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Defining Quantum Dynamical Entropy

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Abstract
We propose an elementary definition of the dynamical entropy for a discrete-time quantum dynamical system. We apply our construction to classical dynamical systems and to the shift on a quantum spin chain. In both cases the expected results are obtained.

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1. Introduction

We propose a new definition of dynamical entropy for discrete-time quantum dynamical systems. The basic ingredient is the notion of finite partition of unity. Such partitions can be evolved with the dynamics and composed among themselves to yield finer and finer, but still finite, partitions. A finite partition, together with the dynamics, can be considered as a symbolic dynamics modeling the quantum dynamical system. We can then use the standard notion of von Neumann entropy as a measure of the dynamical entropy. This is very reminiscent of the classical situation where finer and finer partitions of the "phase space" of the system are generated by the dynamics. The Kolmogorov-Sinai entropy then measures the asymptotic entropy production by the dynamics [1,2].

Several constructions of quantum dynamical entropy have been considered. It is well-known that a truly quantum mechanical dynamics behaves badly with respect to locality e.g. a finite dimensional subalgebra together with its single-step evolved algebra will generically generate an infinite dimensional algebra. A first attempt to overcome this difficulty was made in [3]. The basic object was a finite dimensional subalgebra of the centralizer of the invariant state, which is, unfortunately, in most cases limited to the scalars. A general theory, together with quite a number of applications, has been developed, by mapping the system onto classical models and computing the dynamical entropy as the supremum of the classical entropies [4–6]. This approach is technically very involved and in order to apply it to specific models, it is often necessary to satisfy strong conditions.

Decompositions of unity, similar to our's, appeared in [7]. This proposal however, assumed strong time invariance conditions on the partition, producing hereby a severe conflict between locality and invariance.

In section 2, we introduce the notion of partition of unity and of dynamical entropy H. We also describe the symbolic dynamics generated by the partition. The aim of section 3 is to show that, for classical dynamical systems, H is the Kolmogorov-Sinai invariant. As a second application we consider the shift on a quantum spin chain with the $d \times d$ matrices as single site observables and show that, up to a term $\log d$, H coincides with the usual von Neumann mean entropy of a shift invariant state.

2. Defining dynamical entropy

A discrete-time dynamical system consists in a C*-algebra $\mathcal{A}$, an automorphism Θ of $\mathcal{A}$ and a state ω on $\mathcal{A}$ which is left invariant by the dynamics: $\omega \circ \Theta = \omega$. We will also need to specify a unital $*$-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$, globally invariant under Θ. The elements of $\mathcal{A}_0$ will play the role of smooth or local elements. In the examples of sections 3 and 4 a natural choice for $\mathcal{A}_0$ is obvious. We don't know however how to specify $\mathcal{A}_0$ for a completely general system.

A *finite partition of size k of unity* is a set $\mathbf{X} = \{x_1, x_2, \dots x_k\}$ of elements of $\mathcal{A}_0$ satisfying:

$$\sum_{i=1}^{k} x_i^* x_i = \mathbb{1}. \tag{2.1}$$

We can compose two partitions $\mathbf{X} = \{x_1, x_2, \dots x_k\}$ and $\mathbf{Y} = \{y_1, y_2, \dots y_\ell\}$ to get a new partition $\mathbf{X} \circ \mathbf{Y} = \{x_i y_j \mid i = 1, 2, \dots k, \quad j = 1, 2, \dots \ell\}$. Also, if $\mathbf{X}$ is a partition of unity $\Theta(\mathbf{X}) = \{\Theta(x_1), \Theta(x_2), \dots \Theta(x_k)\}$ is again a partition.

To any partition $\mathbf{X}$ of size k we associate a $k \times k$ density matrix $\rho[\mathbf{X}]$ with (i, j) matrix elements

$$\rho[\mathbf{X}]_{i,j} = \omega(x_j^* x_i), \quad i, j = 1, 2, \dots k. \tag{2.2}$$

We now define the dynamical entropy in terms of the von Neumann entropy of the density matrices generated by the partition $\mathbf{X}$ and the shift Θ. More precisely, let $\mathbf{X}$ be any finite partition. $H_{(\omega,\Theta)}(\mathbf{X})$ is defined as:

$$\begin{aligned} H_{(\omega,\Theta)}(\mathbf{X}) &= \limsup_m \ \frac{1}{m} S(\rho[\Theta^{m-1}(\mathbf{X}) \circ \cdots \Theta(\mathbf{X}) \circ \mathbf{X}]) \\ &= \limsup_m \ \frac{1}{m} \operatorname{Tr} \eta \left(\rho[\Theta^{m-1}(\mathbf{X}) \circ \cdots \Theta(\mathbf{X}) \circ \mathbf{X}]\right) \leq \log k. \end{aligned} \tag{2.3}$$

η is the standard entropy function on $[0, 1]$:

$$\eta(t) = \begin{cases} -t \log t & \text{for } 0 < t \leq 1; \\ 0 & \text{for } t = 0. \end{cases} \tag{2.4}$$

The *dynamical entropy* $H_{(\omega,\Theta)}$ is obtained by taking the supremum over all partitions of unity:

$$H_{(\omega,\Theta)} = \sup_{\mathbf{X}} \ H_{(\omega,\Theta)}(\mathbf{X}). \tag{2.5}$$

We will now give an interpretation of $\rho[\Theta^{m-1}(\mathbf{X}) \circ \cdots \Theta(\mathbf{X}) \circ \mathbf{X}]$ in terms of either a quantum stochastic process on $\mathcal{M}_k$ or of a spin chain in quantum statistical mechanics. Therefore the construction of above can be considered to be a symbolic dynamics for $(\mathcal{A}, \Theta, \omega)$ with the shift as single step time evolution. So, choose a fixed partition $\mathbf{X}$ of size k of the identity and let $\mathcal{M}_k$ denote the algebra of complex $k \times k$ matrices. An arbitrary element $A \in \mathcal{M}_k \otimes \mathcal{A}$ is in fact a $k \times k$ matrix $[a_{ij}]$ with entries in $\mathcal{A}$. We can now consider the map

$$\Gamma_{\mathbf{X}} : \mathcal{M}_k \otimes \mathcal{A} \to \mathcal{A} : A \mapsto \sum_{i,j=1}^{k} x_i^* a_{ij} x_j. \tag{2.6}$$

Lemma 2.1. $\Gamma_{\mathbf{X}}$ *extends to a unity preserving completely positive map from* $\mathcal{M}_k \otimes \mathcal{A}$ *to* $\mathcal{A}$.

Proof : In order to prove that $\Gamma_{\mathbf{X}}$ is completely positive , we have to show that for each choice of $p = 1, 2, \ldots$ and $A^\ell \in \mathcal{M}_k \otimes \mathcal{A}$, $\ell = 1, 2, \ldots p$ the $p \times p$ matrix $[\Gamma_{\mathbf{X}}((A^\ell)^* A^m)]$ is positive. Now,

$$\begin{aligned}
[\Gamma_{\mathbf{X}}((A^\ell)^* A^m)] &= \left[\sum_{i,j=1}^{k} x_i^* \left(\sum_{r=1}^{k} (a_{r,i}^\ell)^* a_{r,j}^m \right) x_j \right] \\
&= \sum_{r=1}^{k} \left[\left(\sum_{i=1}^{k} a_{r,i}^\ell x_i \right)^* \left(\sum_{j=1}^{k} a_{r,j}^m x_j \right) \right] \\
&\geq 0.
\end{aligned}$$

Clearly $\Gamma_{\mathbf{X}}$ is unity preserving due to the fact that $\mathbf{X}$ is a partition of unity.

■

As Θ is an automorphism of $\mathcal{A}$, $\Gamma_{(\mathbf{X},\Theta)} = \Theta \circ \Gamma_{\mathbf{X}}$ is also a unity preserving completely positive map. The map $\Gamma_{(\mathbf{X},\Theta)}$, tensored with suitable identity maps on $\mathcal{M}_k$, can be iterated to yield maps

$$\Gamma_{(\mathbf{X},\Theta)}^{(m)} : (\mathcal{M}_k)^{\otimes m} \otimes \mathcal{A} \to \mathcal{A} : \Gamma_{(\mathbf{X},\Theta)}^{(m)} = \Gamma_{(\mathbf{X},\Theta)} \circ (\mathrm{id} \otimes \Gamma_{(\mathbf{X},\Theta)}) \circ \cdots (\mathrm{id} \otimes \mathrm{id} \otimes \cdots \Gamma_{(\mathbf{X},\Theta)}). \tag{2.7}$$

The maps $\Gamma_{(\mathbf{X},\Theta)}^{(m)}$ can now be used to construct a state $\omega_{\mathbf{X}}$ on the half-chain algebra $(\mathcal{M}_k)^{\mathbb{N}} = \otimes_{i=0}^{\infty} \mathcal{M}_k$. On local elements $M \in \mathcal{M}_k^{[0,m-1]}$, $\omega_{\mathbf{X}}$ is given by:

$$\omega_{\mathbf{X}}(M) = \omega(\Gamma_{(\mathbf{X},\Theta)}^{(m)}(M \otimes \mathbb{1}_{\mathcal{A}})). \tag{2.8}$$

The necessary positivity, normalization and compatibility conditions for $\omega_{\mathbf{X}}$ are satisfied because it is a composition of unity preserving completely positive maps.

The construction (2.8) of a state on a spin chain is in fact the starting point for studying the class of C*-finitely correlated states on quantum spin chains [8]. Alternatively (2.8) could be considered to specify the multitime correlation functions of a stochastic process on $\mathcal{M}_d$. The reduced density matrices $\rho_{\mathbf{X}}^{(m)}$, corresponding to the restriction of $\omega_{\mathbf{X}}$ to the first m tensor factors of $(\mathcal{M}_k)^{\mathbb{N}}$ can easily be computed. Let $M \in \mathcal{M}_k^{\otimes[0,m-1]}$, and let $\mathbf{i}$ denote the multiindex $(i_0, i_1, \ldots i_{m-1})$, then:

$$\begin{aligned}
&\mathrm{Tr}_{(\mathbb{C}^k)^{\otimes m}}\, \rho_{\mathbf{X}}^{(m)} M = \omega_{\mathbf{X}}(M) = \omega(\Gamma_{(\mathbf{X},\Theta)}^{(m)}(M \otimes \mathbb{I}_{\mathcal{A}})) = \\
&\sum_{\mathbf{i},\mathbf{j}} M_{\mathbf{i},\mathbf{j}}\, \omega\left(\Theta(x_{i_0}^*)\Theta^2(x_{i_1}^*)\cdots\Theta^m(x_{i_{m-1}}^* x_{j_{m-1}})\cdots\Theta^2(x_{j_1})\Theta(x_{j_0})\right) = \\
&\sum_{\mathbf{i},\mathbf{j}} M_{\mathbf{i},\mathbf{j}}\, \omega\left(x_{i_0}^*\Theta(x_{i_1}^*)\cdots\Theta^{m-1}(x_{i_{m-1}}^* x_{j_{m-1}})\cdots\Theta(x_{j_1})x_{j_0}\right).
\end{aligned}$$

Therefore:

$$(\rho_{\mathbf{X}}^{(m)})_{(i_0,\ldots i_{m-1}),(j_0,\ldots j_{m-1})} = \omega\left(x_{j_0}^*\Theta(x_{j_1}^*)\cdots\Theta^{m-1}(x_{j_{m-1}}^* x_{i_{m-1}})\cdots\Theta(x_{i_1})x_{i_0}\right),$$

or, using the notation of (2.2):

$$\rho_{\mathbf{X}}^{(m)} = \rho[\Theta^{m-1}(\mathbf{X}) \circ \cdots \Theta(\mathbf{X}) \circ \mathbf{X}]. \tag{2.9}$$

The state $\omega_{\mathbf{X}}$, defined in (2.8) is generally not shift invariant as

$$\omega(y) \neq \sum_{i=1}^{k} \omega(x_i^* y x_i), \quad y \in \mathcal{A}. \tag{2.10}$$

We can therefore, in general, not ascertain the existence of $\lim_m \frac{1}{m} S(\rho_{\mathbf{X}}^{(m)})$ in (2.3). In specific examples however, using an appropriate choice of $\mathcal{A}_0$, $\omega_{\mathbf{X}}$ approaches sufficiently fast a shift invariant state and this guarantees the existence of the limit. In [7] attention was restricted to those partitions of unity for which equality in (2.10) holds. This is an extremely strong condition in a truly quantum mechanical situation where the reduced density matrices ρ_ω^Λ don't commute amongst themselves.

3. Kolmogorov-Sinai entropy

Consider a probability space $(\mathrm{M}, \mathcal{F}, \mu)$, M is a measurable space, $\mathcal{F}$ the σ-algebra of measurable subsets of M and μ a probability measure on M. We are also given an automorphism T, which is a one to one transformation of M, such that both T and T^{-1} map $\mathcal{F}$ into itself and

$$\mu(A) = \mu(T(A)) = \mu(T^{-1}(A)), \quad A \in \mathcal{F}.$$

If $\mathbf{C} = \{C_1, C_2, \dots C_k\}$ is a partition of M into disjoint, measurable subsets, the entropy $h(\mathbf{C})$ of $\mathbf{C}$ is computed as

$$h(\mathbf{C}) = \sum_{i=1}^{k} \eta(\mu(C_i)), \tag{3.1}$$

η defined as in (2.4). If $\mathbf{C}$ and $\mathbf{D}$ are two partitions of M, $\mathbf{C} \vee \mathbf{D}$ is the partition of M generated by $\mathbf{C}$ and $\mathbf{D}$. Choosing a fixed partition $\mathbf{C}$, it can be shown that the following limit exists:

$$h_{(\mu,T)}(\mathbf{C}) = \lim_{m\to\infty} \frac{1}{m}\, h\left(T^{m-1}(\mathbf{C}) \vee \cdots T(\mathbf{C}) \vee \mathbf{C}\right). \tag{3.2}$$

The Kolmogorov-Sinai entropy $h_{(\mu,T)}$ [1] is defined to be:

$$h_{(\mu,T)} = \sup_{\mathbf{C}}\; h_{(\mu,T)}(\mathbf{C}). \tag{3.3}$$

We now apply the scheme of section 2 to this particular case. Let $\mathcal{A} = \mathcal{L}^\infty(\mathrm{M}, \mathcal{F}, \mu)$ and let $\mathcal{A}_0$ be the subalgebra of $\mathcal{A}$ consisting of all step functions i.e. all finite linear combinations of characteristic functions of measurable subsets of M. The map T on M defines an automorphism Θ of $\mathcal{A}$ through the relation

$$\Theta(f) = f \circ T^{-1}, \quad f \in \mathcal{L}^\infty(\mathrm{M}, \mathcal{F}, \mu). \tag{3.4}$$

Clearly $\mathcal{A}_0$ is globally invariant under Θ. Furthermore, the state $\omega(f) = \int d\mu\, f$ is invariant under Θ.

Theorem 3.1. *With the notation of above,*

$$H_{(\omega,\Theta)} = h_{(\mu,T)}.$$

Proof : We first show that $H_{(\omega,\Theta)} \leq h_{(\mu,T)}$. Let $\mathbf{X} = \{x_1, x_2, \dots x_k\}$ be a partition of unity in the sense of (2.1). As each x_i is a step function we can find a finite partition $\mathbf{C} = \{C_1, C_2, \dots C_\ell\}$ of M into disjoint sets and complex numbers ψ_i^α, such that:

$$x_i = \sum_{\alpha=1}^{\ell} \psi_i^\alpha \, \chi_{C_\alpha}, \quad i = 1, 2, \dots k,$$

χ_C denoting the characteristic function of a measurable set C. $\mathbf{X}$ is a partition of unity iff for all α

$$\sum_{i=1}^{k} \overline{\psi_i^\alpha} \, \psi_i^\alpha = 1. \tag{3.5}$$

We then compute the reduced density matrices $\rho_{\mathbf{X}}^{(m)}$ of (2.7):

$$(\rho_{\mathbf{X}}^{(m)})_{(i_0,\dots i_{m-1}),(j_0,\dots j_{m-1})} =$$

$$(\rho[\Theta^{m-1}(\mathbf{X}) \circ \cdots \Theta(\mathbf{X}) \circ \mathbf{X}])_{(i_0,\dots i_{m-1}),(j_0,\dots j_{m-1})} =$$

$$\sum_{\alpha_0,\dots\alpha_{m-1}} \overline{\psi_{j_0}^{\alpha_0}}\,\overline{\psi_{j_1}^{\alpha_1}} \cdots \overline{\psi_{j_{m-1}}^{\alpha_{m-1}}}\, \psi_{i_0}^{\alpha_0}\, \psi_{i_1}^{\alpha_1} \cdots \psi_{i_{m-1}}^{\alpha_{m-1}}\, \omega\left(\Theta^{m-1}(\chi_{C_{\alpha_{m-1}}}) \cdots \Theta(\chi_{C_{\alpha_1}})\chi_{C_{\alpha_0}}\right) =$$

$$\sum_{\alpha_0,\dots\alpha_{m-1}} \overline{\psi_{j_0}^{\alpha_0}}\,\overline{\psi_{j_1}^{\alpha_1}} \cdots \overline{\psi_{j_{m-1}}^{\alpha_{m-1}}}\, \psi_{i_0}^{\alpha_0}\, \psi_{i_1}^{\alpha_1} \cdots \psi_{i_{m-1}}^{\alpha_{m-1}}\, \mu\left(T^{m-1}(C_{\alpha_{m-1}}) \cap \cdots T(C_{\alpha_1}) \cap C_{\alpha_0}\right).$$

Due to condition (3.5), the vector $\Psi^\alpha \in \mathbb{C}^k$, with components $(\psi_1^\alpha, \psi_2^\alpha, \dots \psi_k^\alpha)$ is normalized. Hence $(\rho^{(m)})_{\mathbf{X}}$ now be written as:

$$\rho_{\mathbf{X}}^{(m)} = \sum_{\alpha_0,\dots\alpha_{m-1}} \mu\left(T^{m-1}(C_{\alpha_{m-1}}) \cap \cdots C_{\alpha_0}\right) \,|\, \Psi^{\alpha_0} \otimes \cdots \Psi^{\alpha_{m-1}} \rangle \langle \Psi^{\alpha_0} \otimes \cdots \Psi^{\alpha_{m-1}} \,| \,.$$

By [9], formula (2.4), we have:

$$S(\rho_{\mathbf{X}}^{(m)}) \le \sum_{\alpha_0,\dots\alpha_{m-1}} \eta\left(\mu\left(T^{m-1}(C_{\alpha_{m-1}}) \cap \cdots C_{\alpha_0}\right)\right) = h\left(T^{m-1}(\mathbf{C}) \vee \cdots T(\mathbf{C}) \vee \mathbf{C}\right).$$

Dividing both sides by m, and taking limes suprema and suprema on both sides we obtain

$$H_{(\omega,\Theta)} \le h_{(\mu,T)}.$$

The Kolmogorov-Sinai entropy is reached by taking the supremum of the $H_{(\omega,\Theta)}(\mathbf{X})$ over a subclass of the partitions of the unity, namely those consisting in characteristic functions only.

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4. Dynamical entropy for the shift on a quantum spin chain

As the second application we consider a quantum dynamical system $(\mathcal{A}, \Theta, \omega)$ where $\mathcal{A} = (\mathcal{M}_d)^{\mathbb{Z}}$ is the spin chain with single site algebra $\mathcal{M}_d$. The dynamics Θ is given by the right shift and ω is an arbitrary translation invariant state on $\mathcal{A}$. For $\mathcal{A}_0$ we choose the local observables i.e.

$$\mathcal{A}_0 = \bigcup_{\substack{\Lambda \subset \mathbb{Z} \\ \#(\Lambda) < \infty}} (\mathcal{M}_d)^{\otimes \Lambda}.$$

For any finite $\Lambda \subset \mathbb{Z}$ the restriction of ω to $(\mathcal{M}_d)^{\otimes \Lambda}$ is given by a reduced density matrix ρ_ω^Λ

$$\omega(A) = \mathrm{Tr}_{\mathbb{C}^{\otimes \Lambda}}\, \rho_\omega^\Lambda\, A, \quad A \in (\mathcal{M}_d)^{\otimes \Lambda}.$$

The entropy $S^\Lambda(\omega)$ of ω restricted to Λ is defined to be

$$S^\Lambda(\omega) = \mathrm{Tr}\, \eta(\rho_\omega^\Lambda), \tag{4.1}$$

and the entropy density $\sigma(\omega)$ of ω is

$$\sigma(\omega) = \lim_{\Lambda \to \mathbb{Z}} \frac{1}{\#(\Lambda)} S^\Lambda(\omega), \tag{4.2}$$

where the limit can be taken in the sense of van Hove.

Theorem 4.1. *With the notation of above,*

$$H_{(\omega,\Theta)} = \sigma(\omega) + \log d.$$

Proof : Choose a size k partition $\mathbf{X} = \{x_1, x_2, \ldots x_k\} \subset \mathcal{A}_0$ of unity. By locality the x_i belong to some finite volume algebra. As ω is translation invariant we can always assume that the x_i belong to $(\mathcal{M}_d)^{\otimes[0,\ell-1]}$ for some $\ell = 1, 2, \ldots$. Let $\{e_1, e_2, \ldots e_k\}$ be an orthonormal basis for $\mathbb{C}^k$ and denote by w the mapping:

$$w : (\mathbb{C}^d)^{\otimes \ell} \to \mathbb{C}^k \otimes (\mathbb{C}^d)^{\otimes \ell} : \phi \mapsto \sum_{i=1}^k e_k \otimes x_k \phi.$$

It is straightforward to check that w^* is given by

$$w^* : \mathbb{C}^k \otimes (\mathbb{C}^d)^{\otimes \ell} \to (\mathbb{C}^d)^{\otimes \ell} : \sum_{i=1}^k e_i \otimes \phi_i \mapsto \sum_{i=1}^k x_i \phi_i.$$

Condition (2.1), which expresses that $\mathbf{X}$ is a partition of unity is equivalent to w being an isometry, $w^* w = \mathbb{1}$. The reduced m-site density matrices $\rho_{\mathbf{X}}^{(m)}$ of the state $\omega_{\mathbf{X}}$ on $(\mathcal{M}_k)^{\mathbb{N}}$ can be expressed in terms of the reduced $(m+\ell-1)$-point density matrices $\rho_\omega^{(m+\ell-1)}$ of the state ω on the chain $(\mathcal{M}_d)^{\mathbb{Z}}$. If $X \in (\mathcal{M}_k)^{\otimes m}$, $\Gamma^{(m)}_{(\mathbf{X},\Theta)}(X \otimes \mathbb{I}_{\mathcal{A}})$ will actually belong to $(\mathcal{M}_d)^{\otimes(m+\ell-1)}$ and:

$$\begin{aligned}
&\omega_{\mathbf{X}}(X) = \mathrm{Tr}_{(\mathbb{C}^k)^{\otimes m}}\, \rho^{(m)}_{\mathbf{X}}\, X = \\
&\omega(\Gamma^{(m)}_{(\mathbf{X},\Theta)}(X \otimes \mathbb{I}_{\mathcal{A}})) = \mathrm{Tr}_{(\mathbb{C}^d)^{\otimes(m+\ell-1)}}\, \rho_\omega^{(m+\ell-1)}\, \Gamma^{(m)}_{(\mathbf{X},\Theta)}(X \otimes \mathbb{I}) = \\
&\mathrm{Tr}_{(\mathbb{C}^d)^{\otimes(m+\ell-1)}}\, \rho_\omega^{(m+\ell-1)}\, (w^{(m)})^*\, X \otimes (\mathbb{I}_d)^{\otimes(m+\ell-1)}\, w^{(m)}.
\end{aligned}$$

In this formula, $w^{(m)}$ maps

$$(\mathbb{C}^d)^{\otimes(m+\ell-1)} \to (\mathbb{C}^k)^{\otimes m} \otimes (\mathbb{C}^d)^{\otimes(m+\ell-1)}.$$

Up to a reshuffling of tensor factors $\mathbb{C}^d$ and $\mathbb{C}^k$,

$$w^{(m)} = (\mathbb{I}_k^{\otimes(m-1)} \otimes \mathbb{I}_d^{\otimes(m-1)} \otimes w) \cdots (\mathbb{I}_k \otimes \mathbb{I}_d \otimes w \otimes \mathbb{I}_d^{\otimes(m-2)})(w \otimes \mathbb{I}_d^{\otimes(m-1)}).$$

Clearly $w^{(m)}$ is also isometric and

$$\begin{aligned}
\rho^{(m)}_{\mathbf{X}} &= \mathrm{Tr}_{(\mathbb{C}^d)^{\otimes(m+\ell-1)}}\, w^{(m)}\, \rho_\omega^{(m+\ell-1)}\, (w^{(m)})^* \\
&= \mathrm{Tr}_{(\mathbb{C}^d)^{\otimes(m+\ell-1)}}\, R^{(m)}.
\end{aligned}$$

The density matrix $R^{(m)}$ on $(\mathbb{C}^k)^{\otimes m} \otimes (\mathbb{C}^d)^{\otimes(m+\ell-1)}$ has, up to multiple eigenvalues 0, the same spectrum as $\rho_\omega^{(m+\ell-1)}$. From [10], Theorem 2 (c), we then get

$$S(\rho^{(m)}_{\mathbf{X}}) \le S^{(m+\ell-1)}(\omega) + (m+\ell-1)\log d.$$

Dividing both sides by m, taking limsup for $m \to \infty$ and the supremum over all finite partitions of unity, we obtain

$$H_{(\omega,\Theta)} \le \sigma(\omega) + \log d.$$

To get the converse inequality, consider the partition $\mathbf{X} = \{\frac{1}{\sqrt{d}} e_{ij} \mid i,j = 1,2,\ldots d\}$ of unity, where the e_{ij} are matrix units in $\mathcal{M}_d$ living say at the site 0 of the chain $(\mathcal{M}_d)^{\mathbb{Z}}$. One can easily compute the reduced m-site density matrix $\rho^{(m)}_{\mathbf{X}}$ on the half chain $(\mathcal{M}_{d^2})^{\mathbb{N}}$, writing $\mathcal{M}_{d^2} = \mathcal{M}_d \otimes \mathcal{M}_d$:

$$\rho^{(m)}_{\mathbf{X}} = \rho^{(m)}_\omega \otimes (\frac{1}{d}\mathbb{I}_d)^{\otimes m},$$

and therefore $S(\rho^{(m)}_{\mathbf{X}}) = S^{(m)}(\omega) + m \log d$.

$\blacksquare$

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