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DUAL PAIRS OF QUANTUM SPACES

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Abstract

We discuss the notion of "dual riemannian quantum space" corresponding to the "algebraic condition" in Alain Connes' definition of non-commutative Poincaré duality. We show that the DeRham complex of a dual riemannian quantum space is the homomorphic image of a skew tensor product, this leading to a natural definition of "biconnection".

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Dual pairs of quantum spaces

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Alain Connes initially defined the "non-commutative differential geometry" of a (cohomologically finite-dimensional) algebra A by specifying a d^+-summable K-cycle (H, D) of A [1][2] - a procedure generalizing to the non-commutative frame the classical situation where classical differential geometry is coded in the Dirac operator. The couples $(A, (H, D))$ of an algebra and a d^+-summable K-cycle do not however fully deserve to be considered as "non-commutative riemannian manifolds" because they still lack the important feature of "Poincaré duality"[1] generalized to the non-commutative frame - a concept which was (astonishingly!) discovered whilst looking at the standard model [3] - and which arises by considering couples (A', A'') of algebras and $A' - A''$-bimodules (in other terms $A' \otimes A''$-modules) under particular conditions (generalizing the classical situation) ensuring the isomorphim of the homology of A' and the cohomology of A'' - see **1** (iii) below. In this note we study the structure (rebaptized dual riemannian quantum space - see **1** (ii) below) corresponding to the "algebraic condition" in Alain Connes' definition of Poincaré duality. The "topological condition" recalled under (3), below will not be used in what follows.

1. Definitions. Terminology.

(i) A *riemannian quantum space* is a couple $(A, \mathbf{H})$ of a unital*-algebra A and a K-cyle $\mathbf{H} = (H, D, \chi)$ over A. The quantum space $(A, \mathbf{H})$ is d-dimensional whenever the K-cyle $\mathbf{H}$ is d^+-summable.

(ii) A (d-dimensional) *dual riemannian quantum space* is a riemannian quantum space $(A' \otimes A'', \mathbf{H})$, where $A' \otimes A''$ is the algebraic tensor product of the unital*-algebras A', A'' (with respective units $\mathbb{1}'$ and $\mathbb{1}''$); and where $\mathbf{H} = (H, D, \chi)$ is a (d^+-summable) K-cyle of $A' \otimes A''$ fullfilling the algebraic condition:

$$[[D, \underline{a' \otimes \mathbb{1}}], \ \underline{\mathbb{1} \otimes a''}] = 0 \quad , \ a' \in A', \ a'' \in A'', \tag{1}$$

(In order to alleviate notation, we shall consider A' and A'' as acting on H by commuting representations $a' \to \underline{a'} = \underline{a' \otimes \mathbb{1}}$ and $a'' \to \underline{a''} =$

[1] We propose below to call these couples "riemannian quantum spaces".

$\underline{1\!\!1 \otimes a''}$, then reading

$$[[D,\ \underline{a'}],\ \underline{a''}] = 0 \quad ,\ a' \in A',\ a'' \in A'') \,. \tag{2}$$

(iii) A d-dimensional *riemannian quantum manifold* is a dual riemannian quantum space $(A' \otimes A'', \mathbf{H})$ whose d^+-summable K-cyle $\mathbf{H} = (H, D, \chi)$ has its "Hochschild obstruction" vanishing under the operator B, i.e.

$$Tr_\omega \left\{ \chi \underline{a}_0 [D, \underline{a}_1] \ldots [D, \underline{a}_k] |D|^{-k} \right\} = 0 \quad ,\ a_0, a_1, \ldots, a_k \in A. \tag{3}$$

(iv) The quantum space $(A, \mathbf{H})$, $\mathbf{H} = (H, D, \chi)$, is *self-dual* whenever the operator algebra $\{\underline{a}, a \in A\}$ is commutative, and one has:

$$[[D,\ \underline{a}],\ \underline{b}] = 0 \quad ,\ a, b \in A. \tag{4}$$

The *gauge group of the dual riemannian quantum space* $(A' \otimes A'', \mathbf{H})$ is by definition the product $\mathcal{G}' \times \mathcal{G}''$ of the respective gauge group $\mathcal{G}'\mathcal{G}''$ (=groups of unitaries of A', resp. A'').

2. Remarks

(a) There is a bijection between riemannian quantum spaces $(A' \otimes A'', \mathbf{H})$, and pairs $(A', \mathbf{H}), (A'', \mathbf{H})$ of riemannian quantum spaces with a common d^+-summable K-cycle $\mathbf{H} = (H, D, \chi)$ for which $\underline{a'}$ and $\underline{a''}$ commute for all $a' \in A'$ and $a'' \in A''$. The bijection is effected via the correspondence $\underline{a' \otimes a''} \leftrightarrow \underline{a'} \cdot \underline{a''}$, in accordance with the former identifications $\underline{a'} = \underline{a' \otimes 1\!\!1''}$ and $\underline{a''} = \underline{1\!\!1' \otimes a''}$.

(b) With $(A' \otimes A'', \mathbf{H})$ a dual riemannian quantum space we have that:

$$[\pi_D(\Omega A'),\ \pi_D(A'')] = [\pi_D(\Omega A''),\ \pi_D(A')] = 0, \tag{5}$$

(c) There is a bijection between self-dual riemannian quantum spaces $(A, \mathbf{H})$ and riemannian quantum spaces $(A \otimes A, \mathbf{H})$ with A commutative.

(d) The couple of the algebra of smooth functions over a compact spin^c riemannian manifold, and its Dirac K-cycle is (the archetype of) a self-dual quantum space.

Proof:

(a): obvious.

(b): follows from (1a) via the definition of π_D.

(c): obvious.

(d): follows from the fact that the a commute with the $\underline{\gamma}(\mathbf{d}a)$.

Our first problem is the definition of connections of quantum manifolds (or, for that matter of dual riemannian quantum spaces). The passage from riemannian quantum spaces involving one algebra to dual quantum spaces involving a commuting[2] couple of algebras (A', A'') (i.e. their tensor product $A = A' \otimes A''$ together with its tensorial splitting) raises questions as to the maintenance in this enlarged[3] frame of the previous philosophy pertaining to riemannian quantum spaces (and of the latter's implications on connections, the quantum Yang-Mills algorithm, etc.). Taking the tensor product $A = A' \otimes A''$ as the new "basic" algebra, one is at first tempted to consider $(\Omega A, \delta)$ as the "basic" differential algebra. However the latter does not contain the information of the commutativity of A' and A'' and of the vanishing of commutators (1a) in the representation π_D. This information is incorporated in the following definitions:

3. Definitions. Let A' and A'' be unital *-algebras with $A = A' \otimes A''$ and with $(\Omega A, \delta)$ the unital differential envelope of A; and consider the following subsets of ΩA^1:[4]

$$\mathbf{S}' = \left\{ [\delta(a' \otimes 1\!\!1''),\ 1\!\!1' \otimes a''] \in \Omega A^1 ; a' \in A', a'' \in A'' \right\} = -\mathbf{S}'' \tag{6}$$

resp.

$$\mathbf{S}'' = \left\{ [\delta(1\!\!1' \otimes a''),\ a' \otimes 1\!\!1''] \in \Omega A^1 ; a' \in A', a'' \in A'' \right\}. \tag{7}$$

We denote by $\mathbf{j}_P$ the ideal of ΩA generated by $\mathbf{S}'$ (or, for that matter, by $\mathbf{S}''$), and define the *Poincaré ideal* as:

$$\mathbf{k}_P = \mathbf{j}_P + \delta \mathbf{j}_P. \tag{8}$$

[2]in the representation π_D.

[3]enlarged in view of Remark see.

[4]Note that $\mathbf{S}' = -\mathbf{S}''$ because $\delta\,[1\!\!1' \otimes a'', a' \otimes 1\!\!1''] = 0 = [\delta(1\!\!1' \otimes a''),\ a' \otimes 1\!\!1''] + [1\!\!1' \otimes a'', \delta(a' \otimes 1\!\!1'')]$.

This definition implies:

4. Lemma. *With* $(A, \mathbf{H})$, $A = A' \otimes A''$, $\mathbf{H} = (H, D, \chi)$, $\mathbf{j}_P$, *and* $\mathbf{k}_P$ *as in the previous definitions we have that:*

(i): $\mathbf{j}_P$ *is a graded ideal of* ΩA.

(ii): $\mathbf{k}_P$ *is a graded differential ideal of* ΩA, *with* $\mathbf{k}_P^n = \mathbf{j}_P^n + \delta \mathbf{j}_P^{n-1}$, *hence* $\Omega A / \mathbf{k}_P$ *is a* $\mathbf{N}$*-graded differential algebra, with differential* d *obtained from that of* ΩA *by passage to the quotient through the ideal* $\mathbf{k}_P$.

(iii): *One has* $[\delta(a' \otimes \mathbb{1}''), \delta(\mathbb{1}' \otimes a'')] \in \mathbf{k}_P^2$, $a' \in A', a'' \in A''$.

(iv): *The ideal* $\mathbf{k}_P$ *is generated by the elements* $[\delta(a' \otimes \mathbb{1}''), \mathbb{1}' \otimes a'']$ *and* $[\delta(a' \otimes \mathbb{1}''), \delta(\mathbb{1}' \otimes a'')]$, $a' \in A', a'' \in A''$.

(v): *If in addition* $(A, \mathbf{H})$, $A = A' \otimes A''$, $\mathbf{H} = (H, D, \chi)$, *is a dual riemannian quantum space, the quantum DeRham complex* ΩA_D *is a quotient of* $\Omega A / \mathbf{k}_P$ *as a* $\mathbf{N}$*-graded differential algebra.*

Proof:

(i): $\mathbf{j}_P$ is indeed generated by homogeneous elements.

(ii): cf. [1].

(iii): We have, since $\delta^2 = 0$:

$$\delta\,[\mathbb{1}' \otimes a'',\ \delta(a' \otimes \mathbb{1}'')] = [\delta(\mathbb{1}' \otimes a''),\ \delta(a' \otimes \mathbb{1}'')]\,. \tag{9}$$

(iv): $\mathbf{j}_P$ consists of linear combinations of elements of the type $\omega\,[\delta(a' \otimes \mathbb{1}''), \mathbb{1}' \otimes a'']\,\Psi$, $a' \in A'$, $a'' \in A''$, $\omega, \Psi \in \Omega A$, thus $\mathbf{k}_P = \mathbf{j}_P + \delta \mathbf{j}_P$ consists of linear combinations of elements of the latter type plus elements of the type

$$\begin{aligned} (\delta\omega)\,[\delta(a' \otimes \mathbb{1}''), \mathbb{1}' \otimes a''] \quad & \Psi^{\pm}\omega\,[\delta(a' \otimes \mathbb{1}''),\ \delta(a' \otimes \mathbb{1}'')] \\ & \Psi^{\pm}(\delta\omega)\,[\delta(a' \otimes \mathbb{1}''), \mathbb{1}' \otimes a'']\,\Psi, \end{aligned} \tag{10}$$

whence the result by (iii).

(v): The definition of a dual riemannian quantum space entails that the ideal $\mathbf{j}_P$ is contained in the kernel $\mathbf{K}^*$ of π_D, hence the graded differential ideal $\mathbf{k}_P$ is contained in the graded differential ideal $\mathbf{J} = \mathbf{K}^* + \delta\mathbf{K}^*$. Thus ΩA_D is a quotient of $\Omega A/\mathbf{k}_P$ as a $\mathbf{N}$-graded differential algebra.

The fact which enlightens the issues raised above is now the isomorphism $(\Omega A/\mathbf{k}, \delta) \cong (\Omega A_\otimes,\ \delta_\otimes)$ - cf. **6**. - and the ensuing fact that the skew tensor product $\Omega A' \widehat{\otimes} \mu \Omega A''$ "sits above" $\Omega_D A$ - cf. **7**. - and thus is a natural receptacle for connections. The relationship between $\Omega A' \widehat{\otimes},\ \Omega A''$ and the differential envelope ΩA of $A' \widehat{\otimes},\ A''$ has been studied in [4], we recall the results here:

5. Reminder (differential envelope of tensor products cf. [4]). Let A' and A'' be unital (real or complex) *-algebras, with respective units $\mathbb{1}'$ and $\mathbb{1}''$, and with respective unital differential envelopes $(\Omega A', \delta')$ and $(\Omega A'', \delta'')$. Let $A = A' \otimes A''$, with unit $\mathbb{1}$ and unital differential envelope $(\Omega A, \delta)$, be the algebraic tensor-product of A' and A". Set $\Omega A_\otimes = \Omega A' \widehat{\otimes} \Omega A''$, with differential $\delta_\otimes$, for the skew tensor product of $\Omega A'$ and $\Omega A''$: i.e. we have, for $\omega', \psi' \in A', \omega'', \psi'' \in A''$:

$$\left\{ \begin{array}{l} \Omega A_\otimes^n = \sum_{p+q=n} \Omega A'^p \otimes \Omega A''^q, \\ \\ (\omega' \otimes \omega'')(\psi' \otimes \psi'') = \omega'\psi' \otimes \omega''\psi'' \quad . \\ \\ \delta_\otimes(\omega' \otimes \omega'') = \delta'\omega' \otimes \omega'' + \omega' \otimes \delta\omega'' \end{array} \right. \tag{11}$$

We then have a split short exact sequence $0 \to \mathbf{k} \to \Omega A \overset{\bar{i}}{\underset{\overset{J}{\leftarrow}}{\to}} \Omega A_\otimes \to 0$, where:

$$\begin{aligned} &\bar{i}\left[\left(a'_0 \widehat{\otimes} a''_0\right) \delta \left(a'_1 \widehat{\otimes} a''_1\right) \ldots \delta \left(a'_n \widehat{\otimes} a''_n\right)\right] \\ &= \left(a'_0 \widehat{\otimes} a''_0\right) \delta_\otimes \left(a'_1 \widehat{\otimes} a''_1\right) \ldots \delta_\otimes \left(a'_n \widehat{\otimes} a''_n\right) \\ &= \left(a'_0 \widehat{\otimes} a''_0\right) \left[\delta' a'_1 \widehat{\otimes} a''_1 + a'_1 \widehat{\otimes} \delta'' a''_1\right] \ldots \left[\delta' a'_n \widehat{\otimes} a''_n + a'_n \widehat{\otimes} \delta'' a''_n\right], \end{aligned} \tag{12}$$

and

$$J\left(\omega' \widehat{\otimes} \omega''\right) = \overline{j'}(\omega')\overline{j''}(\omega''), \quad \omega' \in \Omega A',\ \omega'' \in \Omega A'', \tag{13}$$

where:

$$\begin{aligned}&\overline{j'}\left(a_0'\delta' a_1'\dots\delta' a_n'\right)\\&=\left(a_0'\widehat{\otimes}\mathbb{1}''\right)\delta\left(a_1'\widehat{\otimes}\mathbb{1}''\right)\dots\delta\left(a_n'\widehat{\otimes}\mathbb{1}''\right),\quad a_0',a_1',\dots,a_n'\in A',\end{aligned}\tag{14}$$

$$\begin{aligned}&\overline{j''}\left(a_0''\delta'' a_1''\dots\delta'' a_n''\right)\\&=\left(\mathbb{1}'\widehat{\otimes}a_0''\right)\delta\left(\mathbb{1}'\widehat{\otimes}a_1''\right)\dots\delta\left(\mathbb{1}'\widehat{\otimes}a_n''\right),\quad a_0'',a_1'',\dots,a_n''\in A'',\end{aligned}\tag{15}$$

hence, for $\omega'\in\Omega A',\ \omega''\in\Omega A''$:

$$\begin{cases}\overline{i}\circ\overline{j'}(\omega')=\omega'\widehat{\otimes}\mathbb{1}''\\ \overline{i}\circ\overline{j''}(\omega'')=\mathbb{1}'\widehat{\otimes}\omega''\end{cases}\tag{16}$$

Here $\overline{i}$, $\overline{j'}$ and $\overline{j''}$ are homomorphisms of bigraded differential algebras, whilst J is simply linear injective (neither multiplicative, nor intertwining differentials). As a consequence, the kernel $\mathbf{k}$ of $\overline{i}$ is a bigraded differential ideal of ΩA, with

$$\Omega A^n_{\otimes}=\Omega A^n/\mathbf{k}^n,\tag{17}$$

and one has

$$\mathbf{k}^n=(id-J\overline{i})\Omega A^n_{\otimes}.\tag{18}$$

6. Proposition. *The Poincaré ideal $\mathbf{k}_P$ of Definition 1 coincides with the kernel $\mathbf{k}$ of $\overline{i}$:*

$$\mathbf{k}^n_P=\mathbf{k}^n,\qquad n\in\mathbf{N},\tag{19}$$

Consequently we have the following isomorphism of $\mathbf{N}$-graded differential algebras:

$$(\Omega A/\mathbf{k},\delta)\cong(\Omega A_{\otimes},\ \delta_{\otimes}).\tag{20}$$

7. Corollary. *The canonical map $\psi:\Omega A\to\Omega A_D$ factors through the canonical map $\phi:\Omega A\to\Omega A_{\otimes}$ as $\psi=\xi\circ\phi$, where $\xi:\Omega A_{\otimes}\to\Omega A_D$ is a homomorphism of $\mathbf{N}$-graded differential algebras.*

Proof of the Corollary: We recall that we have, with $\mathbf{K}^*$ the homogeneized kernel of π_D:

$$\Omega A_D=\Omega A/\mathbf{J}=\Omega A/(\mathbf{K}^*+\delta\mathbf{K}^*).\tag{21}$$

Now requirement (1) entails that π_D vanishes on $\mathbf{S}'$, thus on $\mathbf{j}_P$, whence $\mathbf{j}_P \subset \mathbf{K}^*$, $\delta\mathbf{j}_P \subset \delta\mathbf{K}^*$ and $\mathbf{k}_P \subset \mathbf{J}$, whence the claim.

Proof of the Proposition: We first prove the inclusion $\subset$ in (19). Owing to [4](vi) it suffices to check $[\delta(a' \otimes 1\!\!1''),\ 1\!\!1' \otimes a'']$, $[\delta(a' \otimes 1\!\!1''),\ \delta(1\!\!1' \otimes a'')] \in \mathbf{k}$, now we have, by (21):

$$\begin{aligned} \bar{i}\left\{[\delta(a' \otimes 1\!\!1''),\ 1\!\!1' \otimes a'']\right\} &= \left[\bar{i}\delta(a' \otimes 1\!\!1''),\ \bar{i}(1\!\!1' \otimes a'')\right] \\ &= [\delta a' \otimes 1\!\!1'',\ 1\!\!1' \otimes a''] = 0, \end{aligned} \tag{22}$$

but $\bar{i}$ then vanishes also on $[\delta(a' \otimes 1\!\!1''),\ \delta(1\!\!1' \otimes a'')] = \delta[\delta(a' \otimes 1\!\!1''),\ 1\!\!1' \otimes a'']$ because it commutes with δ.

We now prove the inclusion $\subset$ in (19). According to (18) it is enough to check that $(id - J\bar{i})\omega \in \mathbf{K}^n_P$, for all ω of the form:

$$\begin{aligned} \omega &= \left(a'_0 \widehat{\otimes} a''_0\right) \delta\left(a'_1 \widehat{\otimes} a''_1\right) \ldots \delta\left(a'_n \widehat{\otimes} a''_n\right) \\ &= \left(a'_0 \widehat{\otimes} a''_0\right)\left[\delta\left(a'_1 \widehat{\otimes} 1\!\!1\right)\left(1\!\!1' \widehat{\otimes} a''_1\right) + \left(a'_1 \widehat{\otimes} 1\!\!1\right)\delta\left(1\!\!1' \widehat{\otimes} a''_1\right)\right] \ldots \\ &\qquad \ldots\left[\delta\left(a'_n \widehat{\otimes} 1\!\!1\right)\left(1\!\!1' \widehat{\otimes} a''_n\right) + \left(a'_n \widehat{\otimes} 1\!\!1\right)\delta\left(1\!\!1' \widehat{\otimes} a''_n\right)\right] \end{aligned} \tag{23}$$

for $a'_i \in A'$, $a''_i \in A$, $i = 1, \ldots, n$. We compute $J\bar{i}\omega$: we have:

$$\bar{i}\omega = \left(a'_0 \widehat{\otimes} a''_0\right)\left[\delta' a'_1 \widehat{\otimes} a''_1 + a'_1 \widehat{\otimes} \delta'' a''_1\right] \ldots \left[\delta' a'_n \widehat{\otimes} a''_n + a'_n \widehat{\otimes} \delta'' a''_n\right], \tag{24}$$

to which we apply J as in (13): for this we must effect the products (23) so as to get a sum of terms of the form $\omega' \widehat{\otimes} \omega''$, where the ω' are products of primed symbols a'_i and $\delta' a'_i$, whilst the ω'' are products of double-primed symbols a''_j and $\delta'' a''_i$. The product $\overline{j'}(\omega')\overline{j''}(\omega'')$ is then obtained from $\omega'\omega''$ by effecting the changes $a'_i \to a'_i \widehat{\otimes} 1\!\!1''$, $\delta' a'_i \to \delta\left(a'_i \widehat{\otimes} 1\!\!1''\right)$, $a''_j \to 1\!\!1' \widehat{\otimes} a''_j$, $\delta'' a''_i \to \delta\left(1\!\!1' \widehat{\otimes} a''_i\right)$, ending up with an expression which can be turned into the expression in the second line (23) by commutations of primed with double-primed terms: the difference $(id - J\bar{i})\omega$ is thus a sum of products each of which contains a commutator generator of the Poincaré ideal $\mathbf{k}_P$.

We are now in a position to discuss the issue of "connections" of the dual riemannian quantum space $(A = A' \otimes A'',\ \mathbf{H})$, $\mathbf{H} = (H, D, \chi)$.

The quantum forms of A, if we forget about its tensorial decomposition $A = A' \otimes A''$, are the elements of the quantum DeRham complex $(\Omega A_D, \mathbf{d})$, its connections are thus the grade-one $\mathbf{d}$-derivations ∇ of ΩA_D. Now, utilizing the homomorphism $\xi : \Omega A_\otimes \to \Omega A_D$ (cf. **7.**), the latter can be considered as ξ-images of δ-derivations of $\Omega A_\otimes$ ("preconnexions"), this according to the scheme:

8. Lemma. *Let $\phi : (\Omega, \delta) \to (\widetilde{\Omega}, \widetilde{\delta})$ be an epimorphism of $\mathbf{N}$-graded differential algebras, and denote by $Der_\delta \Omega$ the set of grade-one δ-derivations of Ω considered as a right Ω-module (resp. by $Der_{\widetilde{\delta}} \widetilde{\Omega}$ the set of $\widetilde{\delta}$-derivations of $\widetilde{\Omega}$ considered as a right $\widetilde{\Omega}$-module). Then, given $\widetilde{\nabla} \in Der_{\widetilde{\delta}} \widetilde{\Omega}$, there is $\nabla \in Der_\delta \Omega$ such that ϕ intertwines ∇ and $\widetilde{\nabla} : \phi \circ \nabla = \widetilde{\nabla} \circ \phi$. Specifically, for $\widetilde{\nabla} = \widetilde{\delta} + \widetilde{\rho}, \rho' \in \widetilde{\Omega}^1$, one has $\nabla = \delta + \rho$, picking $\rho \in \Omega^1$ such that $\phi\rho = \widetilde{\rho}$.*[5] *The respective curvatures θ and $\widetilde{\theta}$ of ∇ and $\widetilde{\nabla}$ are related by $\widetilde{\theta} = \phi\theta$.*

<u>Proof:</u> The surjectivity of ϕ insures the existence of ρ. One has then, for $\omega \in \Omega$:

$$\begin{aligned} (\phi \circ \nabla)\omega &= \phi(\delta\omega + \rho\omega) = (\phi \circ \delta)\omega + (\phi\rho)(\phi\omega) \\ &= (\widetilde{\delta} \circ \phi)\omega + \widetilde{\rho}(\phi\omega) = (\widetilde{\nabla} \circ \phi)\omega. \end{aligned} \tag{25}$$

One has furthermore:

$$\phi\theta = \phi(\delta\rho + \rho^2) = \widetilde{\delta}(\phi\rho) + (\phi\rho)^2 = \widetilde{\delta}\widetilde{\rho} + \widetilde{\rho}^2 = \theta'. \tag{26}$$

Our problem was to select, amongst all connexions of A, the ones matching the tensorial decomposition $A = A' \otimes A''$. Looking at the preconnections acting on $\Omega A_\otimes = \Omega A' \widehat{\otimes} \Omega A''$, one is now naturally led to the following notion of "biconnection".

9. Definitions. Let $(A = A' \otimes A'',\ \mathbf{H})$, with $\mathbf{H} = (H, D, \chi)$, be a dual riemannian quantum space, and consider the above $\mathbf{N}$-graded differential algebras $(\Omega A_\otimes, \delta)$ and $(\Omega A_D, \mathbf{d})$.

(i): a *preconnection of* $(A, \mathbf{H})$ is a grade-one graded δ-derivation ∇ of the right $\Omega A_\otimes$-module $\Omega A_\otimes$, with *curvature* is $\theta = \nabla^2$. The *preconnection-one form* $\rho \in \Omega A^1_\otimes$ arises by asking $\nabla = \delta + \rho$ (one has thus $\theta = \delta\rho + \rho^2$).

[5] In the expression $\nabla = \delta + \rho$ the symbol ρ has to be interpreted as usual as denoting multiplication from the left by ρ.

The action $\nabla \to \nabla^{\mathbf{u}}$ of the element $\mathbf{u}$ of the gauge group $\mathcal{G}$ of A on the preconnection is by definition:[6]

$$\nabla^{\mathbf{u}}\omega = \mathbf{u}\nabla(\mathbf{u}^\star\omega) \tag{27}$$

(one has then $\nabla^{\mathbf{u}} = \delta + \rho^{\mathbf{u}}$ with

$$\rho^{\mathbf{u}} = \mathbf{u}\rho\mathbf{u}^\star + \mathbf{u}\delta\mathbf{u}^\star). \tag{28}$$

(ii): a *biconnection* is a preconnection of the form $\nabla = \nabla'\widehat{\otimes} id + id\widehat{\otimes}\nabla''$, where ∇' and ∇'' are respective connections of A' and A''.[7] (one has thus $\rho = \rho' \otimes 1\!\!1'' + 1\!\!1' \otimes \rho''$ for the preconnection-one form ρ of ∇)

(iii): a *connection of the dual riemannian quantum space* $(A = A' \otimes A'', \mathbf{H})$ is a grade-one graded δ-derivation ∇ of the right ΩA_D-module ΩA_D which is the image of a biconnection in the homomorphism $\xi : \Omega A_\otimes \to \Omega A_D$.

We check that $\nabla = \nabla'\widehat{\otimes} id + id\widehat{\otimes}\nabla''$ is a preconnection with preconnection-one form $\rho = \rho' \otimes 1\!\!1'' + 1\!\!1' \otimes \rho''$: for homogeneous $\omega', \psi' \in \Omega A', \omega'', \psi'' \in \Omega A''$, we have:

$$\begin{aligned}\nabla(\omega'\widehat{\otimes}\omega'') &= \nabla'\omega'\widehat{\otimes}\omega'' + (-1)^{\partial\omega'}\omega'\widehat{\otimes}\nabla''\omega'' \\ &= (\delta' + \rho')\omega'\widehat{\otimes}\omega'' + (-1)^{\partial\omega'}\omega'\widehat{\otimes}(\delta'' + \rho'')\omega'' \\ &= \delta(\omega'\widehat{\otimes}\omega'') + (\rho' \otimes 1\!\!1'' + 1\!\!1' \otimes \rho'')(\omega'\widehat{\otimes}\omega'').\end{aligned} \tag{29}$$

We then have the following satisfying set of facts:

10. Proposition. *Let* $\nabla = \nabla'\widehat{\otimes} id + id\widehat{\otimes}\nabla'' = \delta + \rho,\ \rho = \rho'\widehat{\otimes}1\!\!1'' + 1\!\!1'\widehat{\otimes}\rho''$, *be a biconnection of* $(A = A' \otimes A'', \mathbf{H})$. *Then*

(i): *The curvature* θ *of* ∇ *is the sum of the curvatures* θ' *and* θ'' *of* ρ' *and* ρ'' *in the following sense: one has:*

$$\nabla^2 = \nabla'^2\widehat{\otimes} id + id\widehat{\otimes}\nabla''^2, \tag{30}$$

$$\theta = \theta'\widehat{\otimes}1\!\!1'' + 1\!\!1'\widehat{\otimes}\theta'', \tag{31}$$

$$\rho^2 = \rho'^2\widehat{\otimes}1\!\!1'' + 1\!\!1'\widehat{\otimes}\rho''^2. \tag{32}$$

[6] $\mathcal{G}$ is the group of unitaries of $A = \Omega A^0_\otimes$.

[7] Thus ∇' is a grade-one δ'-derivation of $\Omega A'$, with $\nabla' = \delta' + \rho', \rho' \in \Omega A'^1_\otimes$, and ∇'' is a grade-one δ''-derivation of $\Omega A''$ with $\nabla'' = \delta'' + \rho'', \rho'' \in \Omega A''^1_\otimes$.

(ii): *The set of biconnections is stable under the gauge group* $\mathcal{G}' \times \mathcal{G}''$ *of the dual riemannian quantum space* $(A = A' \otimes A'', \mathbf{H})$ *one has, for* $\mathbf{u}' \in \mathcal{G}', \mathbf{u}'' \in \mathcal{G}'', \mathbf{u} = \mathbf{u}' \widehat{\otimes} \mathbf{u}''$*:*

$$\nabla^{\mathbf{u}} = \nabla'^{\mathbf{u}'} \widehat{\otimes} id + id \widehat{\otimes} \nabla''^{\mathbf{u}''}, \tag{33}$$

$$\rho^{\mathbf{u}} = \rho'^{\mathbf{u}'} \widehat{\otimes} \mathbb{1}'' + \mathbb{1}' \widehat{\otimes} \rho''^{\mathbf{u}''}. \tag{34}$$

<u>Proof:</u> Check of (32): we have:

$$\begin{aligned} \nabla^2 &= \left(\nabla' \widehat{\otimes} id + id \widehat{\otimes} \nabla''\right)\left(\nabla' \widehat{\otimes} id + id \widehat{\otimes} \nabla''\right) \\ &= \nabla'^2 \widehat{\otimes} id + \nabla' \widehat{\otimes} \nabla'' - \nabla' \widehat{\otimes} \nabla'' + id \widehat{\otimes} \nabla''^2 \\ &= \nabla'^2 \widehat{\otimes} id + id \widehat{\otimes} \nabla''^2 \\ &= \text{(multiplication from the left by) } \theta' \otimes \mathbb{1}'' + \mathbb{1}' \otimes \theta'' \end{aligned} \tag{35}$$

(one can also deduce (33) from (34) via $\theta = \delta\rho + \rho^2$. Check of (34):

$$\begin{aligned} 2\rho^2 = [\rho, \rho] &= \left[\rho' \widehat{\otimes} \mathbb{1}'' + \mathbb{1}' \widehat{\otimes} \rho'',\ \rho' \widehat{\otimes} \mathbb{1}'' + \mathbb{1}' \widehat{\otimes} \rho''\right] \\ &= \left[\rho' \widehat{\otimes} \mathbb{1}'',\ \rho' \widehat{\otimes} \mathbb{1}''\right] + \left[\rho' \widehat{\otimes} \mathbb{1}'',\ \mathbb{1}' \widehat{\otimes} \rho''\right] \\ &\quad + \left[\mathbb{1}' \widehat{\otimes} \rho'',\ \rho' \widehat{\otimes} \mathbb{1}''\right] + \left[\mathbb{1}' \widehat{\otimes} \rho'',\ \mathbb{1}' \widehat{\otimes} \rho''\right], \\ &= [\rho' \rho'] \widehat{\otimes} \mathbb{1}'' + \mathbb{1}' \widehat{\otimes} [\rho'' \rho''] = 2\rho'^2 \widehat{\otimes} \mathbb{1}'' + 2\mathbb{1}' \widehat{\otimes} \rho''^2). \end{aligned} \tag{36}$$

Check of (35): one has:

$$
\begin{aligned}
\nabla^{\mathbf{u}}\left(\omega'\widehat{\otimes}\omega''\right) &= \left(\mathbf{u}'\widehat{\otimes}\mathbf{u}''\right)\nabla\left[\left(\mathbf{u}'^{*}\widehat{\otimes}\mathbf{u}''^{*}\right)\left(\omega'\widehat{\otimes}\omega''\right)\right] \\
&= \left(\mathbf{u}'\widehat{\otimes}\mathbf{u}''\right)\left(\nabla'\widehat{\otimes}id + id\widehat{\otimes}\nabla''\right)\left[\left(\mathbf{u}'^{*}\omega'\widehat{\otimes}\mathbf{u}''^{*}\omega''\right)\right] \\
&= \left(\mathbf{u}'\widehat{\otimes}\mathbf{u}''\right)\left[\nabla'\left(\mathbf{u}'^{*}\omega'\right)\widehat{\otimes}\mathbf{u}''^{*}\omega'' + (-1)^{\partial\omega'}\mathbf{u}'^{*}\omega'\widehat{\otimes}\nabla''\left(\mathbf{u}''^{*}\omega''\right)\right] \\
&= \left(\nabla'^{\mathbf{u}}\omega'\right)\widehat{\otimes}\omega'' + (-1)^{\partial\omega'}\omega'\widehat{\otimes}\nabla''^{\mathbf{u}}\omega'' \\
&= \left(\nabla'^{\mathbf{u}'}\widehat{\otimes}id + id\widehat{\otimes}\nabla''^{\mathbf{u}''}\right)\left(\omega'\widehat{\otimes}\omega''\right)
\end{aligned}
\tag{37}
$$

(other proof)

$$
\begin{aligned}
\rho^{\mathbf{u}} &= \left(\mathbf{u}'\widehat{\otimes}\mathbf{u}''\right)\left(\rho'\widehat{\otimes}\mathbb{1}'' + \mathbb{1}'\widehat{\otimes}\rho''\right)\left(\mathbf{u}'^{*}\widehat{\otimes}\mathbf{u}''^{*}\right) \\
&\quad + \left(\mathbf{u}'\widehat{\otimes}\mathbf{u}''\right)\left(\delta'\mathbf{u}'^{*}\widehat{\otimes}\mathbf{u}''^{*} + \mathbf{u}'^{*}\widehat{\otimes}\delta''\mathbf{u}''^{*}\right) \\
&= \left(\mathbf{u}'\rho'\mathbf{u}'^{*}\widehat{\otimes}\mathbb{1}'' + \mathbb{1}'\widehat{\otimes}\mathbf{u}''\rho''\mathbf{u}''^{*}\right) + \left(\mathbf{u}'\delta'\mathbf{u}'^{*}\widehat{\otimes}\mathbb{1}'' + \mathbb{1}'\widehat{\otimes}\mathbf{u}''\delta''\mathbf{u}''^{*}\right) \\
&= \rho'^{\mathbf{u}'}\widehat{\otimes}\mathbb{1}'' + \mathbb{1}'\widehat{\otimes}\rho''^{\mathbf{u}'}
\end{aligned}
\tag{38}
$$

Our next concern is a discussion of duality versus tensor products, defined as follows:

11. Definitions. With $(A_1, \mathbf{H}_1)$, $(A_2, \mathbf{H}_2)$ riemannian quantum spaces, the *(tensor) product of* $(A_1, \mathbf{H}_1)$ *and* $(A_2, \mathbf{H}_2)$ is the riemannian quantum space $(A_1 \otimes A_2,\ \mathbf{H}_1 \otimes \mathbf{H}_2)$, where $\mathbf{H}_1 \otimes \mathbf{H}_2$ is the tensor product of the K-cycles $\mathbf{H}_1$ and $\mathbf{H}_2$: one has thus, with $\mathbf{H}_1 = (H_1, D_1, \chi_1)$, $\mathbf{H}_2 = (H_2, D_2, \chi_2)$, and

$\mathbf{H}_1 \otimes \mathbf{H}_2 = (H, D, \chi),\ a_1 \in A_1,\ a_2 \in A_2$:

$$\begin{cases} H = H_1 \otimes H_2 \\ \\ \chi = \chi_1 \otimes \chi_2 \\ \\ \underline{a_1 \otimes a_2} = \underline{a}_1 \otimes \underline{a}_2 \\ \\ D \ = D_1 \hat{\otimes} id_2 + id_1 \hat{\otimes} D_2 \\ \\ \quad\ = D_1 \otimes id_2 + \chi_1 \otimes D_2 \end{cases} \tag{39}$$

12. Proposition. *The (tensor) product of two dual riemannian quantum spaces* $(A_1' \otimes A_1'', H_1), (A_2' \otimes A_2'', H_2)$ *is a dual riemannian quantum space* $((A_1' \otimes A_2') \otimes (A_1'' \otimes A_2''), H_1 \otimes H_2)$.

Proof: With $(A_1 = A_1' \otimes A_1'', \mathbf{H}_1),\ \mathbf{H}_1 = (H_1, D_1, \chi_1),\ (A_2 = A_2' \otimes A_2'', \mathbf{H}_2)\mathbf{H}_2 = (H_2, D_2, \chi_2)$, dual riemannian quantum spaces, we consider the riemannian quantum space $(A_1 \otimes A_2, H_1 \otimes H_2)$ tensor product of $(A_1,\ H_1)$ and $(A_2,\ H_2)$ cf. [4].

With $a_1' \in A_1',\ a_1'' \in A_1'',\ a_2' \in A_2',\ a_2'' \in A_2''$, we want to check that one has:

$$\left[\left[D, \underline{a_1'} \otimes \underline{a_2'}\right],\ \underline{a_1''} \otimes \underline{a_2''}\right] = 0 \tag{40}$$

We have, since $\underline{a_1'}$ is even:

$$(D_1 \otimes id + \chi_1 \otimes D_2)\left(\underline{a_1'} \otimes \underline{a_2'}\right) = D_1 \underline{a_1'} \otimes \underline{a_2'} + \chi_1 \underline{a_1'} \otimes D_2 \underline{a_2'}, \tag{41}$$

$$\left(\underline{a_1'} \otimes \underline{a_2'}\right)(D_1 \otimes id + \chi_1 \otimes D_2) = \underline{a_1'} D_1 \otimes \underline{a_2'} + \chi_1 \underline{a_1'} \otimes \underline{a_2'} D_2, \tag{42}$$

hence

$$\left[D, \underline{a_1'} \otimes \underline{a_2'}\right] = \left[D_1, \underline{a_1'}\right] \otimes \underline{a_2'} + \chi_1 \underline{a_1'} \otimes \left[D_2, \underline{a_2'}\right], \tag{43}$$

and further

$$\left[D, \underline{a_1'} \otimes \underline{a_2'}\right]\left(\underline{a_1''} \otimes \underline{a_2''}\right) = \left[D_1, \underline{a_1'}\right] \underline{a_1''} \otimes \underline{a_2' a_2''} + \chi_1 \underline{a_1' a_1''} \otimes \left[D_2, \underline{a_2'}\right] \underline{a_2''}, \tag{44}$$

$$\left(\underline{a_1''} \otimes \underline{a_2''}\right)\left[D, \underline{a_1'} \otimes \underline{a_2'}\right] = \underline{a_1''}\left[D_1, \underline{a_1'}\right] \otimes \underline{a_2'' a_2'} + \chi_1 \underline{a_1'' a_1'} \otimes \underline{a_2''}\left[D_2, \underline{a_2'}\right], \tag{45}$$

hence, since $\underline{a'_1}$ and $\underline{a''_1}$ commute:

$$\left[\left[D, \underline{a'_1 \otimes a'_2}\right], \ \underline{a''_1 \otimes a''_2}\right] = \left[\left[D_1, \underline{a'_1}\right], \ \underline{a''_1}\right] \otimes \underline{a'_2 a''_2} + \chi_1 \underline{a'_1 a''_1} \otimes \left[\left[D_2, \underline{a'_2}\right], \ \underline{a''_2}\right], \tag{46}$$

vanishing by the duality property of the systems 1 and 2.

A first type of dual riemannian quantum spaces is obtained as follows.

13. Proposition. *The (tensor) product* $(A_1 \otimes A_2,\ H_1 \otimes H_2)$ *of two riemannian quantum spaces* (A_1, H_1) *and* (A_2, H_2) *is a dual riemannian quantum spaces.*

Proof: We have

$$\begin{aligned} \left[D, \underline{a_1 \otimes 1\!\!1_2}\right] &= \left[D_1 \otimes id_2 + \chi_1 \otimes D_2, \underline{a_1} \otimes id_2\right] \\ &= \left[D_1, \underline{a_1}\right] \otimes id_2 + \left[\chi_1, \underline{a_1}\right] \otimes D_2 \\ &= \left[D_1, \underline{a_1}\right] \otimes id_2 \end{aligned} \tag{47}$$

hence

$$\begin{aligned} \left[\left[D, \underline{a_1 \otimes 1\!\!1_2}\right], \underline{1\!\!1_2 \otimes a_1}\right] &= \left[\left[D_1, \underline{a_1}\right] \otimes id_2, id_1 \otimes \underline{a_2}\right] \\ &= \left[D_1, \underline{a_1}\right] \otimes \underline{a_2} - \left[D_1, \underline{a_1}\right] \otimes \underline{a_2} = 0 \end{aligned} \tag{48}$$

Other examples of dual riemannian quantum spaces abund in the theory of Von Neumann algebras whose “bilaterality”, as displayed on their standard representations, is essentially an expression of the fact that (at least in a supersymmetric situation [5]) they are (non-commutative) self dual riemannian quantum spaces. We will return to this question in a later publication.

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