TAMING GRIFFITHS' SINGULARITIES: INFINITE DIFFERENTIABILITY OF QUENCHED CORRELATION FUNCTIONS[1]

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Abstract

We prove infinite differentiability of the magnetization and of all quenched correlation functions for disordered spin systems at high temperature or strong magnetic field in the presence of Griffiths' singularities. We also show uniqueness of the Gibbs state and exponential decay of truncated correlation functions with probability one. Our results are obtained through new simple modified high temperature or low activity expansions whose convergence can be displayed by elementary probabilistic arguments. Our results require no assumptions on the probability distributions of the random parameters, except for the obvious one of no percolation of infinite couplings, and, in the strong field situation, for the also obvious requirement that zero magnetic fields do not percolate.

1 INTRODUCTION

In 1969 Griffiths [1] considered the statistical mechanics of a random ferromagnetic Ising model, with Hamiltonian given by

$$H \;=\; -\sum_{\langle xy\rangle} J_{xy}\,\sigma_x\,\sigma_y \;+\; h\sum_x \sigma_x \tag{1.1}$$

where $x \in \mathbb{Z}^d$, $\langle xy\rangle$ denotes a pair of nearest neighbor sites in $\mathbb{Z}^d$, $\sigma_x = \pm 1$, and the couplings $\mathbf{J} = \{J_{xy} \geq 0\}_{\langle xy\rangle}$ are taken as identically distributed random variables. He pointed out that for the site diluted model, i.e., $J_{xy} = J\xi_x\xi_y$, where the independent random variables ξ_x are 1 or 0 with probability p and *1-p* respectively, the quenched magnetization, considered as a function of $z = \mathrm{e}^{\beta h}$, displayed a non-analytical behavior at $z = 1$ for values of the inverse temperature β at which the system has neither long-range order nor spontaneous magnetization (see also [2],[3]). His arguments should apply to a large class of ferromagnetic models; in particular, if the couplings $J_{xy} > 0$ are independent identically distributed random variables, which may assume with non zero probability arbitrarily large values, these singularities should occur for every value of the temperature. At the origin of this behavior is the fact that even if, with probability one, the infinite system is not ordered as whole, there are, also with probability one, infinitely many arbitrarily large regions inside which the system is strongly correlated.

This phenomenon is now recognized to be a regular feature in the statistical mechanics of disordered systems not just of the type discussed above. It has the unpleasant consequence that the usual high temperature or low activity expansions, the standard tools for obtaining exponential decay of correlation functions (and also existence and uniqueness of the thermodynamical limit), fail to converge.

In this paper we consider a class of systems whose typical representative is an Ising model in $\mathbb{Z}^d$ whose Hamiltonian is given in a finite volume $\Lambda \subset \mathbb{Z}^d$ by

$$H_\Lambda \;=\; -\sum_{\langle xy\rangle\in\Lambda^*} J_{xy}\,\sigma_x\,\sigma_y \;+\; \mathcal{B}\sum_{x\in\Lambda} h_x\,\sigma_x \;+\; h\sum_{x\in\Lambda}\sigma_x \tag{1.2}$$

where the couplings $\mathbf{J} = \{J_{xy},\ \langle xy\rangle \in \mathbb{Z}^{d^*}\}$ and the external fields $\mathbf{h} = \{h_x,\, x \in \mathbb{Z}^d\}$ are independent families of independent identically distributed (within each family) random variables; we allow the random variable J_{xy} to take also the value $+\infty$. We use the notation $\Lambda^* = \{\langle xy\rangle\,;x,y \in \Lambda\}$. If $\mathcal{B} = 0$, the model may be used to describe a spin glass or a random ferromagnet; if the $J_{xy} \equiv J > 0$, we have the random field Ising model.

For such a model, we prove that at high temperature or at strong field $\mathcal{B}$, in spite of the non-analyticity pointed out by Griffiths, the magnetization, or more generally all quenched correlation functions, are infinitely differentiable functions of the uniform external field h. We also show uniqueness of the Gibbs state and exponential decay of truncated correlation functions with probability one. Our results require no assumptions on the probability distributions of J_{xy} and h_x, except for the obvious requirement of no percolation of infinite *couplings (e.g., $\mathbb{P}\{J_{xy} = +\infty\}$ small), and, in the strong field situation, for the also obvious requirement that zero magnetic fields do not percolate (e.g., $\mathbb{P}\{h_x = 0\}$ small).*

To prove these results, we develop a modified high temperature/low activity expansion whose convergence can be displayed through simple and elementary probabilistic arguments. A key new feature of these expansions is their simplicity.

Our methods can be applied to any lattice model in classical statistical mechanics. For models with finite range interaction, bounded spins and independence of the random parameters, the application is straightforward.

Boundary conditions may be introduced in the usual way. Given Λ, we define its boundary $\partial\Lambda$ and its external boundary $\partial\Lambda^+$ by

$$\partial\Lambda \;=\; \{\langle xy\rangle \in \mathbb{Z}^{d^*};\;\; x \in \Lambda,\; y \notin \Lambda\}\,, \tag{1.3}$$

$$\partial\Lambda^+ \;=\; \{y \in \mathbb{Z}^d;\;\; \langle xy\rangle \in \partial\Lambda \;\text{ for some }\; x \in \Lambda\}\,. \tag{1.4}$$

A boundary condition on Λ is a a map $\chi:\ \partial\Lambda^+ \to [-1,1]$. It is an external boundary condition if it is a configuration on $\partial\Lambda^+$, i.e., a map $\chi:\ \partial\Lambda^+ \to \{-1,1\}$. If $\chi \equiv 0$ we have free boundary conditions. We set

$$H_\Lambda^\chi(\sigma) \;=\; H_\Lambda(\sigma) - \sum_{\langle xy\rangle\in\partial\Lambda} J_{xy}\,\sigma_x\,\chi_y\,. \tag{1.5}$$

Finite volume thermal averages of local observables (i.e., functions of a finite number of spins) at fixed $\mathbf{J}$ and $\mathbf{h}$, with boundary condition χ, are defined by

$$\langle A\rangle_\Lambda^\chi \;=\; \frac{\sum_\sigma\, A(\sigma)\,\mathrm{e}^{-\beta H_\Lambda^\chi(\sigma)}}{Z_\Lambda^\chi} \quad\text{with}\quad Z_\Lambda^\chi \;=\; \textstyle\sum_\sigma\,\mathrm{e}^{-\beta H_\Lambda^\chi(\sigma)}\,, \tag{1.6}$$

the sums running over all configurations σ in Λ; β being the inverse temperature. If some $J_{xy} = +\infty$ we take limits in (1.6). In the case of free boundary conditions we will simply write $\langle A\rangle_\Lambda$. When necessary we will make explicit the dependence on the uniform external field h.

The truncated or connected finite volume correlation function of two local observables $A(\sigma)$ and $B(\sigma)$, with boundary condition χ, is defined by :

$$\langle A;B\rangle_\Lambda^\chi \;=\; \langle AB\rangle_\Lambda^\chi - \langle A\rangle_\Lambda^\chi\,\langle B\rangle_\Lambda^\chi\,. \tag{1.7}$$

More generally, given a state $\prec\ \succ$ on an algebra of local observables, we define the truncated correlation function (Ursell function) of n local observables $A_1,\ldots,A_n$ by (e.g.,[4])

$$\prec A_1;A_2;\ldots;A_n \succ \;=\; \sum_{\mathcal{P}}(-1)^{|\mathcal{P}|-1}(|\mathcal{P}|-1)!\prod_{P\in\mathcal{P}} \prec \prod_{p\in P} A_p \succ\,, \tag{1.8}$$

where the sum runs over all partitions $\mathcal{P}$ of $\{1,\ldots,n\}$. We recall

$$\prec A_1; A_2; \ldots; A_n \succ \;=\; \frac{\partial^n}{\partial s_1 \partial s_2 \ldots \partial s_n} \ln \prec \exp\left(\sum_{i=1}^{n} s_i A_i\right) \succ \Big|_{s_1=\ldots=s_n=0} . \tag{1.9}$$

Given a local observable A we set $\|A\| = \sup_\sigma |A(\sigma)|$, and denote by $\operatorname{supp} A$ the support of A, that is, the (finite) set of $x \in \mathbb{Z}^d$ such that $A(\sigma)$ depends non-trivially on σ_x .

The precise statements of our results are presented in the two theorems below which consider separately the two situations , high temperature or strong field, to which our methods apply. We will use $p_c^b(d)$ and $p_c^s(d)$ to denote the critical probabilities for bond and site percolation in $\mathbb{Z}^d$, respectively. Recall (e.g., [5])

$$\begin{aligned} p_c^b(1) \;&=\; p_c^s(1) \;=\; 1 \,, \\ \frac{1}{2} \;=\; p_c^b(2) \;&<\; p_c^s(2) \;<\; 1 \,, \\ 0 \;<\; \frac{1}{2d-1} \;<\; p_c^b(d) \;&\leq\; p_c^s(d) \;<1 \quad \text{for } d \geq 3 \,. \end{aligned} \tag{1.10}$$

We will use the ℓ^1 norm in $\mathbb{Z}^d$:

$$\|x\|_1 \;=\; \sum_{i=1}^{d} |x_i| \;;$$

distances in $\mathbb{Z}^d$ will be measured with respect to this norm. Given $X, Y \subset \mathbb{Z}^d$, $d(X,Y)$ will denote the distance between X and Y; notice that in the ℓ^1 norm

$$d(X,Y) \;=\; \min\{|G|\,; G \subset \mathbb{Z}^{d^*} \text{ connecting } X \text{ and } Y\} \,. \tag{1.11}$$

More generally, if $X_1, \ldots, X_n \subset \mathbb{Z}^d$, we set

$$d(X_1,\ldots,X_n) \;=\; \min\{|G|\,; G \subset \mathbb{Z}^{d^*} \text{ connecting } X_1,\ldots,X_n\} \,. \tag{1.12}$$

Here by $G \subset \mathbb{Z}^{d^*}$ connecting $X_1, \ldots, X_n$ we mean that for each $i,j \in \{1,\ldots,n\}$, $i \neq j$, we can find $\langle x_1 y_1 \rangle$, … , $\langle x_\ell y_\ell \rangle \in G$ with $x_1 \in X_i$ and $y_\ell \in X_j$, such that for each $k = 1,\ldots,\ell-1$ we have either $x_{k+1} = y_k$ or we can find $t \in \{1,\ldots,n\}$ so that $x_{k+1}, y_k \in X_t$.

If A and B are local observables, we will write $d(A,B)$ for the distance between the supports of A and B, i.e., $d(\operatorname{supp} A, \operatorname{supp} B)$. We will also write $d(A, x_1, \ldots, x_n)$ for $d(\operatorname{supp} A, x_1, \ldots, x_n)$.

We start with the high temperature case. *In this case we fix arbitrary* $\mathcal{B} \in \mathbb{R}$, $\mathbf{h} \in \mathbb{R}^{\mathbb{Z}^d}$ *and* $h \in \mathbb{R}$ *in (1.2); only* $\mathbf{J}$ *is random (all our estimates will be uniform in* $\mathcal{B}$, $\mathbf{h}$ *and* h*).* For a given $\delta > 0$ we set $p_\delta = \mathbb{P}\{|J_{xy}| > \delta\}$ and $J_{xy}^{(\delta)} = J_{xy}\, \mathbb{I}_{\{|J_{xy}| \leq \delta\}}$. We also set $p_\infty = \mathbb{P}\{J_{xy} = +\infty\}$; notice $\lim_{\delta \to \infty} p_\delta = p_\infty$.

THEOREM 1.1 (High Temperature Regime) *If $p_\infty < p_c^b(d)$ there exists $\beta_1 = \beta_1(d) > 0$, such that:*

(i) *For all $0 < \beta < \beta_1$ we can find $C = C(\beta) < \infty$ and $m = m(\beta) > 0$, such that for any two local observables A and B and any finite Λ containing their supports, we have*

$$\mathbb{E}\left(|\langle A; B\rangle_\Lambda^\chi|\right) \;\; \leq \;\; C\,|supp\,A|\,\|A\|\,\|B\|\,\mathrm{e}^{-m\,d(A,B)}\;, \tag{1.13}$$

for all $\mathcal{B} \in \mathbb{R}$, $\mathbf{h} \in \mathbb{R}^{\mathbf{Z}^d}$, $h \in \mathbb{R}$ and any boundary condition χ on Λ.

(ii) *There exists a set $\mathcal{J}$ of realizations of the random couplings with $\mathbb{P}\{\mathbf{J} \in \mathcal{J}\} = 1$, and for each $0 < \beta < \beta_1$ we can choose $\mu = \mu(\beta) > 0$ with $\lim_{\beta\to 0}\mu(\beta) = \infty$, such that if $\mathbf{J} \in \mathcal{J}$ and $0 < \beta < \beta_1$, then for all $\mathcal{B} \in \mathbb{R}$, $\mathbf{h} \in \mathbb{R}^{\mathbf{Z}^d}$ and $h \in \mathbb{R}$:*

(a) *For any two local observables A and B, any finite Λ containing their supports, and any boundary condition χ on Λ, we have*

$$|\langle A; B\rangle_\Lambda^\chi| \;\; \leq \;\; D_A\,\|A\|\,\|B\|\,\mathrm{e}^{-\mu\,d(A,B)}\;, \tag{1.14}$$

for some $D_A = D(supp\,A, \mathbf{J}, \beta) < \infty$.

(b) *For every local observable A, the thermodynamical limit*

$$\langle A\rangle \;\; \equiv \;\; \lim_{\Lambda\to\mathbf{Z}^d}\,\langle A\rangle_\Lambda^{\chi_\Lambda} \tag{1.15}$$

exists and is independent of the boundary condition χ_Λ used in each finite volume Λ. In particular, there is a unique Gibbs state.

(iii) *For all $0 < \beta < \beta_1$, $\mathcal{B} \in \mathbb{R}$, $\mathbf{h} \in \mathbb{R}^{\mathbf{Z}^d}$ and $h \in \mathbb{R}$, the quenched expectation $\mathbb{E}(\langle A\rangle\,(h))$ of a local observable A is an infinitely differentiable function of the uniform external field h. In particular, for each $n = 1, 2, ...$ there exists a constant $C_n < \infty$, depending only on C, m and n, such that*

$$\mathbb{E}\left(|\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n}\rangle|\right) \;\; \leq \;\; C_n\,|supp\,A|\,\|A\|\,\exp\left\{-\tfrac{2m}{(n+1)!}\,d(A, x_1, \ldots, x_n)\right\} \tag{1.16}$$

for all local observables A and $x_1, \ldots, x_n \in \mathbb{Z}^d$, and

$$\frac{\partial^n}{\partial h^n}\mathbb{E}\,\langle A\rangle \;\; = \;\; (-\beta)^n \sum_{x_1,\ldots,x_n\in\mathbf{Z}^d} \mathbb{E}\,\langle A; \sigma_{x_1}; ...; \sigma_{x_n}\rangle\;. \tag{1.17}$$

REMARK 1.2 *If $p_\infty < \frac{1}{2d-1}$ and we pick $\delta > 0$ so $p_\delta < \frac{1}{2d-1}$, then for any inverse temperature β such that*

$$\bar{\rho}_{\delta,\beta} \;\; \equiv \;\; (\mathbb{E}(\mathrm{e}^{4\beta|J_{xy}^{(\delta)}|}) \;-\; 1) \;+\; p_\delta \;\; < \;\; \frac{1}{2d-1}\,, \tag{1.18}$$

we can take

$$m \;=\; -\;\log\left((2d-1)\,\bar\rho_{\delta,\beta}\right) \tag{1.19}$$

and

$$C \;=\; 4d\left[(2d-1)\left(1-(2d-1)\,\bar\rho_{\delta,\beta}\right)\right]^{-1}. \tag{1.20}$$

in (1.13).

Remark 1.2 was discussed by Klein [6].

We now turn to the strong field case. We will denote by $p_c^{(2)}(d)$ the critical probability for site percolation on the lattice $\mathbb{Z}_2^d$, which has for vertices the subset of $\mathbb{Z}^d$ (also denoted by $\mathbb{Z}_2^d$) consisting of all sites $x \in \mathbb{Z}^d$ with $\|x\|_1$ an even integer, and for edges the collection

$$\mathbb{Z}_2^{d^*} \;=\; \{\,[xy]\,;\; x,y \in \mathbb{Z}_2^d \;\text{ with }\; \|x-y\|_1 = 2\,\}\,; \tag{1.21}$$

notice that each site in $\mathbb{Z}_2^d$ has $2d^2$ nearest neighbors, i.e., it belongs to $2d^2$ edges. It is easy to see that $p_c^{(2)}(1) = 1$ and, if $d \geq 2$,

$$\frac{1}{2d^2-1} \;<\; p_c^{(2)}(d) \;\leq\; p_c^s(d)\,. \tag{1.22}$$

For each $x \in \mathbb{Z}^d$ and $\mathcal{B} > 0$ (we can take $\mathcal{B} > 0$ in (1.2) without loss of generality) we define

$$Y_{\mathcal{B},x} \;=\; Y_{\mathcal{B},x}(\mathbf{J},\mathbf{h}) \;=\; \mathcal{B}\,|h_x| \;-\; 6\textstyle\sum_{y:\,\|y-x\|_1=1}|J_{xy}|\;, \tag{1.23}$$

and for $\delta \geq 0$ we set $Y_{\mathcal{B},x}^{(\delta)} = Y_{\mathcal{B},x}\,\mathbb{1}_{\{Y_{\mathcal{B},x}>\delta\}}$ and $q_{\mathcal{B},\delta} = \mathbb{P}\{Y_{\mathcal{B},x} \leq \delta\}$. We have $\lim_{\mathcal{B}\to\infty} q_{\mathcal{B},\delta} = q_\infty$ for any $\delta \geq 0$, where

$$q_\infty \;= \mathbb{P}\left\{\{h_x = 0\} \cup \left(\cup_{y:\,\|y-x\|_1=1}\{J_{xy} = +\infty\}\right)\right\} \;\leq\; \mathbb{P}\{h_x = 0\} \;+\; 2d\,p_\infty\,. \tag{1.24}$$

Notice that $q_\infty < p_c^{(2)}(d)$ implies $p_\infty < p_c^b(d)$.

THEOREM 1.3 (Strong Field Regime) *If $q_\infty < p_c^{(2)}(d)$, then for each $\beta > 0$ we can find $\mathcal{B}_1(\beta,d) < \infty$, monotonically decreasing in β, and $\epsilon(\beta,d) > 0$, such that:*

(i) *For any $\beta > 0$ and $\mathcal{B} > \mathcal{B}_1(\beta,d)$, we can find $C = C(\beta,\mathcal{B}) < \infty$ and $m = m(\beta,\mathcal{B}) > 0$, such that for any two local observables A and B and any finite Λ containing their supports, we have (1.13) for any $|h| < \epsilon(\beta,d)$ and any boundary condition χ on Λ.*

(ii) *there exists a set Ω of realizations of the random parameters $(\mathbf{J},\mathbf{h})$, with $\mathbb{P}\{(\mathbf{J},\mathbf{h}) \in \Omega\} = 1$, and for each $\beta > 0$ and $\mathcal{B} > \mathcal{B}_1(\beta,d)$ we can choose $\mu = \mu(\beta,\mathcal{B}) > 0$, with $\lim_{\mathcal{B}\to\infty}\mu(\beta,\mathcal{B}) = \infty$, such that if $(\mathbf{J},\mathbf{h}) \in \Omega$, $\beta > 0$, $\mathcal{B} > \mathcal{B}_1(\beta,d)$ and $|h| < \epsilon(\beta,d)$, then the conclusions (a) and (b) in Theorem 1.1(ii) are true with $D_A = D(\operatorname{supp} A,\mathbf{J},\mathbf{h},\beta,\mathcal{B}) < \infty$.*

(iii) *For any $\beta > 0$, $\mathcal{B} > \mathcal{B}_1(\beta,d)$ and $|h| < \epsilon(\beta,d)$ the conclusions of Theorem 1.1(iii) hold.*

REMARK 1.4 *If $q_\infty < \frac{1}{2(2d^2-1)}$ and we pick $\mathcal{B} > 0$ such that $q_{\mathcal{B},0} < \frac{1}{2(2d^2-1)}$, then for any inverse temperature β and $h \in \mathbb{R}$ such that*

$$\bar{\theta}_{\mathcal{B},\beta,h} \;\equiv\; 3\mathbb{E}(\mathrm{e}^{-2\beta(Y^{(0)}_{\mathcal{B},x}-2|h|)}) \;+\; q_{\mathcal{B},0} \;<\; \frac{1}{2(2d^2-1)}\,, \tag{1.25}$$

we can take

$$m \;=\; -\,\frac{1}{2}\log\Big(2(2d^2-1)\,\bar{\theta}_{\mathcal{B},\beta,h}\Big) \tag{1.26}$$

and

$$C \;=\; 8d^3(2d+1)^2\,\Big[(2d^2-1)^{\frac{3}{2}}(2\bar{\theta}_{\mathcal{B},\beta,h})^{\frac{1}{2}}(1-2(2d^2-1)\,\bar{\theta}_{\mathcal{B},\beta,h})\Big]^{-1} \tag{1.27}$$

in (1.13).

REMARK 1.5 *If the J_{xy} are bounded, say $|J_{xy}| \le M < \infty$, then we only need $q_\infty < p^s_c(d)$ in Theorem 1.3. In this case, if $q_\infty < \frac{1}{2d-1}$ we can take*

$$m \;=\; -\,\log\Big((2d-1)\,\hat{\theta}_{\mathcal{B},\beta,h}\Big) \tag{1.28}$$

and

$$C \;=\; 4d\,\Big[(2d-1)\,(1-(2d-1)\,\hat{\theta}_{\mathcal{B},\beta,h})\Big]^{-1}\,\hat{\theta}_{\mathcal{B},\beta,h}\,, \tag{1.29}$$

if $\hat{\theta}_{\mathcal{B},\beta,h} < \frac{1}{2d-1}$, where $\hat{\theta}_{\mathcal{B},\beta,h}$ is defined as in (1.25), but with $Y_{\mathcal{B},x}$ replaced by $\hat{Y}_{\mathcal{B},x} = \mathcal{B}|h_x| - 6M$.

The first rigorous results controlling the effect of Griffiths' singularities were obtained by Olivieri, Perez and Rosa Jr. [7], who studied the Ising ferromagnet with random couplings ($J_{xy} \ge 0$, $h_x \equiv 0$, $h = 0$), and showed exponential decay of correlation functions in the presence of Griffiths' singularities if $\mathbb{E}(J_{xy}) < \infty$. These same results were obtained by Perez [8] without the finite moment requirement.

Exponential decay of truncated correlation functions and uniqueness of the Gibbs state for the class of models described by (1.2), for small β or large $\mathcal{B}$, were obtained by Berretti [9] with strong restrictions on the probability distributions of the random parameters ($\mathbb{E}(\mathrm{e}^{a|J_{xy}|}) < \infty$ for all $a > 0$; $\mathbb{P}\{h_x = 0\} = 0$). Fröhlich and Imbrie [10], through an intrincate analysis of partially resummed high temperature/low activity expansions were able to obtain these results under less restrictive assumptions on the probability distributions of the relevant random parameters ($|J_{xy}| < \infty$ with a slowly decaying distribution, e.g. a Cauchy distribution, for small β, and $\mathbb{P}\{h_x = 0\} = 0$ for large $\mathcal{B}$). Bassalygo and Dobrushin [11] proved uniqueness of the Gibbs state for small β with no assumptions on the probability distributions if $|J_{xy}| < \infty$. The small β behavior of long range spin glasses has been studied by Fröhlich and Zegarlinski [12] and Zegarlinski [13].

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2 THE EXPANSIONS

In this section we will prove parts (i) and (ii) of Theorems 1.1 and 1.3. Our strategy may be summarized as follows: given a realization $(\mathbf{J}, \mathbf{h})$ and a set $\mathcal{S}$ of bonds\sites, we perform a high temperature\low activity expansion outside $\mathcal{S}$. We then show that if $\mathcal{S}$ is taken to be the appropriate *singular set*, characterized by either J_{xy} being large (in the high temperature case) or $\mathcal{B}h_x$ being small (in the strong field case), and we are in the situation when a bond\site has low probability of belonging to $\mathcal{S}$ (i.e., small β\large $\mathcal{B}$), we get decay after either averaging in $(\mathbf{J}, \mathbf{h})$ (part (i)) or by picking $(\mathbf{J}, \mathbf{h})$ in a set of probability one by a Borel-Cantelli argument (part (ii)).

To deal with truncated correlation functions we use the duplication trick. We thus consider two non-interacting copies of the original system, i.e., a new spin system with configurations $\tilde{\sigma} = \left\{\tilde{\sigma}_x = (\sigma_x, \sigma'_x);\ x \in \mathbb{Z}^d\right\}$, $\sigma_x, \sigma'_x \in \{-1, +1\}$, and Hamiltonian $\tilde{H}_\Lambda(\tilde{\sigma})$, where for any function $F(\sigma)$ we set

$$\tilde{F}_\Lambda(\tilde{\sigma}) \;=\; F_\Lambda(\sigma) + F_\Lambda(\sigma') \quad . \tag{2.1}$$

The set of all configurations of the duplicated system in a given region $\Lambda \subset \mathbb{Z}^d$ will be denoted by $\mathcal{C}(\Lambda)$. Finite volume thermal averages of an observable $C(\tilde{\sigma})$ in the duplicated system, with boundary condition χ *(same for both copies)*, are given by

$$\langle\!\langle C \rangle\!\rangle_\Lambda^\chi \;=\; \frac{\sum_{\tilde{\sigma}\in\mathcal{C}(\Lambda)} C(\tilde{\sigma})\ \mathrm{e}^{-\beta\tilde{H}_\Lambda^\chi(\tilde{\sigma})}}{\tilde{Z}_\Lambda^\chi} \quad \text{with} \quad \tilde{Z}_\Lambda^\chi \;=\; \sum_{\tilde{\sigma}\in\mathcal{C}(\Lambda)} \mathrm{e}^{-\beta\tilde{H}_\Lambda^\chi(\tilde{\sigma})} \quad . \tag{2.2}$$

Truncated correlation functions of the original system may be expressed as ordinary correlation functions of the duplicated system through the identity

$$\langle A; B\rangle_\Lambda^\chi \;=\; \frac{1}{2}\left\langle\!\!\left\langle \hat{A}\ \hat{B} \right\rangle\!\!\right\rangle_\Lambda^\chi , \tag{2.3}$$

where to every observable A of the original system we associate an observable $\hat{A}$ of the duplicated system by setting

$$\hat{A}(\tilde{\sigma}) \;=\; A(\sigma) - A(\sigma') \,. \tag{2.4}$$

2.1 The High Temperature Expansion

The following notion will play an important role in our expansion: a self- avoiding *bond* walk ω from a site x to another site y, written $\omega : x \to y$, is a finite sequence $\langle x_1 y_1\rangle$, ... , $\langle x_n y_n\rangle$ of bonds in $(\mathbb{Z}^d)^*$, such that:

1. $x_1 \;=\; x$ and $y_n \;=\; y$.

2. $x_{i+1} = y_i$ for $i = 1, \ldots, n-1$.

3. $x_i \neq x_j$ if $i \neq j$.

For such ω we set $|\omega| = n$. We define $\mathcal{W}_{xy} = \{\omega : x \to y\}$ and set $\mathcal{W}_x = \bigcup_{y \in \mathbf{Z}^d} \mathcal{W}_{xy}$. In addition, given two local observables A and B, we write $\mathcal{W}_{AB} = \{\omega : x \to y : x \in \operatorname{supp} A\,,\ y \in \operatorname{supp} B\}$.

We have the following high temperature expansion for fixed $\mathbf{J}, \mathbf{h}, h$ and B in (1.2).

THEOREM 2.1 *Given $\mathcal{S} \subset \mathbb{Z}^{d^*}$ let*

$$\rho_{xy} = \rho_{xy}(\mathcal{S}, \mathbf{J}, \beta) = \begin{cases} \xi_{xy} & \textit{if the bond } \langle xy \rangle \in \mathbb{Z}^{d^*} \setminus \mathcal{S}\,, \\ 1 & \textit{if the bond } \langle xy \rangle \in \mathcal{S}\,, \end{cases} \tag{2.5}$$

where

$$\xi_{xy} = \xi_{xy}(\mathbf{J}, \beta) = \mathrm{e}^{4\,\beta\,|J_{xy}|} - 1 \ . \tag{2.6}$$

Then for any local observables A and B, any finite Λ containing their supports, and any boundary condition χ on Λ, we have

$$|\langle A; B \rangle_\Lambda^\chi| \leq 2\,\|A\|\,\|B\| \sum_{\omega \in \mathcal{W}_{AB}} \prod_{\langle xy \rangle \in \omega} \rho_{xy} \ . \tag{2.7}$$

Proof: We start by redefining the Hamiltonian (1.2) as

$$H_\Lambda = -\textstyle\sum_{\langle xy \rangle \in \Lambda^*} (J_{xy}\,\sigma_x\,\sigma_y + |J_{xy}|) + B \sum_{x \in \Lambda} h_x\,\sigma_x + h \sum_{x \in \Lambda} \sigma_x \,, \tag{2.8}$$

which differs from (1.2) by a harmless subtraction of an overall constant. We set

$$T_G(\sigma) = -\textstyle\sum_{\langle xy \rangle \in G} (J_{xy}\,\sigma_x\,\sigma_y + |J_{xy}|) \ , \tag{2.9}$$

for any $G \subset \mathbb{Z}^{d^*}$; if $G \subset \Lambda^*$ and χ is a boundary condition on Λ, we set

$$T_G^\chi(\sigma) = T_G(\sigma) - \sum_{\langle xy \rangle \in \partial\Lambda} J_{xy}\,\sigma_x\,\chi_y \ , \tag{2.10}$$

so

$$H_\Lambda^\chi = T_{\Lambda^*}^\chi + V_\Lambda \,, \tag{2.11}$$

with

$$V_\Lambda = \mathcal{B} \textstyle\sum_{x \in \Lambda} h_x\,\sigma_x + h \sum_{x \in \Lambda} \sigma_x \ . \tag{2.12}$$

For $\langle xy \rangle \in \mathbb{Z}^{d^*}$ we define

$$E_{xy}(\tilde{\sigma}) = \mathrm{e}^{\beta\,J_{xy}\left(\sigma_x\,\sigma_y + \sigma'_x\,\sigma'_y\right) + 2\,\beta\,|J_{xy}|} - 1 \ ; \tag{2.13}$$

notice

$$0 \;\leq\; E_{xy}(\tilde{\sigma}) \;\leq\; \xi_{xy} \;\;, \tag{2.14}$$

the nonnegativity coming from the subtraction in (2.8).

We now perform a high temperature expansion in $\Lambda^* \setminus \mathcal{S}$ only. We can write

$$\begin{aligned}
\left\langle\!\left\langle \hat{A}\hat{B} \right\rangle\!\right\rangle_{\Lambda}^{\chi} &= \frac{1}{\tilde{Z}_{\Lambda}^{\chi}} \sum_{\tilde{\sigma}} \hat{A}\,\hat{B}\,\mathrm{e}^{-\beta(\tilde{T}_{\Lambda^*\cap\mathcal{S}}^{\chi}+\tilde{V}_{\Lambda})} \prod_{\langle xy\rangle\in\Lambda^*\setminus\mathcal{S}} (E_{xy}+1) \\
&= \frac{1}{\tilde{Z}_{\Lambda}^{\chi}} \sum_{\tilde{\sigma}} \hat{A}\,\hat{B}\,\mathrm{e}^{-\beta(\tilde{T}_{\Lambda^*\cap\mathcal{S}}^{\chi}+\tilde{V}_{\Lambda})} \sum_{G\subset\Lambda^*\setminus\mathcal{S}} \prod_{\langle xy\rangle\in G} E_{xy} \;\;.
\end{aligned} \tag{2.15}$$

Due to the invariance of the Hamiltonian of the duplicated system under the exchange $\sigma \leftrightarrow \sigma'$, any $G \subset \Lambda^* \setminus \mathcal{S}$ such that the graph $G \cup \mathcal{S}$ does not contain some $\omega \in \mathcal{W}_{AB}$ gives a zero contribution in (2.15). We can thus restrict the sum to those G of the form $G = \omega_{\mathcal{S}} \cup G'$, where $\omega \in \mathcal{W}_{AB}$, $\omega_{\mathcal{S}} = \omega \setminus \mathcal{S}$, and $G' \subset \mathcal{G}_{\mathcal{S},\omega} = (\Lambda^* \setminus \mathcal{S}) \setminus \omega$. Thus

$$\begin{aligned}
\left|\left\langle\!\left\langle \hat{A}\hat{B} \right\rangle\!\right\rangle_{\Lambda}^{\chi}\right| &\leq 4\,\|A\|\,\|B\|\,\frac{1}{\tilde{Z}_{\Lambda}^{\chi}} \sum_{\tilde{\sigma}} \mathrm{e}^{-\beta(\tilde{T}_{\Lambda^*\cap\mathcal{S}}^{\chi}+\tilde{V}_{\Lambda})} \sum_{\omega\in\mathcal{W}_{AB}} \prod_{\langle xy\rangle\in\omega_{\mathcal{S}}} E_{xy} \sum_{G'\subset\mathcal{G}_{\mathcal{S},\omega}} \prod_{\langle x'y'\rangle\in G'} E_{x'y'} \\
&\leq 4\,\|A\|\,\|B\|\,\frac{1}{\tilde{Z}_{\Lambda}^{\chi}} \sum_{\omega\in\mathcal{W}_{AB}} \prod_{\langle xy\rangle\in\omega_{\mathcal{S}}} \xi_{xy} \sum_{\tilde{\sigma}} \mathrm{e}^{-\beta(\tilde{T}_{\Lambda^*\cap\mathcal{S}}^{\chi}+\tilde{V}_{\Lambda})} \sum_{G'\subset\mathcal{G}_{\mathcal{S},\omega}} \prod_{\langle x'y'\rangle\in G'} E_{x'y'} \\
&\leq 4\,\|A\|\,\|B\|\,\frac{1}{\tilde{Z}_{\Lambda}^{\chi}} \sum_{\omega\in\mathcal{W}_{AB}} \prod_{\langle xy\rangle\in\omega_{\mathcal{S}}} \xi_{xy} \sum_{\tilde{\sigma}} \mathrm{e}^{-\beta(\tilde{T}_{\Lambda^*\cap\mathcal{S}}^{\chi}+\tilde{V}_{\Lambda})} \prod_{\langle xy\rangle\in\omega_{\mathcal{S}}} (E_{xy}+1) \sum_{G'\subset\mathcal{G}_{\mathcal{S},\omega}} \prod_{\langle x'y'\rangle\in G'} E_{x'y'} \\
&= 4\,\|A\|\,\|B\|\,\frac{1}{\tilde{Z}_{\Lambda}^{\chi}} \sum_{\omega\in\mathcal{W}_{AB}} \prod_{\langle xy\rangle\in\omega_{\mathcal{S}}} \xi_{xy}\, \tilde{Z}_{\Lambda}^{\chi} \\
&= 4\,\|A\|\,\|B\| \sum_{\omega\in\mathcal{W}_{AB}} \prod_{\langle xy\rangle\in\omega_{\mathcal{S}}} \xi_{xy} \qquad (2.16) \\
&= 4\,\|A\|\,\|B\| \sum_{\omega\in\mathcal{W}_{AB}} \prod_{\langle xy\rangle\in\omega} \rho_{xy} \;\;, \qquad (2.17)
\end{aligned}$$

where we used (2.14) and (2.5).

(2.7) now follows from (2.17) and (2.3). ∎

Proof of Theorem 1.1(i): If $p_\infty < \frac{1}{2d-1}$, we pick $\delta > 0$ such that $p_\delta < \frac{1}{2d-1}$, and take

$$\mathcal{S} \;=\; \{\langle xy\rangle \in \mathbb{Z}^{d^*};\; |J_{xy}| > \delta\} \;\;; \tag{2.18}$$

so $p_\delta = \mathbb{P}\{\langle xy\rangle \in \mathcal{S}\}$. In this case we have $\mathbb{E}(\rho_{xy}) = \bar{\rho}_{\delta,\beta}$ given in (1.18). If we now take expectations in (2.7) we get

$$\mathbb{E}\left(|\langle A;B\rangle_{\Lambda}^{\chi}|\right) \;\leq\; 2\,\|A\|\,\|B\| \sum_{\omega\in\mathcal{W}_{AB}} \bar{\rho}_{\delta,\beta}^{|\omega|} \tag{2.19}$$

A standard argument now gives (1.13) with (1.19) and (1.20). (See [6]).

In the more general case when we only have $p_\infty < p_c^b(d)$, we choose $\delta > 0$ such that $p_\delta < p_c^b(d)$, and take $\mathcal{S}$ as in (2.18). Let $\tau(\langle xy\rangle) = 1\!\!\mathrm{I}_{\mathbb{Z}^{d*}\setminus\mathcal{S}}(\langle xy\rangle)$ for any $\langle xy\rangle \in \mathbb{Z}^{d^*}$, and set $\tau(\omega) = \sum_{\langle xy\rangle\in\omega}\tau(\langle xy\rangle)$. It follows from (2.16) and (2.6) that

$$|\langle A;B\rangle_\Lambda^\chi| \;\leq\; 2\,\|A\|\,\|B\| \sum_{\omega\in\mathcal{W}_{AB}} \xi^{\tau(\omega)} \;, \tag{2.20}$$

where

$$\xi \;=\; \xi_{\beta,\delta} \;=\; \mathrm{e}^{4\,\beta\,\delta} - 1 \;. \tag{2.21}$$

To estimate (2.20) we use a result of Kesten [14]. Given $r > 0$, let us define the events

$$\mathcal{E}_n(x) \;=\; \{\text{there exists } \omega\in\mathcal{W}_x \text{ such that } |\omega|\geq n \text{ and } \tau(\omega) < rn\}\;. \tag{2.22}$$

Proposition 5.8 in [14] states that, if $p_\delta < p_c^b(d)$, we can pick r for which there exist constants constants $b > 0$ and C_1 (we will use $C_1, C_2, \ldots$ to denote finite constants) such that

$$\mathbb{P}(\mathcal{E}_n(x)) \;\leq\; C_1\,\mathrm{e}^{-b\,n} \tag{2.23}$$

for all $x\in\mathbb{Z}^d$ and all $n = 1, 2, \ldots$. We now define the random variable

$$\bar{n}(x) \;=\; \sup\{n\,;\, \mathcal{E}_n(x) \text{ occurs}\}\;, \tag{2.24}$$

and we have from (2.23) that

$$\mathbb{P}\{\bar{n}(x) \;\geq\; n\} \;\leq\; C_2\,\mathrm{e}^{-b\,n}\;. \tag{2.25}$$

We also define

$$\bar{n}(A) \;=\; \max\{\bar{n}(x)\,;\; x\in\operatorname{supp} A\}\;, \tag{2.26}$$

notice

$$\mathbb{P}\{\bar{n}(A) \;\geq\; n\} \;\leq\; C_2\,|\operatorname{supp} A|\,\mathrm{e}^{-b\,n}\;. \tag{2.27}$$

Thus

$$\begin{aligned}
\mathbb{E}\,(|\langle A;B\rangle_\Lambda^\chi|) \;&\leq\; 2\,\|A\|\,\|B\|\left(\mathbb{P}\{\bar{n}(A)\geq d(A,B)\} \;+\; \mathbb{E}\big(\sum_{\omega\in\mathcal{W}_{AB}}\xi^{\tau(\omega)}\,;\;\bar{n}(A) < d(A,B)\big)\right)\\
&\leq\; 2\,\|A\|\,\|B\|\left(\mathbb{P}\{\bar{n}(A)\geq d(A,B)\} \;+\; \sum_{\omega\in\mathcal{W}_{AB}}\xi^{r|\omega|}\right)\\
&\leq\; 2\,\|A\|\,\|B\|\,|\operatorname{supp} A|\,\Big(C_2\,\mathrm{e}^{-bd(A,B)} \;+\; C_3\,((2d-1)\,\xi^r)^{d(A,B)}\Big)\;,
\end{aligned} \tag{2.28}$$

by (2.20) and (2.22) - (2.27), if β is such that

$$\xi^r \;<\; \frac{1}{2d-1}\;. \tag{2.29}$$

If we now pick $\beta_1 > 0$ by

$$\left(e^{4\beta_1\delta} - 1\right)^r = \frac{1}{2d-1} , \tag{2.30}$$

it is clear from (2.21) that (2.29) holds for all $\beta < \beta_1$, in which case (1.13) follows from (2.28). ∎

Proof of Theorem 1.1(ii)(a): If $0 < \beta < \beta_1$, it also follows from (2.20) and (2.29) that for all $\mathcal{B} \in \mathbb{R}$, $\mathbf{h} \in \mathbb{R}^{\mathbf{Z}^d}$ and $h \in \mathbb{R}$,

$$\begin{aligned} |\langle A; B\rangle_\Lambda^\chi| &\le 2\,\|A\|\,\|B\| \left(\mathbb{1}_{\{\bar{n}(A)\ge d(A,B)\}} + \mathbb{1}_{\{\bar{n}(A)<d(A,B)\}} \sum_{\omega\in\mathcal{W}_{AB}} \xi^{\tau(\omega)} \right) \\ &\le 2\,\|A\|\,\|B\| \left(\mathbb{1}_{\{\bar{n}(A)\ge d(A,B)\}} + \mathbb{1}_{\{\bar{n}(A)<d(A,B)\}} \sum_{\omega\in\mathcal{W}_{AB}} \xi^{r|\omega|} \right) \\ &\le 2\,\|A\|\,\|B\| \left(\mathbb{1}_{\{\bar{n}(A)\ge d(A,B)\}} + \mathbb{1}_{\{\bar{n}(A)<d(A,B)\}} C_3 \left((2d-1)\,\xi^r\right)^{d(A,B)} \right) \end{aligned} \tag{2.31}$$

So let $\mathcal{J}$ be defined by

$$\mathcal{J} = \{\mathbf{J};\ \bar{n}(x) < \infty \ \text{ for all } \ x \in \mathbb{Z}^d\} . \tag{2.32}$$

It follows from (2.25) and the Borel Cantelli Lemma that $\mathbb{P}\{\mathbf{J} \in \mathcal{J}\} = 1$. But for $\mathbf{J} \in \mathcal{J}$ (1.14) follows from (2.31), with

$$\mu = -\log((2d-1)\,\xi^r) , \tag{2.33}$$

so $\lim_{\beta\to 0} \mu = \infty$ by (2.21). ∎

2.2 The Low Activity Expansion

We now define a self- avoiding *site* walk ν from a site x to another site y, written $\nu : x \rightsquigarrow y$, as a finite sequence $x_1, x_2, \ldots, x_n$ of sites in $\mathbb{Z}^d$, such that:

1. $x_1 = x$ and $x_n = y$.
2. $\| x_{i+1} - x_i \|_1 = 1$ for $i = 1, \ldots, n$.
3. $x_i \ne x_j$ if $i \ne j$.

For such ν we set $|\nu| = n$. We define $\mathcal{N}_{xy} = \{\nu : x \rightsquigarrow y\}$ and set $\mathcal{N}_x = \bigcup_{y\in\mathbf{Z}^d} \mathcal{N}_{xy}$. In addition, given two local observables A and B, we write $\mathcal{N}_{AB} = \{\nu : x \rightsquigarrow y \ : \ x \in \operatorname{supp} A ,\ y \in \operatorname{supp} B\}$.

Self-avoiding site walks $\hat{\nu}$ on the lattice $\mathbb{Z}_2^d$ are defined in the same way, except that we now require $\|x_{i+1} - x_i\|_1 = 2$. We define $\hat{\nu} : x \hat{\rightsquigarrow} y$, $\hat{\mathcal{N}}_{xy}$ and $\hat{\mathcal{N}}_x$ as above.

We have the following low activity expansion for fixed $\mathbf{J}, \mathbf{h}, h$ and $\mathcal{B}$ in (1.2).

THEOREM 2.2 *Given $\mathcal{S} \subset \mathbb{Z}^d$ let*

$$\theta_x = \theta_x(\mathcal{S}, \mathbf{J}, \mathbf{h}, h, \mathcal{B}, \beta) = \begin{cases} \zeta_x & \text{if } x \in \mathbb{Z}^d \setminus \mathcal{S} , \\ 1 & \text{if } x \in \mathcal{S} , \end{cases} \tag{2.34}$$

where

$$\zeta_x = \zeta_x(\mathbf{J}, \mathbf{h}, h, \mathcal{B}, \beta) = 3\,\mathrm{e}^{-2\beta\,(Y_{\mathcal{B},x} - 2|h|)} , \tag{2.35}$$

where $Y_{\mathcal{B},x}$ is given in (1.23). Then for any local observables A and B, any finite Λ containing their supports, and any boundary condition χ on Λ, we have

$$|\langle A; B\rangle_\Lambda^\chi| \leq 2\,\|A\|\,\|B\| \sum_{\nu \in \mathcal{N}_{AB}} \prod_{x \in \nu} \theta_x . \tag{2.36}$$

Proof: For best visualization of the expansion steps, we introduce the variables $\eta = \{\eta_x, ; x \in \mathbb{Z}^d\}$, where each $\eta_x \in \{0, 1\}$ is given by

$$\eta_x = \frac{(\operatorname{sgn} h_x)\,\sigma_x + 1}{2} , \tag{2.37}$$

where $\operatorname{sgn} u = 1$ if $u \geq 0$ and $\operatorname{sgn} u = -1$ otherwise. The Hamiltonian (1.2) written in terms of the new variables (after a subtraction of an overall constant) reads:

$$H_\Lambda(\eta) = -4 \sum_{\langle xy\rangle \in \Lambda^*} K_{xy}\,\eta_x\,\eta_y + 2 \sum_{x \in \Lambda} \left(\mathcal{B}|h_x| + (\operatorname{sgn} h_x)\,h + \sum_{y \in \Lambda:\, \langle xy\rangle \in \Lambda^*} K_{xy} \right) \eta_x , \tag{2.38}$$

where $K_{xy} = (\operatorname{sgn} h_x)\,(\operatorname{sgn} h_y)\,J_{xy}$. If χ is a boundary condition on Λ, we have (after subtracting a harmless boundary term)

$$H_\Lambda^\chi(\eta) = H_\Lambda(\eta) + 2 \sum_{\langle xy\rangle \in \partial\Lambda} K_{xy}\,\eta_x |\chi_y| . \tag{2.39}$$

Given a configuration $\tilde\eta = (\eta, \eta')$ of the duplicated system, we set

$$G_{\tilde\eta} = \Big\{ x \in \mathbb{Z}^d ;\ \eta_x + \eta'_x > 0 \Big\} ; \tag{2.40}$$

and say that a configuration $\tilde\eta$ is compatible with $G \subset \mathbb{Z}^d$, and write $\tilde\eta \prec G$, if $G_{\tilde\eta} = G$. We rewrite $\left\langle\!\left\langle \hat A \hat B \right\rangle\!\right\rangle_\Lambda^\chi$ as

$$\begin{aligned} \left\langle\!\left\langle \hat A \hat B \right\rangle\!\right\rangle_\Lambda^\chi &= \frac{1}{\tilde Z_\Lambda^\chi} \sum_{\tilde\eta} \hat A\,\hat B\,\mathrm{e}^{-\beta \tilde H_\Lambda^\chi(\tilde\eta)} \\ &= \frac{1}{\tilde Z_\Lambda^\chi} \sum_{G \subset \Lambda} \sum_{\tilde\eta \prec G} \hat A\,\hat B\,\mathrm{e}^{-\beta \tilde H_G^{\Lambda,\chi}(\tilde\eta)} , \end{aligned} \tag{2.41}$$

where

$$H^{\Lambda,\chi}_G(\eta) \;=\; -4\sum_{\langle xy\rangle\in G^*} K_{xy}\,\eta_x\,\eta_y \;+ \qquad (2.42)$$
$$2\sum_{x\in G}\left(\mathcal{B}|h_x| + (\operatorname{sgn} h_x)\,h + \sum_{y\in\Lambda:\,\langle xy\rangle\in\Lambda^*} K_{xy} + \sum_{y\in\partial\Lambda^+:\,\langle xy\rangle\in\partial\Lambda} K_{xy}\,|\chi_y|\right)\eta_x \;.$$

We now perform a low activity expansion in $\Lambda\setminus\mathcal{S}$ only. Again, due to the invariance of the Hamiltonian of the duplicated system under the exchange $\eta\leftrightarrow\eta'$, we can restrict the sum in (2.41) to those to those G of the form $G=\nu_{\mathcal{S}}\cup G'$, where $\nu\in\mathcal{N}_{\Lambda,AB}=\{\nu'\in\mathcal{N}_{AB}\,;\;\nu'\subset\Lambda\}$, $\nu_{\mathcal{S}}=\nu\setminus\mathcal{S}$ and $G'\subset\mathcal{G}_{\mathcal{S},\nu}=\Lambda\setminus\nu_{\mathcal{S}}$. Thus

$$\left|\left\langle\!\left\langle\hat A\hat B\right\rangle\!\right\rangle^{\chi}_{\Lambda}\right| \;\le\; 4\,\|A\|\,\|B\|\,\frac{1}{\tilde Z^{\chi}_{\Lambda}}\sum_{\nu\in\mathcal{N}_{\Lambda,AB}}\;\sum_{G'\subset\mathcal{G}_{\mathcal{S},\nu}}\;\sum_{\tilde\eta\prec\nu_{\mathcal{S}}\cup G'} \mathrm{e}^{-\beta\tilde H^{\Lambda,\chi}_{\nu_{\mathcal{S}}\cup G'}(\tilde\eta)}$$
$$=\; 4\,\|A\|\,\|B\|\,\frac{1}{\tilde Z^{\chi}_{\Lambda}}\sum_{\nu\in\mathcal{N}_{\Lambda,AB}}\;\sum_{G'\subset\mathcal{G}_{\mathcal{S},\nu}}\;\sum_{\tilde\eta\prec\nu_{\mathcal{S}}\cup G'}\;\prod_{x\in\nu_{\mathcal{S}}} \mathrm{e}^{-\beta\,\tilde\Gamma^{\Lambda,\chi}_{x}(\tilde\eta)}\;\mathrm{e}^{-\beta\tilde H^{\Lambda,\chi}_{G'}(\tilde\eta)} \qquad (2.43)$$

where

$$\Gamma^{\Lambda,\chi}_x(\eta) \;=\; -4\sum_{y\in\Lambda:\,\langle xy\rangle\in\Lambda^*} K_{xy}\,\eta_x\,\eta_y \;+ \qquad (2.44)$$
$$2\left(\mathcal{B}|h_x| + (\operatorname{sgn} h_x)\,h + \sum_{y\in\Lambda:\,\langle xy\rangle\in\Lambda^*} K_{xy} + \sum_{y\in\partial\Lambda^+:\,\langle xy\rangle\in\partial\Lambda} K_{xy}\,|\chi_y|\right)\eta_x \;.$$

Since we have

$$\tilde\Gamma^{\Lambda,\chi}_x(\tilde\eta) \;\ge\; 2(Y_{\mathcal{B},x}-2|h|) \quad \text{if} \quad \eta_x+\eta'_x>0 \;, \qquad (2.45)$$

we get

$$\left|\left\langle\!\left\langle\hat A\hat B\right\rangle\!\right\rangle^{\chi}_{\Lambda}\right| \;\le\; 4\,\|A\|\,\|B\|\,\frac{1}{\tilde Z^{\chi}_{\Lambda}}\sum_{\nu\in\mathcal{N}_{\Lambda,AB}}\;\prod_{x\in\nu_{\mathcal{S}}} 3\mathrm{e}^{-2\beta\,(Y_{\mathcal{B},x}-2|h|)}\sum_{G'\subset\mathcal{G}_{\mathcal{S},\nu}}\;\sum_{\tilde\eta\prec G'} \mathrm{e}^{-\beta\tilde H^{\Lambda,\chi}_{G'}(\tilde\eta)}$$
$$\le\; 4\,\|A\|\,\|B\|\sum_{\nu\in\mathcal{N}_{\Lambda,AB}}\;\prod_{x\in\nu_{\mathcal{S}}} 3\mathrm{e}^{-2\beta\,(Y_{\mathcal{B},x}-2|h|)} \;, \qquad (2.46)$$

since

$$\tilde Z^{\chi}_{\Lambda} \;=\; \sum_{G\subset\Lambda}\sum_{\tilde\eta\prec G}\mathrm{e}^{-\beta\tilde H^{\Lambda,\chi}_{G}(\tilde\eta)} \;\ge\; \sum_{G'\subset\mathcal{G}_{\mathcal{S},\nu}}\;\sum_{\tilde\eta\prec G'}\mathrm{e}^{-\beta\tilde H^{\Lambda,\chi}_{G'}(\tilde\eta)} \qquad (2.47)$$

and

$$\sum_{\tilde\eta\prec\nu_{\mathcal{S}}} 1 \;=\; 3^{|\nu_{\mathcal{S}}|} \;. \qquad (2.48)$$

(2.36) now follows from (2.46), (2.35) and (2.34). □

Proof of Theorem 1.3(i): If the J_{xy} are bounded, so we are in the situation described in Remark 1.5, the $\{\hat{Y}_{\mathcal{B},x}\,;\ x \in \mathbb{Z}^d\}$ are independent random variables. In the general case the $\{Y_{\mathcal{B},x}\,;\ x \in \mathbb{Z}^d\}$ are not independent, since $Y_{\mathcal{B},x}$ and $Y_{\mathcal{B},y}$ may not be independent if $\|x-y\|_1 = 1$, but the $\{Y_{\mathcal{B},x}\,;\ x \in \mathbb{Z}_2^d\}$ are independent random variables. Clearly, (2.36) implies

$$|\langle A; B\rangle_\Lambda^\chi| \ \le\ 2\,\|A\|\,\|B\|\,(2d+1)^2 \sum_{\hat{\nu}\in\hat{\mathcal{N}}_{AB}} 2^{|\hat{\nu}|} \prod_{x\in\hat{\nu}} \theta_x \ , \tag{2.49}$$

where

$$\hat{\mathcal{N}}_{AB} = \{\hat{\nu} : x \hat{\rightsquigarrow} y \ :\ x \in \ \mathrm{supp}_2 A\,,\ y \in \ \mathrm{supp}_2 B\} \tag{2.50}$$

with

$$\mathrm{supp}_2\, C \ = \ \{x \in \mathbb{Z}_2^d;\ d(x, \mathrm{supp}\, C) \le 1\}\,; \tag{2.51}$$

notice that each $\hat{\nu} \in \hat{\mathcal{N}}_{AB}$ replaces at most $(2d+1)^2 2^{|\hat{\nu}|-1}$ ν's in $\mathcal{N}_{AB}$.

If $q_\infty < \frac{1}{2(2d^2-1)}$, we pick $\mathcal{B} > 0$ such that $q_{\mathcal{B},0} < \frac{1}{2(2d^2-1)}$, and define

$$\mathcal{S} \ = \ \{x \in \mathbb{Z}^d;\ Y_{\mathcal{B},x} \le 0\} \ ; \tag{2.52}$$

In this case Theorem 1.3(i) (with (1.25), (1.26) and (1.27)) follows immediately from (2.49).

If $q_\infty < p_c^{(2)}(d)$, we again use Proposition 5.8 in [14], which holds also for site percolation in lattices like $\mathbb{Z}^d$ and $\mathbb{Z}_2^d$. More precisely, for $\mathbb{Z}_2^d$, say, let $\{\tau_q(x)\,;\ x \in \mathbb{Z}_2^d\}$ be independent identically distributed $\{0,1\}$-valued Bernoulli random variables, with $q = \mathbb{P}\{\tau_q(x) = 0\}$. Given $r > 0$, we define the events

$$\mathcal{E}_{q,n}(x) \ = \ \{\text{there exists } \ \hat{\nu} \in \hat{\mathcal{N}}_x \ \text{ such that } \ |\hat{\nu}| \ge n \ \text{ and } \ \tau_q(\hat{\nu}) < rn\}\ , \tag{2.53}$$

where $\tau_q(\hat{\nu}) = \sum_{x\in\hat{\nu}} \tau_q(x)$. Since $\mathcal{E}_{q,n}(x)$ is increasing in q, it follows from Proposition 5.8 in[14] that given $\tilde{q} < p_c^{(2)}(d)$, we can pick r for which there exist constants constants $b > 0$ and $C_1 < \infty$ such that

$$\mathbb{P}(\mathcal{E}_{q,n}(x)) \ \le \ C_1\,\mathrm{e}^{-b\,n} \tag{2.54}$$

for all $q \le \tilde{q}$, $x \in \mathbb{Z}_2^d$ and $n = 1, 2, \ldots$.

So given $q_\infty < p_c^{(2)}(d)$ we pick $\tilde{q}$, $q_\infty < \tilde{q} < p_c^{(2)}(d)$, and the corresponding r, b and C_1 in (2.54). For a given inverse temperature γ we define $\delta_\gamma > 0$ and $\mathcal{B}_1(\gamma) \ge 0$ by

$$(3\,\mathrm{e}^{-\gamma\,\delta_\gamma})^r \ = \ \frac{1}{2(2d^2-1)} \tag{2.55}$$

and

$$\mathcal{B}_1(\gamma) \ = \ \inf\{\mathcal{B} \ge 0\,;\ \ q_{\mathcal{B},\delta_\gamma} \le \tilde{q}\}\,; \tag{2.56}$$

notice $q_{\mathcal{B},\delta_\gamma} \ \le\ \tilde{q}$ for all $\mathcal{B} \ >\ \mathcal{B}_1(\gamma)$. We take

$$\mathcal{S} \ = \ \mathcal{S}_\gamma \ = \ \{x \in \mathbb{Z}^d\,;\ Y_{\mathcal{B}_1(\gamma),x} < \delta_\gamma\} \tag{2.57}$$

and $\tau^{(\gamma)}(x) = \mathbb{1}_{\mathbf{Z}^d\setminus\mathcal{S}_\gamma}(x)$, so the $\{\tau^{(\gamma)}(x);\ x \in \mathbb{Z}_2^d\}$ are independent identically distributed Bernoulli random variables with $\mathbb{P}\{\tau^{(\gamma)}(x) = 0\} \leq \tilde{q}$; in particular we have (2.54).

Let $\beta \geq \gamma$, $\mathcal{B} > \mathcal{B}_1(\gamma)$ and $x \notin \mathcal{S}_\gamma$; we have

$$Y_{\mathcal{B},x} \;=\; (\mathcal{B} - \mathcal{B}_1(\gamma))\,|h_x| + Y_{\mathcal{B}_1(\gamma),x} \;\geq\; \frac{\mathcal{B}}{\mathcal{B}_1(\gamma)}\delta_\gamma\,, \tag{2.58}$$

so (recall (2.35))

$$\zeta_x \;\leq\; 3\,\mathrm{e}^{-2\,\beta\,(\frac{\mathcal{B}}{\mathcal{B}_1(\gamma)}\delta_\gamma - 2|h|)}\,. \tag{2.59}$$

If we now set $\epsilon_\gamma = \frac{1}{4}\delta_\gamma$ and take $|h| \leq \epsilon_\gamma$, we get

$$\zeta_x \;\leq\; 3\,\mathrm{e}^{-\beta\,\frac{2\mathcal{B}-\mathcal{B}_1(\gamma)}{\mathcal{B}_1(\gamma)}\delta_\gamma} \;\leq\; 3\,\mathrm{e}^{-\beta\,\frac{\mathcal{B}}{\mathcal{B}_1(\gamma)}\delta_\gamma} \;<\; \left(\frac{1}{2(2d^2-1)}\right)^{\frac{1}{r}}\,, \tag{2.60}$$

the last inequality following from (2.55).

Thus for $\beta \geq \gamma$, $\mathcal{B} > \mathcal{B}_1(\gamma)$ and $|h| \leq \epsilon_\gamma$ it follows from (2.49) that

$$|\langle A;B\rangle_\Lambda^\chi| \;\leq\; 2\,\|A\|\,\|B\|\,(2d+1)^2 \sum_{\hat{\nu}\in\hat{\mathcal{N}}_{AB}} 2^{|\hat{\nu}|}\left(3\,\mathrm{e}^{-\beta\,\frac{\mathcal{B}}{\mathcal{B}_1(\gamma)}\delta_\gamma}\right)^{\tau^{(\gamma)}(\hat{\nu})}\,. \tag{2.61}$$

Theorem 1.3(i) now follows from (2.61) in the same way as Theorem 1.1(i) followed from (2.20). ∎

Proof of Theorem 1.3(ii)(a): For each $\gamma > 0$ the argument used to prove Theorem 1.1(ii)(a) now gives a set Ω_γ of realizations of the random parameters $(\mathbf{J},\mathbf{h})$, with $\mathbb{P}\{(\mathbf{J},\mathbf{h}) \in \Omega_\gamma\} = 1$, such that for each $(\mathbf{J},\mathbf{h}) \in \Omega_\gamma$ conclusion (a) in Theorem 1.3(ii) holds for all inverse temperatures $\beta \geq \gamma$. Now let $\Omega = \cup_{0<\gamma\in\mathbb{Q}}\Omega_\gamma$, clearly $\mathbb{P}\{(\mathbf{J},\mathbf{h}) \in \Omega\} = 1$ and now for each $(\mathbf{J},\mathbf{h}) \in \Omega$ conclusion (a) in Theorem 1.3(ii) holds for all inverse temperatures $\beta > 0$. ∎

2.3 Thermodynamical Limits and Uniqueness of the Gibbs State

LEMMA 2.3 *Let us fix $\mathbf{J},\mathbf{h},\mathcal{B}$ and h in (1.2). Suppose that for a given inverse temperature β and a local observable A, we can find $\mu > 0$ and $D_A < \infty$, such that for any other local observable B, any finite Λ containing their supports, and any any boundary condition χ on Λ, we have*

$$|\langle A;B\rangle_\Lambda^\chi| \;\leq\; D_A\,\|A\|\,\|B\|\,\mathrm{e}^{-\mu\,d(A,B)}\,. \tag{2.62}$$

Then for any finite Λ containing the support of A and any boundary condition χ on Λ, we have

$$|\langle A\rangle_\Lambda^\chi - \langle A\rangle_\Lambda| \;\leq\; \beta\,D_A\,\|A\| \sum_{\langle xy\rangle\in\partial\Lambda} |J_{xy}|\,\mathrm{e}^{-\mu\,d(A,x)}\,. \tag{2.63}$$

Proof: It follows from the Fundamental Theorem of Calculus that for any finite Λ containing the support of A and any boundary condition χ on Λ, we have

$$\langle A\rangle_{\Lambda}^{\chi} - \langle A\rangle_{\Lambda} = \int_0^1 \frac{d}{ds} \langle A\rangle_{\Lambda}^{s\chi}\, ds = \beta \int_0^1 \sum_{\langle xy\rangle \in \partial\Lambda} J_{xy}\, \chi_y\, \langle A; \sigma_x\rangle_{\Lambda}^{s\chi}\, ds\,. \tag{2.64}$$

(2.63) is now an immediate consequence of (2.62) and (2.64). ∎

LEMMA 2.4 *Let us fix* $\mathbf{J}, \mathbf{h}, \mathcal{B}$ *and* h *in (1.2). Suppose that for a given inverse temperature* β *we can find a sequence* $\{\Lambda_n\}$ *of finite subsets of* $\mathbb{Z}^d$*, with the property that every* $x \in \mathbb{Z}^d$ *is eventually in* Λ_n*, such that for any local observable* A *we have*

$$\lim_{n\to\infty} \sup_{\chi} \left| \langle A\rangle_{\Lambda_n}^{\chi} - \langle A\rangle_{\Lambda_n} \right| = 0\,, \tag{2.65}$$

where the supremum is taken over all external boundary conditions χ *on* Λ_n*. Then there exists a unique Gibbs state at this inverse temperature* β*, and for every local observable* A *the thermodynamical limit*

$$\langle A\rangle \equiv \lim_{\Lambda\to\mathbf{Z}^d} \langle A\rangle_{\Lambda}^{\chi_\Lambda} \tag{2.66}$$

exists and is independent of the boundary condition χ_Λ *used in each finite volume* Λ*.*

Proof: Let Φ be a Gibbs state at inverse temperature β; for every local observable A and finite Λ containing its support, the DLR equations give

$$\Phi(A) = \int \langle A\rangle_{\Lambda}^{\sigma|_{\partial\Lambda+}}\, d\Phi(\sigma)\,. \tag{2.67}$$

Thus

$$|\Phi(A) - \langle A\rangle_{\Lambda}| \le \sup_{\chi} |\langle A\rangle_{\Lambda}^{\chi} - \langle A\rangle_{\Lambda}|\,, \tag{2.68}$$

so the lemma follows. ∎

Recall $p_\infty = \mathbb{P}\{J_{xy} = +\infty\}$ and $p_\delta = \mathbb{P}\{|J_{xy}| > \delta\}$. For $R > 0$ set

$$B_R = \{x \in \mathbb{Z}^d;\ \|x\|_1 \le R\}\,. \tag{2.69}$$

LEMMA 2.5 *Suppose* $p_\infty < p_c^b(d)$*. Then there exist a finite number* $\kappa > 0$ *and a set* $\mathcal{J}'$ *of realizations of the random couplings with* $\mathbb{P}\{\mathbf{J} \in \mathcal{J}'\} = 1$*, such that for any* $R > 0$ *and* $\mathbf{J} \in \mathcal{J}'$ *we can find a finite subset* $\Lambda_R = \Lambda_{R,\mathbf{J}}$ *of* $\mathbb{Z}^d$ *with* $B_R \subset \Lambda_R$ *and* $|J_{xy}| \le \kappa$ *for all* $\langle xy\rangle \in \partial\Lambda_R$*.*

Proof: Since $p_\infty < p_c^b(d)$, we can find a finite number $\kappa > 0$ such that $p_\kappa < p_c^b(d)$. Given $x, y \in \mathbb{Z}^d$, we will say that $x \leftrightarrow y$ if there exists a self-avoiding bond walk $\omega : x \to y$

with $|J_{x'y'}| > \kappa$ for each $\langle x'y'\rangle \in \omega$; for $x \in \mathbb{Z}^d$ we call $\mathcal{C}_x = \{y \in \mathbb{Z}^d ;\ x \leftrightarrow y\}$ the cluster of x. As $p_\kappa < p_c^b(d)$, we can find a set $\mathcal{J}'$ of realizations of the random couplings with $\mathbb{P}\{\mathbf{J} \in \mathcal{J}'\} = 1\} = 1$, such that for $\mathbf{J} \in \mathcal{J}'$ there are no infinite clusters. For any $R > 0$ and $\mathbf{J} \in \mathcal{J}'$ the set

$$\Lambda_R \quad = \quad B_R \cup \{\cup_{x\in B_R} \mathcal{C}_x\} \ . \tag{2.70}$$

is finite and clearly satisfies the desired properties. ∎

Proof of Part (ii)(b) of Theorems 1.1 and 1.3: In both case we have $p_\infty < p_c^b(d)$, so we use Lemma 2.5 to pick κ and $\mathcal{J}'$ and choose $\Lambda_R = \Lambda_{R,\mathbf{J}}$ for each $R > 0$ and $\mathbf{J} \in \mathcal{J}'$. Now let $\mathbf{J}, \mathbf{h}, \mathcal{B}$, h and β be as in part (ii)(a) of either Theorem 1.1 or 1.3, with $\mathbf{J} \in \mathcal{J}'$ also. Let A be a local observable with $\operatorname{supp} A \subset B_S$ for some $S < \infty$. It follows from (1.14) and Lemma 2.3 that for any $R > S$ we have

$$\begin{aligned}
\sup_{\chi} \left| \langle A\rangle^{\chi}_{\Lambda_R} - \langle A\rangle_{\Lambda_R} \right| \quad &\le \quad \kappa\beta\, D_A\, \|A\| \sum_{\langle xy\rangle\in\partial\Lambda_R} \mathrm{e}^{-\mu\, d(A,x)} \\
&\le \quad \kappa\beta\, D_A\, \|A\|\, \mathrm{e}^{\mu\, S} \sum_{\langle xy\rangle\in\partial\Lambda_R} \mathrm{e}^{-\mu\, \|x\|_1} \\
&\le \quad \kappa\beta\, D_A\, \|A\|\, \mathrm{e}^{\mu\, S} \left(2d \sum_{x\in\mathbf{Z}^d} \mathrm{e}^{-\frac{\mu}{2}\|x\|_1} \right) \mathrm{e}^{-\frac{\mu}{2} R} \, .
\end{aligned} \tag{2.71}$$

The desired conclusion now follows from Lemma 2.4. ∎

3 INFINITE DIFFERENTIABILITY

In this section we will prove part (iii) of Theorems 1.1 and 1.3, i.e., the infinite differentiability of of all quenched correlation functions. In particular we will obtain the infinite differentiability of the magnetization with probability one, since it follows from ergodicity that

$$\lim_{\Lambda\to\infty} \frac{1}{|\Lambda|} \sum_{x\in\Lambda} \langle \sigma_x\rangle \quad = \quad \mathbb{E}\, \langle \sigma_0\rangle \tag{3.1}$$

with probability one.

LEMMA 3.1 *Let $\langle\ \rangle_\Lambda$ be a random state on the algebra of local observables with support in the set $\Lambda \subset \mathbb{Z}^d$, such that there exist $C < \infty$ and $m > 0$ for which*

$$\mathbb{E}\left(|\langle A;B\rangle_\Lambda|\right) \quad \le \quad C\, |supp\, A|\, \|A\|\, \|B\|\, \mathrm{e}^{-m\, d(A,B)} \tag{3.2}$$

for any two local observables A and B with support in Λ. Then there exist constants $C_n < \infty$, $n = 1, 2, \ldots$, depending only on C, m and n, such that

$$\mathbb{E}\left(|\langle A;\sigma_{x_1};\ldots;\sigma_{x_n}\rangle_\Lambda|\right) \quad \le \quad C_n\, |supp\, A|\, \|A\|\, \exp\{-m_n\, d(A,x_1,\ldots,x_n)\} \tag{3.3}$$

for all local observables A with support in Λ and all $x_1, \ldots, x_n \in \Lambda$, where

$$m_n \;=\; \frac{2m}{(n+1)!} \; . \tag{3.4}$$

Proof: The proof proceeds by induction on n. If $n=1$ (3.3) is just a special case of (3.2).

Now suppose (3.3) is true for all $n' \leq n-1$ and let A be a local observable with support in Λ and $x_1, \ldots, x_n \in \Lambda$. Using Lemma 3.9 in [4] we have the reduction

$$\begin{aligned} &\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n} \rangle_\Lambda \;= \\ &\qquad \langle \sigma_{x_1} A; \sigma_{x_2}; \ldots; \sigma_{x_n} \rangle_\Lambda - \sum_{P \subset \{2,\ldots,n\}} \left\langle A; \sigma_{x_{p_1}}; \ldots; \sigma_{x_{p_k}} \right\rangle_\Lambda \left\langle \sigma_{x_1}; \sigma_{x_{q_1}}; \ldots; \sigma_{x_{q_{n-k-1}}} \right\rangle_\Lambda , \end{aligned} \tag{3.5}$$

where $P = \{p_1, \ldots, p_k\}$ and $\{q_1, \ldots, q_{n-k-1}\} = \{2, \ldots, n\} \setminus P$. Using Holder's inequality and the induction hypothesis, we have that for any $P \subset \{2, \ldots, n\}$

$$\begin{aligned} &\mathbb{E}\left(\left| \left\langle A; \sigma_{x_{p_1}}; \ldots; \sigma_{x_{p_k}} \right\rangle_\Lambda \left\langle \sigma_{x_1}; \sigma_{x_{q_1}}; \ldots; \sigma_{x_{q_{n-k-1}}} \right\rangle_\Lambda \right| \right) \\ &\quad \leq \left(\mathbb{E}\left(\left| \left\langle A; \sigma_{x_{p_1}}; \ldots; \sigma_{x_{p_k}} \right\rangle_\Lambda \right|^{\frac{n-1}{k}} \right) \right)^{\frac{k}{n-1}} \left(\mathbb{E}\left(\left| \left\langle \sigma_{x_1}; \sigma_{x_{q_1}}; \ldots; \sigma_{x_{q_{n-k-1}}} \right\rangle_\Lambda \right|^{\frac{n-1}{n-k-1}} \right) \right)^{\frac{n-k-1}{n-1}} \\ &\quad \leq C'_n |\mathrm{supp}\, A| \, \|A\| \exp\left\{ -\tfrac{k}{n-1} m_k \, d(A, x_{p_1}, \ldots, x_{p_k}) \right\} \exp\left\{ -\tfrac{n-k-1}{n-1} m_{n-k-1} \, d(x_1, x_{q_1}, \ldots, x_{q_{n-k-1}}) \right\} \\ &\quad \leq C'_n |\mathrm{supp}\, A| \, \|A\| \exp\left\{ -m_{n-1} \, d(A \cup x_1, x_2, \ldots, x_n) \right\} \; , \end{aligned} \tag{3.6}$$

where we used $A \cup x_1$ for $\mathrm{supp}\, A \cup x_1 = \mathrm{supp}\, \sigma_{x_1} A$, since

$$d(A, x_{p_1}, \ldots, x_{p_k}) \, + \, d(x_1, x_{q_1}, \ldots, x_{q_{n-k-1}}) \;\; \geq \;\; d(A \cup x_1, x_2, \ldots, x_n) \; . \tag{3.7}$$

By C'_n, C''_n, ... we denote finite constants depending only on n, C and m. It thus follows from (3.5), with the induction hypothesis applied to the first term on the right hand side and (3.6) to each term in the sum, that

$$\mathbb{E}\left(|\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n} \rangle_\Lambda| \right) \;\; \leq \;\; C''_n \, |\mathrm{supp}\, A| \, \|A\| \, \exp\left\{ -m_{n-1} \, d(A \cup x_1, x_2, \ldots, x_n) \right\} \; . \tag{3.8}$$

We need one more reduction formula, which follows from Lemma 3.9 and formula (1.2) in [4]:

$$\begin{aligned} &\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n} \rangle_\Lambda \;= \\ &\qquad \langle A; \sigma_{x_1} \cdots \sigma_{x_n} \rangle_\Lambda - \sum_{\substack{P \subset \{1,\ldots,n\} \\ P \neq \emptyset, \{1,\ldots,n\}}} \left\langle A; \sigma_{x_{p_1}}; \ldots; \sigma_{x_{p_k}} \right\rangle_\Lambda \left\langle \sigma_{x_{q_1}} \cdots \sigma_{x_{q_{n-k}}} \right\rangle_\Lambda \; , \end{aligned} \tag{3.9}$$

where $P = \{p_1, \ldots, p_k\}$ and $\{q_1, \ldots, q_{n-k}\} = \{1, \ldots, n\} \setminus P$. Taking expectations and using the induction hypothesis we get

$$\mathbb{E}\left(|\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n} \rangle_\Lambda| \right) \;\; \leq \;\; C'''_n \, |\mathrm{supp}\, A| \, \|A\| \, \exp\left\{ -m_{n-1} \, d(A, \{x_1, x_2, \ldots, x_n\}) \right\} \; . \tag{3.10}$$

(3.3) now follows from (3.8), applied to each of the n cyclical permutations of $x_1, x_2, \ldots, x_n$, and (3.10), since a moment of reflexion shows that

$$\sum_{i=1}^{n} d(A \cup x_i, x_1, \ldots, \hat{x}_i, \ldots, x_n) \ + \ d(A, \{x_1, x_2, \ldots, x_n\}) \ \geq \ d(A, x_1, \ldots, x_n) \, . \tag{3.11}$$

∎

Proof of Part (iii) of Theorems 1.1 and 1.3: For any $\mathbf{J} \in \mathbb{R}^{\mathbf{Z}^{d^*}}$, $\mathbf{h} \in \mathbb{R}^{\mathbf{Z}^d}$, $\mathcal{B} \in \mathbb{R}$, and $h_1, h_2 \in \mathbb{R}$, $\beta > 0$, finite region $\Lambda \subset \mathbb{Z}^d$ and local observable A with support in Λ, it is easy to see that

$$\begin{aligned} \langle A; \sigma_{x_1}; \ldots; \sigma_{x_n} \rangle_\Lambda (h_2) \ - \ \langle A; \sigma_{x_1}; \ldots; \sigma_{x_n} \rangle_\Lambda (h_1) \\ = \ -\beta \int_{h_1}^{h_2} \sum_{x \in \Lambda} \langle A; \sigma_{x_1}; \ldots; \sigma_{x_n}; \sigma_x \rangle_\Lambda (h) \, dh \end{aligned} \tag{3.12}$$

for any $n = 0, 1, \ldots$ and $x_1, \ldots, x_n \in \mathbb{Z}^d$.

If our parameters satisfy the hypotheses of part (iii) of either Theorem 1.1 or Theorem 1.3, (1.16) follows from Lemma 3.1. Moreover, we can take expectations in (3.12), use Lemma 3.1 and take the thermodynamical limit to obtain

$$\begin{aligned} | \, \mathbb{E}(\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n} \rangle (h_2)) \ - \ \mathbb{E}(\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n} \rangle (h_1)) \, | \\ \leq \ \beta \, C_{n+1} \, |\mathrm{supp}\, A| \, \|A\| \, |h_2 - h_1| \sum_{x \in \mathbf{Z}^d} \exp \{ -m_{n+1} \, d(A, x_1, \ldots, x_n, x) \} \ , \end{aligned} \tag{3.13}$$

so we can conclude that each $\mathbb{E}(\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n} \rangle (h))$ is a continuous function of h. It follows that

$$\sum_{x_1, \ldots, x_n \in \mathbf{Z}^d} \mathbb{E}(\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n} \rangle (h))$$

is also a continuous function of h since

$$\sum_{x_1, \ldots, x_n, x \in \mathbf{Z}^d} \exp \{ -m_{n+1} \, d(A, x_1, \ldots, x_n, x) \} \ < \ \infty \, . \tag{3.14}$$

Once more we take expectations and the thermodynamical limit in (3.12), using the continuity of $\mathbb{E}(\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n}; \sigma_x \rangle (h))$ in h, (3.3) and the bounded convergence theorem, obtaining

$$\begin{aligned} \mathbb{E}(\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n} \rangle (h_2)) \ - \ \mathbb{E}(\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n} \rangle (h_1)) \\ = \ -\beta \int_{h_1}^{h_2} \sum_{x \in \mathbf{Z}^d} \mathbb{E}(\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n}; \sigma_x \rangle (h)) \, dh \, . \end{aligned} \tag{3.15}$$

The Fundamental Theorem of Calculus now tells us that

$$\frac{\partial}{\partial h} \mathbb{E}(\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n} \rangle (h)) \ = \ \sum_{x \in \mathbf{Z}^d} \mathbb{E}(\langle A; \sigma_{x_1}; \ldots; \sigma_{x_n}; \sigma_x \rangle (h)) \, . \tag{3.16}$$

Since

$$\frac{\partial^n}{\partial h^n}\mathbb{E}(\langle A\rangle_\Lambda(h)) = (-\beta)^n \sum_{x_1,\ldots,x_n\in\Lambda} \mathbb{E}(\langle A;\sigma_{x_1};\ldots;\sigma_{x_n}\rangle_\Lambda(h)) , \qquad (3.17)$$

a similar argument using (3.14) gives the infinite differentiability of $\mathbb{E}(\langle A\rangle(h))$ and (1.17). ∎

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