DISTRIBUTION OF CLOSED ORBITS FOR LINEAR AUTOMORPHISMS OF TORI

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Abstract. In this paper we study the distribution properties of periodic orbits for the linear hyperbolic automorphisms of the d-torus. We first obtain an explicit expression of the dynamical zeta function and prove general equidistribution results similar to those obtained for Axiom A flows. We then study in detail some families of periodic orbits living on invariant prime lattices: they have the property that the integral of any character along any single orbit can be reduced to a number theoretic exponential sum over a finite field. This fact enables us to obtain explicit estimates on their asymptotic distributional properties.

1. Introduction.

A number of interesting results on the distribution of periodic orbits for hyperbolic dynamical systems have recently been obtained pursuing strong analogies with general results of number theory and probability theory.

Indeed, using the Ruelle zeta function [18] for a hyperbolic flow in the same way as the Riemann zeta function is used to prove the prime number theorem (that is through Wiener-Ikehara Tauberian theorem, see, e.g., [11]), Parry and Pollicott [15] proved a 'prime orbit theorem', saying that the number $\pi(x)$ of periodic orbits γ with (minimal) period $p(\gamma) \leq x$ is asymptotic to e^{hx}/hx as $x \to \infty$, where h is the topological entropy of the flow. Moreover, if $\int_\gamma f$ denotes the (normalized) integral of the continuous function f along γ, then $(1/\pi(x)) \sum_{p(\gamma)\leq x} \int_\gamma f \to \int f\, d\mu$ as $x \to \infty$, where μ is the invariant probability measure of maximal entropy (see [16] and references therein).

Other types of theorems yield the equidistribution of periodic orbits for hyperbolic flows in the sense of probability theory [12], [10]. For instance, under certain conditions and for any $\epsilon > 0$, $\mathrm{Prob}\{|\int_\gamma f - \int f\, d\mu| > \epsilon\} \to 0$ as $p(\gamma) \to \infty$, which of course improves the former result. This means that, except for a small (tending to zero) fraction, long periodic orbits are nearly uniformly distributed according to μ.

In what follows we shall study the distribution properties of periodic orbits for a particular example: the linear hyperbolic automorphism of the d-torus. Besides proving general results as those mentioned above (Section 2), we shall concentrate on some particular families of periodic orbits living on invariant prime lattices (Section 3). These enjoy the nice properties that the integrals of characters over any single orbit can be reduced to (some combinations of) number theoretic exponential sums, like generalized Gauss sums or Kloosterman sums, over a finite field (see [20]). This makes possible to obtain explicit estimates on their asymptotic distributional properties.

We point out that the statistical properties of the periodic orbits turn out to be crucial in the investigation of the quantum counterpart of this dynamical system [8], [9]. In particular there is an intriguing connection between the spatial distribution of certain families of closed orbits (see Section 3) and the localization properties of the eigenfunctions of the quantum propagator in the classical limit [4], [5].

2. The algebraic hyperbolic automorphisms of the d-torus.

We consider the discrete time dynamical system $F : M \to M$ where M is the $d-$torus $\boldsymbol{T}^d = \mathbb{R}^d/Z^d$ (points on $\boldsymbol{T}^d$ are denoted by $x = (x_1, \ldots, x_d) \in [0,1]^d$) and F is the hyperbolic automorphism of $\boldsymbol{T}^d$ generated by the $d \times d$ matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1d} \\ a_{21} & a_{22} & \ldots & a_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ a_{d1} & a_{d2} & \ldots & a_{dd} \end{pmatrix} \tag{2.1}$$

where $a_{ij} \in Z$, $i,j = 1 \ldots d$, and A has real spectrum $\Lambda = \{\lambda_1, \ldots, \lambda_d\}$ such that $\lambda_1 \lambda_2 \ldots \lambda_d = 1$. The Lebesgue measure μ is invariant because $\det A = 1$. The above condition on the eigenvalues makes this dynamical system an Anosov one and hence, in particular, ergodic and mixing with respect to μ.

Now, an orthonormal basis in $L^2(\boldsymbol{T}^d, d\mu)$ is given by the set

$$\{T(n) = e^{2\pi i <n,x>} | n \in Z^d J\} \tag{2.2}$$

and A acts on points $x = (x_1, \ldots, x_d)$ and on suitably smooth functions

$$f(x) = \sum_{n \in Z^d} f_n \, T(n)$$

respectively as:

$$\begin{cases} Ax = ((a_{11}x_1 + \ldots a_{1d}x_d)(\mathrm{mod}\, 1), \ldots, (a_{d1}x_1 + \ldots a_{dd}x_d)(\mathrm{mod}\, 1)) \\ f(Ax) = \sum_{n \in Z^d} f_n \, T(A^t n) \end{cases} \tag{2.3}$$

where A^t is the transposed matrix of A (notice that we have used the same symbol A to denote the matrix A and the map F; this will be repeatedly done in what follows without fear of confusion).

Now, if A is a continuous map of a compact metric space X then $h(A)$, the topological entropy, satisfies the restricted variational principle:

$$h(A) = \sup_{\mu \in \mathcal{M}_A(X)} h_\mu(A) \tag{2.4}$$

where $\mathcal{M}_A(X)$ the set of the probability measures on the Borel σ-algebra of X which are A-invariant and $h_\mu(A)$ is the measure theoretic entropy (see, e.g., [13]). Moreover, A is *intrinsically ergodic* if there exist a unique $\mu \in \mathcal{M}_A(X)$ such that $h(A) = h_\mu(A)$. In this case μ is called the intrinsic measure of A, or maximal entropy measure of A. Thus, for a linear automorphism of the torus the Haar measure, i.e. the Lebesgue one, is actually an intrinsic measure. We have the

Theorem 2.1 (Sinai) *Let $A : \boldsymbol{T}^d \to \boldsymbol{T}^d$ be a linear automorphism with eigenvalues $\lambda_1, \dots \lambda_d$. Let μ be the Haar measure of T^d. Then*

$$h(A) = h_\mu(A) = \log \prod_{i\,:\,|\lambda_i|>1} |\lambda_i| \tag{2.5}$$

Proof. See, e.g., [13] p. 265.

Consider again the general situation of a continuous map A of a compact metric space X. We say that $x \in X$ is a *periodic point* of A, of period n, if it is a *fixed point* of A^n, i.e. $A^n x = x$. We denote by Fix_n the set of such points. It is easy to see that for linear automorphism of the torus the set of periodic points of A is dense in $\boldsymbol{T}^d$, because it coincides with the subset of $\boldsymbol{T}^d$ formed by all points having rational coordinates. More generally we have the

Theorem 2.2 (Bowen-Sinai) *Every topologically mixing hyperbolic homeomorphism $A : X \to X$ is intrinsically ergodic. If μ denotes its intrinsic measure* *then for every continuous map $f : X \to I\!R$:*

$$\int_X f d\mu = \lim_{n\to\infty} \frac{1}{\# Fix_n} \sum_{x\in Fix_n} f(x) \tag{2.6}$$

and A has topological entropy

$$h = \lim_{n\to\infty} \frac{1}{n} \log \# Fix_n \tag{2.7}$$

Proof. See [13], p.254.

For the linear automorphism of the d-torus we have the additional result:

Proposition 2.1.

$$\# Fix_n = |\det(A^n - 1)| = \left| \prod_{i=1}^{d} (\lambda_i^n - 1) \right| \tag{2.8}$$

Remark. Notice that from (2.7) and (2.8) one immediately recovers (2.5).

Proof. The periodic points of period n of the map $A : \boldsymbol{T}^d \to \boldsymbol{T}^d$ are given by the solution of the linear system

$$(A^n - 1)\; x \;=\; k, \qquad x \in I\!R^d,\; k \in Z^d \tag{2.9}$$

It is easy to realize that for k varying in Z^d the above system determines a regular lattice L_n in the covering space $I\!R^d$ so that the density of periodic points belonging to Fix_n is given by the inverse of the volume of the fundamental domain Δ_n of L_n. On the other hand, if $\{e_1, \ldots, e_d\}$ denotes the basis of the lattice L_n and each vector e_i is expressed as $e_i = (r_{1i}, \ldots, r_{di})$ then the volume $v(\Delta_n)$ of Δ_n is given by

$$v(\Delta_n) = |\det r_{ij}| \tag{2.10}$$

Now, a basis $\{e_1, \ldots, e_d\}$ of the lattice L_n can be easily obtained by solving eq. (2.9) for d distinct and linearly independent vectors $k_1 = (k_{11}, \ldots, k_{1d}), \ldots, k_d = (k_{d1}, \ldots, k_{dd})$, $k_{ij} \in Z$, such that

$$\det K \equiv \det \begin{vmatrix} k_{11} & \dots & k_{1d} \\ \vdots & \ddots & \vdots \\ k_{d1} & \dots & k_{dd} \end{vmatrix} = 1 \tag{2.11}$$

This yields

$$\upsilon(\Delta_n) = \frac{1}{\Gamma^d} \det \begin{vmatrix} D_{11} & \dots & D_{1d} \\ \vdots & \ddots & \vdots \\ D_{d1} & \dots & D_{dd} \end{vmatrix} \tag{2.12}$$

where $\Gamma = |\det(A^n - 1)|$ and D_{ij} is the determinant of the $d \times d$ matrix obtained by inserting the vector k_i in the place of the j-th column of the matrix $A^n - 1$ (Cramer's rule). Finally, a tedious but trivial calculation where the basic properties of the determinant are exploited gives

$$\upsilon(\Delta_n) = \frac{\Gamma^{d-1} \det K}{\Gamma^d} \tag{2.13}$$

Hence, from (2.11) and (2.13) one obtains

$$\#Fix_n = \frac{1}{\upsilon(\Delta_n)} = |\det(A^n - 1)| \tag{2.14}$$

Q.E.D.

Let us now restrict ourselves to symplectic matrices, i.e. d is even, say $d = 2r$, and $A^t J A = J$, where $J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}$, and E denotes the $r \times r$ identity matrix, so that if $\lambda \in \Lambda$ then also $1/\lambda \in \Lambda$. The following result extends an earlier result for $d = 2$ [I].

Corollary 2.1. *Let A be symplectic and $\Lambda = \{\lambda_1, \dots, \lambda_r, 1/\lambda_1, \dots, 1/\lambda_r\}$ where $\lambda_i > 1$, $i = 1, \dots, r$. Then,*

$$\#Fix_n = \prod_{i=1}^{r} \left(\lambda_i^n + \lambda_i^{-n} - 2\right) \tag{2.15}$$

Proof.

$$\left|\prod_{i=1}^{d} (\lambda_i^n - 1)\right| = \prod_{i=1}^{r} (\lambda_i^n - 1) \prod_{i=1}^{r} \left(1 - \lambda_i^{-n}\right) = \prod_{i=1}^{r} \left(\lambda_i^n + \lambda_i^{-n} - 2\right)$$

Q.E.D.

We now consider the following dynamical zeta function [21]:

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{z^n}{n} \# Fix_n \tag{2.16}$$

By virtue of Theorem 2.2. the series converges for $|z| < e^{-h}$. Moreover, from Proposition 2.1 it easily follows that for any algebraic hyperbolic automorphisms of the d-torus $\zeta(z)$ is a rational function with real zeroes and poles. In the simpler case of an orientation preserving symplectomorphism, from Corollary 2.1 one obtains the following expression:

Proposition 2.2. *Under the assumptions of Corollary 2.1 we have*

$$\zeta(z) = \prod_{\substack{i_1 \neq \ldots \neq i_{r_1} \neq j_1 \neq \ldots \neq j_{r_2} \\ r_1 + r_2 + r_3 = r}} \frac{1}{\left(1 - \lambda_{i_1} \ldots \lambda_{i_{r_1}} \lambda_{j_1}^{-1} \ldots \lambda_{j_{r_2}}^{-1} \cdot z\right)^{(-2)^{r_3}}} \tag{2.17}$$

where the product is over distinct $(r_1 + r_2)$*-tuples.*

Observe that the above zeta function has 2^r simple real poles located at $z = \left(\lambda_{i_1} \ldots \lambda_{i_{r_1}} \lambda_{j_1}^{-1} \ldots \lambda_{j_{r_2}}^{-1}\right)^{-1}$ where $i_1 \neq \ldots \neq i_{r_1} \neq j_1 \neq \ldots \neq j_{r_2}$ and $r_1 + r_2 = r$. In particular, for $r_2 = 0$ one gets the first pole $z = (\lambda_1 \ldots \lambda_r)^{-1} = e^{-h}$. All the other poles and zeroes have multiplicities larger than one. In particular, for $r_1 = r_2 = 0$ one gets the most degenerate zero (or pole, depending on the parity of r) located at $z = 1$, with multiplicity 2^r.

Remarks.

1) For the case $d = 2$ $(r = 1)$ one finds the expression

$$\zeta(z) = \frac{(1 - z)^2}{(1 - \lambda z)(1 - z/\lambda)}$$

where λ is the largest eigenvalue of A [7].

2) If instead of the zeta function (2.16) one considers the following Fredholm determinant:

$$d(z) = \exp - \sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{x \in Fix_n} \frac{1}{|\det(A^n(x) - 1)|}$$

then using Proposition 2.1 one gets the expression (valid for any algebraic hyperbolic automorphism A of the d-torus): $d(z) = 1 - z$. In particular, this implies the otherwise easily proven result that correlation functions for analytic observables decay faster than any exponential (see [19]).

We now deduce some consequences of the above results. For simplicity' sake we shall give the arguments for the case of the hyperbolic symplectomorphism of the d-torus, with $d = 2r$, but all the results given in the rest of this Section can be easily extended to the general setting.

Denote by P the set of all primitive (i.e., non repeated) periodic orbits of A and by $\pi'(n)$ and $\pi(x)$ the number of them whose period is n and less or equal to x respectively, i.e.

$$\pi'(n) = \#\{\gamma \in P \,:\, p(\gamma) = n\} \qquad \pi(x) = \#\{\gamma \in P \,:\, p(\gamma) \leq x\} \tag{2.18}$$

Proposition 2.3.

$$\pi'(n) \sim \frac{e^{nh}}{n} \tag{2.19}$$

and

$$\pi(x) \sim \frac{e^h}{e^h - 1} \cdot \frac{e^{xh}}{x} \tag{2.20}$$

where $f(t) \sim g(t)$ means that $f(t)/g(t) \to 1$ when $t \to \infty$.

Remark. Proposition 2.2 is the analogue of the *prime orbit theorem* proved in the context of Axiom A flows (see [16]).

Proof. The strategy is to gain insight on the distribution of closed orbits out of the meromorphy domain of $\zeta(z)$. ¿From Proposition 2.2 we have that

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{e^h}{1 - e^h z} + g(z)$$

where $g(z)$ is analytic in $\{z \,:\, |z| < e^{\epsilon - h}\}$ for some $\epsilon > 0$. The rest of the proof makes use of the Wiener-Ikehara Tauberian theorem and proceeds exactly in the same way as in the proof for subshifts of finite type ([16], p.100). Q.E.D.

We now prove that closed orbits exhibit a regularity in a spatial sense. In particular, we show that they are equidistributed on the average with respect to the Lebesgue measure μ.

Let μ_γ be the measure defined by

$$\mu_\gamma = \frac{1}{p(\gamma)} \sum_{k=0}^{p(\gamma)-1} \delta_{A^k(x)} \qquad x \in \gamma \tag{2.24}$$

and set $\int_\gamma f = \int_{\boldsymbol{T}^d} f d\mu_\gamma$. Then we have the

Proposition 2.4. *For every continuous map $f : \boldsymbol{T}^d \to I\!\!R$:*

$$\frac{1}{\pi'(n)} \sum_{p(\gamma)=n} \int_\gamma f \longrightarrow \int_{\boldsymbol{T}^d} f d\mu \quad \text{as} \quad x \to \infty \tag{2.25}$$

Remark. This result is an easy consequence of more general results proved in [16]. Nevertheless, for the convenience of the reader, we shall give a simple direct proof based on Theorem 2.2 and Proposition 2.1.

Proof. We first write the number of fixed points of period n as

$$\#Fix_n = \sum_{l|n} l \cdot \pi'(l) \tag{2.26}$$

where $l|n$ means that l divides n. Form Proposition 2.1 we then have

$$\sum_{l|n} l \cdot \pi'(l) \sim e^{nh} \tag{2.27}$$

On the other hand

$$\pi'(n) = \frac{1}{n} \left(\sum_{l|n} l \cdot \pi'(l) - \sum_{l|n,\, l<n} l \cdot \pi'(l) \right)$$

which, together with (2.27), gives

$$\pi'(n) \sim \frac{e^{nh}}{n}(1 - C_n) \tag{2.28}$$

where C_n is of order strictly smaller than one. Analogously,

$$\sum_{x \in Fix_n} f(x) = \sum_{l|n} l \cdot \sum_{p(\gamma)=l} \int_\gamma f \tag{2.29}$$

¿From Theorem 2.2 we then have

$$\sum_{l|n} l \cdot \sum_{p(\gamma)=l} \int_\gamma f \sim e^{nh} \int_{\boldsymbol{T}^d} f d\mu \tag{2.30}$$

On the other hand

$$\sum_{p(\gamma)=n} \int_\gamma f = \frac{1}{n}\Big(\sum_{l|n} l \cdot \sum_{p(\gamma)=l} \int_\gamma f - \sum_{l|n,\ l<n} l \cdot \sum_{p(\gamma)=l} \int_\gamma f \Big)$$

which, together with (2.30), gives

$$\sum_{p(\gamma)=n} \int_\gamma f \sim \frac{e^{nh}}{n}(1 - C_n) \int_{\boldsymbol{T}^2} f d\mu, \quad \text{as} \quad n \to \infty \tag{2.31}$$

where C_n is as in (2.28). To conclude the proof it now suffices to divide (2.31) by (2.28). Q.E.D.

Proposition 2.4 yields a result of uniform distribution on the average. The next result is an analogue of the weak law of large numbers and improves Proposition 2.4.

Proposition 2.5. *For any $\epsilon > 0$ and for any continuous function f*

$$\lim_{n \to \infty} \left(\frac{\#\{\gamma \,:\, p(\gamma) = n,\ \ |\int_\gamma f - \int_{\boldsymbol{T}^d} f d\mu| > \epsilon\}}{\pi'(n)} \right) = 0 \tag{2.32}$$

Proof. For the sake of simplicity we shall use the notation $\int_{\boldsymbol{T}^d} f d\mu = \bar{f}$. For any $k \in Z_+$ set

$$m_k f(x) = \frac{1}{k} \sum_{l=0}^{k-1} f(A^l x) \tag{2.33}$$

Now, given $\delta > 0$, introduce the sets

$$\begin{aligned} S_{\delta,n} &= \{\gamma \,:\, p(\gamma) = n,\ \ \left| \int_\gamma f - \bar{f} \right| \leq \sqrt{\delta}\} \\ R_{\delta,n,k} &= \{\gamma \,:\, p(\gamma) = n,\ \ \int_\gamma |m_k f - \bar{f}| \leq \sqrt{\delta}\} \end{aligned} \tag{2.34}$$

We have

$$\left|\int_\gamma f - \bar f\right| \le \left|\int_\gamma (f - m_k f)\right| + \int_\gamma |m_k f - \bar f|$$

On the other hand it is obvious that $\int_\gamma (f - m_k f) = 0$, for any $k \in Z_+$; and thus

$$S_{\delta,n} \supset R_{\delta,n,k} \Longrightarrow R^c_{\delta,n,k} \supset S^c_{\delta,n} \quad \text{for any} \quad \delta > 0,\ k \in Z_+ \tag{2.35}$$

The ergodicity of A implies that, for μ-almost every $x \in \boldsymbol{T}^d$,

$$\lim_{k\to\infty} m_k f(x) = \bar f \tag{2.36}$$

Hence, for any continuous f, by the Lebesgue dominated convergence theorem we can find a $k_0 > 0$ such that

$$\int_{\boldsymbol{T}^d} |m_{k_0} f(x) - \bar f| d\mu \le \frac{\delta}{2} \tag{2.37}$$

Set

$$\int_{\boldsymbol{T}^d} f d\mu_n := \frac{1}{\pi'(n)} \sum_{p(\gamma)=n} \int_\gamma f$$

Then Proposition 2.4 entails that $d\mu_n$ converges vaguely to the Lebesgue measure. This implies the existence of $n_0(\delta) > 0$ such that for any $n > n_0$

$$\int_{\boldsymbol{T}^d} |m_{k_0} f(x) - \bar f| d\mu_n \le \delta \tag{2.38}$$

Hence by the Chebychev inequality we obtain

$$\frac{\#\{\gamma\,:\,p(\gamma) = n,\ \ \int_\gamma |m_{k_0} f - \bar f| \ge \sqrt{\delta}\}}{\pi'(n)} \le \sqrt{\delta} \tag{2.39}$$

and therefore, by the second of (2.34):

$$\frac{\# R^c_{\delta,n,k_0}}{\pi'(n)} \le \sqrt{\delta}, \qquad n > n_0 \tag{2.40}$$

If we now put $\delta = 1/k^2$ then from (2.35) we find

$$\frac{\#\{\gamma\,:\,p(\gamma) = n,\ \ |\int_\gamma f - \int_{\boldsymbol{T}^d} f d\mu| > \frac{1}{k}\}}{\pi'(n)} \le \frac{1}{k}, \qquad n > n_0 \tag{2.41}$$

so that, given $\epsilon > 0$, we have for $k > 1/\epsilon$ and $n > n_0(k)$,

$$\frac{\#\{\gamma \,:\, p(\gamma) = n, \;\; |\int_\gamma f - \int_{\boldsymbol{T}^d} f d\mu| > \epsilon\}}{\pi'(n)} \leq \frac{1}{k} \tag{2.42}$$

and the assertion follows by taking $k \to \infty$ and consequently $n \to \infty$. Q.E.D.

3. Periodic orbits living on prime lattices

We now turn to study the asymptotic properties of the probability measures μ_γ (cf. (2.24)) supported on *single* periodic orbits γ's of A when $p(\gamma) \to \infty$. To this end we first remark the following fact. Consider a sequence $\{\gamma_k\}$ of periodic orbits of $A : \boldsymbol{T}^d \to \boldsymbol{T}^d$ ordered with non-decreasing periods. Then $\{\mu_{\gamma_k}\}$ converges weakly to μ as $k \to \infty$ if and only if [1]

$$\int_{\gamma_k} e^{2\pi i <n,x>} \to \int e^{2\pi i <n,x>} d\mu \tag{3.1}$$

for any $n \in Z^d$. Thus, it suffices to consider integrals of characters.

Let $A \in GL(d, Z)$ with $\det A = 1$, we denote with $f(x) \in Z[x]$ the corresponding characteristic polynomial. Given an integer p, we denote by $\mathbb{L}_p$ the invariant subset of $\boldsymbol{T}^d$ given by all the rational points with denominator p, i.e.

$$\mathbb{L}_p = \{x \in \boldsymbol{T}^d \;|px \in Z^d \;\} \tag{3.2}$$

$\mathbb{L}_p$ is invariant under the action of A, so that any point in $\mathbb{L}_p$ is periodic with period $\leq p^d$. This means that $\mathbb{L}_p$ splits into periodic orbits (which in general may have different periods) of A. As it will become more clear in the sequel, to study the structure of periodic orbits living on $\mathbb{L}_p$ amounts to perform with a modulo p arithmetic.

In what follows we shall only consider lattices $\mathbb{L}_p$ for primes p such that $f(x)$ completely splits over Z_p, that is $f(x) = \prod_{j=1}^d (x - \lambda_j)$, $\lambda_j \in Z_p$, and call such a prime *completely splitting*.

As a consequence of Chebotarev's theorem [3], the set of such primes p as positive density among all primes. One can actually give an estimate from below of such a density in terms of the order of the Galois group associated to the polynomial $f(x)$ [3].

Given a completely splitting prime p, we denote by

$$\Lambda = \{\lambda_1, \cdots, \lambda_d\} \tag{3.3}$$

the set of the eigenvalues of A, which in general are not all distinct and satisfy $\lambda_1 \lambda_2 .... \lambda_d = 1$. In the case $d = 2r$ and $A \in SL(r, Z)$, i.e. A symplectic, $\lambda_j \in \Lambda$ implies $\lambda_j^{-1} \in \Lambda$.

For any $j = 1, ..., d$, we denote by T_j the order of λ_j in the multiplicative group Z_p^*, that is $\lambda_j^{\mathrm{T}_j} = 1$ and $\lambda_j^l \neq 1,\ 0 < l < \mathrm{T}_j$. The set $\mathcal{T}$ of *elementary periods* is then given by

$$\mathcal{T} = \{\mathrm{T}_1, \mathrm{T}_2, ..., \mathrm{T}_d\} \tag{3.4}$$

which we implicitly consider as ordered: $\mathrm{T}_i \leq \mathrm{T}_{i+1}$, $i = 1, \ldots, d-1$.

We also define the set $\mathcal{M}$ of *elementary degeneracies* by:

$$\mathcal{M} = \{m_1, \cdots, m_d\} \tag{3.5}$$

where, $\forall j \in \{1, \cdots, d\}$,

$$\mathrm{T}_j = \frac{p-1}{m_j} \tag{3.6}$$

Remark. The possibility of estimating the behaviour of the elementary degeneracies m_j's as functions of p (for fixed A) is intimately related to the Artin conjecture, saying briefly that any non square integer $a \in Z$ generates the cyclic multiplicative subgroup Z_p^* for an infinite number of primes p [17]. Some numerical and heuristic results regarding the two dimensional situation and its application to the statistical behaviour of the eigenvalues of the corresponding quantum system can be found in [8],[9].

For simplicity' sake and without loss of generality, we will restrict to the case where all the eigenvalues are simple over $\mathbb{C}$. It is then easy to see that this property holds true over Z_p for all primes but a finite number. Indeed, $f(x)$ has no multiple roots in $\mathbb{C}$ if and only if $(f, f') = 1$, i.e. there exist $q_1, r_1 \in \mathbb{Q}[x]$ such that $q_1 f + r_1 f' = 1$. This entails that there exist $q_2, r_2 \in Z[x]$ and $n \in Z,\ n \neq 0$, such that $q_2 f + r_2 f' = n$. Clearly this equation also shows that $(f, f') = 1$ in Z_p for all primes p but a finite number.

Under these assumptions the action of the map A on $L_p \simeq Z_p^d$ is diagonalizable, namely there exist $v_j \in Z_p^d,\ j = 1, \cdots, d$ such that any $x \in Z_p^d$ can be written as $x = \sum_{j\in\mathcal{D}} \alpha_j v_j$, where $\mathcal{D} \subseteq \{1, \ldots, d\}$ is such that $0 \neq \alpha_j \in Z_p$ for any $j \in \mathcal{D}$, and

$$Ax = \sum_{j\in\mathcal{D}} \alpha_j \lambda_j v_j \tag{3.7}$$

For any $x \neq \{0\}$, we denote by $\mathrm{T}(x)$ the corresponding period, i.e. the least positive integer (> 1) such that $A^{\mathrm{T}(x)} x = x$. Clearly, if $x = \sum_{j\in\mathcal{D}} \alpha_j v_j$ then

$$\mathrm{T}(x) = l.c.m.\{\, \mathrm{T}_j \ :\ j \in \mathcal{D}\} \tag{3.8}$$

The set $\mathcal{P}$ of all *admissible periods* is then given by

$$\mathcal{P} = \{\, \mathrm{T} \;=\; l.c.m.\,(\mathrm{T}_{j_1}, \mathrm{T}_{j_2}, ..., \mathrm{T}_{j_k} \ :\ (j_1, ..., j_k) \subset (1, ..., d))\,\} \tag{3.9}$$

T_1 is the smallest period, whereas $\mathrm{T}_{max} = \mathrm{l.c.m.}\{\mathrm{T}_1, \ldots, \mathrm{T}_d\} \leq (p-1)^d$ is the largest one among the admissible periods. Clearly, the natural ordering of $\mathcal{P}$ induces a filtration of the lattice L_p in subsets of points with common period.

Remark. In the two dimensional case, it is known ([14], [4]) that all points in $\mathbb{L}_p \setminus \{0\}$ share the same period: $\mathrm{T}(x) = \mathrm{T} = (p-1)/m$ (for inert primes p, $\mathrm{T}(x) = \mathrm{T} = (p+1)/m$, [14]), so that $\mathcal{P}$ is made out of a single point. The number $m = (p-1)/\mathrm{T}$ corresponds exactly to the degeneracy of the eigenvalues of the associated quantum propagator [4].

In order to study the spatial distribution of a given periodic orbit, we are concerned in evaluating the average of an arbitrary (non trivial) character. Given

p as above, $x \in Z_p^d$ and $n \in Z^d$, we define

$$I_p(x,n) = \frac{1}{\mathrm{T}(x)} \sum_{s=0}^{\mathrm{T}(x)-1} e^{\frac{2\pi i}{p} < n, A^s x >} \tag{3.10}$$

or, equivalently,

$$I_p(x,n) = \frac{1}{\mathrm{T}(x)} \sum_{s=0}^{\mathrm{T}(x)-1} e^{\frac{2\pi i}{p} \sum_{j=1}^{d} \beta_j \lambda_j^s} \tag{3.11}$$

where $\beta_j = \alpha_j < n, v_j > \in Z_p$.

We shall reduce the above expression to (some combinations of) generalized sums over finite fields of the type:

$$\sum_{x \in Z_p} \chi(f(x)) \psi(g(x))$$

where χ is a non trivial multiplicative character of order $d|(p-1)$ of Z_p^* (i.e. χ^d is equal to the trivial character χ_0), ψ a non trivial additive character of Z_p and $f(x), g(x) \in Z_p[x]$ are given algebraic functions, for instance polynomials, over Z_p.

The reference result is then the following:

Theorem 3.1. *Let χ,ψ be a multiplicative character $\neq \chi_0$ of order d with $d|(p-1)$, and a non trivial additive character, respectively, of Z_p. Let $f(x) \in Z_p[x]$ admit m distinct roots, and let $g(x) \in Z_p[x]$ have degree n. Suppose that either $(d, deg(f)) = (n,p) = 1$, or, more generally, that the polynomials $y^d - f(x)$ and $z^p - z - g(x)$ are absolutely irreducible (i.e. irreducible over any finite algebraic extension of Z_p). Then*

$$|\sum_{x \in Z_p} \chi(f(x)) \psi(g(x))| \leq (m+n-1)p^{1/2}$$

Proof. See, e.g., [20] page 45.

This result has been obtained by A. Weil as a consequence of the validity of the Riemann hypothesis for curves over finite fields and important extensions to cases where f, g are rational functions over arbitrary finite fields has been achieved by P. Deligne [6].

Examples.

1) For $f(x) = g(x) = x$ we have the generalized *Gauss sum* $g(\psi, \chi)$, and, by direct application of Theorem 3.1, $|g(\psi, \chi)| \leq \sqrt{p}$.

2) Setting $f(x) = x^2 - 4ab$, $g(x) = x$, χ the (non trivial) character of order two (Legendre symbol) and $\psi = \exp\left(\frac{2\pi i x}{p}\right)$, we find the *Kloosterman sum*

$$Kl(p,a,b) = \sum_{x \in Z_p} \left(\frac{x^2 - 4ab}{p}\right) \exp \frac{2\pi i}{p} x = \sum_{x \in Z_p^*} \exp \frac{2\pi i}{p}(ax + bx^{-1})$$

Again, from Theorem 3.1, one has the estimate $|Kl(p,a,b)| \leq 2\sqrt{p}$.

The next result is useful in reducing to the previous case some sums over cyclic subgroups.

Lemma 3.1. *$\forall \lambda \in Z_p^*$, denote $\Lambda_\lambda =< \lambda >$ the cyclic subgroup generated by λ. Let $\#\Lambda_\lambda = \frac{p-1}{m} = \mathrm{T}$. If $f : Z_p \longrightarrow \mathbb{C}$ is any complex valued function, then*

$$\sum_{s=0}^{\mathrm{T}-1} f(\lambda^s) = \frac{1}{m} \sum_{j=0}^{m-1} \sum_{x \in {Z_p}^*} \chi_j(x) f(x)$$

where $\{\chi_0, \cdots, \chi_{m-1}\}$ is a set of multiplicative characters of order m.

Proof. See the Appendix in [4].

3.1. Ideal lines and ideal orbits.

As a first step toward the understanding of the distribution of a generic periodic orbit, we now consider those orbits which lay entirely on invariant sublattices of the form $\Gamma_j = Z_p v_j = \{u v_j \mid u \in Z_p\} \subset \mathbb{L}_p$ for $j = 1, \ldots, d$, where $v_j \in Z_p^d$ is an eigenvector of A. In analogy with the two dimensional case, we call Γ_j and $\gamma \subset \Gamma_j$ *ideal line* and *ideal orbits*, respectively ([2],[4]). Notice that any ideal line is almost completely equidistributed, in the following sense: for any $n \in Z^d$

such that $< n, v_j > \neq 0 \bmod p$, we have

$$\sum_{x \in \Gamma_j} e^{\frac{2\pi i}{p} <n,x>} = \sum_{u \in Z_p} e^{\frac{2\pi i}{p} u <n,v_j>} = 0 \tag{3.12}$$

Define

$$K_j \equiv K_j(p) = \{n \in Z^d \setminus \{0\} \; : < n, v_j >= 0 \bmod p\}$$

for $j = 1, \ldots, d$, and

$$k_j = \inf_{n \in K_j} |n|$$

where $|n|$ denotes the euclidean length. We then have the following,

Lemma 3.2. *Let A be irreducible.*

$$k(p) = \inf_j k_j \to \infty \quad as \quad p \to \infty$$

where the limit is taken over an arbitrary increasing sequence of completely splitting primes.

Proof. Assume that there exists an infinite sequence of primes $\{p_l\}$ such that $k(p_l)$ remains bounded as $l \to \infty$.

By eventually restricting to a subsequence, we can assume that there exist a fixed $n \in Z^d$ such that for each l there exist an eigenvector $v_j \in Z^d_{p_l}$ such that $< v_j, n >= 0 \bmod p_l$.

Given $l \in Z_+$, let us denote by $M_l \subset Z^d_{p_l}$ the Z_{p_l}-module generated by $\{B^s n\}_{s \in Z}$, where $B = A^t$. M_l is a non trivial submodule of dimension $1 \leq r_l \leq d-1$. In particular, there exist a submodule $N_l \supset M_l$ with the same dimension r_l and another non trivial submodule R_l such that $Z^d_{p_l} = N_l \bigoplus R_l$.

Since this decomposition holds for an infinite number of primes and since n has been chosen independent of l, a simple argument shows in fact the existence of a non trivial B-invariant Z-module $M \subset Z^d$, in contradiction with the irreducibility hypothesis. Q.E.D.

Proposition 3.1. *For any completely splitting prime p and for any $x \in \Gamma_j$ ($x \neq \{0\}$)*

$$|I_p(x,n)| = \begin{cases} \leq \frac{m_j}{\sqrt{p}}, & \text{if } n \notin K_j; \\ 1, & \text{otherwise} \end{cases}$$

Remark.

1) If A is irreducible, then Lemma 3.2 and the above result implies that any sequence $\gamma_k \subset \mathbb{L}_{p_k}$ of periodic ideal orbits, where $\{p_k\}$ is any increasing sequence of completely splitting primes such that $m_k/\sqrt{p_k} \to 0$ as $k \to \infty$, becomes equidistributed according to μ as $k \to \infty$.
2) If A is reducible, e.g. $A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$, where $B \in GL(r,Z)$, $C \in GL(s,Z)$ with $r+s=d$, then there is an invariant decomposition $\boldsymbol{T}^d = \boldsymbol{T}^r \times \boldsymbol{T}^s$. In this case, one can also find families of periodic orbits which are confined to invariant tori of dimension $< d$. The same technique can then be used to describe the equidistribution of the such periodic orbits with respect the Haar measure supported on these invariant tori.

Proof. Let $x \in \Gamma_j$, i.e. $x = \alpha v_j$, $\alpha \in Z_p^*$. If $n \in K_j$ then, clearly, $I_p(x,n) = 1$. On the other hand, if $\beta_j = \alpha < n, v_j > \neq 0 \bmod p$, using Lemma 3.1 we find

$$\begin{aligned} I_p(x,n) &= \frac{1}{\mathrm{T}_j} \sum_{s=0}^{\mathrm{T}_j - 1} e^{\frac{2\pi i}{p}\beta_j \lambda_j^s} = \frac{1}{\mathrm{T}_j m_j} \sum_{x \in Z_p^*} e^{\frac{2\pi i}{p}\beta_j x^{m_j}} \\ &= \frac{1}{\mathrm{T}_j\, m_j} \sum_{k=0}^{m_j - 1} \sum_{x \in Z_p^*} \chi_k(x) e^{\frac{2\pi i}{p}\beta_j x} = \frac{1}{\mathrm{T}_j\, m_j} \sum_{k=0}^{m_j - 1} g(\psi_{\beta_j}, \chi_j) \end{aligned} \tag{3.16}$$

where we have used the notation $\psi_a(x) = \exp\left(\frac{2\pi i a x}{p}\right)$, $a \in Z_p$. The first statement then follows from the estimate on the absolute value of an arbitrary Gauss sum. Q.E.D.

3.2. Generic orbits.

Having proved the equidistribution of ideal orbits, we now turn to the generic situation. The simplest cases to treat are those corresponding to a *completely non resonant* situation and/or a *completely resonant* situation.

Let $x \in \mathbb{L}_p$ and write again $x = \sum_{j\in\mathcal{D}} \alpha_j v_j$, where $\mathcal{D} \subseteq \{1,\dots,d\}$ and $0 \neq \alpha_j \in Z_p$ for any $j \in \mathcal{D}$.

Definition 3.1. *A point $x \in \mathbb{L}_p$ as above is called* completely non resonant *if* $(\mathrm{T}_i, \mathrm{T}_j) = 1$ *for all $i, j \subset \mathcal{D}$, i.e. all the cyclic subgroups generated by λ_j's with $j \in \mathcal{D}$ are distinct with trivial intersections.*

Proposition 3.2. *Let $x \in \mathbb{L}_p$ be completely non resonant and $n \in Z^d \backslash \cap_{j\in\mathcal{D}} K_j$, then*

$$|I_p(x,n)| \leq \frac{1}{\sqrt{p}} \min_{j\in\mathcal{D}'} m_j$$

where $\mathcal{D}' \subseteq \mathcal{D}$ is such that $\beta_j = \alpha_j < n, v_j > \neq 0$ for $j \in \mathcal{D}'$.

Proof. Set $k = \#\mathcal{D}$. Let $\{\lambda_1,\dots,\lambda_k\}$ be the eigenvalues corresponding to v_j, $j \in \mathcal{D}$, and let $\{\mathrm{T}_1,\dots,\mathrm{T}_k\}$, $\{m_1,\dots,m_k\}$ be the corresponding periods and degeneracies, respectively. We have $\mathrm{T}(x) = \mathrm{T} = \mathrm{T}_1\mathrm{T}_2...\mathrm{T}_k$. Set moreover $\mathrm{T}_0 = \mathrm{T}_1\mathrm{T}_2\dots\mathrm{T}_{k-1}$.

Note that $(\mathrm{T}_0, \mathrm{T}_k) = 1$ and therefore $\{\lambda_k^{l\mathrm{T}_0}\}_{l=0}^{\mathrm{T}_k-1} = \{\lambda_k^s\}_{s=0}^{\mathrm{T}_k-1}$. We then get

$$\begin{aligned}
I_p(x,n) &= \frac{1}{\mathrm{T}} \sum_{s=0}^{T-1} e^{\frac{2\pi i}{p}[\beta_1\lambda_1^s + \dots \beta_k\lambda_k^s]} \\
&= \frac{1}{\mathrm{T}} \sum_{s=0}^{\mathrm{T}_0-1} \sum_{l=0}^{\mathrm{T}_k-1} e^{\frac{2\pi i}{p}[\beta_1\lambda_1^{(s+l\mathrm{T}_0)} + \dots + \beta_k\lambda_k^{(s+l\mathrm{T}_0)}]} \\
&= \frac{1}{\mathrm{T}} \sum_{s=0}^{\mathrm{T}_0-1} e^{\frac{2\pi i}{p}(\beta_1\lambda_1^s + \dots + \beta_{k-1}\lambda_{k-1}^s)} \sum_{l=0}^{\mathrm{T}_k-1} e^{\frac{2\pi i}{p}\beta_k\lambda_k^s\lambda_k^{l\mathrm{T}_0}} \\
&= \frac{1}{\mathrm{T}} \sum_{s=0}^{\mathrm{T}_0-1} e^{\frac{2\pi i}{p}(\beta_1\lambda_1^s + \dots + \beta_{k-1}\lambda_{k-1}^s)} \frac{1}{m_k} \sum_{l=0}^{m_k-1} g(\psi_{\beta_k\lambda_k^s}, \chi_l)
\end{aligned} \tag{3.17}$$

where, again, $g(\psi_a, \chi) = \sum_{x\in Z_p^*} \chi(x) e^{\frac{2\pi i}{p}ax}$ denotes the Gauss sum. If now $\beta_k \neq 0$ we obtain

$$|I_p(x,n)| \leq \frac{\mathrm{T}_0}{\mathrm{T}}\sqrt{p} \leq \frac{m_k}{\sqrt{p}} \tag{3.18}$$

which immediately yields the result. Q.E.D.

Let us now consider the opposite case.

Definition 3.2. *A point $x \in \mathbb{L}_p$ as above is called* completely resonant *if, for all $i, j \subset \mathcal{D}$, $<\lambda_i>=<\lambda_j>=<\lambda>$.*

This definition implies that $\mathrm{T}(x) = \mathrm{T} = (p-1)/m$. Moreover, for any $i, j \subset \mathcal{D}$ there exist two integers s_i, s_j such that $\lambda_i^{s_i} = \lambda_j^{s_j}$. Set again $k = \#\mathcal{D}$. For any $j \in \mathcal{D}$ there exist $\eta_j \in \{-\frac{\mathrm{T}-1}{2}, ..., -1, 0, 1, ..., \frac{\mathrm{T}-1}{2}\}$ such that $\lambda_j = \lambda^{\eta_j}$. We may choose λ and the η_j in such a way that $\eta = 2 \max_{j\in\mathcal{D}} |\eta_j|$ is as small as possible.

Proposition 3.3. *Let $x \in \mathbb{L}_p$ be completely resonant and $n \in Z^d \setminus \cap_{j\in\mathcal{D}} K_j$,* then

$$|I_p(x,n)| \le \frac{m}{\sqrt{p}}\,\eta$$

Proof. We have

$$\begin{aligned}
I_p(x,n) &= \frac{1}{\mathrm{T}} \sum_{s=0}^{\mathrm{T}-1} e^{\frac{2\pi i}{p}[\beta_1 \lambda_1^{m\eta_1} + \dots \beta_k \lambda_k^{m\eta_k}]} \\
&= \frac{1}{\mathrm{T}m} \sum_{x\in Z_p^*} e^{\frac{2\pi i}{p}[\beta_1 x^{m\eta_1} + \dots \beta_k x^{m\eta_k}]} \\
&= \frac{1}{\mathrm{T}\,m} \sum_{j=0}^{m-1} \sum_{x\in Z_p^*} \chi_j(x)\, e^{\frac{2\pi i}{p}[\beta_1 x^{\eta_1} + \dots + \beta_k x^{\eta_k}]}
\end{aligned} \tag{3.19}$$

and the result follows by direct application of Theorem 3.1. Q.E.D.

Remark. In the case $d = 2$ one has $\Lambda = \{\lambda, \lambda^{-1}\}$. Moreover, since any point in $\mathbb{L}_p \setminus \{0\}$ is completely resonant (with $\mathrm{T}(x) = (p-1)/m$) one finds the expression

$$\begin{aligned}
I_p(x,n) &= \frac{1}{\mathrm{T}} \sum_{s=0}^{\mathrm{T}-1} e^{\frac{2\pi i}{p}[\beta_1 \lambda^{s} + \beta_2 \lambda^{-s}]} \\
&= \frac{1}{\mathrm{T}m} \sum_{x\in Z_p^*} e^{\frac{2\pi i}{p}[\beta_1 x^{m} + \beta_2 x^{-m}]} \\
&= \frac{1}{\mathrm{T}\,m} \sum_{j=0}^{m-1} \sum_{x\in Z_p^*} \chi_j(x)\, e^{\frac{2\pi i}{p}[\beta_1 x + \beta_2 x^{-1}]}
\end{aligned} \tag{3.20}$$

and the estimate on the Kloosterman sums can be used to give (see Example 2 above)

$$|I_p(x,n)| \le \frac{2m}{\sqrt{p}}$$

so that any sequence $\gamma_k \subset \mathbb{L}_{p_k}$ of periodic orbits, where $\{p_k\}$ is any increasing sequence of splitting primes such that $m(p_k)/\sqrt{p_k} \to 0$ as $k \to \infty$, becomes equidistributed according to μ as $k \to \infty$ [4],[5].

To conclude, we briefly discuss the more general case, corresponding to *partially resonant* situations. Given an arbitrary point $x \in \mathbb{L}_p \setminus \{0\}$ let us write it again in the form $x = \sum_{j\in\mathcal{D}} \alpha_j v_j$. For any fixed $\lambda = \lambda_k$, $k \in \mathcal{D}$, we decompose $\mathcal{D}$ as follows

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3$$

where, if $\mathrm{T} = \mathrm{T}_k$ denotes the order of $< \lambda >$:

1) for any $j \in \mathcal{D}_1$: $(\mathrm{T}, \mathrm{T}_j) = 1$;
2) for any $j \in \mathcal{D}_2$: $< \lambda_j > = < \lambda >$, $(\mathrm{T}_j = \mathrm{T})$;
3) (*partially resonant*) for any $j \in \mathcal{D}_3$: $< \lambda_j > \neq < \lambda >$ and $< \lambda_j > \cap < \lambda > \neq \emptyset$. In particular, if $j \in \mathcal{D}_3$ there exist $s_1 = s_1(j)$ and $s_2 = s_2(j)$ such that $\lambda^{s_1} = \lambda_j^{s_2}$.

Now, if one can choose λ in such a way that the corresponding set $\mathcal{D}_3$ is empty, then the integral $I_p(n, x)$ can be estimated by suitably combining the previous two cases. More precisely, one can first isolate the non resonant eigenvalues as in (3.17) and then proceed as in (3.19). Finally, the case where the presence of partial resonances cannot be avoided seems to be more involved. Nevertheless the same sort of techniques might be used to reduce $I_p(n, x)$ to suitable exponential sums over Z_p and thus produce, at least in principle, explicit estimates.

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