Field Redefinition Invariance in Quantum Field Theory

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Abstract

We investigate the consequences of field redefinition invariance in quantum field theory by carefully performing nonlinear transformations in the path integral. We first present a "paradox" whereby a $1+1$ free massless scalar theory on a Minkowskian cylinder is reduced to an effectively quantum mechanical theory. We perform field redefinitions both before and after reduction to suggest that one should not ignore operator ordering issues in quantum field theory. We next employ a discretized version of the path integral for a free massless scalar quantum field in d dimensions to show that beyond the usual jacobian term, an infinite series of divergent "extra" terms arises in the action whenever a nonlinear field redefinition is made. The explicit forms for the first couple of these terms are derived. We evaluate Feynman diagrams to illustrate the importance of retaining the extra terms, and conjecture that these extra terms are the exact counterterms necessary to render all physical quantities invariant under field redefinitions. As a byproduct, we see explicitly how these extra terms are essential to understanding why the unphysical practice of dimensional regularization works at all. We indicate how the extra counterterms cancel out unwanted divergent contributions to physical quantities so that the result is *consistent* with simply setting divergences such as $\delta^{(d)}(0)$ equal to zero in evaluations of Feynman diagrams. An exciting possibility is that these extra terms could allow for the presence of anomalies of higher orders in $\hbar$ in quantum field theories with nonlinear symmetries.

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1 Introduction

Perhaps the most primitive of all properties of a quantum field theory is field redefinition invariance. Indeed, experiments yield measurements and these numbers relative to some scale have meaning, but the numbers in no way imply a particular choice of fields. One could go further to argue that in fact although theoretical calculations using an effective quantum field theory yield viable results, there is no proof that another framework (perhaps not even a quantum field theory) could not yield similar or equivalent results.

Under field redefinitions, physically meaningful quantities must remain unchanged. For example, one expects the poles of renormalized propagators to remain the same under a field redefinition. On the other hand, quantities related to choice of fields, such as wave function renormalization factors, are physically insignificant and may change.

In this paper we investigate the consequences of field redefinition invariance in the context of quantum field theories, using as our basic method of attack the consideration of nonlinear point canonical transformations in the Feynman path integral. Our analysis requires a careful handling of the path integral under nonlinear change of variables, and we contend that in spite of the path integral's ubiquitous employment, manipulations involving the path integral must be reevaluated [1].

In the case of quantum mechanics, it has been known for some time that $\mathcal{O}(\hbar^2)$ terms arise in the path integral action upon performing nonlinear point canonical transformations of the coordinates. From the lagrangian point of view this phenomenon is a manifestation of the stochastic nature of the path integral, and is studied from a discretization of the quantum mechanical path integral. It is easy to understand why these so-called "extra" terms are generated since upon quantization, the classical coordinates become quantum operators, and therefore there are nontrivial issues of operator ordering. In fact, it may be seen directly from the hamiltonian point of view that these extra terms are generated by operator ordering [2].

In contrast to the case of quantum mechanics, the standard lore in quan-

[1] We will avoid completely mathematically rigorous questions of existence of path integrals in continuum quantum field theory, and imagine an underlying spacetime lattice with arbitrarily small spacings.

[2] A given prescription for handling the discretized path integral corresponds to a particular operator ordering of the associated hamiltonian. In particular, the midpoint prescription corresponds to Weyl ordering, and in the coordinate representation supports the interpretation of Brownian motion. For a nice review of the Feynman path integral in quantum mechanics, see the work of Grosche [4].

tum field theory dictates that under a field redefinition, the action changes by direct substitution of the change of variables in the action together with inclusion of the jacobian determinant of the transformation. Furthermore, when dimensional regularization is employed, the jacobian determinant does not contribute since upon exponentiation the formally infinite spacetime delta function $\delta^{(d)}(0)$ generated by the trace is set equal to zero. Similarly, there is also a standard argument that any extra terms generated in the path integral, since they are manifestations of operator ordering (i.e. from $[\pi(x), \phi(x)] = -i\hbar\delta^{(d-1)}(0)$), would involve delta functions at zero argument and therefore would vanish by dimensional regularization. The usual justification for ignoring these terms is loosely as follows. One presupposes the existence and necessity of local counterterms in the action, so that jacobians and any other terms resulting from operator ordering, which are local quantities in the action, have the effect of just changing the coefficients of these local counterterms. The argument presumes that one is not interested in the initial values of the counterterms, and therefore there is no need to worry about keeping track of extra terms that could conceivably arise.

Even if the validity of dimensional regularization is not questioned, a solid justification for setting infinite quantities equal to zero is lacking. Certainly from the lattice point of view, the jacobian term, as well as any extra term, is very real and does not vanish since the spacetime delta function at zero argument is just a power of the inverse lattice spacing i.e., $\delta^{(d)}(0) \sim a^{-d}$. Dimensional regularization does in fact fail in cases where the action has some feature depending on the dimensionality of spacetime. Examples are theories with chiral fermions, conformal symmetries, and theories with nontrivial spacetime topology.

Our investigation reveals that upon making nonlinear field redefinitions in the quantum field theory path integral extra terms do appear. In the generic case, an infinite number of divergent "extra" terms appear. We give a detailed account of the method of derivation of these terms and give the explicit form for the first couple of terms. We conjecture that this series of "extra" terms gives the precise counterterms necessary to ensure that physically significant quantities remain unchanged under field redefinitions. Furthermore, inclusion of these terms allows one to understand what is really going on in the procedure of setting $\delta^{(d)}(0)$ and other divergent quantities to zero in dimensional regularization. Especially when dimensional regularization is not applicable, one must take care to include any divergent extra terms that may arise when manipulating fields in the path integral. An important application of this work could possibly be the existence of quantum anomalies of higher orders in $\hbar$ in theories with nonlinear symmetries.

This paper is organized as follows. In section 2, we present a simple "paradox" where quantum field theory meets quantum mechanics, which suggests the existence of extra terms in the quantum field theory action upon change of field variables. We consider the dimensional reduction of a free, massless $1+1$ dimensional real scalar quantum field theory to an effectively quantum mechanical theory, and perform field redefinitions both before and after reduction. In section 3, we investigate the problem of "extra" terms in quantum field theory more directly by making general field redefinitions in a discretized version of the path integral in a massless non-interacting scalar quantum field theory in flat d-dimensional spacetime. We show that an infinite series of extra terms arise upon performing a nonlinear field transformation in the path integral, and give explicit expressions for the first couple of extra terms. In section 4, we turn to Feynman diagrams to calculate corrections to the propagator to the lowest couple of orders in the field redefinition parameter. We discuss the important role of the extra terms derived in the previous section. We conjecture that the extra terms ensure that the pole of the renormalized scalar propagator remains unchanged to all orders, while only the wave function gets renormalized. These calculations enable one to see explicitly that the extra terms are essential to understanding why the unphysical practice of dimensional regularization works at all. Conclusions and further research questions are given in the final section.

2 Kaluza-Klein "Paradox"

An effective laboratory in which to learn about nonlinear point canonical transformations in quantum field theory is that of a 1+1 dimensional real scalar field on a Minkowskian cylinder. By integrating out the angular coordinate one arrives at an effectively quantum mechanical problem, and then the well-established results for nonlinear point canonical transformations in quantum mechanics may be employed. On the other hand, one may use the standard lore to make a nonlinear field redefinition directly in the 1+1 dimensional quantum field theory, and then afterwards integrate out the angular coordinate to arrive again to a quantum mechanics action. One expects that the two approaches should give the same result, so if it is true that no new terms appear in the latter scenario when one performs a field redefinition in the quantum field theory, one must be able to explain what happens to the extra terms that arise along the way in the former scenario. In the following we consider a real massless scalar field $\phi(x,t)$ in a flat $1+1$ Minkowskian spacetime with a periodic spatial coordinate

$$S = -\frac{1}{2}\int d^2x \;\, \eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{2.1}$$

where $\eta_{tt} = 1$, $\eta_{xx} = -1$ and $x \cong x + 2\pi R$.

2.1 Method 1

Using an angular coordinate $\theta = x/R$, the action is

$$S = R\int d\theta dt \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{2R^2}(\partial_\theta\phi)^2\right]. \tag{2.2}$$

One may convert this quantum field theory into an effectively quantum mechanical problem by expanding the scalar field in a Kaluza-Klein-like decomposition

$$\phi(\theta,t) = \sum_{p\in Z} a^{(p)}(t)e^{ip\theta} \tag{2.3}$$

and integrating out the θ dependence in the action. The resulting action is that of a sum of an infinite number of quantum mechanical complex oscillators $a^{(m)}(t)$ with frequencies $\omega_m = \frac{m}{R}$ dependent on mode number m and radius R of the internal dimension. Explicitly,

$$S_0[a] = \sum_{p\in Z}\int dt \left[\sum_{m\in Z}\frac{1}{2}g_{mp}[a]\dot{a}^{(m)}\dot{a}^{(p)} - \frac{1}{2}\frac{p^2}{R^2}g_{mp}[a]a^{(m)}a^{(p)}\right] \tag{2.4}$$

where the metric

$$g_{mp}[a] = 2\pi R\delta_{m,-p} = 2\pi R\delta_{m+p=0}$$

is an infinite-dimensional unit matrix multiplied by $2\pi R$. We may consider a "cutoff" version of this theory, wherein only modes with $|n| \leq N_{\max}$ are included. In this case, $g_{mp}[a]$ is an ordinary $2N_{\max}+1$ by $2N_{\max}+1$ metric defining a Euclidean space of oscillators $a(t)$.

Let us perform a change of variables in this quantum mechanical model. In particular, we wish to make the change of variables corresponding to

$$\phi = F[\varphi] \equiv \varphi + \alpha\varphi^N. \tag{2.5}$$

We expand the new field $\varphi(\theta,t)$ in modes

$$\varphi(\theta,t)=\sum_{p\in Z} b^{(p)}(t)e^{ip\theta}, \tag{2.6}$$

and define the following useful objects

$$\begin{aligned} E_L^{(\ell)}[b] &\equiv \sum_{p_1\in Z}\cdots\sum_{p_L\in Z} b^{(p_1)}\ldots b^{(p_L)}\delta^{\ell-\sum_{i=1}^L p_i=0} && (2.7)\\ B_L[b] &\equiv E_L^{(0)}[b]=\int\frac{d\theta}{2\pi}\varphi(\theta,t)^L, && (2.8)\end{aligned}$$

where $\delta^{\ell=0}$ is one when $\ell=0$ and is zero if $\ell\neq 0$. Note $E_L^{(\ell)}[b]$ involves the product of L b's whose mode numbers add up to ℓ, and $B_L[b]$ involves L b's whose overall sum of mode numbers vanishes. A number of identities, the simplest being

$$\begin{aligned} \sum_{p\in Z} E_M^{(\ell+p)}[b]E_L^{(-p+m)}[b] &= E_{M+L}^{(\ell+m)}[b] && (2.9)\\ \frac{\partial}{\partial b^{(m)}}E_M^{(\ell)}[b] &= ME_{M-1}^{(\ell-m)}[b] && (2.10)\end{aligned}$$

greatly facilitate subsequent calculations. Appendix A contains useful identities and their proofs. The transformation of the modes corresponding to the field redefinition above in equation 2.5 is

$$a^{(p)}\equiv f^{(p)}[b]=b^{(p)}+\alpha E_N^{(p)}[b]. \tag{2.11}$$

To implement this change of coordinates in the quantum mechanical path integral, we must include not only the naive substitution of change of variables in the action and the jacobian determinant, but also the extra term due to the stochastic nature of the path integral. Specifically, after exponentiating the usual jacobian factor, the action becomes

$$S[b]=S_0[b]+S_{\text{jacobian}}[b]+S_{\text{extra}}[b] \tag{2.12}$$

where $S_0[b]$ is the action obtained by direct substitution into the original free action given in equation 2.4, and where the extra term of $\mathcal{O}(\hbar^2)$ is

$$S_{\text{extra}}[b]=-\hbar^2\int dt\sum_{k,\ell,p,n\in Z}\frac{1}{8}g^{\ell n}[b]\Gamma^k_{\ell p}[b]\Gamma^p_{nk}[b]. \tag{2.13}$$

A modified version of the classic derivation of this extra term, which is written in terms of the metric and connection in the new coordinates, is reviewed in section 3.1. It is possible to compute all three contributions to the action $S[b]$ exactly to all orders in the parameter α. The metric in the new coordinates is given by

$$g_{mn}[b] = 2\pi R\left(\delta_{m+n=0} + 2\alpha N E_{N-1}^{(-m-n)}[b] + N^2\alpha^2 E_{2N-2}^{(-m-n)}[b]\right), \qquad (2.14)$$

and the inverse metric, which may be obtained via a recursion equation (see appendix B), is given by

$$g^{km}[b] = \frac{1}{2\pi R}\sum_{j=0}^{\infty}(-\alpha N)^j (j+1) E_{j(N-1)}^{(k+m)}[b]. \qquad (2.15)$$

In terms of this metric, direct substitution of the nonlinear change of variables into the original action yields

$$S_0[b] = \sum_{p\in Z}\int dt\left[\frac{1}{2}\sum_{m\in Z} g_{mp}[b]\dot{b}^{(m)}\dot{b}^{(p)} - \frac{1}{2}\frac{p^2}{R^2}2\pi R f^{(p)}[b] f^{(-p)}[b]\right], \qquad (2.16)$$

where $f^{(p)}[b]$ was given in equation 2.11. The jacobian factor may be evaluated (see appendix B) to give a contribution to the action of

$$\begin{aligned} S_{\text{jacobian}}[b] &= -\frac{i\hbar}{2}\text{Trln} g_{mn}[b]\delta(t-t') \\ &= i\hbar\delta(0)\left[\sum_{k\in Z}1\right]\sum_{j=1}^{\infty}\frac{(-\alpha N)^j}{j}\int dt B_{j(N-1)}, \end{aligned} \qquad (2.17)$$

where we have dropped an infinite constant $-i\hbar\delta(0)\ln[2\pi R][\sum_{k\in Z}1]/2$. Finally, the extra term arising from the stochastic nature of the path integral may be evaluated (see appendix B) to give a contribution to the action of

$$\begin{aligned} S_{\text{extra}}[b] &= -\frac{\hbar^2}{8}\frac{(N-1)^2}{2\pi R}\left[\sum_{k\in Z}1\right]^2\sum_{X=0}^{\infty}(-\alpha N)^{X+2}\begin{pmatrix} X+3 \\ 3\end{pmatrix} \\ &\quad \times\int dt\ B_{X(N-1)+2(N-2)}[b]. \end{aligned} \qquad (2.18)$$

Note the presence of the infinite sum over modes in equation 2.17 and the same sum squared in equation 2.18.

2.2 Method 2

Alternately, we may choose to perform the field redefinition in the original quantum field theory, and then afterwards expand in the Kaluza Klein modes. Upon direct substitution, the field redefinition of equation 2.5 introduces derivative interaction terms into the action

$$S_0[\varphi] = -\frac{1}{2}\int d^2x\ \eta^{\mu\nu}(1+2N\alpha\varphi^{N-1}+N^2\alpha^2\varphi^{2(N-1)})\partial_\mu\varphi(x)\partial_\nu\varphi(x). \quad (2.19)$$

The exponentiated jacobian determinant gives a contribution to the action

$$S_{\text{jacobian}}[\varphi] = -i\hbar\delta^{(2)}(0)\int d^2x\ \ln[1+N\alpha\varphi^{N-1}]. \quad (2.20)$$

Assuming the conventional arguments hold would imply that the total action is given by $S_0[\varphi] + S_{\text{jacobian}}[\varphi]$. We now convert this action to an effectively quantum mechanical one by integrating out the angular coordinate. The terms in the action for the b modes obtained from method 2 will be written with tildes to distinguish them from those of method 1. Upon integrating out the θ dependence, $S_0[\varphi]$ becomes

$$\tilde{S}_0[b] = \int dt \sum_{m\in Z}\sum_{p\in Z}\left[\frac{1}{2}g_{mp}[b]\dot{b}^{(m)}\dot{b}^{(p)} + \frac{mp}{2R^2}g_{mp}[b]b^{(m)}b^{(p)}\right] \quad (2.21)$$

where $g_{mp}^{(b)}$ is exactly the metric appearing in the previous section. Similarly, expanding the logarithm and using equation 2.8, one obtains the expression for the jacobian

$$\tilde{S}_{\text{jacobian}}[b] = 2\pi Ri\hbar\delta^{(2)}(0)\sum_{j=1}^{\infty}\frac{(-\alpha N)^j}{j}\int dt B_{j(N-1)}. \quad (2.22)$$

2.3 Comparison of Methods

If the standard lore holds true, one would expect to find that the two actions for the effective quantum mechanics theory are equivalent, i.e.

$$S_0[b] + S_{\text{jacobian}}[b] + S_{\text{extra}}[b] = \tilde{S}_0[b] + \tilde{S}_{\text{jacobian}}[b]. \quad (2.23)$$

Firstly, we note that using identity A.7 from appendix A allows one to prove the equality of the kinetic terms, i.e. $S_0[b] = \tilde{S}_0[b]$ (see equations 2.16 and 2.21) .

Secondly, comparison of the jacobian factors (equations 2.17 and 2.22) indicates that the following correspondence must hold

$$\sum_{k \in Z} 1 = 2\pi R \;\; \delta(0) \tag{2.24}$$

where the delta function at zero argument on the right hand side of this equation is a delta fuction in the coordinate x, i.e. $\delta(0) = \operatorname*{Lim}_{x \to 0} \delta(x)$. If we express the delta function in terms of the angular coordinate $\theta = x/R$, the R dependence in the above equation drops out. With this identification, we find $S_{\text{jacobian}}[b] = \tilde{S}_{\text{jacobian}}[b]$ up to the infinite constant that we dropped.

Finally, we turn our attention to the most interesting term, namely $S_{\text{extra}}[\varphi]$. In the second method where we performed the nonlinear field redefinition in the quantum field theory using the standard arguments, we did not pick up any such term. Taking the pragmatic point of view that this term can not be rationalized to vanish, we ask the following question "could this term arise as an extra term generated upon nonlinear field redefinition in the original quantum field theory?" Remarkably, it is possible (see appendix B) to perform the resummation of the infinite series in equation 2.18. The resulting term is

$$S_{\text{extra}}[\varphi] = -\frac{\hbar^2}{8} \frac{\left[\sum_{k \in Z} 1\right]^2}{(2\pi R)^2} \alpha^2 N^2 (N-1)^2 \int d^2x \frac{\varphi^{2N-4}}{(1 + \alpha N \varphi^{N-1})^4}. \tag{2.25}$$

We still must contend with the peculiar infinite summations to relate this potential to the two dimensional field theory. Using the result of the comparison of the jacobian terms, i.e. equation 2.24, we conclude that the extra term becomes

$$S_{\text{extra}}[\varphi] = -\frac{\hbar^2}{8} (\delta(0))^2 \alpha^2 N^2 (N-1)^2 \int d^2x \frac{\varphi^{2N-4}}{(1 + \alpha N \varphi^{N-1})^4}. \tag{2.26}$$

Interestingly, note that this term diverges as the square of the $\delta(0)$ in the spatial coordinate, and *not* like the two-dimensional delta function at zero argument. This feature (which persists in d-dimensions quantum field theory where an extra term containing $(\delta^{(d-1)}(0))^2$ appears) occurs for an important reason which becomes evident when one evaluates Feynman diagrams.

We could have elected to present the material in the intermediate step of Method 2 of this exercise in a different, more complete manner. That is, after the field redefinition, we should add to the action an infinite series of counterterms with unspecified coefficients c_ℓ, i.e.

$$\begin{aligned} S[\varphi] &= -\frac{1}{2}\int d^2x\,\eta^{\mu\nu}(1+2N\alpha\varphi^{N-1}+N^2\alpha^2\varphi^{2(N-1)})\partial_\mu\varphi(x)\partial_\nu\varphi(x) \\ &\quad -i\hbar\delta^{(2)}(0)\int d^2x\,\ln[1+N\alpha\varphi^{N-1}]+\int d^2x\,\sum_{\ell=1}^{\infty}c_\ell\varphi^\ell(\varphi). \end{aligned}$$

With this action, one would have to determine the infinite series of unknown coefficients, the c_ℓ's, by calculating physically significant quantities and matching with the original free theory. On the other hand, our exercise has enabled us to pin down the precise counterterm without having to resort to evaluating Feynman diagrams, i.e. the series in c_ℓ is replaced by the simple expression in equation 2.26.

To conclude this section, we see evidence for a single extra term being generated in the $1+1$ dimensional quantum field theory on a Minkowskian cylinder upon making a nonlinear field redefinition. Note that the field theory involved in this example is one with nontrivial spacetime topology. In the following section, we will back up and address instead the issue of extra terms in d dimensional quantum field theories with *trivial* spacetime topology more directly by considering a discretized version of the path integral.

3 Discretization of the Path Integral

The derivation of the extra term in the path integral of quantum mechanics involves discretizing the time coordinate, so one is inclined to consider a discretization of the spacetime coordinates in the quantum field theory case.

3.1 Quantum Mechanics Preliminary

As a warmup to the quantum field theory case, let us review the salient features of the quantum mechanics case. For the most part, we will follow the derivation of the well-known extra term in quantum mechanics by Gervais and Jevicki [1]. We will, however, treat the jacobian term in a different, but completely equivalent way. Consider the simple case of an n-dimensional quantum mechanical particle

$$Z=\int\prod_{a=1}^{n}Dq^a\;\; e^{\frac{i}{\hbar}\int dt\left[\frac{1}{2}\sum_{a=1}^{n}(\dot{q}^a)^2-V_0(q)\right]}. \tag{3.1}$$

In particular, we will assume that there are no time derivatives in the potential. Discretizing the time interval $t_f - t_0$ into N segments of length ϵ such that $t_k = t_0 + \epsilon k$ and $t_f \equiv t_N$, and using the abbreviated notation $q^a(k) = q^a(t_k)$, one arrives at

$$Z = \int \prod_{a=1}^{n} \prod_{k=1}^{N-1} Dq^a(k) \prod_{k=1}^{N-1} e^{\frac{i}{\hbar}\left[\frac{1}{2\epsilon}\sum_{a=1}^{n}(q^a(k+1)-q^a(k))^2 - \epsilon V_0(q(k))\right]}. \tag{3.2}$$

The limit $N \to \infty$ and $\epsilon \to 0$ while $N\epsilon = t_f - t_0$ is implied. A key observation to make is that the one dimensional time differential $dt \propto \epsilon$ carries a single power of ϵ, so any terms of $\mathcal{O}(\epsilon^2)$ in the exponential will vanish in the small ϵ limit. After a field transformation, the only "extra" terms that need be retained in the path integral action are either $\mathcal{O}(\epsilon)$ or possibly divergent.

Under a coordinate redefinition $q^a(t) = F^a(Q(t))$ from the q^a to the Q^i $(i = 1, \ldots n)$, the path integral becomes

$$\begin{aligned} Z &= \int \prod_{i=1}^{n} \prod_{k=1}^{N-1} DQ^i(k) \prod_{k=1}^{N-1} \left[\det g_{ij}(Q(k))\right]^{\frac{1}{2}} && (3.3) \\ &\times \prod_{k=1}^{N-1} e^{\frac{i}{\hbar}\left[\frac{1}{2\epsilon}(F^a(Q(k+1)) - F^a(Q(k)))^2 - \epsilon V_0(F(Q(k)))\right]} && (3.4) \end{aligned}$$

where the metric in the new coordinates is

$$g_{ij}(Q) = \sum_{a=1}^{n} \frac{\partial F^a(Q)}{\partial Q^i} \frac{\partial F^a(Q)}{\partial Q^j}.$$

Choosing the midpoint prescription

$$\begin{aligned} \bar{Q}^i(k) &\equiv \frac{1}{2}\Big(Q^i(k+1) + Q^i(k)\Big) \\ \Delta Q^i(k) &\equiv Q^i(k+1) - Q^i(k) && (3.5) \end{aligned}$$

we Taylor expand the jacobian determinant and the kinetic term about the midpoint, and integrate out terms in ΔQ to put the action back in its original "canonical" form. By rescaling Q^i by $\epsilon^{\frac{1}{2}}$, it is clear that the normalized integral of $\Delta Q^i \Delta Q^j$ is proportional to ϵ, therefore one need only keep terms in the jacobian to quadratic order in ΔQ and terms in the kinetic term to quartic order in ΔQ. Note that since integrating out the fluctuations about the midpoints only brings in positive powers of ϵ, the extra terms in the action will always be less singular than the ϵ^{-1} kinetic term. Explicitly, writing the path integral normalization as

$$\langle 1\rangle = \prod_{k=1}^{N-1} \left[\det g_{ij}(\bar{Q}(k))\right]^{\frac{1}{2}} \times \int \prod_{\ell=1}^{n} \prod_{k=1}^{N-1} \mathcal{D}\Delta Q^\ell(k)\; e^{\frac{i}{\hbar}\frac{1}{2\epsilon} g_{ij}(\bar{Q}(k))\Delta Q^i(k)\Delta Q^j(k)}$$

we may use the relations

$$\begin{aligned} \langle \Delta Q^\ell \Delta Q^m\rangle &= \langle 1\rangle i\hbar\epsilon g^{\ell m} &(3.6)\\ \langle \Delta Q^i \Delta Q^j \Delta Q^k \Delta Q^\ell\rangle &= \langle 1\rangle (i\hbar\epsilon)^2 (g^{ij}g^{k\ell} + g^{ik}g^{j\ell} + g^{i\ell}g^{jk}) &(3.7)\\ \langle \Delta Q^{i_1} \Delta Q^{i_2} \ldots \Delta Q^{i_{2n+1}}\rangle &= 0 &(3.8)\end{aligned}$$

to rewrite the following extra terms which appear in the lagrangian

$$\begin{aligned} \frac{i}{\hbar}\mathcal{L}_{\rm extra}(\bar{Q}) &= \frac{1}{16}\left[g(\bar{Q})^{ij}_{,\ell} g(\bar{Q})_{ij,m} + g(\bar{Q})^{ij} g(\bar{Q})_{ij,\ell m}\right]\Delta Q^\ell \Delta Q^m \\ &\quad + \frac{i}{24\hbar\epsilon} F^a(\bar{Q})_{,i} F^a(\bar{Q})_{,jk\ell} \Delta Q^i \Delta Q^j \Delta Q^k \Delta Q^\ell. \end{aligned}$$

The result is the extra potential introduced in the previous section

$$S_{\rm extra}(\bar{Q}) = -\hbar^2 \int dt \sum_{k,\ell,p,n\in Z} \frac{1}{8} g^{\ell n}(\bar{Q})\Gamma^k_{\ell p}(\bar{Q})\Gamma^p_{nk}(\bar{Q}).$$

3.2 Quantum Field Theory

How does this change when we consider quantum field theory? Let us focus on the simple case of a massless, noninteracting scalar field theory existing in a spacetime of trivial topology. In order to maintain Poincare invariance in quantum field theory, one expects to discretize all spacetime coordinates with the same parameter ϵ. Consider a d-dimensional Euclidean spacetime and discretize [3] the spacetime into N hypercubic volumes of size ϵ^d. The points of the latticized spacetime are labeled by a d-dimensional vector k. The action for a noninteracting massless scalar field $\phi(x)$ in a flat d-dimensional spacetime is

$$Z = \int \prod_k \mathcal{D}\phi_k \prod_k e^{-\frac{1}{\hbar}\epsilon^{d-2}\frac{1}{2}\sum_{i=1}^{2d}\frac{1}{2}(\phi_{k+n_i}-\phi_k)^2} \qquad (3.9)$$

[3]If spacetime is finite volume, then as in the quantum mechanical case we have $N\to\infty$ and $\epsilon\to 0$ as the product $N\epsilon^d$ goes to a finite value. If, on the other hand, spacetime is infinite, N must diverge more rapidly than ϵ^d. One may imagine even that there exists a lower length scale so that ϵ can not be taken all the way to zero.

where the n_i are the unit vectors connecting the vector k to its $2d$ nearest neighbors. In $d = 2$, for example, one may write $n_1 = \hat{x}$, $n_2 = \hat{y}$, $n_3 = -\hat{x}$ and $n_4 = -\hat{y}$. The factor of $1/2$ has been inserted to counter the fact the kinetic term sums over forward and backward directions. Clearly, an equivalent way of writing this action is to drop the factor of $\frac{1}{2}$ and allow the sum to run over only the d independent directions. If spacetime has a boundary where the field values are fixed, the product over lattice sites k excludes these points.

Let us now implement a change of variables $\phi = F[\varphi]$ on the lattice. The integrand above simply becomes

$$\prod_k e^{-\frac{1}{\hbar}\epsilon^{d-2}\sum_{i=1}^{d}\frac{1}{2}(F(\varphi_{k+n_i})-F(\varphi_k))^2}. \tag{3.10}$$

The issue of the measure is slightly more complicated, and before addressing it, it will be advantageous to look ahead to the prescription that we will use to calculate any extra terms.

We will choose an analog to the midpoint prescription. Since the kinetic term is the sum of differences in d directions, the analog is rather a "midpoints" prescription. We define d midpoint variables and d difference variables

$$\bar{\varphi}_{k,n_i} \equiv \frac{1}{2}\left(\varphi_{k+n_i} + \varphi_k\right) \tag{3.11}$$

$$\Delta\varphi_{k,n_i} \equiv \varphi_{k+n_i} - \varphi_k. \tag{3.12}$$

Inverting these equations, one may write the field values at the neighboring lattice sites to k uniquely as

$$\varphi_{k+n_i} = \bar{\varphi}_{k,n_i} + \frac{1}{2}\Delta\varphi_{k,n_i}. \tag{3.13}$$

On the other hand, one may write φ_k asymmetrically by using a single n_i

$$\varphi_k = \bar{\varphi}_{k,n_i} - \frac{1}{2}\Delta\varphi_{k,n_i} \qquad \text{for any } n_i, \tag{3.14}$$

or symmetrically by averaging over all directions

$$\varphi_k = \frac{1}{d}\sum_{i=1}^{d}\left(\bar{\varphi}_{k,n_i} - \frac{1}{2}\Delta\varphi_{k,n_i}\right), \tag{3.15}$$

or in a multitude of other ways since we are using redundant variables. The issue of how to handle a function of φ_k in the action, i.e. whether one writes

$$\mathcal{G}(\varphi_k) = \frac{1}{d}\sum_{i=1}^{d}\mathcal{G}(\bar{\varphi}_{k,n_i} - \frac{1}{2}\Delta\varphi_{k,n_i})$$

versus, for example,

$$\mathcal{G}(\varphi_k) = \mathcal{G}(\frac{1}{d}\sum_{i=1}^{d}(\bar{\varphi}_{k,n_i} - \frac{1}{2}\Delta\varphi_{k,n_i}))$$

could be resolved by choosing the most technically convenient expression, as all forms are formally equivalent. The constraints among the new difference variables will be reflected in the measure, which will ensure the equivalence of these expressions. In writing the measure, the essential question is how to go from the $d\phi_k$ differential (or equivalently the $d\Delta\phi_k$ differential) to the (d) $d\Delta\varphi_{k,n_i}$ differentials. Let us label the d different yet equivalent expressions for φ_k by $\varphi_k^{A_i}$, i.e.

$$\varphi_k = \bar{\varphi}_{k,n_i} - \frac{1}{2}\Delta\varphi_{k,n_i} \equiv \varphi_k^{A_i}. \tag{3.16}$$

After making the change of variables from ϕ_k to φ_k in the measure, we insert a factor of one using delta functions constraints in order to introduce the d equivalent representations of φ_k

$$\begin{aligned} d\phi_k &= d\varphi_k \, \frac{\partial F}{\partial\varphi}(\varphi_k) \qquad (3.17)\\ &= d\varphi_k \, \frac{\partial F}{\partial\varphi}(\varphi_k)\int\prod_{i=1}^{d} d\varphi_k^{A_i}\,\delta(\varphi_k - \varphi_k^{A_i}). \end{aligned}$$

Notice this form is completely symmetric and there is no preferred direction. In practice these integrals will be done by integrating first with respect to φ_k and employing any one of the d delta functions. For example one could use the first delta function as follows

$$\int d\phi_k \mathcal{H}\left[\phi_k\right] = \int\prod_{i=1}^{d} d\varphi_k^{A_i} \times \prod_{j=2}^{d}\delta(\varphi_k^{A_1} - \varphi_k^{A_j}) \times \frac{\partial F}{\partial\varphi}(\varphi_k^{A_1})\mathcal{H}\left[F(\varphi_k^{A_1})\right] \tag{3.18}$$

Let us return to the integral that we must perform

$$Z = \int\prod_k d\phi_k \prod_k e^{-\frac{1}{\hbar}\epsilon^{d-2}\frac{1}{2}\sum_{i=1}^{2d}\frac{1}{2}(\phi_{k+n_i}-\phi_k)^2}$$

$$= \int d\varphi_k \prod_{i=1}^{d} d\varphi_k^{A_i} \times \prod_{j=1}^{d} \delta(\varphi_k - \varphi_k^{A_j})$$
$$\times \prod_k e^{-\frac{1}{\hbar}\left[\epsilon^{d-2}\sum_{i=1}^{d}\frac{1}{2}(F(\varphi_{k+n_i})-F(\varphi_k))^2 - \hbar \ln \frac{\partial F}{\partial \varphi}(\varphi_k)\right]}. \qquad (3.19)$$

We will evaluate this integral by expanding in the d fluctuation variables about the d midpoint variables for each k and using perturbation theory about the canonical Gaussian action. Specifically, we must expand the jacobian and the kinetic terms and integrate out terms in $\Delta\varphi_{k,n_i}$ to put the action back in its original "canonical" form. The first few terms in the expansion of the kinetic term, using a convenient choice for φ_k in each term, are

$$\sum_{i=1}^{d} \frac{1}{2} \left(F[\varphi_{k+n_i}] - F[\varphi_k]\right)^2$$
$$= \frac{1}{2}\sum_{i=1}^{d} \left[\sum_{p=0}^{\infty} \frac{2}{(2p+1)!} \left(\frac{\Delta\varphi_{k,n_i}}{2}\right)^{2p+1} \frac{\partial^{2p+1} F}{\partial \varphi^{2p+1}}|_{\bar\varphi_{k,n_i}}\right]^2$$
$$= \sum_{i=1}^{d} \left[\frac{1}{2} g(\bar\varphi_{k,n_i})(\Delta\varphi_{k,n_i})^2 + \frac{1}{24} F'(\bar\varphi_{k,n_i}) F'''(\bar\varphi_{k,n_i})(\Delta\varphi_{k,n_i})^4 \right.$$
$$\left. + \frac{1}{1152}\left((F'''(\bar\varphi_{k,n_i}))^2 + \frac{3}{5} F'(\bar\varphi_{k,n_i}) F^{(v)}(\bar\varphi_{k,n_i})\right)(\Delta\varphi_{k,n_i})^6 + \mathcal{O}((\Delta\varphi_{k,n_i})^8)\right],$$

where the first term is the usual canonical kinetic term in the new coordinates. All terms but the first will be integrated out. The first terms in the expansion of the logarithm of the jacobian are

$$\ln F'(\varphi_k) = \ln F'(\bar\varphi_{k,n_i}) - \frac{1}{2}\frac{F''(\bar\varphi_{k,n_i})}{F'(\bar\varphi_{k,n_i})}\Delta\varphi_{k,n_i}$$
$$+ \left(-\frac{1}{8}\frac{F''(\bar\varphi_{k,n_i})^2}{(F'(\bar\varphi_{k,n_i}))^2} + \frac{1}{8}\frac{F'''(\bar\varphi_{k,n_i})}{F'(\bar\varphi_{k,n_i})}\right)(\Delta\varphi_{k,n_i})^2$$
$$+ \left(-\frac{1}{24}\frac{(F''(\bar\varphi_{k,n_i}))^3}{(F'(\bar\varphi_{k,n_i}))^3} + \frac{1}{16}\frac{F''(\bar\varphi_{k,n_i})F'''(\bar\varphi_{k,n_i})}{(F'(\bar\varphi_{k,n_i}))^2} - \frac{1}{48}\frac{F^{(iv)}(\bar\varphi_{k,n_i})}{F'(\bar\varphi_{k,n_i})}\right)(\Delta\varphi_{k,n_i})^3$$
$$+ \left(-\frac{1}{64}\frac{(F''(\bar\varphi_{k,n_i}))^4}{(F'(\bar\varphi_{k,n_i}))^4} + \frac{1}{32}\frac{(F''(\bar\varphi_{k,n_i}))^2 F'''(\bar\varphi_{k,n_i})}{(F'(\bar\varphi_{k,n_i}))^3} - \frac{1}{128}\frac{(F'''(\bar\varphi_{k,n_i}))^2}{(F'(\bar\varphi_{k,n_i}))^2}\right.$$
$$\left. -\frac{1}{96}\frac{F''(\bar\varphi_{k,n_i})F^{(iv)}(\bar\varphi_{k,n_i})}{(F'(\bar\varphi_{k,n_i}))^2} + \frac{1}{384}\frac{F^{(v)}(\bar\varphi_{k,n_i})}{F'(\bar\varphi_{k,n_i})}\right)(\Delta\varphi_{k,n_i})^4 + \mathcal{O}((\Delta\varphi_{k,n_i})^5).$$

As in the quantum mechanics case, all but the first expressions in the expansion of the kinetic and jacobian terms will be integrated out to put the path integral into the canonical form. The path integral normalization, denoted with bra-ket notation, is as follows

$$
\begin{aligned}
\langle 1\rangle &= \int \prod_k d\varphi_k \prod_{i=1}^{d} d\varphi_k^{A_i} \prod_{j=1}^{d} \delta(\varphi_k - \varphi_k^{A_i}) && (3.20)\\
&\quad\times \prod_k e^{-\frac{1}{\hbar}\epsilon^{d-2}\sum_{i=1}^{d}\frac{1}{2}g(\bar{\varphi}_{k,n_i})\left(\Delta\varphi_{k,n_i}\right)^2} e^{\ln\frac{\partial F}{\partial\varphi}(\bar{\varphi}_k)} \\
&= \int \prod_{i=1}^{d} d\varphi_k^{A_i} \prod_{j=2}^{d} \delta(\varphi_k^{A_1} - \varphi_k^{A_i}) && (3.21)\\
&\quad\times \prod_k e^{-\frac{1}{\hbar}\epsilon^{d-2}\sum_{i=1}^{d}\frac{1}{2}g(\bar{\varphi}_{k,n_i})\left(\Delta\varphi_{k,n_i}\right)^2} e^{\ln\frac{\partial F}{\partial\varphi}(\bar{\varphi}_k^{A_1})},
\end{aligned}
$$

where the bar in the argument of the logarithm indicates that there are no fluctuations in that term. In other words, the leading jacobian term is a function of a midpoint variable and is thus inert under the integration. Making a linear change to the proper integration variables for integrating out the fluctuations, i.e.

$$
d\varphi_k^{A_i} = -\frac{1}{2}d\Delta\varphi_{k,n_i}, \tag{3.22}
$$

the integral becomes

$$
\begin{aligned}
\langle 1\rangle &= (-\frac{1}{2})^d \int \prod_{i=1}^{d} d\Delta\varphi_{k,n_i} \prod_{j=2}^{d} \delta(\bar{\varphi}_{k,n_1} - \bar{\varphi}_{k,n_i} - \frac{1}{2}\Delta\varphi_{k,n_1} + \frac{1}{2}\Delta\varphi_{k,n_i}) \\
&\quad\times \prod_k e^{-\frac{1}{\hbar}\epsilon^{d-2}\sum_{i=1}^{d}\frac{1}{2}g(\bar{\varphi}_{k,n_i})\left(\Delta\varphi_{k,n_i}\right)^2} e^{\ln\frac{\partial F}{\partial\varphi}(\bar{\varphi}_{k,n_1})}.
\end{aligned}
$$

Temporarily using the abbreviation

$$
a_i = \frac{1}{2\hbar}\epsilon^{d-2} g(\bar{\varphi}_{k,n_i})
$$

one finds upon evaluation

$$
\langle 1\rangle = \sqrt{\frac{\pi}{\sum_{i=1}^{d} a_i}} \times e^{\ln\frac{\partial F}{\partial\varphi}(\bar{\varphi}_{k,n_1})} \times \exp\left[4\left(\frac{(\sum_{i=1}^{d} a_i\bar{\varphi}_{k,n_i})^2}{\sum_{i=1}^{d} a_i} - \sum_{i=1}^{d} a_i\bar{\varphi}_{k,n_i}^2\right)\right]. \tag{3.23}
$$

In the limit of vanishing epsilon, all barred variables reach the same value, which we will denote $\bar{\varphi}_k$. This has the consequence that the quantity in the square brackets above goes to zero. Integrals over fluctuations may be obtained in the usual way by differentiation with respect to the a_i. One can easily show that the quantity in square brackets above is of no consequence to these calculations, and hence we will drop it here. From the expression for $\langle 1 \rangle$, one deduces

$$\langle \Delta\varphi_{k,n_i}^{2m} \rangle = \langle 1 \rangle \frac{(2m-1)!! \quad \hbar^m \epsilon^{(2-d)m}}{\left(\sum_{i=1}^d (\frac{\partial F}{\partial \varphi}(\bar{\varphi}_{k,n_i}))^2\right)^m}, \tag{3.24}$$

$$\langle \Delta\varphi_{k,n_i}^{2m} \, \Delta\varphi_{k,n_j}^{2p} \rangle = \langle 1 \rangle \frac{(2(m+p)-1)!! \quad \hbar^{m+p} \epsilon^{(2-d)(m+p)}}{\left(\sum_{i=1}^d (\frac{\partial F}{\partial \varphi}(\bar{\varphi}_{k,n_i}))^2\right)^m \left(\sum_{j=1}^d (\frac{\partial F}{\partial \varphi}(\bar{\varphi}_{k,n_j}))^2\right)^p} \tag{3.25}$$

and so on, while integrals over odd powers of $\Delta\varphi_{k,n_i}$ vanish. Symbolically, we have the result

$$\langle \Delta\varphi^{2m} \rangle = \langle 1 \rangle (2m-1)!! \left(\frac{\hbar \epsilon^{2-d}}{d F'[\bar{\varphi}]^2} \right)^m. \tag{3.26}$$

Note that unlike the case of quantum mechanics, in quantum field theory an infinite number of terms potentially contribute.

Taking the $\mathcal{O}(\Delta\varphi_{k,n_i}^2)$ terms coming from the jacobian and the $\mathcal{O}(\Delta\varphi_{k,n_i}^4)$ terms coming from the kinetic term, one finds in the discretized action a term

$$S_{\text{extra}}^{(1)} = \epsilon^d \frac{\hbar^2}{8} \frac{1}{d^2} \sum_{i=1}^d \frac{(F''[\varphi_{k,n_i}])^2}{(F'[\varphi_{k,n_i}])^4} \epsilon^{2-2d}.$$

Thus the first extra term (corresponding to the extra $\mathcal{O}(\hbar^2)$ term in quantum mechanics) that arises in the quantum field theory upon a change of variables is

$$S_{\text{extra}}^{(1)}[\varphi] = \int d^d x \frac{\hbar^2}{8d} \frac{F''[\varphi]^2}{F'[\varphi]^4} \epsilon^{2-2d}. \tag{3.27}$$

Note that in contrast to the situation in quantum mechanics, the extra term $S_{\text{extra}}^{(1)}[\varphi]$ in d dimensional field theory contains a *divergent* quantity ϵ^{2-2d} or equivalently $(\delta^{(d-1)}(0))^2$. In four dimensions, this term is the $(\delta^{(3)}(0))^2$ term previously mentioned by Dowker and Mayes [3], Charap [6] and Suzuki and Hattori [5], especially in the context of chiral dynamics. The second extra term comes from the $\mathcal{O}((\Delta\varphi_{k,n_i})^4)$ term in the jacobian expansion together with the $\mathcal{O}((\Delta\varphi_{k,n_i})^6)$ term in the kinetic term expansion

$$\begin{aligned} S^{(2)}_{\text{extra}}[\varphi] &= \frac{\hbar^3}{192d^2}\int d^dx \,\left[\frac{F^{(5)}[\varphi]}{(F'[\varphi])^5}+6\frac{F''[\varphi]F^{(4)}[\varphi]}{(F'[\varphi])^6}+\frac{26}{3}\frac{(F'''[\varphi])^2}{(F'[\varphi])^6}\right.\\ &\left. -18\frac{(F''[\varphi])^2F'''[\varphi]}{(F'[\varphi])^7}+9\frac{(F''[\varphi])^4}{(F'[\varphi])^8}\right]\epsilon^{4-3d}. \end{aligned} \tag{3.28}$$

More generally, the j-th extra term in the action will be $\mathcal{O}(\hbar^{j+1})$, have d dependence like d^{-j} and diverge as $\epsilon^{2j-d(j+1)}$.

Although the analysis in this section applies to a theory on a trivial spacetime, it is interesting to compare these results to those for the scalar field living on a cylindrical spacetime that we considered in the previous section. In the case of a field redefinition of the form $\phi = F[\varphi] \equiv \varphi + \alpha\varphi^N$, the first extra term is

$$S^{(1)}_{\text{extra}}[\varphi] = \int d^dx \frac{\hbar^2}{8d}\epsilon^{2-2d}\frac{\alpha^2(N-1)^2N^2\varphi^{2N-4}}{(1+\alpha N\varphi^{N-1})^4}. \tag{3.29}$$

Comparing this to the extra term generated in the Kaluza Klein example (see equation 2.25) , we see that the first term above gives precisely that found in the Kaluza Klein example, provided we identify ϵ^{-2} with $\delta(0)^2$, take into account the overall minus sign in going from Euclidean to Minkowskian signature, and finally we set the d in the denominator equal to 1 *not* 2. The justification for this last proviso will become clear shortly. But first one may ask, "what about the string of extra terms that we have found in this section?" In the Kaluza Klein example, there was only evidence of a single extra term, not a string of extra terms. What is going on? We argue that this is an indication that the correct treatment of theories on nontrivial spacetime topologies involves a different discretization than the one performed here. For example, if a dimension is compactified, Poincare invariance of the total space is lost and one should use different ϵ factors as necessary in discretizing the spacetime. In the example of the scalar field on the Minkowskian cylinder, one would use ϵ_1 and ϵ_2 to discretize the space. If one is allowed to take the limit of the epsilons going to zero separately, then the problem is effectively reduced to the quantum mechanical case (hence $d = 1$), and only one term will be generated since θ is integrated out.

4 Feynman Diagram Analysis

To illustrate the important role of the divergent extra terms that arise in a quantum field theory after nonlinear field redefinitions are made, we turn to explicit Feynman diagram calculations. In the process on exploring this issue, we will gain insight into what is really happening when one employs the unphysical procedure of dimensional regularization. We will work with the real, massless, noninteracting d-dimensional scalar field theory considered in the previous section

$$Z = \int \mathcal{D}\phi \; e^{-\frac{1}{\hbar}\int d^dx \frac{1}{2}\partial^\mu\phi\partial_\mu\phi}. \tag{4.1}$$

and will take the concrete example of a field redefinition involving a cubic term

$$\phi = F[\varphi] \equiv \varphi + \alpha\varphi^3. \tag{4.2}$$

To properly analyze this situation using Feynman diagrams, we will evaluate the Feynman diagrams using the lattice action. This is necessary since the continuum limit must be taken in a consistent manner. In particular, we must use the correct form for the propagator and for the derivative interactions. Extracting lattice Feynman rules for the non-derivative interactions is trivial. Under the above nonlinear transformation, the previous section implies that the non-derivative part of the action consists of the "usual" jacobian along with an infinite series of extra terms. The usual lattice jacobian term is

$$S_{\mathcal{J}}[\bar{\varphi}] = \sum_x \epsilon^d \left[-\hbar\epsilon^{-d}\ln\left(1+3\alpha\bar{\varphi}^2\right)\right],$$

and the first couple of extra terms are

$$S^{(1)}_{\text{extra}}[\bar{\varphi}] = \sum_x \epsilon^d \left[\frac{9\hbar^2\alpha^2}{2d}\epsilon^{2-2d}\frac{\bar{\varphi}^2}{(1+3\alpha\bar{\varphi}^2)^4}\right]$$

and

$$S^{(2)}_{\text{extra}}[\bar{\varphi}] = \sum_x \epsilon^d \left[\frac{\hbar^3\alpha^2}{4d^2}\epsilon^{4-3d}\left(\frac{27}{(1+3\alpha\bar{\varphi}^2)^8} - \frac{27}{(1+3\alpha\bar{\varphi}^2)^7} + \frac{13}{2\,(1+3\alpha\bar{\varphi}^2)^6}\right)\right].$$

Extracting the lattice Feynman rules for the derivative terms requires a little more work. (The apparent continuum expression is

$$S_0[F[\varphi]] = \int d^dx \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi\left(1+3\alpha\varphi^2\right)^2.)$$

For example, the derivative part of the discretized action contains the term, with slight notational changes from the previous section (to avoid confusion due to the fact that we perform a Fourier transform),

$$S[\varphi] = \epsilon^{d-2} \sum_{x\in\epsilon Z^d} \frac{1}{2} \sum_{\mu=1}^{d} \left(\varphi_{x+\epsilon\hat{n}_\mu} - \varphi_x\right)^2$$

where n_μ is a unit vector in the μ direction. Inserting the Fourier transformed field

$$\varphi_x = \int d^d p \tilde{\varphi}(p) e^{ip\cdot x}$$

into the action, and taking care to note that the momentum integrations cover only the first Brillioun zones

$$-\frac{\pi}{\epsilon} \le p_\mu \le \frac{\pi}{\epsilon} \qquad \mu = 1 \dots d,$$

we find

$$\begin{aligned} & \frac{1}{\epsilon^2} \sum_{x\in\epsilon Z^d} \frac{1}{2} \sum_{\mu=1}^{d} \left(\varphi_{x+\epsilon\hat{n}_\mu} - \varphi_x\right)^2 \\ = & \sum_{x\in\epsilon Z^d} \frac{1}{2\epsilon^2} \int d^d p \int d^d p' \tilde{\varphi}(p)\tilde{\varphi}(p') e^{i(p+p')\cdot x} \left(e^{ip\cdot n_\mu \epsilon} - 1\right)\left(e^{ip'\cdot n_\mu \epsilon} - 1\right) \\ = & \frac{(2\pi)^d}{\epsilon^2} \frac{1}{2} \int d^d p \tilde{\varphi}(p)\tilde{\varphi}(-p) \left|e^{ip\cdot n_\mu \epsilon} - 1\right|^2 \end{aligned}$$

This is the only term both quadratic in φ and order $\mathcal{O}(\alpha^0)$ in the action, and our unrenormalized lattice propagator for φ is thus

$$\Delta_\epsilon(p) = \left[\sum_{\mu=1}^{d} \frac{1}{\epsilon^2} \left|e^{ip\cdot n_\mu \epsilon} - 1\right|^2\right]^{-1} = \frac{1}{p^2} + \frac{\epsilon^2}{12} \frac{\sum_{\mu=1}^{d} p_\mu^4}{p^4} + \cdots. \tag{4.3}$$

We only expand the propagator here to make contact with the usual form of the continuum propagator. Note that taking the usual form for the propagator assumes that the components of the momentum are very small in magnitude compared to the natural cutoff dictated by the lattice spacing, i.e. taking the usual continuum propagator means that for each component you stay away from the edges of the Brillioun zone. We will not truncate the propagator in this way as there is not a meaningful expansion in ϵ for all values of p. We stress that the small ϵ limit does not imply the usual $1/p^2$

form for the propagator from the lattice point of view, unless one is willing to introduce an ad hoc cutoff Λ which is much smaller than the natural cutoff $\frac{\pi}{\epsilon}$.

In a similar manner, one may derive the Feynman vertices for the non-quadratic derivative interactions. For the four-field interaction one finds

$$\lambda_{\epsilon}^{(4)}(p,k) = \frac{12i\alpha}{\epsilon^2}\sum_{\mu=1}^{d}\left(e^{ip\cdot n_\mu \epsilon}-1\right)\left(e^{ik\cdot n_\mu \epsilon}-1\right), \tag{4.4}$$

while for the six-field interaction one finds

$$\lambda_{\epsilon}^{(6)}(p,k) = \frac{216i\alpha^2}{\epsilon^2}\sum_{\mu=1}^{d}\left(e^{ip\cdot n_\mu \epsilon}-1\right)\left(e^{ik\cdot n_\mu \epsilon}-1\right). \tag{4.5}$$

In evaluating Feynman diagrams, it proves expedient to make use of the fact that

$$\lambda_{\epsilon}^{(4)}(p,-p) = 12i\alpha\left[\Delta_\epsilon(p)\right]^{-1}$$

and

$$\lambda_{\epsilon}^{(6)}(p,-p) = 216i\alpha^2\left[\Delta_\epsilon(p)\right]^{-1}.$$

The lattice Feynman rules extracted from the φ theory are illustrated in Figure 1. These Feynman rules taken by themselves would appear to describe an interacting scalar field theory, but knowing their origin, we are aware that this theory is simply a disguised free theory. This complicated action for φ should yield results physically equivalent to the trivial free action for ϕ. In particular, one would expect that the pole of the renormalized φ propagator should be equal to the original physical pole.

The parameter α is not necessarily a small parameter and we have presented the exact action to $\mathcal{O}(\hbar^2)$, but now we will treat α as small and compute the order $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^2)$ corrections to the lattice propagator. One may verify that for sufficiently small α, that perturbation theory about $\varphi = 0$ makes sense.

At $\mathcal{O}(\alpha)$, there are three graphs (see figure 2) contributing to the propagator with external momentum k. The divergent contribution coming from the derivative interaction loop is precisely canceled by an equal but opposite contribution from the mass graph arising from the jacobian term, i.e.

$$\frac{1}{2}\int\frac{d^dp}{(2\pi)^d}\int\frac{d^dp'}{(2\pi)^d}\delta^{(d)}(p+p')i\hbar\Delta_\epsilon(p)\lambda_\epsilon^{(4)}(p,p') + 6\alpha\hbar\epsilon^{-d}$$

$$= -\frac{1}{2}\int \frac{d^dp}{(2\pi)^d} i\hbar\Delta_\epsilon(0) 12 i\alpha(\Delta_\epsilon(p))^{-1} + 6\alpha\hbar\epsilon^{-d}$$

$$= -6\alpha\hbar \int \frac{d^dp}{(2\pi)^d} + 6\alpha\hbar\epsilon^{-d} = 0. \tag{4.6}$$

The remaining less divergent diagram from the derivative-interaction loop, where the derivatives are not acting on the loop but on the external legs with momentum k, gives

$$\frac{1}{2}\lambda_\epsilon^{(4)}(k,-k)\int \frac{d^dp}{(2\pi)^d} i\hbar\Delta_\epsilon(p) = -6\alpha\hbar(\Delta_\epsilon(k))^{-1}\int \frac{d^dp}{(2\pi)^d}\Delta_\epsilon(p) \tag{4.7}$$

which is harmless in that it is an overall wavefunction renormalization. This cancellation illustrates that the role of the usual jacobian term with its diverging factor, i.e. ϵ^{-d} on the lattice or $\delta^{(d)}(0)$ in the continuum, is to cancel off the divergent contribution to the propagator which would have shifted the value of the mass. There is no need to think of $\delta^{(d)}(0)$ or ϵ^{-d} being set zero, or as being absorbed into some redefinition.

The situation is more interesting at order $\mathcal{O}(\alpha^2)$, where the so-called extra terms begin to arise. Including the vertex arising from the first "extra" term we found in the previous section, there are seven graphs (see figure 3) which potentially contribute to a mass shift. Figure 3 does not include graphs which would affect only wave function renormalization. Graphs (d) and (e) cancel each other as one would expect from the cancellation at $\mathcal{O}(\alpha)$. Graphs (a), (b) and (c) may be shown to add up to

$$-i72\alpha^2\hbar^2\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\Delta_\epsilon(k_1). \tag{4.8}$$

Note that a change of integration variables from the k_i to $\ell_i \equiv \epsilon k_i$ reveals the degree of divergence of this integral as $\epsilon \to 0$ to be ϵ^{2-2d}, since all ϵ dependence leaves both the integrand and the limits of integration and goes into a prefactor to the integral.

The lattice "setting sun" diagram (f) is much more complicated to evaluate

$$-i\hbar^2\frac{1}{2}\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\delta^{(d)}(k_1+k_2+k_3)\Delta_\epsilon(k_1)\Delta_\epsilon(k_2)\Delta_\epsilon(k_3)$$
$$\times\frac{1}{3}\left[\lambda_\epsilon^{(4)}(k_1,k_2)\lambda_\epsilon^{(4)}(k_1,k_2) + \text{cyclic in } (k_1,k_2,k_3)\right].$$

More explicitly, the integral is

$$
\begin{aligned}
= \quad & i24\alpha^2\hbar^2\epsilon^2 \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\delta^{(d)}(k_1+k_2+k_3=0) \\
& \times \frac{\left[\left(\sum_{\mu=1}^d \left(e^{ik_1\cdot n_\mu \epsilon}-1\right)\left(e^{ik_2\cdot n_\mu\epsilon}-1\right)\right)^2 + \text{cyclic in } (k_1,k_2,k_3)\right]}{\sum_{\nu=1}^d \left|e^{ik_1\cdot n_\nu\epsilon}-1\right|^2 \sum_{\rho=1}^d \left|e^{ik_2\cdot n_\rho\epsilon}-1\right|^2 \sum_{\sigma=1}^d \left|e^{ik_3\cdot n_\sigma\epsilon}-1\right|^2},
\end{aligned}
$$

where the integration domains cover the first Brillioun zones, i.e. $-\frac{\pi}{\epsilon} \le k_\mu \le +\frac{\pi}{\epsilon}$. The first thing to observe about this integral is that a change of integration variables from the k_i to $\ell_i \equiv \epsilon k_i$ reveals that it will diverge like ϵ^{2-2d} in the limit $\epsilon \to 0$. The exact evaluation of this integral is prohibitive, since there is no meaningful expansion in ϵ. Although we have not been able to exactly evaluate this integral, all indications support that this integral does not exactly cancel the contributions from the graphs (a) through (e), and that the extra term graph (g) is necessary to ensure that all contributions add up to zero at $\mathcal{O}(\alpha^2)$ to keep the propagator's pole intact. Note that the extra (counter)term

$$
-i\frac{9}{d}\alpha^2\hbar^2\epsilon^{2-2d}
$$

has the correct divergence structure in ϵ to cancel off the $\mathcal{O}(\alpha^2)$ term that otherwise would shift the physical mass.

To lend credibility to the assertion that graph (g) is necessary to maintain the pole of the renormalized propagator, let us obviate the technical difficulties mentioned above by evaluating the diagrams using the "standard" limiting forms for the propagator and the four-field vertex. The integral above for the "setting sun" diagram becomes

$$
\begin{aligned}
= \quad & i24\alpha^2\hbar^2 \int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\delta^{(d)}(k_1+k_2+k_3) \\
& \times \frac{\left[(k_1\cdot k_2)^2+(k_2\cdot k_3)^2+(k_3\cdot k_1)^2\right]}{k_1^2\,k_2^2\,k_3^2}
\end{aligned}
$$

where the limits of integration are unchanged. A careful evaluation of this integral yields

$$
= i36\alpha^2\hbar^2\left[\int \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{1}{k_1^2} \;+\; \int_{|(k_1+k_2)_\mu|\le\frac{\pi}{\epsilon}} \frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{1}{k_1^2}\right] \tag{4.9}
$$

where the integration region in the second term in restricted from the previous integrals. Furthermore, given that we are integrating an even function in each component of both k_1 and k_2, the restricted integration region may be rewritten to give the following result

$$i72\alpha^2\hbar^2 \prod_{i=1}^{d} \left[\int_{-\frac{\pi}{\epsilon}}^{\frac{\pi}{\epsilon}} dk_i \int_{-\frac{\pi}{\epsilon}}^{\frac{\pi}{\epsilon}} dp_i - 2\int_0^{\frac{\pi}{\epsilon}} dp_i \left(\frac{\pi}{\epsilon} - p_i \right) \right] \frac{1}{p^2}. \tag{4.10}$$

One of the 2^d resulting integrals, namely the one where each component of each momenta is integrated over the entire region, precisely cancels the correspondingly approximated integral of equation 4.8, i.e.

$$i72\alpha^2\hbar^2 \int d^dk \int d^dp \frac{1}{p^2}.$$

The remaining $2^d - 1$ integrals give a nonzero contribution that must be canceled by the extra term, however we will not carry out the many integrals necessary to show this precise cancellation. We have indicated the role of the extra term as a mechanism for canceling off the potential mass shift, using the $1/p^2$ limit of the propagator and $p \cdot q$ limit of the four-field vertex; however, theoretically it should be possible to show this with the strict lattice forms of the propagator and the vertex.

Since physically significant quantities must be invariant under field redefinition, this process must continue, with successive extra terms properly canceling off any would-be shifts in the physical mass. It is encouraging that a cursory look at the divergences at the next order in α go like ϵ^{4-3d} which is precisely the divergence with which the extra term at $\mathcal{O}(\alpha^3)$ appears. We conjecture that proper inclusion of the extra terms will ensure that the pole of the propagator remains at its physical value, and more generally that all physically significant quantities will remain unchanged via the same cancellation mechanism.

5 Conclusion

Using a discretized version of the path integral for a free massless scalar quantum field in d dimensions, we have shown that beyond the usual jacobian term, an infinite series of divergent "extra" terms arises in the action whenever a nonlinear field redefinition is made. The explicit forms for the first couple of these terms were given. We have argued that in the strictest sense these terms should not be neglected, since for example one role of these terms is to cancel off contributions which would change the value of physical quantities such as the pole of the renormalized propagator.

Using a different method than discretization of the path integral, i.e. dimensional reduction down to a quantum mechanical model where known results could be exploited, we analyzed a $1+1$ dimensional free massless scalar theory on a Minkowskian cylinder, and saw that only a single term appeared (as one finds in quantum mechanics). The fact that a single term arose in this case is a consequence of the nontrivial spacetime topology which would affect a discretization and reduce the problem to two essentially one dimensional cases. The role that spacetime topology plays in the discretization of the path integral will be explored further in future articles [11].

Let us clarify the effect of these divergent extra terms which arise upon making nonlinear field redefinitions. Consider a quantum field theory with some number of bare parameters. Physically significant renormalized quantities in this original theory are functions of the bare parameters and the cutoff in the original action. At least in principal, the bare parameters may be fixed by knowledge of experimental data. Suppose one now performs a nonlinear field redefinition, and computes the renormalized quantities as a function of the same bare parameters, the cutoff and any field redefinition parameters. If the divergent extra terms are neglected, the computed values of some physical quantities will differ from those previously computed. We argue that the extra terms generated in the field transformation are precisely what is necessary to cancel the divergences that one normally sweeps into the ad hoc addition of counterterms and/or the redefinition of the bare parameters. From this point of view, what we have presented in this paper are the exact counterterms for the quantum field theory. Thus, it is important to include these terms if one is making nonlinear field redefinitions in a theory where values of some coefficients are known.

Feynman diagrammatic calculations have enabled us to see explicitly that the extra terms are essential to understanding why the unphysical practice of dimensional regularization works at all, at least in this context. From the pragmatist's point of view where $\delta^{(d)}(0) \sim a^{-d}$ blows up as the lattice spacing goes to zero, this work is especially pleasing since we have shown that it is not necessary to make even a formal assertion that $\delta^{(d)}(0) = 0$. We have indicated how the divergent extra counterterms cancel out unwanted divergent contributions to physical quantities so that the result is *consistent* with simply setting these divergences equal to zero in evaluations of Feynman diagrams. Note that this is completely a different matter than setting factors such as $\delta^{(d)}(0)$ equal to zero in the action! This cancellation mechanism by the extra terms is of course valid whether or not dimensional regularization is applicable. In instances where dimensional regularization is not valid, care should be taken that extra terms be included properly in calculations involving nonlinear field redefinitions.

A further reason for interest in these so-called extra terms is the exciting possibility that these extra terms that we have found could in fact allow for the presence of anomalies of higher orders in $\hbar$ in quantum field theories with nonlinear symmetries. This issue is currently under investigation, and could potentially be of extreme importance to, for example, gravity theories and currently popular theories possessing extended conformal $\mathcal{W}$ algebra symmetries. Work on these issues is in progress [11].

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We would like to thank Cliff Burgess and Joe Polchinski for reading a preliminary version of this paper and for insightful comments. This research was supported in part by Robert A. Welch Foundation, NSF Grant PHY 9009850 (UT Austin) and Department of Education grant DE-FG05-87ER40367 (Vanderbilt).

A Useful identities involving modes

In this appendix, we prove some identities used in obtaining results in this paper. Consider the expansion in modes of a function $\varphi(\theta, t)$ with a periodic coordinate θ

$$\varphi(\theta, t) = \sum_{p \in Z} b^{(p)}(t) e^{ip\theta}. \tag{A.1}$$

We call p the mode number. Define the following object which is multilinear in N modes and where the sum of the mode numbers is ℓ

$$E_N^{(\ell)}[b] \equiv \sum_{p_1 \in Z} \dots \sum_{p_N \in Z} b^{(p_1)} \dots b^{(p_N)} \delta^{\ell - \sum_{i=1}^N p_i = 0}. \tag{A.2}$$

Note the special case when the sum of the mode numbers is zero

$$E_N^{(0)}[b] = \int \frac{d\theta}{2\pi} \varphi(\theta, t)^N \equiv B_N[b]. \tag{A.3}$$

One may straightforwardly relate the sum over a product of two E's to a single E in the following way

$$
\begin{aligned}
&\sum_{p\in Z} E_M^{(\ell+p)}[b]E_N^{(-p+m)}[b] \\
=&\sum_{p\in Z} \sum_{\substack{p_1,\dots p_M\in Z\\ \ell_1,\dots \ell_N\in Z}} b^{(p_1)}\dots b^{(p_N)}b^{(\ell_1)}\dots b^{(\ell_N)}\delta^{p+\ell-\sum_{i=1}^M p_i=0}\delta^{-p+m-\sum_{j=1}^N \ell_j=0} \\
=&\sum_{p\in Z}\sum_{p_1\in Z}\dots\sum_{p_{M+N}\in Z} b^{(p_1)}\dots b^{(p_{M+N})}\delta^{p+\ell-\sum_{i=1}^M p_i=0}\delta^{-p+m-\sum_{j=M+1}^{M+N} p_j=0} \\
=&\sum_{p_1\in Z}\dots\sum_{p_{M+N}\in Z} b^{(p_1)}\dots b^{(p_{M+N})}\delta^{\ell+m-\sum_{j=1}^{M+N} p_j=0} \\
=&E_{M+N}^{(\ell+m)}[b].
\end{aligned}
\tag{A.4}
$$

Consider three E's where the number of modes may be different and where the sum of all mode numbers is zero

$$
\begin{aligned}
&\sum_{\ell\in Z}\sum_{n\in Z}\sum_{p\in Z} E_L^{(\ell+n)}[b]\ \ E_M^{(-\ell+p)}[b]E_N^{(-p-n)}[b] \\
=&\sum_{n\in Z}\sum_{p\in Z} E_{L+M}^{(n+p)}[b]\ \ E_N^{(-p-n)}[b] \\
=&\left[\sum_{n\in Z} 1\right]\sum_{p'\in Z} E_{L+M}^{(p')}[b]\ \ E_N^{(-p')}[b] \\
=&\left[\sum_{n\in Z} 1\right] B_{L+M+N}[b].
\end{aligned}
\tag{A.5}
$$

Finally consider the following trace of j E_M's where the sum of all mode numbers is zero (summation convention assumed)

$$
\begin{aligned}
\mathrm{tr}(E_M^j) &= E_M^{(-n_j+n_1)}[b]E_M^{(-n_1+n_2)}[b]\dots E_M^{(-n_{j-1}+n_j)}[b] \\
&= E_{2M}^{(-n_j+n_2)}[b]E_M^{(-n_2+n_3)}[b]\dots E_M^{(-n_{j-1}+n_j)}[b] \\
&= \left[\sum_{n\in Z} 1\right] B_{j(N-1)}[b].
\end{aligned}
\tag{A.6}
$$

This identity is useful when evaluating the jacobian determinant in section 2 of this paper.

A final identity which is useful in comparing the kinetic terms in the example of section 2 is

$$
\begin{aligned}
& \sum_{p\in Z} p^2 E_N^{(p)} E_1^{(-p)} \\
= & \sum_{p\in Z}\sum_{\ell_1\in Z}\cdots\sum_{\ell_N\in Z} p^2 b^{(\ell_1)}\dots b^{(\ell_N)} b^{(-p)} \delta^{p-\sum_{i=1}^N \ell_i=0} \\
= & \sum_{\ell_1\in Z}\cdots\sum_{\ell_N\in Z}\left(\sum_{i=1}^N \ell_i\right)^2 b^{(\ell_1)}\dots b^{(\ell_N)} b^{(-\sum_{i=1}^N \ell_i)} \\
= & \sum_{\ell_1\in Z}\cdots\sum_{\ell_N\in Z}\left(\sum_{i=1}^N (\ell_i)^2 + \sum_{i\neq j}\ell_i\ell_j\right) b^{(\ell_1)}\dots b^{(\ell_N)} b^{(-\sum_{i=1}^N \ell_i)} \\
= & \sum_{\ell_1\in Z}\cdots\sum_{\ell_N\in Z}\left(N(\ell_N)^2 + N\ell_N \sum_{i=1}^{N-1}\ell_i\right) b^{(\ell_1)}\dots b^{(\ell_N)} b^{(-\sum_{i=1}^N \ell_i)} \\
= & N\sum_{\ell_1\in Z}\cdots\sum_{\ell_N\in Z}\ell_N\left(\sum_{i=1}^{N-1}\ell_i\right) b^{(\ell_1)}\dots b^{(\ell_N)} b^{(-\sum_{i=1}^N \ell_i)} \\
= & -N\sum_{p\in Z}\sum_{\ell_1\in Z}\cdots\sum_{\ell_N\in Z}\ell_N p b^{(\ell_1)}\dots b^{(\ell_N)} b^{(p)} \delta^{(-p-\sum_{i=1}^N \ell_i=0)} \\
= & -N\sum_{p\in Z}\sum_{\ell_1\in Z}\cdots\sum_{\ell_{N-1}\in Z}\sum_{\ell\in Z}\ell p b^{(\ell_1)}\dots b^{(\ell_{N-1})} b^{(\ell)} b^{(p)} \delta^{(-p-\ell-\sum_{i=1}^{N-1} \ell_i=0)} \\
= & -N\sum_{p\in Z}\sum_{\ell\in Z}\ell p E_1^{(p)} E_1^{(\ell)} E_{N-1}^{(-\ell-p)}. \qquad \text{(A.7)}
\end{aligned}
$$

B Details of Kaluza Klein calculation

The metric in terms of the $b^{(m)}$ modes in the Kaluza Klein calculation is

$$
\begin{aligned}
g_{mn}[b] &= 2\pi R(\delta_{m+n=0} + 2\alpha N E_{N-1}^{(-m-n)}[b] + N^2\alpha^2 E_{2N-2}^{(-m-n)}[b]) \\
&= 2\pi R(1+\alpha N E_{N-1}[b])^2_{m+n},
\end{aligned}
$$

where 1_{m+n} is shorthand for $\delta_{m+n=0}$, $(E_M)_{m+n}$ is shorthand for $E_M^{(-m-n)}$, and $(E_M^2)_{m+n}$ is shorthand for $E_M^{(-m-p)}E_M^{(p-n)}$. Subtracting out an infinite constant due to the $2\pi R$, we compute the jacobian contribution

$$
\mathrm{Trln}[\frac{g_{mn}[b]}{2\pi R}]\delta(t-t') = 2\mathrm{Trln}(1+\alpha N E_{N-1}[b])_{m+n}\delta(t-t')
$$

$$
\begin{aligned}
&= -2\mathrm{Tr}\sum_{j=1}^{\infty}\frac{(-\alpha N)^j}{j}(E^j_{N-1}[b])_{m+n}\delta(t-t') \\
&= -2\delta(0)\sum_{j=1}^{\infty}\frac{(-\alpha N)^j}{j}\int dt\left[E^{(-m+n_1)}_{N-1}[b]\cdots E^{(n_{j-1}+m)}_{N-1}[b]\right] \\
&= -2\delta(0)\sum_{j=1}^{\infty}\frac{(-\alpha N)^j}{j}\left[\sum_{k\in Z}1\right]\int dt\, B_{j(N-1)}[b]. \qquad \text{(B.1)}
\end{aligned}
$$

We now illustrate how the inverse metric is obtained to all orders in α via a recursion equation. Write

$$
g^{km}[b] = \frac{1}{2\pi R}\sum_{Q=0}^{\infty} f_Q(\alpha)E_Q^{(k+m)}[b]. \qquad \text{(B.2)}
$$

and compute $g^{km}[b]g_{mn}[b] = \delta^{k-n=0}$ as follows

$$
\begin{aligned}
\delta^{k-n=0} &= \sum_{Q=0}^{\infty} f_Q(\alpha)E_Q^{(k-n)}[b] + 2\alpha N\sum_{Q=0}^{\infty} f_Q(\alpha)E_Q^{(k+m)}[b]E_{N-1}^{(-m-n)}[b] \\
&\quad +\alpha^2N^2\sum_{Q=0}^{\infty} f_Q(\alpha)E_Q^{(k+m)}[b]E_{N-2}^{(-m-n)}[b] \\
&= \sum_{Q=0}^{\infty} f_Q(\alpha)E_Q^{(k-n)}[b] + 2\alpha N\sum_{Q=0}^{\infty} f_Q(\alpha)E_{Q+N-1}^{(k-n)}[b] \\
&\quad +\alpha^2N^2\sum_{Q=0}^{\infty} f_Q(\alpha)E_{Q+2N-2}^{(k-n)}[b] \\
&= \sum_{Q=0}^{N-2} f_Q(\alpha)E_Q^{(k-n)}[b] + \sum_{Q=0}^{N-2}\left(f_{Q+N-1}(\alpha)+2\alpha N f_Q(\alpha)\right)E_{Q+N-1}^{(k-n)}[b] \\
&\quad +\sum_{Q=0}^{\infty}\Big(f_{Q+2N-2}(\alpha)+2\alpha N f_{Q+N-1}(\alpha)+\alpha^2N^2 f_Q(\alpha)\Big)E_{Q+2N-2}^{(k-n)}[b].
\end{aligned}
$$

The first sum on the righthand side after the last equal side must be a Kroeneker delta function, and the other two sums must vanish. From the first sum, one finds $f_0(\alpha) = 1$ and $f_1(\alpha) = \cdots = f_{N-2}(\alpha) = 0$. From the second sum, one finds $f_{N-1}(\alpha) = -2\alpha N$ and $f_N(\alpha) = \cdots = f_{2N-3}(\alpha) = 0$. From the third sum, it is clear that $f_Q(\alpha) = 0$ if Q is not divisible by $N-1$. Furthermore, making the ansatz

$$
f_{(N-1)q}(\alpha) = (-\alpha N)^q C_q,
$$

we may determine the unknown function C_q by plugging into the third sum. Explicitly, the condition that the third sum vanishes is

$$f_{(N-1)(2+q)}(\alpha) = -2\alpha N f_{(N-1)(1+q)}(\alpha) - \alpha^2 N^2 f_{(N-1)q}(\alpha),$$

and taking into account the boundary condition $C_0 = 1$, this leads to the solution

$$f_{(N-1)q}(\alpha) = (-\alpha N)^q (q+1).$$

Thus, we obtain the inverse metric to all orders in the parameter α

$$g^{km}[b] = \frac{1}{2\pi R} \sum_{j=0}^{\infty} (-\alpha N)^j (j+1) E^{(k+m)}_{j(N-1)}[b]. \tag{B.3}$$

Using this expression, the connection coefficients are computed to be

$$\Gamma^k_{\ell p}[b] = -\sum_{j=0}^{\infty} (-\alpha N)^{j+1} (N-1) E^{(k-p-\ell)}_{j(N-1)+N-2}[b]. \tag{B.4}$$

Note that $\Gamma^k_{\ell p}[b]$ is dependent only on the total mode number $k-p-\ell$ so it will prove useful to write $\Gamma^k_{\ell p}[b] = \Gamma^{k-p-\ell}[b]$. The metric and inverse metric also depend only on the sum of their indices. Using these expressions, the extra term may be found by computing

$$\begin{aligned}
& \sum_{k,\ell,p,n\in Z} \frac{1}{8} g^{\ell n}[b] \Gamma^k_{\ell p}[b] \Gamma^p_{nk}[b] \\
= & \sum_{k,\ell,p,n\in Z} \frac{1}{8} g^{\ell n}[b] \Gamma^{k-\ell-p}[b] \Gamma^{p-n-k}[b] \\
= & \sum_{k\in Z} 1 \sum_{\ell,p,n\in Z} \frac{1}{8} g^{\ell n}[b] \Gamma^{-\ell-p}[b] \Gamma^{p-n}[b] \\
= & \left[\sum_{k\in Z} 1\right]^2 \frac{(N-1)^2}{8\cdot 2\pi R} \sum_{L,M,J=0}^{\infty} (-\alpha N)^{L+M+J+2} (L+1) E^{(0)}_{(M+J+L)(N-1)+2(N-2)}[b] \\
= & \left[\sum_{k\in Z} 1\right]^2 \frac{(N-1)^2}{8\cdot 2\pi R} \sum_{L,M,J=0}^{\infty} (-\alpha N)^{L+M+J+2} \left(\frac{L+M+J}{3} + 1\right) \\
& \qquad \times B_{(M+J+L)(N-1)+2(N-2)}[b] \\
= & \left[\sum_{k\in Z} 1\right]^2 \frac{(N-1)^2}{8\cdot 2\pi R} \sum_{X=0}^{\infty} \mathcal{P}_3(X) (-\alpha N)^{X+2} \left(\frac{X}{3} + 1\right) B_{X(N-1)+2(N-2)}[b]
\end{aligned}$$

$$= \frac{(N-1)^2}{8 \cdot 2\pi R} \left[\sum_{k \in Z} 1 \right]^2 \sum_{X=0}^{\infty} (-\alpha N)^{X+2} \begin{pmatrix} X+3 \\ 3 \end{pmatrix} B_{X(N-1)+2(N-2)}[b]. \qquad \text{(B.5)}$$

We have used the fact that $\mathcal{P}_3(X) = \frac{1}{2}(X+1)(X+2)$ is the number of ways to write X as the sum of 3 non-negative integers. To rewrite this expression in terms of φ, we use equation A.3

$$\begin{aligned}
&= \frac{(N-1)^2}{48 \cdot 2\pi R} \left[\sum_{k \in Z} 1 \right]^2 \sum_{X=0}^{\infty} (-\alpha N)^{X+2} (X+1)(X+2)(X+3) \\
&\quad \times \int \frac{d\theta}{2\pi} \varphi^{X(N-1)+2(N-2)}[b] \\
&= \frac{(N-1)^2}{48 \cdot 2\pi R} \left[\sum_{k \in Z} 1 \right]^2 \int \frac{d\theta}{2\pi} \frac{1}{(-\alpha N)} \varphi^{2N-4} \frac{d^3}{d(\varphi^{N-1})^3} \sum_{X=0}^{\infty} (-\alpha N \varphi^{N-1})^{X+3} \\
&= \frac{(N-1)^2}{48 \cdot 2\pi R} \left[\sum_{k \in Z} 1 \right]^2 \int \frac{d\theta}{2\pi} \frac{1}{(-\alpha N)} \varphi^{2N-4} \frac{d^3}{d(\varphi^{N-1})^3} \left[\frac{(-\alpha N \varphi^{N-1})^3}{1 + \alpha N \varphi^{N-1}} \right] \\
&= \frac{1}{8} \frac{\alpha^2 N^2 (N-1)^2}{2\pi R} \left[\sum_{k \in Z} 1 \right]^2 \int \frac{d\theta}{2\pi} \frac{(\varphi^{N-2})^2}{(1 + \alpha N \varphi^{N-1})^4}. \qquad \text{(B.6)}
\end{aligned}$$

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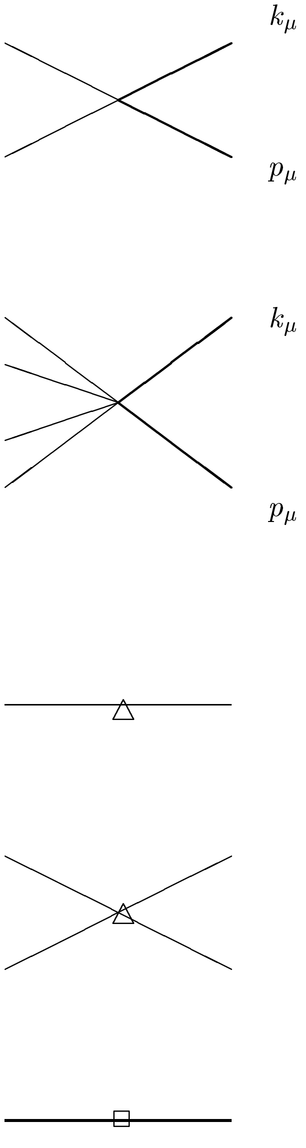

$$\left[\sum_{\mu=1}^{d} \frac{1}{\epsilon^2} \left| e^{ip\cdot n_\mu \epsilon} - 1 \right|^2 \right]^{-1}$$

$$\frac{12i\alpha}{\epsilon^2} \sum_{\mu=1}^{d} \left(e^{ik\cdot n_\mu \epsilon} - 1 \right) \left(e^{ip\cdot n_\mu \epsilon} - 1 \right)$$

$$\frac{216i\alpha^2}{\epsilon^2} \sum_{\mu=1}^{d} \left(e^{ik\cdot n_\mu \epsilon} - 1 \right) \left(e^{ip\cdot n_\mu \epsilon} - 1 \right)$$

$$6\alpha\hbar\delta^{(d)}(0)$$

$$-108\alpha^2\hbar\delta^{(d)}(0)$$

$$-i\frac{9}{d}\alpha^2\hbar^2\epsilon^{2-2d}$$

Figure 1: Feynman rules to order $\mathcal{O}(\alpha^2)$

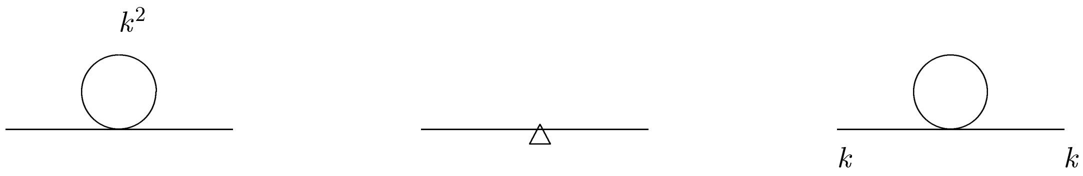

Figure 2: $\mathcal{O}(\alpha)$ Corrections to the Propagator

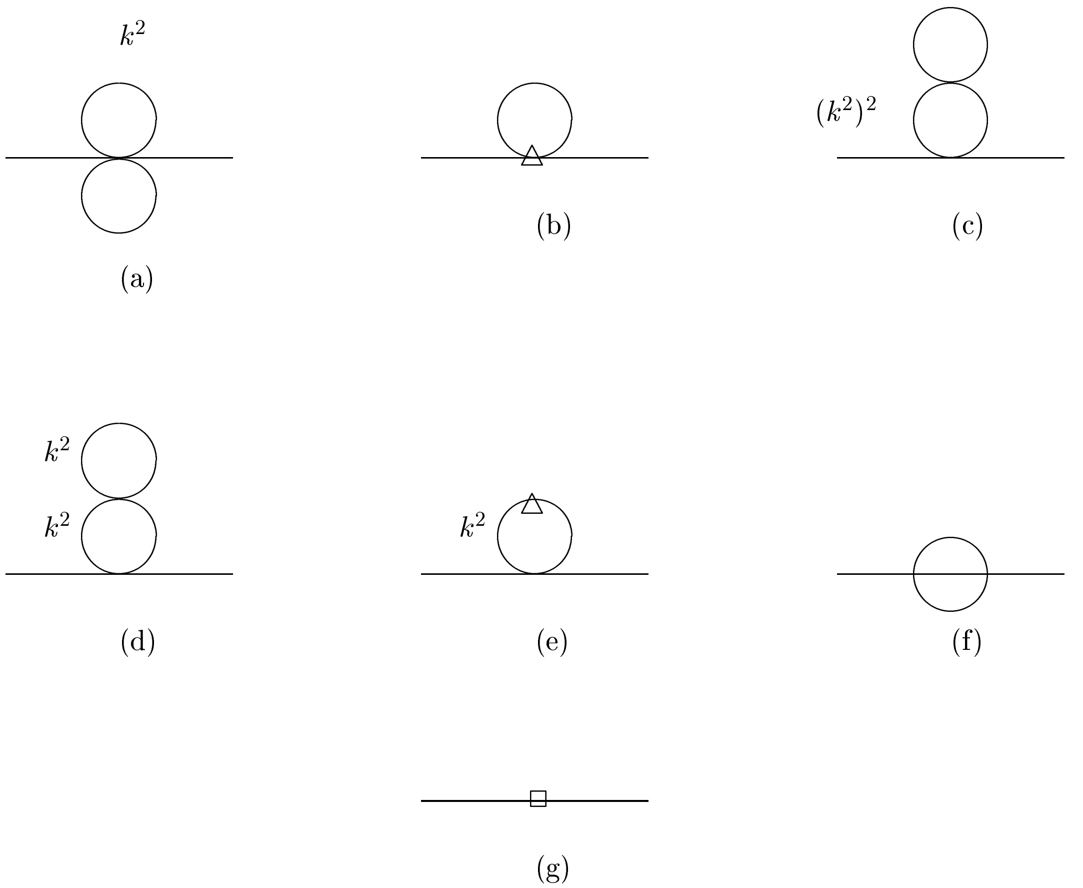

Figure 3: $\mathcal{O}(\alpha^2)$ corrections to the propagator