_n} v_l$ on both sides needs showing

$$1 = \sum_{\substack{F=(l_1,\ldots,l_{n-1})\\ \text{o-forest}}} \frac{u_{l_1}\ldots u_{l_{n-1}}}{(a_{\text{total}}-a_0^F)\ldots(a_{\text{total}}-a_{n-2}^F)} \tag{II.15}$$

This last identity is shown by induction on n. Remark that here F is a complete tree covering I_n. If $n=1$, the only forest is the empty one, its contribution is the empty product i.e. 1.

The induction step form n to $n+1$ needs an argument similar to the former case:

$$\sum_{\substack{F=(l_1,\ldots,l_n)\\ \text{o-forest on } I_n}} \frac{u_{l_1}\ldots u_{l_n}}{(a_{\text{total}}-a_0^F)\ldots(a_{\text{total}}-a_{n-1}^F)} =$$

$$\sum_{F_1=(l_1)} \frac{u_{l_1}}{a_{\text{total}}} \sum_{\substack{F_2=(l_2,\ldots,l_n)\\ \text{st } F=(F_1,F_2) \text{ is}\\ \text{an o-forest on } I_{n+1}}} \frac{u_{l_2}\ldots u_{l_n}}{(a_{\text{total}}-a_1^F)\ldots(a_{\text{total}}-a_{n-1}^F)} \quad . \tag{II.16}$$

The link $l_1 = \{\alpha,\beta\}$ being fixed, this determines a partition $J_n = \big\{\{\alpha,\beta\}\big\} \cup \big\{\{i\}/i \in I_{n+1},\ i\neq\alpha,\ i\neq\beta\big\}$ for which we can repeat the treatment of the partition $\mathcal{D}$ in the preceding case: introduce variables $\overline{u}_{\{ab\}} = \sum_{i\in a,\ j\in b} u_{\{ij\}}$ for a and b in J_n, and induced o-forests $\overline{F} = (\bar{l}_1,\ldots,\bar{l}_{n-1})$ on J_n for any o-forest $F_2 = (l_2,\ldots,l_n)$ on I_n such that $F=(F_1,F_2)$ be an o-forest on I_{n+1}. Remark that

$$\overline{a}_{\text{total}} \overset{\text{def}}{=} \sum_{\substack{\{ab\}\subset J_n\\ a\neq b}} \overline{u}_{\{ab\}} = \left(\sum_{l\in\mathcal{P}_{n+1}} u_l\right) - u_{\{\alpha\beta\}} = a_{\text{total}} - a_1^F \quad , \tag{II.17}$$

and for $0\le\rho\le n-2$, $\overline{a}_\rho^{\overline{F}} = a_{\rho+1}^F - a_1^F$, therefore $\overline{a}_{\text{total}} - \overline{a}_\rho^{\overline{F}} = (a_{\text{total}} - a_1^F) - (a_{\rho+1}^F - a_1) = a_{\text{total}} - a_{\rho+1}^F$.

Hence

$$\sum_{\substack{F_2=(l_2,\ldots,l_n)\\ \text{st } F=(F_1,F_2) \text{ is}\\ \text{an o-forest on } I_{n+1}}} \frac{u_{l_2}\ldots u_{l_n}}{(a_{\text{total}}-a_1^F)\ldots(a_{\text{total}}-a_{n-1}^F)} =$$

$$\sum_{\substack{\overline{F}=(\bar{l}_1,\ldots,\bar{l}_{n-1})\\ \text{o-forest on } J_n}} \frac{\overline{u}_{\bar{l}_1}\ldots\overline{u}_{\bar{l}_{n-1}}}{(\overline{a}_{\text{total}}-\overline{a}_0^{\overline{F}})\ldots(\overline{a}_{\text{total}}-\overline{a}_{n-2}^{\overline{F}})} \tag{II.18}$$

but the last sum is 1 by the induction hypothesis since $|J_n| = n$. Finally in (II.15) there remains

$$\sum_{l_1} \frac{u_{l_1}}{a_{\text{total}}} = \frac{a_{\text{total}}}{a_{\text{total}}} = 1 \quad . \tag{II.19}$$

This completes the proof of Lemma II.2 . ■

Proof of theorem II.1:

We can return now to the Brydges-Kennedy forest formula and first prove it for any complex numbers u_l, $l \in \mathcal{P}_n$ such that for all subset X of $\mathcal{P}_n$, $\sum_{l \in X} u_l \neq 0$. In fact we can rewrite the right hand side of (II.1) using o-forests instead of u-forests. Let $\mathfrak{F}$ be a u-forest and choose an ordering $F = (l_1, \ldots, i_\tau)$ of its elements. We can slice the integral according to the ordering of the h_l's thereby giving a contribution for $\mathfrak{F}$

$$\sum_{\sigma \in \mathfrak{S}_\tau} \int_{1 > h_{l_{\sigma(1)}} > \ldots > h_{l_{\sigma(\tau)}} > 0} \left(\prod_{\nu=1}^{\tau} dh_{l_\nu} \right) \left(\prod_{\nu=1}^{\tau} u_{l_\nu} \right) \exp\left(\sum_{l \in \mathcal{P}_n} h_l^{\mathfrak{F}}(\mathbf{h}).u_l \right) \quad , \tag{II.20}$$

$\mathfrak{S}_\tau$ is the permutation group on τ elements. For any of its elements σ let us denote the o-forest $(l_{\sigma(1)}, \ldots, l_{\sigma(\tau)})$ by F_σ. Let us define, if F is an o-forest, the function $h_{\{ij\}}^F = h_{l_\nu}$ where ν is the largest index in the ordering provided by F of the links appearing in the unique path of F connecting i and j. If no such path exists we again put by convention $h_{\{ij\}}^F = 0$.

Thus the contribution of $\mathfrak{F}$ becomes

$$\sum_{\sigma \in \mathfrak{S}_\tau} \int_{1 > h_{l_{\sigma(1)}} > \ldots > h_{l_{\sigma(\tau)}} > 0} \left(\prod_{\nu=1}^{\tau} dh_{l_\nu} \right) \left(\prod_{\nu=1}^{\tau} u_{l_\nu} \right) \exp\left(\sum_{l \in \mathcal{P}_n} h_l^{F_\sigma}(\mathbf{h}).u_l \right) \tag{II.21}$$

and the right hand side of (II.1)

$$I = \sum_{\substack{F=(l_1,\ldots,l_\tau) \\ \text{o-forest}}} \int_{1 > h_{l_1} > \ldots > h_{l_\tau} > 0} \left(\prod_{\nu=1}^{\tau} dh_{l_\nu} \right) \left(\prod_{\nu=1}^{\tau} u_{l_\nu} \right) \exp\left(\sum_{l \in \mathcal{P}_n} h_l^F(\mathbf{h}).u_l \right) \ . \tag{II.22}$$

Now we perform the change of variables $t_1 = h_{l_1} - h_{l_2}, \ldots, t_{\tau-1} = h_{l_{\tau-1}} - h_{l_\tau}$, $t_\tau = h_{l_\tau}$ then

$$I = \sum_{\substack{F=(l_1,\ldots,l_\tau) \\ \text{o-forest}}} u_{l_1} \ldots u_{l_\tau} \int_{\substack{t_1 \geq 0, \ldots, t_\tau \geq 0 \\ t_1 + \ldots + t_\tau \leq 1}} dt_1 \ldots dt_\tau \ \exp\left(\sum_{\nu=1}^{\tau} t_\nu a_\nu^F \right) \quad . \tag{II.23}$$

There shows up the a_ν^F when collecting in the exponential the $u_{\{ij\}}$'s multiplied by a given t_ν. Since $s_\nu = t_\nu + t_{\nu+1} + \ldots + t_\tau$, the corresponding pairs $\{ij\}$ are those for which all the links appearing in the unique path of F connecting i and j have an index at most equal to ν, i.e. $\{ij\}/\nu$. Since $a_0^F = 0$, we can rewrite (II.23) as

$$I = \sum_{\substack{F=(l_1,\ldots,l_\tau) \\ \text{o-forest}}} u_{l_1} \ldots u_{l_\tau} \int_{\substack{t_0 \geq 0, \ldots, t_\tau \geq 0 \\ t_0 + \ldots + t_\tau = 1}} dt_0 \ldots dt_\tau \ \exp\left(\sum_{\nu=0}^{\tau} t_\nu a_\nu^F \right) \quad , \tag{II.24}$$

then from lemma II.1 we obtain

$$I = \sum_{\substack{F=(l_1,\ldots,l_\tau)\\ \text{o-forest}}} u_{l_1}\ldots u_{l_\tau} \cdot \left(\sum_{\nu=0}^{\tau} \frac{e^{a_\nu^F}}{\prod_{\substack{\mu=0\\ \mu\neq\nu}}^{\tau}(a_\nu^F - a_\mu^F)}\right) \ . \tag{II.25}$$

Then if we substitute for the indeterminates u_l the complex numbers we have now (this is allowed since the factors in the denominators $a_\nu^F - a_\mu^F$ are of the form $\pm\sum_{l\in X} u_l \neq 0$) and for the v_l the complex numbers $\exp(u_l)$, the lemma II.2 just proves (II.1). We can remove the condition $\sum_{l\in X} u_l \neq 0$ of being in a finite number of hyperplanes of $\mathbb{C}^{\mathcal{P}_n}$, by density and continuity of both sides of (II.1). To prove it for an arbitrary commutative Banach algebra $\mathcal{B}$ we need only note that the identity we are about to show, is of the form $f(u_1,\ldots,u_k)=0$, where the function f is analytic, its value for u's in a commutative Banach algebra being defined by its power series expansion. Thus proving it for all complex numbers is enough to entail the vanishing of the coefficients of the power series and to prove the identity for any u's in $\mathcal{B}$. ■

Proof of theorem II.2:

We recall the notion of connected set function. If X is a finite set and ψ is a map from $\mathcal{P}(X)$, the power set of X, to a commutative algebra $\mathcal{B}$, the connected function of ψ is $\psi_c : \mathcal{P}(X) \to \mathcal{B}$ defined inductively by

$$\forall C \subset X,\ \ \psi(C) = \sum_{\substack{\Pi=\{C_1,\ldots,C_k\}\\ \text{partition of } C}} \prod_{i=1}^{k} \psi_c(C_i) \quad , \quad \psi_c(\emptyset) = 1 \ . \tag{II.26}$$

Let $X = I_n$, $\mathcal{B}$ be the field of complex numbers, $\forall l \in \mathcal{P}_n, \epsilon_l \stackrel{\text{def}}{=} e^{u_l} - 1$, and let $\psi(C) = \prod_{\{ij\}\subset C}(1+\epsilon_{\{ij\}})$. We make also the assumption that for any non empty subset Q of $\mathcal{P}_n$, $\sum_{l\in Q} u_l \neq 0$. It is well known [B1] that

$$\psi_c(C) = \sum_{\mathfrak{G} \text{ graph connecting } C} (\prod_{l\in\mathfrak{G}} \epsilon_l) \tag{II.27}$$

("connecting C" means that for every $i \neq j$ in C there exists a path in $\mathfrak{G}$ going from i to j, the empty graph connects C if $|C| \leq 1$).

We claim that

$$\psi_c(C) = \sum_{\mathfrak{T} \text{ tree connecting } C} \left(\prod_{l\in\mathfrak{T}} \int_0^1 dw_l\right)\left(\prod_{l\in\mathfrak{T}} u_l\right) \exp\left(\sum_{\{ij\}\subset C} w^{\mathfrak{T}}_{\{ij\}}(\mathbf{w}).u_{\{ij\}}\right) \quad . \qquad \text{(II.28)}$$

This is trivial if $|C| \leq 1$. If $|C| \geq 2$, let $\mathfrak{T}$ be a non trivial tree connecting C. We associate the sequence $\mathcal{C}^{\mathfrak{T}} = (C^{\mathfrak{T}}_\nu)_{\nu\in\mathbb{N}}$ of disjoint subsets of C such that $C^{\mathfrak{T}}_\nu$ is the ν-th layer of $\mathfrak{T}$. Note that $C^{\mathfrak{T}}_0 = \{r_C\}$, $\cup_{\nu\in\mathbb{N}} C^{\mathfrak{T}}_\nu = C$, and $\exists \nu_{\mathfrak{T}} \in \mathbb{N}$ such that $\nu \leq \nu_{\mathfrak{T}} \;\Rightarrow\; C^{\mathfrak{T}}_\nu \neq \emptyset$ and $\nu > \nu_{\mathfrak{T}} \;\Rightarrow\; C^{\mathfrak{T}}_\nu = \emptyset$; a sequence $\mathcal{C} = (C_\nu)_{\nu\in\mathbb{N}}$ sharing those three properties will be called *admissible*.

We will compute in the right hand side of (II.28) the contribution of the trees corresponding to a given sequence $\mathcal{C}^{\mathfrak{T}} = \mathcal{C}$. Let $\{ij\}$ be a pair in C with $i \in C_{\nu_i}$, $j \in C_{\nu_j}$ and $\nu_i \geq \nu_j$. The only way $w^{\mathfrak{T}}_{\{ij\}}(\mathbf{w})$ can be non zero is that $\nu_i = \nu_j$ or $\nu_i = \nu_j + 1$. For $\nu_i = \nu_j$ we just get a factor $e^{u_{\{ij\}}} = 1 + \epsilon_{\{ij\}}$, since $w^{\mathfrak{T}}_{\{ij\}}(\mathbf{w}) = 1$.

But if $\nu_i = \nu_j + 1$, let i' be the element of C_{ν_j} such that $\{ii'\} \in \mathfrak{T}$; i' is the *ancestor* of i. Let us perform the integration explicitly in $w_{\{ii'\}}$. The pairs l such that $w^{\mathfrak{T}}_l(\mathbf{w}) \stackrel{\text{def}}{=} w_{\{ii'\}}$ are the pairs $\{ij\}$ with $j \in C_{\nu_i - 1}$, the corresponding contribution is

$$\int_0^1 dw_{\{ii'\}}\; u_{\{ii'\}}.\exp\left(w_{\{ii'\}}.\sum_{j\in C_{\nu_i-1}} u_{\{ij\}}\right) = u_{\{ii'\}} \;.\; \frac{\exp\left(\sum_{j\in C_{\nu_i-1}} u_{\{ij\}}\right) - 1}{\sum_{j\in C_{\nu_i-1}} u_{\{ij\}}} \quad . \qquad \text{(II.29)}$$

Now, building a tree with prescribed $\mathcal{C}^{\mathfrak{T}}$ means linking the points of $C^{\mathfrak{T}}_1$ to the root r_C, then for each i in $C^{\mathfrak{T}}_2$ choosing an ancestor i' in $C^{\mathfrak{T}}_1$, then for each i in $C^{\mathfrak{T}}_3$ choosing an ancestor i' in $C^{\mathfrak{T}}_2$, and so on, until we exhaust C. The choice of ancestor for points in the same layer $C^{\mathfrak{T}}_\nu$ are completely independent operations. Summing the contributions (II.29) for all choices of ancestors for i is simply

$$\begin{aligned}\sum_{i'\in C_{\nu_i-1}} u_{\{ii'\}} \;.\; \frac{\exp\left(\sum_{j\in C_{\nu_i-1}} u_{\{ij\}}\right) - 1}{\sum_{j\in C_{\nu_i-1}} u_{\{ij\}}} &= \exp\left(\sum_{j\in C_{\nu_i-1}} u_{\{ij\}}\right) - 1 \\ &= \prod_{j\in C_{\nu_i-1}} (1+\epsilon_{\{ij\}}) \;- 1 = \sum_{\substack{J\subset C_{\nu_i-1}\\ J\neq\emptyset}} \left(\prod_{j\in J} \epsilon_{\{ij\}}\right) \quad . \end{aligned} \qquad \text{(II.30)}$$

Therefore the right hand side of (II.28) is

$$\sum_{\substack{\text{admissible}\\ \text{sequence } \mathcal{C}}} \prod_{\nu\geq 1} \left(\left(\prod_{i\in C_\nu}\left(\sum_{\substack{J\subset C_{\nu-1}\\ J\neq\emptyset}} \prod_{j\in J} \epsilon_{\{ij\}}\right)\right) \times \left(\prod_{\{ij\}\subset C_\nu} (1+\epsilon_{\{ij\}})\right)\right) \quad , \qquad \text{(II.31)}$$

and is to be compared with (II.27).

Note that, in a connected graph $\mathfrak{G}$, we can define the distance $d_{\mathfrak{G}}(i,j)$ between two points i and j by the minimal number of links in a path in $\mathfrak{G}$ going from i to j. Given a preferred point in C, namely the root r_C, and a graph $\mathfrak{G}$ connecting C, we can define an admissible sequence $\mathcal{C}^{\mathfrak{G}} = (C_\nu^{\mathfrak{G}})_{\nu\in\mathbb{N}}$, by letting $C_\nu^{\mathfrak{G}} \stackrel{\text{def}}{=} \{i \in C / d_{\mathfrak{G}}(i, r_C) = \nu\}$. This is consistent with our previous definition since a tree is obviously a special case of graph.

We just have now to convince the reader that for a given admissible $\mathcal{C}$

$$\prod_{\nu\geq 1}\left(\left(\prod_{i\in C_\nu}\Big(\sum_{\substack{J\subset C_{\nu-1}\\ J\neq\emptyset}}\prod_{j\in J}\epsilon_{\{ij\}}\Big)\right)\times\Big(\prod_{\{ij\}\subset C_\nu}(1+\epsilon_{\{ij\}})\Big)\right) = \sum_{\substack{\mathfrak{G}\text{ graph}\\ \text{connecting } C\\ \text{such that } \mathcal{C}^{\mathfrak{G}}=\mathcal{C}}}\prod_{l\in\mathfrak{G}}\epsilon_l \quad . \qquad \text{(II.32)}$$

Note that building a graph $\mathfrak{G}$ with $\mathcal{C}^{\mathfrak{G}} = \mathcal{C}$ is first linking the points i of C_1 to the root r_C, then for each i in C_2 choosing a *non empty* subset $J_i \subset C_1$ of points j to which i will be linked, then for i in C_3 we again choose a non empty subset $J_i \subset C_2$ of points j to be linked with, and so on. Having built such a "skeleton" we, as a final step, have complete freedom to add all the links $\{ij\}$, for i and j in the same C_ν, that we want. Remark that in such a graph $\mathfrak{G}$, there is no link $\{ij\}$ with $i \in C_{\nu_i}$, $j \in C_{\nu_j}$ and $\nu_j + 1 < \nu_i$, for we would have $d_{\mathfrak{G}}(r_C, i) \leq d_{\mathfrak{G}}(r_C, j) + 1 = \nu_j + 1 < \nu_i = d_{\mathfrak{G}}(r_c, i)$, a contradiction.

Now (II.28) is proven, so the right hand side of (II.2) is

$$\sum_{\substack{\Pi=\{C_1,\ldots,C_k\}\\ \text{partition of } I_n}}\prod_{i=1}^{k}\psi_c(C_i) = \psi(I_n) = \prod_{l\in\mathcal{P}_n}(1+\epsilon_l) = \exp\Big(\sum_{l\in\mathcal{P}_n} u_l\Big) \quad . \qquad \text{(II.33)}$$

Finally we can remove the conditions $\sum_{l\in Q} u_l \neq 0$ and take the u_l in any commutative Banach algebra $\mathcal{B}$, as we did for the proof of theorem II.1. ∎

III. A first Generalization: The Taylor forest formulas

In practice we not only need "algebraic" cluster expansions, but also "Taylor" cluster expansion, in which one typically makes a Taylor expansion with integral remainder to interpolate between a coupled situation (parameters set to 1) and an uncoupled one (parameters set to 0).

For instance a "pair of cubes" cluster expansion [R1] boils down to applying the following interpolation formula:

$$f(1,\ldots,1) = \sum_{I\subset I_n} \left(\prod_{i\in I}\int_0^1 dh_i\right)\left(\left(\prod_{i\in I}\frac{\partial}{\partial x_i}\right) f\right)(\psi_I(\mathbf{h})) \tag{III.1}$$

where $f(x_1,\ldots,x_n)$ is a smooth function on $\mathbb{R}^n$, and $\psi_I(\mathbf{h})$ is the vector of coordinates $(x_1,\ldots,x_n)$ defined by $x_i = 0$ if $i \notin I$ and $x_i = h_i$ if $i \in I$. Such a formula is easy to prove.

The "Taylor" tree cluster expansion describes a similar interpolation formula. Let $\mathcal{S}$ be the space of smooth functions from $\mathbb{R}^{\mathcal{P}_n}$ to an arbitrary Banach space $\mathcal{V}$. An element of $\mathbb{R}^{\mathcal{P}_n}$ will be generally denoted by $\mathbf{x} = (x_l)_{l\in\mathcal{P}_n}$. The vector with all entries equal to 1 will be denoted by $\mathbb{1}$. Applied to an element H of $\mathcal{S}$, we can state two different Taylor forest formulas depending on which of theorem II.1 or II.2 we use for its derivation.

Theorem III.1 (The Brydges-Kennedy Taylor forest formula)

$$H(\mathbb{1}) = \sum_{\mathfrak{F}\ \mathrm{u-forest}} \left(\prod_{l\in\mathfrak{F}}\int_0^1 dh_l\right)\left(\left(\prod_{l\in\mathfrak{F}}\frac{\partial}{\partial x_l}\right) H\right)(X_{\mathfrak{F}}^{BK}(\mathbf{h})) \quad . \tag{III.2}$$

Here $X_{\mathfrak{F}}^{BK}(\mathbf{h})$ is the vector $(x_l)_{l\in\mathcal{P}_n}$ of $\mathbb{R}^{\mathcal{P}_n}$ defined by $x_l = h_l^{\mathfrak{F}}(\mathbf{h})$, which is the value at which we evaluate the complicated derivative of H.

and

Theorem III.2 (The rooted Taylor forest formula)

$$H(\mathbb{1}) = \sum_{\mathfrak{F}\ \mathrm{u-forest}} \left(\prod_{l\in\mathfrak{F}}\int_0^1 dw_l\right)\left(\left(\prod_{l\in\mathfrak{F}}\frac{\partial}{\partial x_l}\right) H\right)(X_{\mathfrak{F}}^{r}(\mathbf{w})) \quad . \tag{III.3}$$

Here $X_{\mathfrak{F}}^{r}(\mathbf{w})$ is the vector $(x_l)_{l\in\mathcal{P}_n}$ of $\mathbb{R}^{\mathcal{P}_n}$ defined by $x_l = w_l^{\mathfrak{F}}(\mathbf{w})$, which is the value at which we evaluate the complicated derivative of H.

Proof: These formulas look more general than the algebraic forest formulas (II.1) and (II.2) (obtained by letting $H(\mathbf{x}) = \exp(\sum_{l\in\mathcal{P}_n} x_l u_l)$) but are in fact a consequence of them. For any integer $p \geq 0$, let $\mathcal{S}_p$ be the space of polynomial functions in the x_l's of total degree at most p. $\mathcal{S}_p$ can be dressed in a finite dimensional Banach space structure. Let $\mathcal{B}_p$ be the algebra of linear operators on $\mathcal{S}_p$ generated by the derivations $\partial/\partial x_l$ for $l \in \mathcal{P}_n$, equipped with the operator norm. $\mathcal{B}_p$ is a finite dimensional *commutative* Banach algebra. If we consider the $u_l \stackrel{\text{def}}{=} \partial/\partial x_l$ in $\mathcal{B}_p$ we have the right to use the multiple forest formula of the preceding section.

We then remark that, in $\mathcal{S}_p$, $\exp(\lambda\partial/\partial x_l)$ is the translation operator by the vector $-\lambda e_l$, where $(e_l)_{l\in\mathcal{P}_n}$ is the canonical base of $\mathbb{R}^{\mathcal{P}_n}$ and the exponential is defined by its power series, which converges because $\partial/\partial x_l$ is a *bounded* operator on $\mathcal{S}_p$. It reduces to a translation simply because Taylor's formula is finite and exact for polynomials.

Since $\exp(\sum_{l\in\mathcal{P}_n}\lambda_l\partial/\partial x_l)$ is the translation by $\lambda = -\sum_{l\in\mathcal{P}_n}\lambda_l e_l$ i.e. the operator on $\mathcal{S}_p$ which maps the polynomial function H to $H' : \mathbf{x} \mapsto H(\mathbf{x}-\lambda)$, we have proven (III.2) for any H in $\mathcal{S}_p$. But p is arbitrary so we have proven the formula for any polynomial function H. By density of the polynomials in the Frechet space $\mathcal{S}$ for the C^∞ topology and the continuity relatively to that topology of both sides of (III.2), we extend the validity of the equation to all H in $\mathcal{S}$. We have proved formula (III.2) but the argument is the same for formula (III.3), just replace "h" by "w". ■

IV. A second generalization: The jungle formulas

Let $m \geq 1$ be an integer, and $(u_l^k)_{l\in\mathcal{P}_n}$, $1 \leq k \leq m$, be m families of $n(n-1)/2$ elements of a commutative Banach algebra $\mathcal{B}$. An *m-jungle* is a sequence $\mathcal{F} = (\mathfrak{F}_1, \ldots, \mathfrak{F}_m)$ of u-forests on I_n such that $\mathfrak{F}_1 \subset \ldots \subset \mathfrak{F}_m$.

The multiple forest formula states:

Theorem IV.1 (The algebraic Brydges-Kennedy jungle formula)

$$\exp\Big(\sum_{\substack{l\in\mathcal{P}_n\\ 1\leq k\leq m}} u_l^k\Big) = \sum_{\substack{\mathcal{F}=(\mathfrak{F}_1,\ldots,\mathfrak{F}_m)\\ m-\text{jungle}}} \Big(\prod_{l\in\mathfrak{F}_m}\int_0^1 dh_l\Big)\Big(\prod_{k=1}^m\Big(\prod_{l\in\mathfrak{F}_k\setminus\mathfrak{F}_{k-1}} u_l^k\Big)\Big)$$

$$.\ \exp\Big(\sum_{k=1}^m\sum_{l\in\mathcal{P}_n} h_l^{\mathcal{F},k}(\mathbf{h}).u_l^k\Big) \qquad \text{(IV.1)}$$

where $\mathfrak{F}_0 = 0$ by convention, $\mathbf{h}$ is the vector $(h_l)_{l\in\mathfrak{F}_m}$ and the functions $h^{\mathcal{F},k}_{\{ij\}}(\mathbf{h})$ are defined in the following manner:

- *If i and j are not connected by $\mathfrak{F}_k$ let $h^{\mathcal{F},k}_{\{ij\}}(\mathbf{h}) = 0$.*
- *If i and j are connected by $\mathfrak{F}_k$ but not by $\mathfrak{F}_{k-1}$ let*

$$h^{\mathcal{F},k}_{\{ij\}}(\mathbf{h}) = \inf\Big\{h_l, l\in L_{\mathfrak{F}_k}\{ij\}\cap(\mathfrak{F}_k\backslash\mathfrak{F}_{k-1})\Big\}$$

(recall that $L_{\mathfrak{F}}\{ij\}$ is the unique path in the forest $\mathfrak{F}$ connecting i to j).

- *If i and j are connected by $\mathfrak{F}_{k-1}$ let $h^{\mathcal{F},k}_{\{ij\}}(\mathbf{h}) = 1$.*

Proof: By induction. The case $m = 1$ was treated in Section II. For the induction step from m to $m+1$, we sum over the last forest $\mathfrak{F}_{m+1}$:

$$\sum_{\substack{\mathcal{F}=(\mathfrak{F}_1,\ldots,\mathfrak{F}_{m+1})\\ (m+1)-\text{jungle}}}\left(\prod_{l\in\mathfrak{F}_{m+1}}\int_0^1 dh_l\right)\left(\prod_{k=1}^{m+1}\Big(\prod_{l\in\mathfrak{F}_k\backslash\mathfrak{F}_{k-1}} u^k_l\Big)\right)\exp\Big(\sum_{k=1}^{m+1}\sum_{l\in\mathcal{P}_n} h^{\mathcal{F},k}_l(\mathbf{h}).u^k_l\Big)$$

$$= \sum_{\substack{\mathcal{F}'=(\mathfrak{F}_1,\ldots,\mathfrak{F}_m)\\ m-\text{jungle}}}\left(\prod_{l\in\mathfrak{F}_m}\int_0^1 dh_l\right)\left(\prod_{k=1}^{m}\Big(\prod_{l\in\mathfrak{F}_k\backslash\mathfrak{F}_{k-1}} u^k_l\Big)\right)\exp\Big(\sum_{k=1}^{m}\sum_{l\in\mathcal{P}_n} h^{\mathcal{F}',k}_l(\mathbf{h}').u^k_l\Big)$$

$$\cdot\sum_{\substack{\mathfrak{F}_{m+1}\ \text{u-forest}\\ \mathfrak{F}_m\subset\mathfrak{F}_{m+1}}}\left(\prod_{l\in\mathfrak{F}_{m+1}\backslash\mathfrak{F}_m}\int_0^1 dh_l\right)\left(\prod_{l\in\mathfrak{F}_{m+1}\backslash\mathfrak{F}_m} u^{m+1}_l\right)\exp\Big(\sum_{l\in\mathcal{P}_n} h^{\mathcal{F},m+1}_l(\mathbf{h}).u^{m+1}_l\Big)\quad\text{(IV.2)}$$

where $\mathbf{h} = (h_l)_{l\in\mathfrak{F}_{m+1}}$, $\mathbf{h}' = (h_l)_{l\in\mathfrak{F}_m}$ and we have noted that if $1\le k\le m$ then $h^{\mathcal{F},k}_l(\mathbf{h}) = h^{\mathcal{F}',k}_l(\mathbf{h}')$. To perform the summation over $\mathfrak{F}_{m+1}$, we will use the forest formula of section II and our favorite argument of forgetting the details of the tree structure up to $\mathfrak{F}_m$, to concentrate on what $\mathfrak{F}_{m+1}$ brings as new connections between the existing clusters. We introduce the partition $\mathcal{D}$ of I_n created by $\mathfrak{F}_m = \{l_1,\ldots,l_\nu\}$ and the u-forest on $\mathcal{D}$, $\overline{\mathfrak{F}}_{m+1} = \{\bar{l}_1,\ldots,\bar{l}_{\tau-\nu}\}$ induced by $\mathfrak{F}_{m+1}\backslash\mathfrak{F}_m = \{l_{\nu+1},\ldots,l_\tau\}$ with $\nu\le\tau$. The definitions are the same as in the proof of Lemma II.2 except that we have u-forests instead of o-forests.

For a link $\{ab\}$ between two elements a and b of $\mathcal{D}$, let $\overline{u}_{\{ab\}} = \sum_{i\in a,j\in b} u^{m+1}_{\{ij\}}$. Summing over $\mathfrak{F}_{m+1}$, u-forest on I_n containing $\mathfrak{F}_m$, with the "propagators" $u^{m+1}_{\{ij\}}$ is the same as summing over the u-forests $\overline{\mathfrak{F}}_{m+1}$ on $\mathcal{D}$ with the "propagators" $\overline{u}_{\{ab\}}$ i.e.

$$\sum_{\overline{\mathfrak{F}}_{m+1}\ \text{u-forest on}\ \mathcal{D}}\left(\prod_{\bar{l}\in\overline{\mathfrak{F}}_{m+1}}\int_0^1 d\overline{h}_{\bar{l}}\right)\left(\prod_{\bar{l}\in\overline{\mathfrak{F}}_{m+1}}\overline{u}_{\bar{l}}\right)\exp\left(\sum_{\substack{\{ab\}\subset\mathcal{D}\\ a\neq b}}\overline{h}^{\overline{\mathfrak{F}}_{m+1}}_{\{ab\}}(\overline{\mathbf{h}}).\overline{u}_{\{ab\}}\right)$$

$$= \sum_{\substack{\mathfrak{F}_{m+1}\ \text{u-forest on } I_n \\ \text{such that } \mathfrak{F}_m \subset \mathfrak{F}_{m+1}}} \left(\prod_{l \in \mathfrak{F}_{m+1} \setminus \mathfrak{F}_m} \int_0^1 dh_l \right) \left(\prod_{l \in \mathfrak{F}_{m+1} \setminus \mathfrak{F}_m} u_l^{m+1} \right) \exp\left(\sum_{\substack{\{ij\} \in \mathcal{P}_n \\ \{ij\} \sharp \mathcal{D}}} h_{\{ij\}}^{\mathcal{F},m+1}(\mathbf{h}).u_{\{ij\}}^{m+1} \right) \tag{IV.3}$$

where $\{ij\}\sharp\mathcal{D}$ means "i and j are in different components of $\mathcal{D}$". By the forest formula of section II, the left hand side of the last equality is

$$\exp\left(\sum_{\substack{\{ab\} \subset \mathcal{D} \\ a \neq b}} \overline{u}_{\{ab\}} \right) \tag{IV.4}$$

that is

$$\exp\left(\sum_{\substack{\{ij\} \in \mathcal{P}_n \\ \{ij\} \sharp \mathcal{D}}} u_{\{ij\}}^{m+1} \right) \quad . \tag{IV.5}$$

The right hand side is almost the partial sum we want to perform on $\mathfrak{F}_{m+1}$ in (IV.2), there misses

$$\exp\left(\sum_{\substack{\{ij\} \in \mathcal{P}_n \\ \{ij\} / \mathcal{D}}} h_{\{ij\}}^{\mathcal{F},m+1}(\mathbf{h}).u_{\{ij\}}^{m+1} \right) \quad . \tag{IV.6}$$

The sum is over all pairs $\{ij\}$ in I_n such that $\{ij\}/\mathcal{D}$ i.e. i and j are connected by $\mathfrak{F}_m$. But then the definition of the functions $h_l^{\mathcal{F},k}(\mathbf{h})$ tells us $h_{\{ij\}}^{\mathcal{F},m+1}(\mathbf{h}) = 1$; so the missing factor becomes

$$\exp\left(\sum_{\substack{\{ij\} \in \mathcal{P}_n \\ \{ij\} \sharp \mathcal{D}}} u_{\{ij\}}^{m+1} \right) \quad . \tag{IV.7}$$

In conclusion

$$\sum_{\substack{\mathfrak{F}_{m+1}\ \text{u-forest on } I_n \\ \text{st } \mathfrak{F}_m \subset \mathfrak{F}_{m+1}}} \left(\prod_{l \in \mathfrak{F}_{m+1} \setminus \mathfrak{F}_m} \int_0^1 dh_l \right) \left(\prod_{l \in \mathfrak{F}_{m+1} \setminus \mathfrak{F}_m} u_l^{m+1} \right) \exp\left(\sum_{l \in \mathcal{P}_n} h_l^{\mathcal{F},m+1}(\mathbf{h}).u_l^{m+1} \right)$$

$$= \exp\left(\sum_{l \in \mathcal{P}_n} u_l^{m+1} \right) \quad . \tag{IV.8}$$

The sum over the m-jungle $\mathcal{F}'$ is now $\exp(\sum_{k=1}^{m} \sum_{l \in \mathcal{P}_n} u_l^k)$ by the induction hypothesis, and the right hand side of (IV.2) is finally summed to $\exp(\sum_{k=1}^{m+1} \sum_{l \in \mathcal{P}_n} u_l^k)$ as wanted. ■

We can generalize in a similar way the rooted forest formula. Given an m-jungle $\mathcal{F}=(\mathfrak{F}_1,\ldots,\mathfrak{F}_m)$, we introduce the notation $\mathbf{w}$ for the vector $(w_l)_{l\in\mathfrak{F}_m}$, and $w^{\mathcal{F},k}_{\{ij\}}(\mathbf{w})$ for the functions defined by:

- If i and j are not connected by $\mathfrak{F}_k$ let $w^{\mathcal{F},k}_{\{ij\}}(\mathbf{w})=0$.

- If i and j are connected by $\mathfrak{F}_{k-1}$ let $w^{\mathcal{F},k}_{\{ij\}}(\mathbf{w})=1$.

- If i and j are connected by $\mathfrak{F}_k$ but not by $\mathfrak{F}_{k-1}$, let r_C be the root of the cluster of $\mathfrak{F}_k$ containing i and j, define $l_k(i)$ to be the number of links of $\mathfrak{F}_k\backslash\mathfrak{F}_{k-1}$ in the unique path that goes from i to r_C, and similarly for j. Now if $|l_k(i)-l_k(j)|\geq 2$ put $w^{\mathcal{F},k}_{\{ij\}}(\mathbf{w})=0$, if $l_k(i)=l_k(j)$ put $w^{\mathcal{F},k}_{\{ij\}}(\mathbf{w})=1$, and if $l_k(i)=l_k(j)+1$ for instance, take $w^{\mathcal{F},k}_{\{ij\}}(\mathbf{w})=w_l$ where l is the first link of $\mathfrak{F}_k\backslash\mathfrak{F}_{k-1}$ on the path that goes from i to r_C. It is about the same definition as for theorem II.2 but we take only into account the links in $\mathfrak{F}_k\backslash\mathfrak{F}_{k-1}$.

We can now state

Theorem IV.2 (The algebraic rooted jungle formula)

$$\exp\Big(\sum_{\substack{l\in\mathcal{P}_n\\1\leq k\leq m}} u^k_l\Big)=\sum_{\substack{\mathcal{F}=(\mathfrak{F}_1,\ldots,\mathfrak{F}_m)\\m-\text{jungle}}}\Big(\prod_{l\in\mathfrak{F}_m}\int_0^1 dw_l\Big)\Big(\prod_{k=1}^m\Big(\prod_{l\in\mathfrak{F}_k\backslash\mathfrak{F}_{k-1}} u^k_l\Big)\Big)$$

$$.\ \exp\Big(\sum_{k=1}^m\sum_{l\in\mathcal{P}_n} w^{\mathcal{F},k}_l(\mathbf{w}).u^k_l\Big)\ . \tag{IV.9}$$

Proof: The deduction of the jungle formula from the forest formula is the same as in the Brydges-Kennedy case. Remark that the property that was used was the following. Once $\mathfrak{F}_m$ is fixed as well as the partition $\mathcal{D}$ of I_n it creates, the numbers $w^{\mathcal{F},m+1}_{\{ij\}}(\mathbf{w})$ are unchanged if we allow i to travel freely in the component a_i of $\mathcal{D}$ it belongs to, and the same for j in a_j. Furthermore $w^{\mathcal{F},m+1}_{\{ij\}}(\mathbf{w})=\overline{w}^{\overline{\mathfrak{F}}_{m+1}}_{\{a_i a_j\}}(\overline{\mathbf{w}})$ where $\overline{\mathfrak{F}}_{m+1}$ is the forest on $\mathcal{D}$ induced by $\mathfrak{F}_{m+1}$, and $\overline{\mathbf{w}}=(\overline{w}_{\overline{l}})_{\overline{l}\in\overline{\mathfrak{F}}_{m+1}}$ is defined by $\overline{w}_{\overline{l}}=w_l$ where l is the unique link in $\mathfrak{F}_{m+1}\backslash\mathfrak{F}_m$ inducing the link $\overline{l}$ in $\mathcal{D}$. The functions $\overline{w}^{\overline{\mathfrak{F}}}_{\overline{l}}(\overline{\mathbf{w}})$ are defined by the same algorithm as in theorem II.2 but with $\mathcal{D}$ instead of I_n as a point set, and with the following *induced choice of root*. If $\overline{C}$ is a non empty subset or cluster of $\mathcal{D}$, $\overline{C}=\{a_1,\ldots,a_\mu\}$, let $C=\cup_{\nu=1}^{\mu}a_\nu$. It is a non empty subset of I_n and already has a chosen root r_C. Then $\exists!\ \nu, 1\leq\nu\leq\mu$ such that $r_C\in a_\nu$. So take $\overline{r}_{\overline{C}} = a_\nu \in \overline{C}$ to be the *induced root* of $\overline{C}$, that will get involved in the definition of the $\overline{w}_{\overline{l}}^{\overline{\mathfrak{F}}}(\overline{\mathbf{w}})$.

Finally, and this is the last ingredient that allows us to copy the precedent proof, we have forced $w_{\{ij\}}^{\mathcal{F},m+1}(\mathbf{w}) = 1$ if i and j fall in the same component of $\mathcal{D}$ so that we once again recover the missing factor

$$\exp\left(\sum_{\substack{\{ij\}\in\mathcal{P}_n \\ \{ij\}\sharp\mathcal{D}}} u_{\{ij\}}^{m+1} \right) \quad . \tag{IV.10}$$

■

The reader must have noticed along this proof that we used a fairly general recipe to obtain a jungle formula from a forest formula. We also have a recipe to get Taylor interpolation formulas from algebraic ones: simply apply them to differentiation operators for the u_l's.

Let $\mathcal{P}_{n,m} = \mathcal{P}_n \times I_m$ and $\mathcal{S}$ be the space of smooth functions from $\mathrm{I\!R}^{\mathcal{P}_{n,m}}$ to an arbitrary Banach space $\mathcal{V}$. An element of $\mathrm{I\!R}^{\mathcal{P}_{n,m}}$ will be generally denoted by $\mathbf{x} = (x_l^k)_{(l,k)\in\mathcal{P}_{n,m}}$. The vector with all entries equal to 1 will be noted $\mathbb{1}$. H is an arbitrary element of $\mathcal{S}$. We can state the following theorems, whose proofs are rewritings of that of theorem III.1.

Theorem IV.3 (The Brydges-Kennedy Taylor jungle formula)

$$H(\mathbb{1}) = \sum_{\substack{\mathcal{F}=(\mathfrak{F}_1,\ldots,\mathfrak{F}_m) \\ m-\text{jungle}}} \left(\prod_{l\in\mathfrak{F}_m} \int_0^1 dh_l \right) \left(\left(\prod_{k=1}^{m} \Big(\prod_{l\in\mathfrak{F}_k\setminus\mathfrak{F}_{k-1}} \frac{\partial}{\partial x_l^k} \Big) \right) H \right) \left(X_{\mathcal{F}}^{BK}(\mathbf{h}) \right) \quad . \tag{IV.11}$$

Here $X_{\mathcal{F}}^{BK}(\mathbf{h})$ is the vector $(x_l^k)_{(l,k)\in\mathcal{P}_{n,m}}$ of $\mathrm{I\!R}^{\mathcal{P}_{n,m}}$ defined by $x_l^k = h_l^{\mathcal{F},k}(\mathbf{h})$, which is the value at which we evaluate the complicated derivative of H.

and

Theorem IV.4 (The rooted Taylor jungle formula)

$$H(\mathbb{1}) = \sum_{\substack{\mathcal{F}=(\mathfrak{F}_1,\ldots,\mathfrak{F}_m) \\ m-\text{jungle}}} \left(\prod_{l\in\mathfrak{F}_m} \int_0^1 dw_l \right) \left(\left(\prod_{k=1}^{m} \Big(\prod_{l\in\mathfrak{F}_k\setminus\mathfrak{F}_{k-1}} \frac{\partial}{\partial x_l^k} \Big) \right) H \right) \left(X_{\mathcal{F}}^{r}(\mathbf{w}) \right) \quad . \tag{IV.12}$$

Here $X^r_{\mathcal{F}}(\mathbf{w})$ is the vector $(x^k_l)_{(l,k)\in\mathcal{P}_{n,m}}$ of $\mathbb{R}^{\mathcal{P}_{n,m}}$ defined by $x^k_l = w^{\mathcal{F},k}_l(\mathbf{w})$, which is the value at which we evaluate the complicated derivative of H.

Let us conclude this section by recalling the important positivity property of the Brydges-Kennedy forest and jungle formulas.

Theorem IV.5 (Positivity) *Let $\mathcal{F}$ be a m-jungle, and M^k, $1 \le k \le m$ be any sequence of m positive symmetric n by n matrices with entries m^k_{ij}. Then the interpolated matrices $M^k(\mathbf{h})$ with entries $m^k_{ij}(\mathbf{h}) = h^{\mathcal{F},k}_{ij}(\mathbf{h}) m^k_{ij}$ if $i<j$ and $m^k_{ii}(\mathbf{h}) = m^k_{ii}$ are all positive.*

Proof: The definition of $M^k(\mathbf{h})$ only involves the forests $\mathfrak{F}_{k-1}$ and $\mathfrak{F}_k$ and the h_l's for l in $\mathfrak{F}_k$. Note that, for every link l, $h^{\mathcal{F},k}_l(\mathbf{h}) = h^{\mathfrak{F}_k}_l(\overline{\mathbf{h}})$, the right hand side is the $h^{\mathfrak{F}_k}_l(\overline{\mathbf{h}})$ function of the simple forest formalism of Section II, where $\overline{\mathbf{h}} = (\overline{h}_l)_{l\in\mathfrak{F}_k}$ with $\overline{h}_l = h_l$ if $l \in \mathfrak{F}_k \backslash \mathfrak{F}_{k-1}$ and $\overline{h}_l = 1$ if $l \in \mathfrak{F}_{k-1}$. In fact, for all points i and j connected by $\mathfrak{F}_{k-1}$, i.e. such that $L_{\mathfrak{F}_k}\{ij\} \subset \mathfrak{F}_{k-1}$, we have

$$h^{\mathfrak{F}_k}_l(\overline{\mathbf{h}}) = \inf\{\overline{h}_l,\ l \in L_{\mathfrak{F}_k}\{ij\}\} = \inf\{1\} = 1 \stackrel{\text{def}}{=} h^{\mathcal{F},k}_l(\mathbf{h}) \quad . \tag{IV.13}$$

So we just need to prove the positivity in the simple forest formalism. This has already been done by Brydges in [B2], but we give here a proof along the lines of our derivation of the formula (II.1).

Suppose we have a forest $\mathfrak{F}$ with parameters $\mathbf{h} = (h_l)_{l\in\mathfrak{F}}$ between 0 and 1, and a positive symmetric matrix M with entries m_{ij} that is interpolated by $M(\mathbf{h})$ with entries $m_{ij}(\mathbf{h}) = m_{ij} h^{\mathfrak{F}}_l(\mathbf{h})$. Let us number the links of $\mathfrak{F}$ into an o-forest $F = (l_1, \ldots . l_\tau)$ in order that $1 \ge h_{l_1} \ge \ldots \ge h_{l_\tau} \ge 0$. We perform the same change of variables as in the formula (II.23). If $i \ne j$ let us denote by $U_{\{ij\}}$ the matrix with zero entries except at the locations (i,j) and (j,i) where we put 1. Let M_{diag} be the matrix with entries $m_{ii}\delta_{ij}$ the diagonal part of M. Then

$$M(\mathbf{h}) = M_{\text{diag}} + \sum_{l\in\mathcal{P}_n} h^F_l(\mathbf{h}).U_l m_l \tag{IV.14}$$

and for the same reason as (II.23)

$$M(\mathbf{h}) = M_{\text{diag}} + \sum_{\nu=1}^{\tau} t_\nu M^F_\nu \tag{IV.15}$$

where $M^F_\nu = \sum_{\{ij\}/\nu} U_{\{ij\}} m_{ij}$, and finally

$$M(\mathbf{h}) = (1 - \sum_{\nu=1}^{\tau} t_\nu) M_{\rm diag} + \sum_{\nu=1}^{\tau} t_\nu (M^F_\nu + M_{\rm diag}) \quad . \tag{IV.16}$$

Now note that for $1 \leq \nu \leq \tau$, $t_\nu \geq 0$ and $\sum_{\nu=1}^{\tau} t_\nu = h_{l_1} \leq 1$ so $1 - \sum_{\nu=1}^{\tau} t_\nu \geq 0$, and $M^F_\nu + M_{\rm diag}$ is a positive symmetric matrix. Indeed let us fix ν and consider the o-forest $F_\nu = (l_1, \ldots, l_\nu)$ completed at stage ν from the given o-forest F. We can reorder the base vectors so that M looks like a matrix with blocks corresponding to indexes in the same component of F_ν. Taking only the diagonal blocks and putting the others to 0 we get exactly the matrix $M^F_\nu + M_{\rm diag}$. It is clear that such a process preserves the positivity of the matrix. $M(\mathbf{h})$, a sum of positive matrices with non negative multipliers, is positive. ■

Note that the rooted formulas **do not preserve positivity** as can be seen on simple examples, with 4 points for instance.

V. Some concrete examples

V.A) Gaussian measures perturbed by a small interaction

In constructive field theory cluster expansions are typically used to perform the thermodynamic limit of a field theory with cutoffs and a small local interaction $I_\Lambda(\phi)$ in a finite volume Λ. The cluster expansion expresses the partition function as a polymer gas with hard core conditions (see e.g. [R2]). The set I_n is then made of a partition of Λ into (unit size) cubes, and clusters are subsets of such cubes.

For instance consider the free bosonic Gaussian measure $d\mu_C$ in $\mathbb{R}^d$ defined by a covariance C with ultraviolet cutoff and good decrease at infinity. A standard example is

$$C(x,y) = \frac{1}{(4\pi)^{d/2}} \int_1^{+\infty} \frac{d\alpha}{\alpha^{d/2}} e^{-\alpha m^2 - |x-y|^2/4\alpha} \tag{V.A.1}$$

so that

$$\hat{C}(p) = \frac{e^{-(p^2+m^2)}}{p^2+m^2} \tag{V.A.2}$$

We perturb this free theory by adding an interaction such as $e^{-g\int_\Lambda \phi^4(x)} dx$, and we want to perform the thermodynamic limit $\Lambda \to \infty$, that is to define and study an intensive quantity such as the pressure

$$p = \lim_{\Lambda \to \mathbb{R}^d} \frac{1}{|\Lambda|} \log Z(\Lambda) \ , \tag{V.A.3}$$

where the partition function $Z(\Lambda)$ in a finite volume Λ is

$$Z(\Lambda) = \int d\mu_C(\phi) e^{-g \int_\Lambda \phi^4(x) dx} \ , \tag{V.A.4}$$

Let us explain how the Taylor formula (III.2) performs the task of rewriting the partition function as a dilute gas of clusters with hard core interaction. We write $\Lambda = \cup_{i \in I_n} b_i$, where each b_i is a unit cube, and define χ_b as the characteristic function of b, and $\chi_\Lambda = \sum_{i \in I_n} \chi_{b_i}$. Since the interaction lies entirely within Λ, the covariance C in (V.A.4) can be replaced by $C_\Lambda = \chi_\Lambda(x) C(x,y) \chi_\Lambda(y)$ without changing the value of $Z(\Lambda)$. Moreover C_Λ can be interpolated, defining for $l = \{i, j\} \in \mathcal{P}_n$

$$C_\Lambda((x_l)_{l \in \mathcal{P}_n})(x,y) = \sum_{i=1}^n \chi_{b_i}(x) C(x,y) \chi_{b_i}(y)$$

$$+ \sum_{\{i,j\} \in \mathcal{P}_n} x_{\{ij\}} \big(\chi_{b_i}(x) C(x,y) \chi_{b_j}(y) + \chi_{b_j}(x) C(x,y) \chi_{b_i}(y) \big) \tag{V.A.5}$$

Remark that $C_\Lambda(1, ..., 1) = C_\Lambda$. Now we apply the Taylor formula (III.2) with the function H being the partition function obtained by replacing in (V.A.3-4) the covariance C by $C_\Lambda((x_l)_{l \in \mathcal{P}_n})$. Here it is crucial to use the positivity theorem IV.5, in order for the interpolated covariance to remain positive, hence for the corresponding normalized Gaussian measure to remain well defined. From the rules of Gaussian integration of polynomials, we can compute the effect of deriving with respect to a given x_l parameter, and we obtain that (III.2) in this case takes the form

$$Z(\Lambda) = H(\mathbb{1}) = \sum_{\mathfrak{F}} \int d\mu_{C_\Lambda(X^{BK}_{\mathfrak{F}}(\mathbf{h}))}(\phi) \Big(\prod_{l \in \mathfrak{F}} \int_0^1 dh_l \Big)$$

$$\Big\{ \prod_{l = \{ij\} \in \mathfrak{F}} \int dx dy \chi_{b_i}(x) \chi_{b_j}(y) C(x,y) \frac{\delta}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)} \Big\} e^{-g \int_\Lambda \phi^4(x) dx} \tag{V.A.6}$$

Since both the local interaction and the covariance as a matrix factorize over the clusters of the forest $\mathfrak{F}$, the corresponding contributions in (V.A.6) themselves factorize, which means that (V.A.6) can also be rewritten as a gas of non-overlapping clusters, each of which has an amplitude given by a *tree formula*:

$$Z(\Lambda) = \int d\mu_{C_\Lambda}(\phi) e^{I_\Lambda(\phi)} = \sum_{\substack{\text{sets } \{Y_1,\dots,Y_n\} \\ Y_i \cap Y_j = \emptyset, \cup Y_i = \Lambda}} \prod_{i=1}^{n} A(Y_i) \tag{V.A.7}$$

$$A(Y) = \sum_{\mathfrak{T} \text{ on } Y} \Big(\prod_{l \in \mathfrak{T}} \int_0^1 dh_l\Big) \int d\mu_{C_Y(X_{\mathfrak{T}}^{BK}(\mathbf{h}))}(\phi)$$

$$\Big\{ \prod_{l=\{ij\}\in\mathfrak{T}} \int dx dy \chi_{b_i}(x) \chi_{b_j}(y) C(x,y) \frac{\delta}{\delta\phi(x)} \frac{\delta}{\delta\phi(y)} \Big\} e^{-g\int_Y \phi^4(x)dx} \tag{V.A.8}$$

where b_i and b_j are the two ends of the line l, and the sum is over trees $\mathfrak{T}$ which connect together the set Y, hence have exactly $|Y|-1$ elements (if $|Y|=1$, $\mathfrak{T}=\emptyset$ connects Y). The measure $d\mu_{C_Y(\{X_{\mathfrak{T}}^{BK}(\mathbf{h})\})}(\phi)$ is the normalized Gaussian measure with (positive) covariance

$$C_Y(X_{\mathfrak{T}}^{BK}(\mathbf{h}))(x,y) = \chi_Y(x) \big(h_{\mathfrak{T}}(\mathbf{h})(x,y)\big) C(x,y) \chi_Y(y) \tag{V.A.9}$$

where $h_{\mathfrak{T}}(\mathbf{h})(x,y)$ is 1 if x and y belong to the same cube, and otherwise it is the infimum of the parameters h_l for l in the unique path $L_{\mathfrak{T}}(b(x),b(y))$ which in the tree $\mathfrak{T}$ joins the cube $b(x)$ containing x to the cube $b(y)$ containing y.

Formula (V.A.8) is somewhat shorter than the different formulas of [R1-2], and can be used in the same way to check that given any constant K, for small enough g with Re $g > 0$

$$\sum_{Y \text{ st } 0 \in Y} |A(Y)| K^{|Y|} \leq 1 \tag{V.A.10}$$

Proof of (V.A.10) requires the slightly cumbersome computation of the action of the functional derivatives in (V.A.8) and a bound on the resulting functional integral. The method is identical to [R1-2]. Remark that although the full amplitudes $A(Y)$ defined in (V.A.8) must be identical to those in [R1-2], the subcontributions associated to particular trees are different.

The Mayer expansion defined below allows to deduce from (V.A.10) the existence and e.g. the Borel summability in g of thermodynamic functions such as the pressure p defined by (V.A.3).

B) The Mayer expansion

In the cluster expansion (V.A.7), the condition that the disjoint union of all clusters is Λ is a global annoying constraint. Remark that the polymer amplitudes are translation invariant. In particular the trivial amplitude of a singleton cluster $Y = \{b\}$ is a number A_0 independent of b. Redefining $A_r(Y) = A(Y)/A_0^{|Y|}$ and $Z_r(\Lambda) = Z(\Lambda)/A_0^{|\Lambda|}$ we quotient out all the trivial clusters so that

$$Z_r(\Lambda) = 1 + \sum_{n\geq 1} \sum_{\substack{\text{sets } \{Y_1,\ldots,Y_n\} \\ |Y_i|\geq 2\ ,\ Y_i\cap Y_j=\emptyset}} \prod_{i=1}^{n} A_r(Y_i) \tag{V.B.1}$$

This is the partition function of a polymer gas: the sums over individual polymers would be independent were it not for the hard core constraints $Y_i \cap Y_j = \emptyset$. Adding in an infinite number of vanishing terms, we can replace the sum in (V.B.1) by a sum over ordered sequences $(Y_1, ..., Y_n)$ of polymers with hard core interaction and a symmetrizing factor $1/n!$ coming from the replacement of sets by sequences.

$$Z_r(\Lambda) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\text{sequences } (Y_1,\ldots,Y_n) \\ |Y_i|\geq 2}} \prod_{i=1}^{n} A_r(Y_i) \prod_{1\leq i<j\leq n} e^{-V(Y_i,Y_j)} \tag{V.B.2}$$

where the hard core interaction is $V(X,Y) = 0$ if $X \cap Y = \emptyset$, and $V(X,Y) = +\infty$ if $X \cap Y \neq \emptyset$ To factorize again this formula we cannot apply directly the algebraic Brydges-Kennedy formula (II.1), because the interaction V can be infinite, but the Taylor forest formula (III.2) easily does the job. More precisely we define now I_n as our set of indices, and define $\epsilon^Y_{\{ij\}} = (e^{-V(X_i,X_j)} - 1)$, for $i \neq j$. For a fixed sequence $(Y_1, \ldots, Y_n)$ of polymers, consider the function

$$H((x_l)_{l\in\mathcal{P}_n}) = \prod_{l\in\mathcal{P}_n} (1 + x_l \epsilon^Y_l) \tag{V.B.3}$$

Rewrite (V.B.2) as

$$Z_r(\Lambda) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\text{sequences } (Y_1,\ldots,Y_n) \\ |Y_i|\geq 2}} H(\mathbb{1}) \prod_{i=1}^{n} A_r(Y_i)$$

$$= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{\text{sequences } (Y_1,\ldots,Y_n) \\ |Y_i|\geq 2}} \prod_{i=1}^{n} A_r(Y_i) \sum_{\mathfrak{F}} \Big(\prod_{l\in\mathfrak{F}} \int_0^1 dh_l \Big) \Big(\prod_{l\in\mathfrak{F}} \epsilon^Y_l \Big) \Big(\prod_{l\notin\mathfrak{F}} (1 + h^{\mathfrak{F}}_l(\mathbf{h}) \epsilon^Y_l) \Big)$$

$$= \sum_{n\geq 0} \frac{1}{n!} \Bigg(\sum_{k\geq 1} \frac{1}{k!} \sum_{\substack{\text{sequences } (Y_1,\dots,Y_k) \\ |Y_i|\geq 2}} \Big(\prod_{i=1}^{k} A_r(Y_i) \Big) C^T(Y_1,...,Y_k) \Bigg)^n \tag{V.B.4}$$

where

$$C^T(Y_1,...,Y_k) = \sum_{\substack{G \text{ connected graph} \\ \text{on}\{1,\dots,k\}}} \prod_{l\in G} \epsilon_l^Y$$

$$= \sum_{\mathfrak{T} \text{ tree on } \{1,\dots,k\}} \Big(\prod_{l\in\mathfrak{T}} \int_0^1 dh_l \Big) \Big(\prod_{l\in\mathcal{P}_k,\ l\in\mathfrak{T}} \epsilon_l^Y \Big) \prod_{l\in\mathcal{P}_k,\ l\notin\mathfrak{T}} \left(1 + h_l^{\mathfrak{T}}(\mathbf{h})\epsilon_l^Y\right) \tag{V.B.5}$$

where $h_l^{\mathfrak{T}}(\mathbf{h})$ is, if $l = \{ij\}$, the infimum of the parameters $h_{l'}$ for l' in the unique path $L_{\mathfrak{T}}\{ij\}$ which in the tree $\mathfrak{T}$ joins i to j.

We obtain immediately that

$$\log Z_r(\Lambda) = \sum_{k\geq 1} \frac{1}{k!} \sum_{\substack{\text{sequences } (Y_1,\dots,Y_k) \\ |Y_i|\geq 2}} \Big(\prod_{i=1}^{k} A_r(Y_i) \Big) C^T(Y_1,...,Y_k) \tag{V.B.6}$$

Formulas (V.B.5-6) are more explicit than those used in [R1-2] and have all desired advantages (every tree coefficient forces the necessary links and is bounded by 1, since $|(1 + h_l^{\mathfrak{T}}(\mathbf{h})\epsilon_l^Y)| \leq 1$). They can therefore be used together with (V.A.10) to control in a similar way the thermodynamic function $p = A_0 + \lim_{\Lambda\to\mathbb{R}^d} \frac{1}{|\Lambda|} \log Z_r(\Lambda)$.

C) A single formula for the succession of a cluster and a Mayer expansion

Formula (V.B.6) involves trees and forests on two different kinds of objects. First are the links between boxes in our lattice that make the clusters Y_i. Then we have Mayer links between indexes i in I_k. This leads to complications, if we iterate the process, when doing a renormalization group analysis. So we propose a formula with two types of links but on the same objects, it is a 2-jungle formula. For simplicity we write it in the algebraic setting of section II, i.e. with $Z = \exp(\sum u_l)$ instead of $Z_r(\Lambda)$, and perform the apparently stupid operation $\log\exp(\sum u_l)$!

The links l are between elements of I_n, but now we introduce a new set of objects $\mathcal{D}_n = I_n \times \mathbb{N}$. If $\bar{l}$ is a link $\{(b,k),(b',k')\}$ between different elements of $\mathcal{D}_n$ we let $u_{\bar{l}} = u_{\{bb'\}}$ and $\delta_{\bar{l}} = \delta_{bb'}$ where the second delta is Kroneker's. Let $\mathcal{F} = (\mathfrak{F}_1, \mathfrak{F}_2)$ be a 2-jungle on $\mathcal{D}_n$, we say that it is *nice* if it fulfills the following requirements:

- $\mathfrak{F}_1$ is only made of horizontal links i.e. of the form $\{(b,k),(b',k)\}$;

- $\mathfrak{F}_2\backslash\mathfrak{F}_1$ is only made of vertical links of length 1 i.e. of the form $\{(b,k),(b,k+1)\}$;

- $\mathfrak{F}_2$ is a non empty tree;

- there are no singletons among the clusters of $\mathfrak{F}_1$ in the support of $\mathfrak{F}_2$;

- there is a unique cluster of $\mathfrak{F}_1$ in the support of $\mathfrak{F}_2$ that lies in the 0-th level $I_n\times\{0\}$;

- if we fix a root r of $\mathfrak{F}_2$ in that unique cluster, then for every element (b,k) in the support of $\mathfrak{F}_2$, k is the number of vertical links on the path going from (b,k) to r (this property is independent of the choice of r).

We denote by $k(\mathcal{F})$ the number of clusters of $\mathfrak{F}_1$ in the support of $\mathfrak{F}_2$. We introduce the following lexicographical order on $\mathcal{D}_n$, $(b,k)\preceq(b',k')$ if and only if $k<k'$ or ($k=k'$ and $b\le b'$). This is the order we use to choose a root for each non empty finite subset of $\mathcal{D}_n$ according to the rooted jungle formalism. We let $\mathbf{h}=(h_{\bar{l}})_{\bar{l}\in\mathfrak{F}_1}$ and $\mathbf{w}=(w_{\bar{l}})_{\bar{l}\in\mathfrak{F}_2\backslash\mathfrak{F}_1}$ and we take the compound $(\mathbf{h},\mathbf{w})$ instead of the $\mathbf{h}$ or the $\mathbf{w}$ vectors that were used in (IV.1) and (IV.9). The functions $h_{\bar{l}}^{\mathcal{F},1}$ and $w_{\bar{l}}^{\mathcal{F},2}$ are however defined exactly in the same manner as in section IV. We now claim that

Theorem V.C.1

$$\log Z=\sum_{\substack{\mathcal{F}=(\mathfrak{F}_1,\mathfrak{F}_2)\\ \text{nice 2-jungle}}}\frac{1}{k(\mathcal{F})}\Big(\prod_{\bar{l}\in\mathfrak{F}_1}u_{\bar{l}}\int_0^1 dh_{\bar{l}}\Big)\Big(\prod_{\bar{l}\in\mathfrak{F}_2\backslash\mathfrak{F}_1}(-\delta_{\bar{l}})\int_0^1 dw_{\bar{l}}\Big)$$

$$\times\exp\Big(\sum_{\bar{l}}h_{\bar{l}}^{\mathcal{F},1}(\mathbf{h},\mathbf{w})u_{\bar{l}}\Big)\ .\ \prod_{\bar{l}\notin\mathfrak{F}_2\backslash\mathfrak{F}_1}\Big(1-w_{\bar{l}}^{\mathcal{F},2}(\mathbf{h},\mathbf{w})\delta_{\bar{l}}\Big)\quad . \tag{V.C.1}$$

Proof:

For $m\in\mathbb{N}$, let $\mathcal{D}_{n,m}=I_n\times\{0,1,\ldots,m\}\subset\mathcal{D}_n$. As in (V.B.6) we have

$$\log Z=\lim_{m\to+\infty}L_m \tag{V.C.2}$$

where

$$L_m=\sum_{s=1}^{m}\frac{1}{s!}\sum_{\substack{\text{sequences }(Y_1,\ldots,Y_s)\\ Y_i\subset I_n,\ |Y_i|\ge 2}}\Big(\prod_{i=1}^{s}A(Y_i)\Big)C^T(Y_1,\ldots,Y_s) \tag{V.C.3}$$

and

$$A(Y)=\sum_{\mathfrak{T}\text{ tree on }Y}\Big(\prod_{l\in\mathfrak{T}}u_l\int_0^1 dh_l\Big)\exp\Big(\sum_{l\text{ in }Y}h_l^{\mathfrak{T}}(\mathbf{h})u_l\Big)\quad . \tag{V.C.4}$$

This time we compute $C^T(Y_1,\dots,Y_s)$ thanks to the rooted forest formalism. Note that

$$\prod_{\{ij\}\subset I_n}(1+\epsilon^Y_{\{ij\}})=\prod_{\{ij\}\subset I_n}\ \prod_{(b,b')\in Y_i\times Y_j}(1-x_{\{ij\}}\delta_{bb'}) \tag{V.C.5}$$

with all the $x_{\{ij\}}$ set to 1. Then we apply the rooted Taylor forest formula III.3 and collect the connected parts. We get

$$C^T(Y_1,\dots,Y_S)=\sum_{\mathfrak{T}\text{ tree on }I_s}\mathcal{C}(s;Y_1,\dots,Y_s;\mathfrak{T}) \tag{V.C.6}$$

where

$$\mathcal{C}(s;Y_1,\dots,Y_s;\mathfrak{T})=\Big(\prod_{l\in\mathfrak{T}}\int_0^1 dw_l\Big)\ .$$

$$\Bigg(\prod_{\{ij\}\in\mathfrak{T}}\Big(\sum_{(b,b')\in Y_i\times Y_j}(-\delta_{bb'})\prod_{\substack{(c,c')\in Y_i\times Y_j\\(c,c')\neq(b,b')}}(1-w^{\mathfrak{T}}_{\{ij\}}(\mathbf{w})\delta_{cc'})\Big)\Bigg)$$

$$\cdot\prod_{\{ij\}\notin\mathfrak{T}}\Big(\prod_{(b,b')\in Y_i\times Y_j}(1-w^{\mathfrak{T}}_{\{ij\}}(\mathbf{w})\delta_{bb'})\Big)\quad . \tag{V.C.7}$$

The only trees $\mathfrak{T}$ giving a non zero contribution are those where for any $i\neq j$ in the same layer of $\mathfrak{T}$, $Y_i\cap Y_j=\emptyset$. In fact for such i and j, $\{ij\}\notin\mathfrak{T}$ and $w^{\mathfrak{T}}_{\{ij\}}(\mathbf{w})=1$ so in (V.C.7) we get a factor

$$\prod_{(b,b')\in Y_i\times Y_j}(1-\delta_{bb'})=e^{-V(Y_i,Y_j)} \tag{V.C.8}$$

forcing the non overlapping condition.

Given a sequence $(Y_1,\dots,Y_s)$ and a tree $\mathfrak{T}$ satisfying that property, we can define for $i\in I_k$, $\overline{Y}_i=Y_i\times\{l^{\mathfrak{T}}(i)\}\subset\mathcal{D}_{n,m}$. Here $l^{\mathfrak{T}}(i)$ is the height of vertex i in the tree $\mathfrak{T}$. The $\overline{Y}_i$ are all disjoint.

For a finite subset $\overline{Y}$ of $\mathcal{D}_n$ let

$$\overline{A}(\overline{Y})=\sum_{\mathfrak{T}\text{ tree on }\overline{Y}}\Big(\prod_{\overline{l}\in\mathfrak{T}}u_{\overline{l}}\int_0^1 dh_{\overline{l}}\Big)\exp(\sum_{\overline{l}\in\overline{Y}}h^{\mathfrak{T}}_{\overline{l}}(\mathbf{h})u_{\overline{l}}\Big) \tag{V.C.9}$$

so that $A(Y_i) = \overline{A}(\overline{Y}_i)$ for every $i \in I_s$. Now define the (unordered) set $\mathcal{O} = \mathcal{O}(Y_1, \ldots, Y_s; \mathfrak{T}) = \{\overline{Y}_1, \ldots, \overline{Y}_s\}$. It is easy to see that $\mathcal{O}$ is a non empty set of disjoint polymers ($|\overline{Y}| \geq 2$ for each $\overline{Y} \in \mathcal{O}$) lying in $\mathcal{D}_{n,m}$, with exactly one element in the ground level $I_n \times \{0\}$, furthermore, the labels k of the occupied levels $I_n \times \{k\}$ form an interval $\{0, 1, \ldots, q\}$, $q \leq |\mathcal{O}| - 1$. An $\mathcal{O}$ verifying these properties is called *admissible.*

We will sum over admissible $\mathcal{O}$'s in (V.C.3)

$$L_m = \sum_{\text{admissible } \mathcal{O}} \frac{1}{|\mathcal{O}|!} \left(\prod_{\overline{Y} \in \mathcal{O}} \overline{A}(\overline{Y}) \right) \sum_{\substack{(Y_1, \ldots, Y_s); \mathfrak{T} \\ \text{st } \mathcal{O} = \mathcal{O}(Y_1, \ldots, Y_s; \mathfrak{T})}} \mathcal{C}(s; Y_1, \ldots, Y_s; \mathfrak{T}) \ . \tag{V.C.10}$$

Knowing the sequence $Y_1, \ldots, Y_s)$, $\mathfrak{T}$ naturally induces a tree on $\mathcal{O}$, $\overline{\mathfrak{T}} = \overline{\mathfrak{T}}(Y_1, \ldots, Y_s; \mathfrak{T})$ by the rule $\{ij\} \in \mathfrak{T}$ if and only if $\{\overline{Y}_i, \overline{Y}_j\} \in \overline{\mathfrak{T}}$. The tree $\overline{\mathfrak{T}}$ has a natural root i.e. the unique $\overline{Y}$ of $\mathcal{O}$ lying in the 0-th level, moreover, for each $\overline{Y}$ the label of the level it belongs to is just its height in the tree $\overline{\mathfrak{T}}$ with the mentioned choice of root. Such a tree will also be called *admissible.*

Let for any tree $\overline{\mathfrak{T}}$ on $\mathcal{O}$

$$U(\mathcal{O}, \overline{\mathfrak{T}}) = \prod_{L \in \overline{\mathfrak{T}}} \int_0^1 dw_L \left(\prod_{\{\overline{Y}\ \overline{Y}'\} \in \overline{\mathfrak{T}}} \left(\sum_{(\overline{b}, \overline{b}') \in \overline{Y} \times \overline{Y}'} (-\delta_{\{\overline{b}\ \overline{b}'\}}) \prod_{\substack{(\overline{c}, \overline{c}') \in \overline{Y} \times \overline{Y}' \\ (\overline{c}, \overline{c}') \neq (\overline{b}, \overline{b}')}} (1 - w^{\overline{\mathfrak{T}}}_{\{\overline{Y}\ \overline{Y}'\}}(\mathbf{w}) \delta_{\{\overline{c}\ \overline{c}'\}}) \right) \right)$$

$$. \prod_{\substack{\{\overline{Y}\ \overline{Y}'\} \subset \mathcal{O} \\ \{\overline{Y}\ \overline{Y}'\} \notin \overline{\mathfrak{T}}}} \prod_{(\overline{b}, \overline{b}') \in \overline{Y} \times \overline{Y}'} (1 - w^{\overline{\mathfrak{T}}}_{\{\overline{Y}\ \overline{Y}'\}}(\mathbf{w}) \delta_{\{\overline{b}\ \overline{b}'\}}) \ . \tag{V.C.11}$$

and

$$N(\mathcal{O}, \overline{\mathfrak{T}}) = \sum_{\substack{(Y_1, \ldots, Y_s); \mathfrak{T} \\ \text{st } \mathcal{O} = \mathcal{O}(Y_1, \ldots, Y_s; \mathfrak{T}) \\ \text{and } \overline{\mathfrak{T}} = \overline{\mathfrak{T}}(Y_1, \ldots, Y_s; \mathfrak{T})}} 1 \ . \tag{V.C.12}$$

It is clear now that

$$L_m = \sum_{\text{admissible } \mathcal{O}} \frac{1}{|\mathcal{O}|!} \left(\prod_{\overline{Y} \in \mathcal{O}} \overline{A}(\overline{Y}) \right) \sum_{\overline{\mathfrak{T}} \text{tree on } \mathcal{O}} U(\mathcal{O}, \overline{\mathfrak{T}}).N(\mathcal{O}, \overline{\mathfrak{T}}) \ . \tag{V.C.13}$$

Let us admit for the moment the following

Lemma V.C.1 *The combinatoric factor $N(\mathcal{O}, \overline{\mathfrak{T}})$ is $(|\mathcal{O}| - 1)!$ for any admissible $\mathcal{O}$ and $\overline{\mathfrak{T}}$.*

Then

$$L_m = \sum_{\substack{\text{admissible } \mathcal{O} \\ \overline{\mathfrak{T}} \text{ tree on } \mathcal{O}}} \frac{1}{|\mathcal{O}|} \left(\prod_{\overline{Y} \in \mathcal{O}} \overline{A}(\overline{Y}) \right) \sum_{\overline{\mathfrak{T}} \text{ tree on } \mathcal{O}} U(\mathcal{O}, \overline{\mathfrak{T}}) \quad . \tag{V.C.14}$$

Besides,

$$\prod_{\overline{Y} \in \mathcal{O}} \overline{A}(\overline{Y}) = \sum_{\mathfrak{F}_1} \left(\prod_{\overline{l} \in \mathfrak{F}_1} u_{\overline{l}} \int_0^1 dh_{\overline{l}} \right) \exp\left(\sum_{\overline{l}} h_{\overline{l}}^{\mathfrak{F}_1}(\mathbf{h}) u_{\overline{l}} \right) \quad , \tag{V.C.15}$$

where we sum over all forests $\mathfrak{F}_1$ made of horizontal links $u_{\overline{l}}$ such that the connected components that are not singletons are exactly the elements of $\mathcal{O}$.

Finally remark that summing over trees $\overline{\mathfrak{T}}$ on $\mathcal{O}$ and, for every $\{\overline{Y}, \overline{Y}'\} \in \overline{\mathfrak{T}}$, over $(\overline{b}, \overline{b}') \in \overline{Y} \times \overline{Y}'$ is the same as summing over forests $\mathfrak{F}_2 \backslash \mathfrak{F}_1$ of vertical links $-\delta_{\overline{l}}$ connecting in a tree the clusters $\overline{Y} \in \mathcal{O}$ formed by $\mathfrak{F}_1$. It is now a matter of (rather unpleasant) checking the definitions to convince oneself that (V.C.14) is just the result of formula (V.C.1), except that the clusters are confined to $\mathcal{D}_{n,m}$. However taking the limit $m \to +\infty$ removes this restriction. ■

Proof of lemma V.C.1 :

Since $\mathcal{O}$ and $\overline{\mathfrak{T}}$ are admissible, there exists a sequence $Y_1, \ldots, Y_s$ and a tree $\mathfrak{T}$ such that $\mathcal{O} = \mathcal{O}(Y_1, \ldots, Y_s; \mathfrak{T})$ and $\overline{\mathfrak{T}} = \overline{\mathfrak{T}}(Y_1, \ldots, Y_s; \mathfrak{T})$. In fact suppose we are given a one-to-one map $\sigma : \mathcal{O} \mapsto I_s$ such that $\sigma(\overline{Y}_1) = 1$, where $\overline{Y}_1$ is the unique element of $\mathcal{O}$ lying in the ground level of $\mathcal{D}_n$. We can construct a sequence $(Y_1^\sigma, \ldots, Y_s^\sigma)$ and a tree $\mathfrak{T}^\sigma$ fulfilling the previous requirements. If π denotes the projection of $\mathcal{D}_n$ on I_n, we let $Y^\sigma_{\sigma(\overline{Y})} = \pi(\overline{Y})$ for all $\overline{Y} \in \mathcal{O}$, and $\mathfrak{T}^\sigma$ be the tree on I_s induced by $\overline{\mathfrak{T}}$ through the correspondence σ.

It is easy to see that every pair $\big((Y_1, \ldots, Y_s); \mathfrak{T}\big)$ we are counting is obtained that way. The interesting thing is that there is a unique σ giving this pair. If σ and τ are two maps such that $\big(({Y_1}^\sigma, \ldots, {Y_s}^\sigma); \mathfrak{T}^\sigma\big) = \big(({Y_1}^\tau, \ldots, {Y_s}^\tau); \mathfrak{T}^\tau\big)$, $\sigma \circ \tau^{-1}$ must preserve the tree $\mathfrak{T}^\sigma = \mathfrak{T}^\tau$, but $\sigma \circ \tau^{-1}(1) = 1$ so it preserves also the root, as a consequence it leaves the layers invariant. But, the corresponding $\overline{Y}_i$ are disjoint contained in the same level of $\mathcal{D}_n$, thus the projections Y_i are distinct and $Y_{\sigma \circ \tau^{-1}(i)} = Y_i$ forces $\sigma \circ \tau^{-1}(i) = i$ inside each layer. In conclusion $\sigma \circ \tau^{-1} = Id$. $N(\mathcal{O}, \overline{\mathfrak{T}})$ is the number of σ's that is $(s-1)!$. ■

If after all that the reader is somewhat sceptical about formula (V.C.1), he might find it amusing to check it for $n = 2$.

Now to use formula (V.C.1) for more clever applications than computing the trivial expression $\log \exp(\sum u_l)$, we state the analogous result for $\log Z(\Lambda)$ in ϕ^4 theory. We introduce a copy of the field ϕ_k for each $k \in \mathbb{N}$, and write Φ for the system $(\phi_k)_{k\in\mathbb{N}}$. Then we define the interaction in $S(\mathfrak{F}_2)$, the support of $\mathfrak{F}_2$, as

$$e^{I(S(\mathfrak{F}_2))}(\Phi) = \prod_{(b,k)\in S(\mathfrak{F}_2)} e^{-\lambda \int_b \phi_k^4(x)dx} \tag{V.C.16}$$

where we have identified the boxes b in Λ with their labels in I_n. To a link $\bar{l} = \{(b,k),(b',k')\}$ we associate the operator

$$C_{\bar{l}} = \delta_{kk'} \int_b dx \int_{b'} dy\ C(x,y) \frac{\delta}{\delta\phi_k(x)} \frac{\delta}{\delta\phi_{k'}(x)} \tag{V.C.17}$$

that will play the role of $u_{\bar{l}}$. $\delta_{\bar{l}}$ is defined in the same fashion as before. Given a 2-jungle $\mathcal{F} = (\mathfrak{F}_1, \mathfrak{F}_2)$ denote by $S_k(\mathfrak{F}_2)$ the k-th slice $S(\mathfrak{F}_2) \cap (I_n \times \{k\})$ of $S(\mathfrak{F}_2)$. We define the following Gaussian measure on the fields $\phi_{k|S_k(\mathfrak{F}_2)}$ such that k is an occupied level of $\mathfrak{F}_2$ (i.e. $S_k(\mathfrak{F}_2) \neq \emptyset$)

$$d\mu_C^{\mathcal{F},\mathbf{h},\mathbf{w}}(\Phi_{|S(\mathfrak{F}_2)}) = \prod_{\text{occupied } k} d\mu_C^{\mathcal{F},\mathbf{h},\mathbf{w},k}(\phi_{k|S_k(\mathfrak{F}_2)}) \tag{V.C.18}$$

where $d\mu_C^{\mathcal{F},\mathbf{h},\mathbf{w},k}(\phi_{k|S_k(\mathfrak{F}_2)})$ has covariance $C^{\mathcal{F},\mathbf{h},\mathbf{w},k}(x,y) = C(x,y)$ if x and y fall in the same b with $(b,k) \in S_k(\mathfrak{F}_2)$; and $C^{\mathcal{F},\mathbf{h},\mathbf{w},k}(x,y) = C(x,y).h^{\mathcal{F},1}_{\{(b,k),(b',k')\}}(\mathbf{h},\mathbf{w})$ if $x \in b$, $y \in b'$, $b \neq b'$ $(b,k) \in S_k(\mathfrak{F}_2)$ and $(b',k) \in S_k(\mathfrak{F}_2)$.

We can now state a formula which performs a cluster and a Mayer expansion in a single move (recall that $|\Lambda| = n$)

Theorem V.C.2

$$\log Z(\Lambda) = n \log A_0 + \sum_{\substack{\mathcal{F}=(\mathfrak{F}_1,\mathfrak{F}_2)\\ \text{nice 2-jungle}}} \frac{A_0^{-|S(\mathfrak{F}_2)|}}{k(\mathcal{F})} \Big(\prod_{\bar{l}\in\mathfrak{F}_1} \int_0^1 dh_{\bar{l}}\Big)\Big(\prod_{\bar{l}\in\mathfrak{F}_2\setminus\mathfrak{F}_1} (-\delta_{\bar{l}}) \int_0^1 dw_{\bar{l}}\Big)$$
$$. \prod_{\bar{l}\notin\mathfrak{F}_2\setminus\mathfrak{F}_1} (1 - w_{\bar{l}}^{\mathcal{F},2}(\mathbf{h},\mathbf{w})\delta_{\bar{l}}) \int d\mu_C^{\mathcal{F},\mathbf{h},\mathbf{w}}(\Phi_{|S(\mathfrak{F}_2)}) \Big(\prod_{\bar{l}\in\mathfrak{F}_1} C_{\bar{l}}\Big) e^{I(S(\mathfrak{F}_2))} \quad . \tag{V.C.19}$$

Proof Use first $\log Z(\Lambda) = n \log A_0 + \log Z_r(\Lambda)$, then repeat exactly the same line of arguments than for Theorem V.C.1. ■

From Theorem V.C.2 one can construct a series for the pressure $\lim_{\Lambda\to\infty} |\Lambda|^{-1} \log Z(\Lambda)$ which is the sum of the trivial term $\log A_0$ plus a sum over nice 2 jungles that extend horizontally over the infinite lattice of all cubes covering $\mathbb{R}^d$, and such that the well defined unique root of $\mathfrak{F}_2$ is the particular cube containing the origin. (This requires to neglect boundary terms of order $|\Lambda|^{(d-1)/d}$ in $\log Z(\Lambda)$). For small λ this series is absolutely convergent.

D) The resolvent expansion

In many situations physical situations (polymers, disordered systems) we want to compute not a full theory but a single Green's function which is expressed as the inverse of some operator. This mathematical situation is formally equivalent to 0-component field theory (or to supersymmetric theories) in which the usual normalizing fermionic or bosonic determinants have been cancelled out.

Consider a finite dimensional operator M on $\mathbb{R}^n$ with matrix elements m_{ij}, and a norm strictly smaller than 1. The operator $A = \frac{1}{1+M}$ is well defined through its power series expansion. Define $M(\mathbf{x})$ as the matrix with elements $x_{\{ij\}} m_{ij}$ if $i \neq j$ and diagonal elements m_{ii} equal to those of M, and $M_{\{ij\}}$ as the matrix with 0 elements except $(M_{\{ij\}})_{ij} = m_{ij}$ and $(M_{\{ij\}})_{ji} = m_{ji}$. The Taylor forest formula (III.2) applied to the operator $H(\mathbf{x}) = \frac{1}{1+M(\mathbf{x})}$ gives

$$A = \frac{1}{1+M} = H(\mathbf{1}) = \sum_{\mathfrak{F}} \Big(\prod_{l\in\mathfrak{F}} \int_0^1 dh_l \Big) \Big(\prod_{l\in\mathfrak{F}} \oint \frac{1}{2\pi i} \frac{dx_l}{x_l^2} \Big) \frac{1}{1 + M(X_{\mathfrak{F}}(\mathbf{h})) + \sum_{l\in\mathfrak{F}} x_l M_l} \tag{V.D.1}$$

where the action of multiple derivatives is for compactness rewritten as a multiple Cauchy integral, and the analyticity radii in (V.D.1) are small enough. This does not look like a very clever rewriting, but it allows to reblock the power series for $\frac{1}{1+M}$ according to the set of sites truly visited in this power series. In particular if we impose a given entry to the matrix A the forest formula (V.D.1) reduces to a tree formula:

$$A_{ij} = \Big(\frac{1}{1+M} \Big)_{ij} = \sum_{\substack{Y\subset\{1,\dots,n\} \\ i\in Y,\ j\in Y}} \sum_{\mathfrak{T} \text{ tree on } Y} \Big(\prod_{l\in\mathfrak{T}} \int_0^1 dh_l \Big)$$

$$\Big(\prod_{l\in\mathfrak{T}} \oint \frac{1}{2\pi i} \frac{dx_l}{x_l^2} \Big) \frac{1}{1_Y + M_Y(X_{\mathfrak{T}}(\mathbf{h})) + \sum_{l\in\mathfrak{T}} x_l M_{Y,l}} \tag{V.D.2}$$

where 1_Y is the identity on R^Y, $M_Y(X_{\mathfrak{T}}(\mathbf{h}))$ is the matrix on R^Y with diagonal entries m_{ii} (for $i \in Y$) and off diagonal entries $h_{ij}^{\mathfrak{T}}(\mathbf{h})m_{ij}$ (for $i \in Y$, $j \in Y$), and $M_{Y,l}$ is the restriction of M_l to indexes in Y.

This kind of formulas, eventually in combination with a large field/small field analysis should be useful for interacting random walks [IM] and presumably for the study of disordered systems.

E) Cluster expansions with large/small field conditions: the $m = 2$ jungle.

Jungle formulas with several levels are interesting when there are various type of links with priority rules between them. The simplest example is a single scale cluster expansion but with so-called small/large field conditions [R2], in which one does *not* want to derive links between cubes of a large field connected component. Indeed large field regions are small for probabilistic reasons, but the vertices of perturbation theory (created by the action of functional derivatives such as those in (V.A.8)) may not be small there, hence it would be dangerous to blindly expand the clusters in the usual way.

We want to study a theory with partition function such as

$$Z(\Lambda) = \sum_{\Gamma \subset \Lambda} Z(\Lambda, \Gamma) \ , \quad Z(\Lambda, \Gamma) = \int d\mu_C(\phi) \chi_\Gamma(\phi) e^{-g\int_\Lambda I(\phi)dx} \ , \tag{V.E.1}$$

where C is a propagator with decay at the unit scale, and $I(\phi)$ is some local interaction. Without describing a precise model we shall assume that the large field condition $\chi_\Gamma(\phi)$ is such that the functional integral $\int d\mu_C(\phi)\chi_\Gamma(\phi)e^{-g\int_\Lambda I(\phi)dx}$ can be bounded by $K^{|\Lambda-\Gamma|}c^{|\Gamma|}$ where K is fixed and c is a small constant that can tend to zero with some coupling constant in $I(\phi)$; one also assumes that the outcome of a functional derivative $\int_b dx \frac{\delta}{\delta\phi(x)}$ acting on $\chi_\Gamma(\phi)e^{-g\int_\Lambda I(\phi)dx}$ gives a small factor but only in the "small field region", hence if $b \notin \Gamma$.

The set I_n is again the set of the cubes which pave our finite volume Λ. We fix a given subset $\Gamma \subset I_n$ and define, for $l = \{ij\} \in \mathcal{P}_n$, $\epsilon_l^\Gamma = 1$ if $b_i \in \Gamma$, $b_j \in \Gamma$ and $\mathrm{dist(b_i, b_j} \le M$. M is some constant that will be fixed to a large value. Otherwise we put $\epsilon_l^\Gamma = 0$. We define also $\eta_l^\Gamma = 1 - \epsilon_{ij}^\Gamma$. Hence $\epsilon_l^\Gamma = 1$ means that b_i and b_j are "large field cubes" which are closer than distance M, and $\eta_l^\Gamma = 1$ means the contrary.

The level two jungle formula will at the first level create connections whose clusters are automatically the "connected components" of Γ in the generalized sense that up to distance M two cubes are considered connected. Then the second level will create ordinary connections with the propagator C. Therefore we introduce a first set of interpolation parameters $\{x_l^1\}$ and define

$$H^\Gamma(\{x_l^1\}) = \prod_{l\in\mathcal{P}_n} (x_l^1 \epsilon_l^\Gamma + \eta_l^\Gamma) \ . \tag{V.E.2}$$

Remark that $H^\Gamma(1,...,1) = 1$, so that we can freely multiply $Z(\Lambda,\Gamma)$ by $H^\Gamma(1,...,1)$ without changing its value. We introduce the second set of parameters $\{x_l^2\}$ on the covariance C exactly as in (V.A.5).

Then the formula (IV.11) with $m = 2$ simply gives:

$$Z(\Lambda,\Gamma) = \sum_{\substack{\mathcal{F}=(\mathfrak{F}_1,\mathfrak{F}_2)\\ 2-\text{jungle}}} \Big(\prod_{l\in\mathcal{F}} \int_0^1 dh_l\Big)\Big(\prod_{l\in\mathfrak{F}_1} \epsilon_l^\Gamma\Big)\Big(\prod_{l\notin\mathfrak{F}_1} \big(\eta_l^\Gamma + \epsilon_l^\Gamma h_l^{\mathcal{F},1}(\mathbf{h})\big)\Big)$$

$$\int d\mu_{C_{\mathcal{F}}(\mathbf{h})}(\phi) \prod_{l=\{ij\}\in\mathfrak{F}_2-\mathfrak{F}_1} \Big\{\int dxdy\chi_{b_i}(x)\chi_{b_j}(y)C(x,y)\frac{\delta}{\delta\phi(x)}\frac{\delta}{\delta\phi(y)}\Big\}\chi_\Gamma(\phi)e^{-g\int_\Lambda I(\phi)dx} \tag{V.E.3}$$

where $d\mu_{C_{\mathcal{F}}(\mathbf{h})}$ is the normalized Gaussian measure with positive covariance

$$C_{\mathcal{F}}(\mathbf{h})(x,y) = h_{l(x,y)}^{\mathcal{F},2}(\mathbf{h})C(x,y) \tag{V.E.4}$$

where $h_{l(x,y)}^{\mathcal{F},2}(\mathbf{h})$ is by definition 1 if x and y belong to the same cube, and is $h_l^{\mathcal{F},2}(\mathbf{h})$ if $l = \{ij\}$, $x \in b_i$ and $y \in b_j$ or $x \in b_j$ and $y \in b_i$.

The only non-zero terms in this formula are those for which the clusters associated to the forest $\mathfrak{F}_1$ are exactly the set of "connected components" Γ_a of the large field region in the generalized sense (for $M = 0$ it gives the ordinary connected components). Indeed they cannot be larger because of the factor $\prod_{l\in\mathfrak{F}_1}\epsilon_l^\Gamma$, nor can they be smaller because of the factor $\prod_{l\notin\mathfrak{F}_1}\big(\eta_l^\Gamma + \epsilon_l^\Gamma h_l^{\mathcal{F},1}(\mathbf{h})\big)$ which is zero if there are some generalized neighbors (for which $\eta_{ij}^\Gamma = 0$) belonging to different clusters (for which $h_{ij}^{\mathfrak{F}}(h) = 0$). Therefore the first forest in this formula simply automatically draws connecting trees of "neighbor links" connecting each such generalized connected component, but in a symmetric way without arbitrary choices. Remark that the factor $\prod_{l\notin\mathfrak{F}_1}\big(\eta_l^\Gamma + \epsilon_l^\Gamma h_l^{\mathcal{F},1}(\mathbf{h})\big)$ is bounded by one as expected for further estimates. Then the second forest of the jungle automatically takes into account the first links, i.e. the existence of large field regions, to draw propagators connections, In this way the units connected by the full forest remain unit cubes instead of being either small field cubes or blocks of large field cubes*. For convergence of the thermodynamic limit one has then simply to check that all connections are summable (this is obvious for the finite range nature of the ϵ_l^Γ links), and that there is a small factor per link of the forest. For ϵ links it comes from the probabilistic factor associated to the presence of the function χ_Γ and for ordinary links, it comes either from the functional derivative localized out of Γ, or from the large distance (at least M) crossed in the case of a link between two large field cubes. This distance induces a small factor through the decay of C. In conclusion there is no need to gain a small factor from functional derivatives localized in the large field region (which is usually impossible anyway), and the whole convergence becomes much more transparent, many combinatoric difficulties being hidden in the jungle formula itself.

A concrete example in which this formula would somewhat simplify the argument is e.g. the single scale expansion of [L]; in [KMR] a three level jungle formula is used, in which the third forest hooks some cubes along the straight paths of the second forest propagators, in order to complete factorization in the context of a Peierls contour argument.

F) Cluster or resolvent expansions with smooth localizations

In some situations (for instance if momentum conservation is important), it may be inconvenient to perform cluster expansions with sharp localization functions such as χ_b in (V.A.8). But with smooth functions there is no naïve factorization in (V.A.8). This is not a serious difficulty and it can be overcome e.g. by an auxiliary expansion (sometimes called a "painting expansion") on the interaction that creates protection belts around the clusters. But in this last section we remark that the Taylor forest formula (III.2) also gives elegant solutions to this type of problems and treat the ϕ^4 interaction again as an example.

Let

$$1 = \sum_b \chi_b^2(x) \tag{V.F.1}$$

* These blocks lead to unpleasant additional sums for where in the blocks functional derivatives really act, etc...

be a smooth partition of unity of $\mathbb{R}^d$ by cubes of unit size. We assume that the support of χ_b is contained in $\{x|\mathrm{dist}(x,b)\leq 1/3\}$. The set of all b's is then further restricted to the finite set $\mathcal{N}$ of all cubes which are at distance zero of our finite volume Λ (hence include a unit corridor around it). Therefore

$$\chi_\Lambda(x)=\chi_\Lambda(x)\sum_{b\in\mathcal{N}}\chi_b^2(x) \tag{V.F.2}$$

so that the corresponding sums are finite from now on.

Let us rewrite the ϕ^4 theory of section V.A in terms of an intermediate ultralocal field:

$$Z(\Lambda)=\int d\mu_C(\phi)e^{-g\int_\Lambda \phi^4(x)dx}=\int d\mu_{C_\Lambda}(\phi)d\nu(V)e^{i\sqrt{g}\int_\Lambda \phi^2(x)V(x)dx} \tag{V.F.3}$$

where $d\nu$ is the normalized Gaussian measure with ultralocal covariance $\delta(x-y)$.

Inserting the identity V.F.2 we get

$$Z(\Lambda)=\int d\mu_C(\phi)d\nu(V)e^{i\sqrt{g}\int_\Lambda dx\sum_{b\in\mathcal{N}}(\phi\chi_b)^2(x)V(x)} \tag{V.F.4}$$

We define now the collection of Gaussian random variables $\{\phi_b(x)\}$, $b\in\mathcal{N}$ distributed according to the measure $d\mu(\{\phi_b\})$ with covariance $C(b,x\,;b',y)=\chi_b(x)C(x,y)\chi_{b'}(y)$, and the collection of Gaussian random variables $\{V_b(x)\}$ distributed with degenerate covariance $\Gamma(b,x\,;b',y)=\delta(x-y)$. We have

$$Z(\Lambda)=\int d\mu(\{\phi_b\})d\nu(\{V_b\})e^{i\sqrt{g}\int_\Lambda dx\sum_{b\in\mathcal{N}}(\phi_b)^2(x)V_b(x)} \tag{V.F.5}$$

(to prove (V.F.5), remark that the right hand side of (V.F.4) and (V.F.5) are both Borel summable functions of g with identical perturbative series). We may now change slightly the boundary condition, dropping the restriction on the interaction range of integration. In other words $Z(\Lambda)$ leads to the same thermodynamic variables than

$$\bar{Z}(\Lambda)=\int d\mu(\{\phi_b\})d\nu(\{V_b\})e^{i\sqrt{g}\sum_{b\in\mathcal{N}}\int dx(\phi_b)^2(x)V_b(x)} \tag{V.F.6}$$

Let us apply the Taylor forest formula (III.2) to $\bar{Z}(\Lambda)$, interpolating both the covariances $C(b,x\,;b',y)$ and $\Gamma(b,x\,;b',y)$, viewed as finite matrices with entries in $\mathcal{N}$ whose elements are operators on $L^2(\mathbb{R}^d)$.

It gives a result very similar to (V.A.7-8):

$$\bar{Z}(\Lambda) = \sum_{\substack{Y_1,\dots,Y_n \\ Y_i\cap Y_j=\emptyset,\cup Y_i=\mathcal{N}}} \prod_{i=1}^{n} A(Y_i)$$

$$A(Y) = \sum_{T \text{ on } Y} (\prod_{l\in T}\int_0^1 dh_l) \int d\mu_{C_Y^T(\mathbf{h})}(\phi) d\nu_{\Gamma_Y^T(\mathbf{h})}(V)$$

$$\Big\{ \prod_{l=\{ij\}\in T} \int dxdy \Big(\chi_{b_i}(x)\chi_{b_j}(y)C(x,y)\frac{\delta}{\delta\phi_{b_i}(x)}\frac{\delta}{\delta\phi_{b_j}(y)} + \delta(x-y)\frac{\delta}{\delta V_{b_i}(x)}\frac{\delta}{\delta V_{b_j}(y)}\Big)\Big\}$$

$$e^{i\sqrt{g}\sum_{b\in Y}\int dx(\phi_b)^2(x)V_b(x)} \tag{V.F.7}$$

where b_i and b_j are the two ends of the line l, and the sum is over trees T which connect together the set Y, hence have exactly $|Y|-1$ elements. The measures $d\mu_{C_Y^T(\mathbf{h})}(\phi)$ and $d\nu_{\Gamma_Y^T(\mathbf{h})}(V)$ are normalized Gaussian measure on the restricted collections of Gaussian random variables $\{\phi_b(x)\}$ and $\{V_b(x)\}$ for $b\in Y$. $d\mu_{C_Y^T(\mathbf{h})}$ has covariance $C_Y^T(\mathbf{h})(b,x\,;b',y) = C(x,y)h_T(b,b')$, where $h^T(b,b)=1$ and for $b\neq b'$, $h^T(b,b')$ is the infimum of all parameters h_l for l in the unique path in T joining the cube b to the cube b'. Similarly $d\nu_{\Gamma_Y^T(\mathbf{h})}(V)$ has covariance $\Gamma_Y^T(b,x\,;b',y) = \delta(x-y)h_T(b,b')$.

Remark that the difference between (V.A.8) and (V.F.7) appears in the term $\delta(x-y)\frac{\delta}{\delta V_{b_i}(x)}\frac{\delta}{\delta V_{b_j}(y)}$ which would be zero if the functions χ_b were sharp characteristic functions. This term corresponds to "emptying" the interaction on the borders of the clusters created so that full factorization occurs, even without sharp localization functions.

Similar formulas can be written for other kind of interactions or for resolvent expansions with smooth localization functions. We leave them to the reader.

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