TOPICS IN ISOTOPIC PAIRS AND THEIR REPRESENTATIONS

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The paper is devoted to the representations of "isotopic pairs" — the objects, by which it maybe convenient to describe some forms of a non-hamiltonian interaction of hamiltonian systems on the quantum level. General concepts are illustrated on simple examples.

The algebraic objects, which will be under a consideration below, describe a non-hamiltonian interaction of hamiltonian systems on the quantum level (see [1] for a classical picture). So their using seems to may be rather convenient for some problems of high energy physics (f.e. of a quantitative derivation of the spontaneous symmetry breaking from "the first principles" of string theory), physics of condensed matter (f.e. of couplings in the high-T_c superconductivity) or psychophysics and informatics of interactive systems.

However, a discussion of possible interesting applications (which certainly may be only more or less speculative and qualitative in present) is avoided in this compact, short paper and the subject is treated purely mathematically in a "topical", "suggestive" manner without a specification of any concrete physical implications: general concepts are illustrated on simple (may be rather mathematically formal) but nevertheless crucial examples.

I. Isotopic pairs: definition and examples

Definition 1 [1]**.** The pair (V_1, V_2) of linear spaces is called *an isotopic pair* iff there are defined two mappings $m_1 : V_2 \otimes \bigwedge^2 V_1 \mapsto V_1$ and $m_2 : V_1 \otimes \bigwedge^2 V_2 \mapsto V_2$ such that the mappings $(X, Y) \mapsto [X, Y]_A = m_1(A, X, Y)$ $(X, Y \in V_1,\ A \in V_2)$ and $(A, B) \mapsto [A, B]_X = m_2(X, A, B)$ $(A, B \in V_2,\ X \in V_1)$ obey the Jacobi identity for all values of a subscript parameter (such operations will be called *isocommutators*

[1]The work was supported in part by C.N.R.S. (France), M.P.I.M. (Germany) and E.S.I. (Austria).

[2]Сдано в журнал "Теоретическая и математическая физика".

and the subscript parameters will be called *isotopies*) and are compatible to each other, i.e. the identities

$$\begin{aligned}[X,Y]_{[A,B]_Z} = & \tfrac{1}{2}([[X,Z]_A,Y]_B + [[X,Y]_A,Z]_B + [[Z,Y]_A,X]_B - \\ & [[X,Z]_B,Y]_A - [[X,Y]_B,Z]_A - [[Z,Y]_B,X]_A)\end{aligned}$$

and

$$\begin{aligned}[A,B]_{[X,Y]_C} = & \tfrac{1}{2}([[A,C]_X,B]_Y + [[A,B]_X,C]_Y + [[C,B]_X,A]_Y - \\ & [[A,C]_Y,B]_X - [[A,B]_Y,C]_X - [[C,B]_Y,A]_X)\end{aligned}$$

$(X,Y,Z \in V_1,\ A,B,C \in V_2)$ hold.

Let's discuss this defintion.

First, it may be considered as a result of an axiomatization of the following construction: let $\mathcal{A}$ be an associative algebra (f.e. any matrix one) and V_1, V_2 be two linear subspaces in it such that V_1 is closed under the isocommutators $(X,Y) \mapsto [X,Y]_A = XAY - YAX$ with isotopies A [2] from V_2, whereas V_2 is closed under the isocommutators $(A,B) \mapsto [A,B]_X = AXB - BXA$ with isotopies X from V_1. It should be marked that if a family (linear space) V_1 of operators forms an isocommutator algebra with a family (linear space) V_2 of admissible isotopies then (V_1,V_2) is an isotopic pair via the so–called "isotopic duality" [3].

Second, one may compare it with the definition of "anti–Jordan pairs" [4] (cf.also the related definition of "anti–Jordan triple systems" in [5]). Namely,

Definition 2 (cf.[4])**.** The pair (V_1,V_2) of linear spaces is called *an anti–Jordan pair* iff there are defined two mappings $m_1 : V_2 \otimes \bigwedge^2 V_1 \mapsto V_1$ and $m_2 : V_1 \otimes \bigwedge^2 V_2 \mapsto V_2$ such that the mappings $(X,Y) \mapsto [X,Y]_A = m_1(A,X,Y)$ $(X,Y \in V_1,\ A \in V_2)$ and $(A,B) \mapsto [A,B]_X = m_2(X,A,B)$ $(A,B \in V_2,\ X \in V_1)$ are compatible to each other in the following manner

$$[X,Y]_{[A,B]_Z} = [[X,Z]_A,Y]_B + [[Z,Y]_A,X]_B - [[X,Y]_B,Z]_A$$

and

$$[A,B]_{[X,Y]_C} = [[A,C]_X,B]_Y + [[C,B]_X,A]_Y - [[A,B]_Y,C]_X$$

$(X,Y,Z \in V_1,\ A,B,C \in V_2)$ hold.

It should be marked that anti–Jordan pairs (anti–Jordan triple systems) are skew–symmetric analogs of the well–known Jordan pairs [6,7] (Jordan triple systems [8,7]).

It can be easily verified that isotopic pairs are always anti–Jordan pairs (to obtain it one should use the Jacobi identity linearized by subscript parameters), and that the anti–Jordan pairs with a multiplication, obeying Jacobi identity if a subscript parameter is fixed in any way, are just the isotopic pairs. So the isotopic pairs may be considered as a particular case of the anti–Jordan pairs. Note that there exist examples of anti–Jordan pairs, which are not isotopic ones [4].

Let's now discuss constructions of isotopic pairs.

First, some interesting isotopic pairs may be obtained by the "isotopic duality" from the so–called Lie $\mathfrak{g}$–bunches, which definition is formulated below.

Definition 3 [3]. Let $\mathfrak{g}$ be a Lie algebra. *A Lie $\mathfrak{g}$–bunch* is a $\mathfrak{g}$–module W such that there exists a $\mathfrak{g}$–equivariant mapping $\mathfrak{g}\otimes\bigwedge^2(W)\mapsto W$, which defines a structure of Lie algebra in W when the first argument is fixed in an arbitrary way.

A lot of examples of Lie $\mathfrak{g}$–bunches is contained in [3]. Now we briefly describe some important examples of isotopic pairs (partially contained in [1]), some of them are constructed from suitable Lie $\mathfrak{g}$–bunches.

Example 1. *The isotopic pair of noncanonically coupled rotators* [1].

Let's consider the Lie $\mathfrak{sl}(2,\mathbb{C})$–bunch in π_1 (the adjoint representation of $\mathfrak{sl}(2,\mathbb{C})$) [3]. It is defined by the next formulas for isocommutators

$$\begin{array}{llll} & [m_{-1},m_0]_{l_0}=m_{-1} & [m_{-1},m_0]_{l_{-1}}=0 & [m_{-1},m_0]_{l_1}=2m_0\\ (1) & [m_1,m_{-1}]_{l_0}=0 & [m_1,m_{-1}]_{l_{-1}}=2m_{-1} & [m_1,m_{-1}]_{l_1}=-2m_1\\ & [m_1,m_0]_{l_0}=m_1 & [m_1,m_0]_{l_{-1}}=2m_0 & [m_1,m_0]_{l_1}=0 \end{array}$$

where l_i $(i=-1,0,1)$ are generators of $\mathfrak{sl}(2,\mathbb{C})$ $([l_i,l_j]=(i-j)l_{i+j})$, m_i $(i=-1,0,1)$ form a basis in π_1 $(l_i(m_j)=(i-j)m_{i+j})$.

$$\begin{array}{llll} & [l_{-1},l_0]_{m_0}=-l_{-1} & [l_{-1},l_0]_{m_{-1}}=0 & [l_{-1},l_0]_{m_1}=-2l_0\\ (2) & [l_1,l_{-1}]_{m_0}=0 & [l_1,l_{-1}]_{m_{-1}}=-2l_{-1} & [l_1,l_{-1}]_{m_1}=2l_1\\ & [l_1,l_0]_{m_0}=-l_1 & [l_1,l_0]_{m_{-1}}=-2l_0 & [l_1,l_0]_{m_1}=0 \end{array}$$

It can be easily verified that isocommutators (1) and (2) define a structure of an isotopic pair in $\pi_1\oplus\pi_1$. We shall denote the first summand by π_1^+ and the second summand by π_1^-.

It is rather convenient to consider the compact real form of the isotopic pair (π_1^+,π_1^-). Let's denote

$$\begin{aligned} l_x=\tfrac{i}{2}(l_1-l_{-1}),\quad l_y=&\tfrac{1}{2}(l_1+l_{-1}),\quad l_z=il_0\\ m_x=\tfrac{i}{2}(m_1-m_{-1}),\quad m_y=&\tfrac{1}{2}(m_1+m_{-1}),\quad m_z=im_0, \end{aligned}$$

then

$$\begin{aligned} [m_i,m_j]_{l_k}=&\delta_{ik}m_j-\delta_{jk}m_i\\ [l_i,l_j]_{m_k}=&\delta_{jk}l_i-\delta_{ik}l_j. \end{aligned}$$

Remark (Hidden $\mathbf{SU}(3)$*–symmetry of noncanonically coupled rotators).* The isotopic pair (π_1^+,π_1^-) of noncanonically coupled rotators admits a hidden $\mathbf{SU}(3)$–symmetry, if one identifies π_1^+ with $\mathbf{SU}(3)$–triplet $\mathbf{3}$ and π_1^- with $\bar{\mathbf{3}}$.

This hidden symmetry clearly manifests itself by dynamical equations on a classical level [1].

Example 2. *The isoquaternionic pair.*

Let's consider the Lie $\mathfrak{sl}(2,\mathbb{C})$–bunch of example 2 of [3]. It is realized in the direct sum $\pi_0\oplus\pi_1$ of the trivial and adjoint representations of $\mathfrak{sl}(2,\mathbb{C})$. If one fixes a basis m_i in π_1 as above and an element c of π_0 then the isocommutators in the Lie $\mathfrak{sl}(2,\mathbb{C})$–bunch will have the form

$$
\begin{aligned}
&[m_{-1},m_0]_{l_0}=0 && [m_{-1},m_0]_{l_{-1}}=0 && [m_{-1},m_0]_{l_1}=-\tfrac12 c\\
&[m_1,m_{-1}]_{l_0}=-\tfrac12 c && [m_1,m_{-1}]_{l_{-1}}=0 && [m_1,m_{-1}]_{l_1}=0\\
&[m_1,m_0]_{l_0}=0 && [m_1,m_0]_{l_{-1}}=\tfrac12 c && [m_1,m_0]_{l_1}=0\\
&[c,m_1]_{l_0}=-m_1 && [c,m_1]_{l_{-1}}=-2m_0 && [c,m_1]_{l_1}=0\\
&[c,m_0]_{l_0}=0 && [c,m_0]_{l_{-1}}=-m_{-1} && [c,m_0]_{l_1}=m_1\\
&[c,m_{-1}]_{l_0}=m_{-1} && [c,m_{-1}]_{l_{-1}}=0 && [c,m_{-1}]_{l_1}=2m_0
\end{aligned}
$$

One may construct the isotopic pair from this Lie $\mathfrak{sl}(2,\mathbb{C})$–bunch in the same manner as in the example 1. This isotopic pair is realized in $(\pi_0\oplus\pi_1,\pi_0\oplus\pi_1)$ and isocommutators have the form

$$
\begin{aligned}
&[m_{-1},m_0]_{l_0}=0 && [m_{-1},m_0]_{l_{-1}}=0 && [m_{-1},m_0]_{l_1}=-\tfrac12 c && [m_{-1},m_0]_d=m_{-1}\\
&[m_1,m_{-1}]_{l_0}=-\tfrac12 c && [m_1,m_{-1}]_{l_{-1}}=0 && [m_1,m_{-1}]_{l_1}=0 && [m_1,m_{-1}]_d=2m_0\\
&[m_1,m_0]_{l_0}=0 && [m_1,m_0]_{l_{-1}}=\tfrac12 c && [m_1,m_0]_{l_1}=0 && [m_1,m_0]_d=m_1\\
&[c,m_1]_{l_0}=-m_1 && [c,m_1]_{l_{-1}}=-2m_0 && [c,m_1]_{l_1}=0 && [c,m_1]_d=0\\
&[c,m_0]_{l_0}=0 && [c,m_0]_{l_{-1}}=-m_{-1} && [c,m_0]_{l_1}=m_1 && [c,m_0]_d=0\\
&[c,m_{-1}]_{l_0}=m_{-1} && [c,m_{-1}]_{l_{-1}}=0 && [c,m_{-1}]_{l_1}=2m_0 && [c,m_{-1}]_d=0
\end{aligned}
$$

$$
\begin{aligned}
&[l_{-1},l_0]_{m_0}=0 && [l_{-1},l_0]_{m_{-1}}=0 && [l_{-1},l_0]_{m_1}=-\tfrac12 d && [l_{-1},l_0]_c=l_{-1}\\
&[l_1,l_{-1}]_{m_0}=-\tfrac12 d && [l_1,l_{-1}]_{m_{-1}}=0 && [l_1,l_{-1}]_{m_1}=0 && [l_1,l_{-1}]_c=2l_0\\
&[l_1,l_0]_{m_0}=0 && [l_1,l_0]_{m_{-1}}=\tfrac12 d && [l_1,l_0]_{m_1}=0 && [l_1,l_0]_c=l_1\\
&[d,l_1]_{m_0}=-l_1 && [d,l_1]_{m_{-1}}=-2l_0 && [d,l_1]_{m_1}=0 && [d,l_1]_c=0\\
&[d,l_0]_{m_0}=0 && [d,l_0]_{m_{-1}}=-l_{-1} && [d,l_0]_{m_1}=l_1 && [d,l_0]_c=0\\
&[d,l_{-1}]_{m_0}=l_{-1} && [d,l_{-1}]_{m_{-1}}=0 && [d,l_{-1}]_{m_1}=2l_0 && [d,l_{-1}]_c=0
\end{aligned}
$$

where d is an element of the π_0–summand of the second space of the isotopic pair.

It is rather natural to consider a compact real form of this isotopic pair. Namely, let's put

$$
\begin{aligned}
l_x&=\tfrac{i}{2}(l_1-l_{-1}), \quad l_y=\tfrac12(l_1+l_{-1}), \quad l_z=il_0\\
m_x&=\tfrac{i}{2}(m_1-m_{-1}), \quad m_y=\tfrac12(m_1+m_{-1}), \quad m_z=im_0,
\end{aligned}
$$

as above. Then the isocommutation relations will have the form

$$
\begin{aligned}
[m_\alpha,m_\beta]_{l_\gamma}&=-\tfrac14\epsilon_{\alpha\beta\gamma}c, & [m_\alpha,m_\beta]_d&=\epsilon_{\alpha\beta\gamma}m_\gamma, & [c,m_\beta]_{l_\alpha}&=\epsilon_{\alpha\beta\gamma}m_\gamma, & [c,m_\alpha]_d&=0,\\
[l_\alpha,l_\beta]_{m_\gamma}&=-\tfrac14\epsilon_{\alpha\beta\gamma}d, & [l_\alpha,l_\beta]_c&=\epsilon_{\alpha\beta\gamma}l_\gamma, & [d,l_\beta]_{m_\alpha}&=\epsilon_{\alpha\beta\gamma}l_\gamma, & [d,l_\alpha]_c&=0.
\end{aligned}
$$

It should be marked that the received real isotopic pair admits a nice realization in terms of the isoquaternions. The isoquaternions are the ordinary ones, supplied by the isomultiplication $(h_1,h_2)\mapsto h_1 *_{h^\circ} h_2=h_1h^\circ h_2$, where h° is a fixed quaternion (an isotopy). Two copies of isoquaternions, which supply each other by isotopies, form the constructed isotopic pair after a transition from the isomultiplication to the isocommutators $[h_1, h_2]_{h^\circ} = h_1 *_{h^\circ} h_2 - h_2 *_{h^\circ} h_1$ (cf. [9] and refs wherein).

Example 3. *The isotopic pairs of noncanonically coupled oscillators.*

Case A [1].

Let's consider a standard representation of the Heisenberg algebra with three generators p, q and r ($[p, q] = r$, $[p, r] = [q, r] = 0$) by 3×3 matrices:

$$p = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The linear space of admissible isotopies is generated by three elements a, b and c, which are represented by matrices

$$a = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The isocommutators have the form

$$\begin{array}{lll} [p, q]_a = 0 & [p, q]_b = 0 & [p, q]_c = r \\ [p, r]_a = r & [p, r]_b = 0 & [p, r]_c = 0 \\ [q, r]_a = 0 & [q, r]_b = -r & [q, r]_c = 0 \end{array}$$

The elements a, b and c are closed themselves under isocommutators with isotopies p, q and r via the "isotopic duality":

$$\begin{array}{lll} [a, b]_p = 0 & [a, b]_q = 0 & [a, b]_r = c \\ [a, c]_p = c & [a, c]_q = 0 & [b, c]_r = 0 \\ [b, c]_p = 0 & [b, c]_q = -c & [a, c]_r = 0 \end{array}$$

The linear spaces generated by p, q, r and a, b, c form an isotopic pair.

Case B.

Let's consider another standard representation of the Heisenberg algebra by differential operators

$$p = \partial_x, \qquad q = x, \qquad r = 1.$$

The linear space of admissible isotopies is generated by three elements a, b and c, which are represented by the same operators as p, q and r, namely

$$a = x, \qquad b = \partial_x, \qquad c = 1.$$

The isocommutators have the form

$$\begin{array}{lll} [p, q]_a = 2q & [p, q]_b = 2p & [p, q]_c = r \\ [p, r]_a = r & [p, r]_b = 0 & [p, r]_c = 0 \\ [q, r]_a = 0 & [q, r]_b = -r & [q, r]_c = 0 \end{array}$$

The elements a, b and c are closed themselves under isocommutators with isotopies p, q and r via the "isotopic duality":

$$\begin{array}{lll} [a,b]_p = -2b & [a,b]_q = -2a & [a,b]_r = -c \\ [a,c]_p = -c & [a,c]_q = 0 & [b,c]_r = 0 \\ [b,c]_p = 0 & [b,c]_q = c & [a,c]_r = 0 \end{array}$$

The linear spaces generated by p, q, r and a, b, c form an isotopic pair.

A general case.

One may also consider a general case with isocommutators

$$\begin{array}{lll} [p,q]_a = 2\varepsilon_1 q & [p,q]_b = 2\varepsilon_1 p & [p,q]_c = \varepsilon_3 r \\ [p,r]_a = \varepsilon_2 r & [p,r]_b = 0 & [p,r]_c = 0 \\ [q,r]_a = 0 & [q,r]_b = -\varepsilon_2 r & [q,r]_c = 0 \end{array}$$

$$\begin{array}{lll} [a,b]_p = 2\tilde\varepsilon_1 b & [a,b]_q = 2\tilde\varepsilon_1 a & [a,b]_r = \tilde\varepsilon_3 c \\ [a,c]_p = \tilde\varepsilon_2 c & [a,c]_q = 0 & [b,c]_r = 0 \\ [b,c]_p = 0 & [b,c]_q = -\tilde\varepsilon_2 c & [a,c]_r = 0 \end{array}$$

where

$$\begin{cases} \varepsilon_1 + \tilde\varepsilon_1 = 0 \\ \varepsilon_2 - \tilde\varepsilon_2 = \varepsilon_1 - \tilde\varepsilon_1 \\ \varepsilon_3\tilde\varepsilon_3 - \varepsilon_2\tilde\varepsilon_2 = 0 \end{cases}$$

Example 4 [1,4]**.**

One of the most interesting examples of isotopic pairs is related to $n\times n$ symmetric and skew–symmetric matrices. Let's consider the simplest case $n=3$. It is rather convenient to introduce the following basises:

$$l_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad l_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad l_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

and

$$m_{xy} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_{yz} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad m_{xz} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$m_{xx} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_{yy} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_{zz} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Elements l_a form a basis in the space of skew–symmetric matrices, whereas m_{ab} form a basis in the space of symmetric matrices. The isocommutators have the form

$$\begin{aligned} [m_{ab}, m_{cd}]_{l_e} &= \epsilon_{ace} m_{bd} + \epsilon_{ade} m_{bc} + \epsilon_{bce} m_{ad} + \epsilon_{bde} m_{ac}, \\ [l_a, l_b]_{m_{cd}} &= \epsilon_{abc} l_d + \epsilon_{abd} l_c, \end{aligned}$$

here ϵ_{abc} is a totally antisymmetric tensor.

It is convenient to consider a complexification of this isotopic pair and a basis l_k $(k=-1,0,1)$, w_j $(j=-2,-1,0,1,2)$, κ in it:

$$\begin{gathered}
l_{-1}=l_y+il_x,\quad l_0=-\,il_z,\quad l_1=l_y-il_x,\\
w_{-2}=\tfrac{1}{2}(m_{yy}-m_{xx})-im_{xy},\quad w_{-1}=\tfrac{1}{2}m_{xz}-\tfrac{i}{2}m_{yz},\\
w_0=\tfrac{1}{6}(m_{xx}+m_{yy}-2m_{zz}),\\
w_1=\tfrac{1}{2}m_{xz}+\tfrac{i}{2}m_{yz},\quad w_2=\tfrac{1}{2}(m_{yy}-m_{xx})+im_{xy},\\
\kappa=\tfrac{1}{2}(m_{xx}+m_{yy}+m_{zz}).
\end{gathered}$$

The writing of explicit formulas for the isocommutators may be considered as an easy but rather tedious exercise for a reader.

Example 5. *An infinite dimensional "geometric" isotopic pair.*

Let $\mathcal{M}$ be an arbitrary smooth manifold, then $(\mathrm{Vect}(\mathcal{M}),C^\infty(\mathcal{M}))$ is an isotopic pair with natural isocommutators [3], namely:

$$\begin{aligned}
[u^a(\xi)\partial_a,v^b(\xi)\partial_b]_{f(\xi)} &= u^a(\xi)\partial_a(f(\xi)v^b(\xi))\partial_b-v^b(\xi)\partial_b(f(\xi)u^a(\xi))\partial_a,\\
[f(\xi),g(\xi)]_{u^a(\xi)\partial_a} &= f(\xi)u^a(\xi)\partial_a g(\xi)-g(\xi)u^a(\xi)\partial_a f(\xi);
\end{aligned}$$

here $\xi\in\mathcal{M}$, $u^a(\xi)\partial_a, v^a(\xi)\partial_a\in\mathrm{Vect}(\mathcal{M})$, $f(\xi),g(\xi)\in C^\infty(\mathcal{M})$.

The most interesting case is $\mathcal{M}=\mathbb{S}^1$. Isocommutators in the isotopic pair $(\mathrm{Vect}(\mathbb{S}^1),C^\infty(\mathbb{S}^1))$ have a nice symmetric form

$$\begin{aligned}
[u(t)\partial_t,v(t)\partial_t]_{f(t)} &= f(t)(u(t)v'(t)-v(t)u'(t))\partial_t,\\
[f(t),g(t)]_{u(t)\partial_t} &= u(t)(f(t)g'(t)-g(t)f'(t)),
\end{aligned}$$

where $t\in\mathbb{S}^1$, $u(t)\partial_t,v(t)\partial_t\in\mathrm{Vect}(\mathbb{S}^1)$, $f(t),g(t)\in C^\infty(\mathbb{S}^1)$.

The question of a description of non–trivial central extensions of this isotopic pair (cf.[10]) is however open.

Remark. The Lie algebra $\mathfrak{aff}(\mathbb{C}^1)$ of Lie group $\mathrm{Aff}(\mathbb{C}^1)$ of affine transformations of $\mathbb{C}^1$ is generated by two elements t and s such that $[t,s]=s$. It has a 2–dimensional strict representation:

$$t\mapsto\begin{pmatrix}1&0\\0&0\end{pmatrix},\qquad s\mapsto\begin{pmatrix}0&1\\0&0\end{pmatrix}.$$

Certainly, one may consider another form of this representation

$$t\mapsto\begin{pmatrix}-1&0\\0&0\end{pmatrix},\qquad s\mapsto\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

It may be straightforwardly verified that images of $\mathfrak{aff}(\mathbb{C}^1)$ under these representations form an isotopic pair. Namely, let's denote the images of elements t and s by $t^\pm$ and $s^\pm$ for the first and second representations, respectively, then

$$\begin{aligned}
[s^+,t^+]_{t^-}&=s^+, &\qquad [s^+,t^+]_{s^-}&=t^+;\\
[s^-,t^-]_{t^+}&=-s^-, &\qquad [s^-,t^-]_{s^+}&=-t^-.
\end{aligned}$$

The obtained symmetry of $t-$ and $s-$variables should be marked.

The isotopic pair generated by $t^{\pm}$ and $s^{\pm}$ admits an infinite number of monomorphisms into the complexification $(\mathrm{Vect}_{\mathbb{C}}(\mathbb{S}^1), C^{\infty}_{\mathbb{C}}(\mathbb{S}^1))$ of the isotopic pair $(\mathrm{Vect}(\mathbb{S}^1), C^{\infty}(\mathbb{S}^1))$. Namely, let's fix a basis $e_n = ie^{int}\partial_t$ in $\mathrm{Vect}_{\mathbb{C}}(\mathbb{S}^1)$ and a basis $f_n = e^{int}$ in $C^{\infty}_{\mathbb{C}}(\mathbb{S}^1)$. The isocommutation relations in the isotopic pair $(\mathrm{Vect}_{\mathbb{C}}(\mathbb{S}^1), C^{\infty}_{\mathbb{C}}(\mathbb{S}^1))$ in the basis $\{e_n, f_m\}$ have the form:

$$[e_n, e_m]_{f_k} = (n-m)e_{n+m+k}, \qquad [f_n, f_m]_{e_k} = (n-m)f_{n+m+k}.$$

The monomorphisms $\tau_{n,m}$ $(n > m)$ of the isotopic pair $(\mathfrak{aff}(\mathbb{C}^1), \mathfrak{aff}(\mathbb{C}^1))$ into $(\mathrm{Vect}_{\mathbb{C}}(\mathbb{S}^1), C^{\infty}_{\mathbb{C}}(\mathbb{S}^1))$ are defined by the formulas

$$t^+ \mapsto \sqrt{n-m}e_m, \quad s^+ \mapsto \sqrt{n-m}e_n, \qquad t^- \mapsto \sqrt{n-m}f_{-n}, \quad s^- \mapsto \sqrt{n-m}f_{-m}.$$

Example 6. *Okubo isotopic pair* (cf.[11]).

Let V be a linear space, V^* be its dual, then (V, V^*) possesses a structure of the isotopic pair, namely

$$\begin{aligned}[X,Y]_A =&(X,A)Y - (Y,A)X\\ [A,B]_X =&(X,B)A - (X,A)B,\end{aligned}$$

where $X, Y \in V,\ A, B \in V^*$.

Note that the Okubo isotopic pair coincides with $(\mathfrak{aff}(\mathbb{C}^1), \mathfrak{aff}(\mathbb{C}^1))$ if $\dim(V) = 2$ and with the pair of the example 1 if $\dim(V) = 3$.

II. Representations of isotopic pairs: definition and examples

Definition 4. *A representation of the isotopic pair* (V_1, V_2) *in the linear space* H is a pair (T_1, T_2) of mappings $T_i : V_i \mapsto \mathrm{End}(H)$ such that

$$\begin{aligned}T_1([X,Y]_A) =&T_1(X)T_2(A)T_1(Y) - T_1(Y)T_2(A)T_1(X),\\ T_2([A,B]_X) =&T_2(A)T_1(X)T_2(B) - T_2(B)T_1(X)T_2(A),\end{aligned}$$

where $X, Y \in V_1,\ A, B \in V_2$.

Below we shall consider two important types of representations of isotopic pairs: split and geometric ones.

2.1. Split representations.

Definition 5. A representation of the isotopic pair (V_1, V_2) in the linear space H is called *split* iff $H = H_1 \oplus H_2$ and

$$\begin{cases} (\forall X \in V_1)\ T_1(X)|_{H_2} = 0,\ T_1(X) : H_1 \mapsto H_2,\\ (\forall A \in V_2)\ T_2(A)|_{H_1} = 0,\ T_2(A) : H_2 \mapsto H_1.\end{cases}$$

Otherwords, operators $T(X)$ and $T(A)$ have the form $\begin{pmatrix}0 & 0\\ * & 0\end{pmatrix}$ and $\begin{pmatrix}0 & *\\ 0 & 0\end{pmatrix}$, respectively.

Example 7. *Highest–weight split representations* (cf.[12,Chapter 10]).

If the isotopic pair is $\mathbb{Z}$–graded (i.e. $V_1=\bigoplus_{i\in\mathbb{Z}}V_{1,i}$, $V_2=\bigoplus_{i\in\mathbb{Z}}V_{2,i}$, $[V_{1,i},V_{1,j}]_{V_{2,k}} \subseteq V_{1,i+j+k}$, $[V_{2,i},V_{2,j}]_{V_{1,k}}\subseteq V_{2,i+j+k}$) and the isotopic pair $(V_{1,0},V_{2,0})$ is trivial (such as for isotopic pairs of all examples below) then one may consider highest–weight split representations for them. The spaces H_1 and H_2 are generated by the highest (vacuum) vectors $|0\rangle_1\in H_1$ and $|0\rangle_2\in H_2$ such that $(\forall X\in V_{1,0})X|0\rangle_1 = \chi_1(X)|0\rangle_2$, $(\forall A\in V_{2,0})X|0\rangle_2=\chi_2(A)|0\rangle_1$, where $\chi_i\in V_{i,0}^*$, $(\forall X\in V_{1,-i})X|0\rangle_1=0$, $(\forall A\in V_{2,-i})X|0\rangle_2=0$ $(i>0)$. Other basal vectors of H_1 and H_2 have the form $A_jX_j\dots A_1X_1|0\rangle_1$ or $A_{j+1}X_jA_j\dots X_1A_1|0\rangle_2$ and $X_jA_j\dots X_1A_1|0\rangle_2$ or $X_{j+1}A_jX_j\dots A_1X_1|0\rangle_1$, respectively, where $X_1,\dots X_j,X_{j+1}\in\bigoplus_{i>0}V_{1,i}$, $A_1,\dots A_j,A_{j+1}\in\bigoplus_{i>0}V_{2,i}$.

Let's consider the case of the isotopic pair of noncanonically coupled rotators explicitely. The space H_1 is spanned by vectors $|n\rangle_1=(l_{-1}m_{-1})^n|0\rangle_1$ and $\left|n+\tfrac12\right\rangle_1=l_{-1}(m_{-1}l_{-1})^n|0\rangle_2$, whereas H_2 is spanned by $|n\rangle_2=(m_{-1}l_{-1})^n|0\rangle_2$ and $\left|n+\tfrac12\right\rangle_2=m_{-1}(l_{-1}m_{-1})^n|0\rangle_1$. The elements l_i and m_i act in H_1 and H_2 in the following manner

$$\begin{aligned}
m_{-1}|n\rangle_1 &= \left|n+\tfrac12\right\rangle_2, & m_{-1}\left|n+\tfrac12\right\rangle_1 &= |n+1\rangle_2\\
m_0|n\rangle_1 &= \alpha|n\rangle_2, & m_0\left|n+\tfrac12\right\rangle_1 &= 0\\
m_1|n\rangle_1 &= 2n\left|n-\tfrac12\right\rangle_2, & m_1\left|n+\tfrac12\right\rangle_1 &= 2(n+1)|n\rangle_2
\end{aligned}$$

$$\begin{aligned}
l_{-1}|n\rangle_2 &= \left|n+\tfrac12\right\rangle_1, & l_{-1}\left|n+\tfrac12\right\rangle_2 &= |n+1\rangle_1\\
l_0|n\rangle_2 &= \beta|n\rangle_1, & l_0\left|n+\tfrac12\right\rangle_2 &= 0\\
l_1|n\rangle_2 &= -2n\left|n-\tfrac12\right\rangle_1, & l_1\left|n+\tfrac12\right\rangle_2 &= -2(n+1)|n\rangle_1
\end{aligned}$$

where $\alpha\beta=1$. So there exist certain compatibility relations between the defining functionals χ_1 and χ_2.

Example 8. *Induced split representations* (cf.[13,Part II, Chapter 13]).

The construction of the example 7 is straightforwardly generalized to receive analogs of induced representations.

Namely, let (V_1,V_2) be an isotopic pair, (V_1°,V_2°) be its isotopic subpair and $T^\circ=(T_1^\circ,T_2^\circ)$ be a representation of (V_1°,V_2°) in the space $H^\circ=H_1^\circ\oplus H_2^\circ$. Then one may consider an induced split representation $T=(T_1,T_2)=\operatorname{Ind}_{(V_1^\circ,V_2^\circ)}^{(V_1,V_2)}(T_1^\circ,T_2^\circ)$ of (V_1,V_2) in the space $H=H_1\oplus H_2$, where H_1 is generated by $A_jX_j\dots A_1X_1|v\rangle_1$ and $A_{j+1}X_jA_j\dots X_1A_1|v\rangle_2$, whereas H_2 is generated by $X_jA_j\dots X_1A_1|v\rangle_2$ and $X_{j+1}A_jX_j\dots A_1X_1|v\rangle_1$ $(X_1,\dots X_j,X_{j+1}\in V_1;\ A_1,\dots A_j,A_{j+1}\in V_2;\ |v\rangle_1\in H_1^\circ$, $|v\rangle_2\in H_2^\circ)$.

Example 9. *Wigner split representations of* $(\mathfrak{aff}(\mathbb{C}^1),\mathfrak{aff}(\mathbb{C}^1))$ [14] (see also [15]).

Let $H=\operatorname{span}\{|n\rangle,\ n\in\mathbb{Z}\}$ be a direct sum $H_1\oplus H_2$, $H_1=\operatorname{span}\{|2n\rangle\,,\ n\in\mathbb{Z}\}$, $H_2=\operatorname{span}\{|2n+1\rangle\,,\ n\in\mathbb{Z}\}$. Let's consider operators $a^\pm: H_1\mapsto H_2$, $b^\pm: H_2\mapsto$

H_1 as follows

$$\begin{aligned} a_1^- \, |2n\rangle =& \sqrt{n+\alpha}\, |2n-1\rangle \\ a_1^+ \, |2n\rangle =& \sqrt{n+\beta}\, |2n+1\rangle \\ a_2^- \, |2n+1\rangle =& \sqrt{n+\beta}\, |2n\rangle \\ a_2^+ \, |2n-1\rangle =& \sqrt{n+\alpha}\, |2n\rangle \end{aligned}$$

The Wigner split representations of the isotopic pair $(\mathfrak{aff}(\mathbb{C}^1), \mathfrak{aff}(\mathbb{C}^1))$ are defined by the following correspondence

$$s^+ \mapsto a_1^-, \quad t^+ \mapsto a_1^+, \qquad s^- \mapsto a_2^+, \quad t^- \mapsto a_2^-.$$

Note that the operators $a^\pm = a_1^\pm \oplus a_2^\pm$ are standard operators of the parastatistics [16,15], so one may construct Wigner (non–split) representations of the isotopic pair $(\mathfrak{aff}(\mathbb{C}^1), \mathfrak{aff}(\mathbb{C}^1))$ in the form

$$s^+ = t^- = a^-, \qquad s^- = t^+ = a^+$$

or to consider Yang ("coordinate space") representations [17] of this isotopic pair, namely

$$s^- = t^+ = q, \qquad s^+ = t^- = \frac{d}{dq} - \frac{\lambda}{q} R,$$

where λ is an arbitrary number and

$$RqR^{-1} = -q, \qquad R\frac{d}{dq}R^{-1} = -\frac{d}{dq}.$$

If $\lambda = 0$ one receives the canonical commutation relations.

2.2. Geometric representations (cf.[13,Part II,Chapter 13]).

It should be marked that the isotopic pair $(\mathrm{Vect}(\mathcal{M}), C^\infty(\mathcal{M}))$ naturally acts in the spaces of geometric objects on $\mathcal{M}$.

It is convenient to consider the case $\mathcal{M} = \mathbb{S}^1$ in more details. The simplest geometric objects on $\mathbb{S}^1$ are λ–differentials $\varphi(t)\,(dt)^\lambda$, which obey the periodicity conditions $\varphi(t+2\pi) = e^{i\mu}\varphi(t)$. Let's denote the space of such differentials by $\mathcal{F}_{\lambda,\mu}$ and fix a basis $\varphi_n = e^{i(n+\mu)}\,(dt)^\lambda$ in its complexification $\mathcal{F}^{\mathbb{C}}_{\lambda,\mu}$. The action of $(\mathrm{Vect}_{\mathbb{C}}(\mathbb{S}^1), C^\infty_{\mathbb{C}}(\mathbb{S}^1))$ in $\mathcal{F}^{\mathbb{C}}_{\lambda,\mu}$ has the form

$$\begin{aligned} e_m(\varphi_n) =& -(n+\mu+\lambda m)\varphi_{n+m}, \\ f_m(\varphi_n) =& \varphi_{n+m}. \end{aligned}$$

To construct a representation of an isotopic pair (V_1, V_2) one may consider a homomorphism of (V_1, V_2) into a suitable "geometric" isotopic pair $(\mathrm{Vect}(\mathcal{M}), C^\infty(\mathcal{M}))$ and pull back any its representation. F.e. representations of $(\mathrm{Vect}_{\mathbb{C}}(\mathbb{S}^1), C^\infty_{\mathbb{C}}(\mathbb{S}^1))$ define representations of the isotopic pair $(\mathfrak{aff}(\mathbb{C}^1), \mathfrak{aff}(\mathbb{C}^1))$ via the monomorphisms $\tau_{m,n}$.

2.3. Some curiosities.

Example 10.1. *A nontrivial irreducible representation of the trivial isotopic pair* $(\mathbb{C}^n, \mathbb{C}^n)$.

The trivial isotopic pair $(\mathbb{C}^n, \mathbb{C}^n)$ admits a nontrivial representation of dimension $n+1$. Namely, let's fix a basis $u_1, \ldots u_n; v_1, \ldots v_n$ in the isotopic pair. The representation π of dimension $n+1$ is defined by the formulas $\pi : u_j \mapsto E_{j,j+1}$, $\pi : v_j \mapsto E_{j+1,j}$, where E_{kl} are standard generators in $\mathrm{Mat}_{n+1}(\mathbb{C})$.

Example 10.2. *Infinite series of irreducible representations of the trivial isotopic pair* $(\mathbb{C}, \mathbb{C})$.

Let's fix a basis e_1, e_2 in the isotopic pair $(\mathbb{C}, \mathbb{C})$ and consider a monomorphism $\rho_n : (\mathbb{C}, \mathbb{C}) \mapsto (\mathbb{C}^n, \mathbb{C}^n)$ defined as $\rho_n : e_1 \mapsto \sum_{i=1}^n u_i$, $\rho_n : e_2 \mapsto \sum_{i=1}^n v_i$. The pull–back $\pi \circ \rho_n$ of the irreducible representation π defines an irreducible representation of $(\mathbb{C}, \mathbb{C})$.

The example 10.2. is generalized immediately on the trivial isotopic pairs $(\mathbb{C}^m, \mathbb{C}^m)$ $(m \geq 2)$, so their irreducible representations are constructed for each dimension.

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