Poisson Distributions for φ^4

by

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ABSTRACT

In an attempt to avoid triviality for relativistic quantum φ_n^4 theories for space-time dimensions $n \geq 5$, and possibly $n = 4$ as well, an additional, nonclassical, nonpolynomial, local potential is included, along with standard factors, in a lattice-regularized formulation of the model. It is argued that if the additional term redistributes the field probability in the manner characteristic of a generalized Poisson distribution, then a nontrivial quantization may be achieved, one which also passes to the correct nontrivial classical theory in the appropriate limit.

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All scalar fields ϕ^4_n in space-time dimensions $n \geq 5$ (and probably $n = 4$ as well) have the feature of being classically nontrivial [1] while their present quantum formulations are trivial (Gaussian) [2]. Such trivial quantum results may be understood as the consequence of forcing nonasymptotically-free field theories into the straightjacket of standard lattice formulations that are inherently consistent with asymptotic freedom. Alternatively stated, trivial quantum results arise for such theories in the continuum limit because in high enough space-time dimensions the non-Gaussian parts of the correlations between fields on neighboring lattice sites are simply too weak to withstand the stronger Gaussian tendencies implicit in the central limit theorem. In this letter we propose an alternative lattice-space formulation for such models—and implicitly for a wide variety of other models as well—that promises to yield nontrivial quantum results, and which should also have the known, nontrivial classical behavior in the appropriate classical limit. In fact, our proposal may well offer alternative, nontrivial quantizations for conventional nontrivial models, such as ϕ^4_n, $n \leq 3$; however, it is for $n \geq 5$ $(n = 4)$ that our method should lead to nontrivial quantizations rather than the physically unacceptable trivial results presently available. In the course of the next few paragraphs, we introduce in several stages the various features that characterize our models.

We begin by showing the equivalence of two distinct versions of the classical theory. Conventionally, one begins with a classical field $\varphi_{c\ell}(x)$, $x \in \mathbb{R}^n$, and an action functional

$$\int \left(h(x)\varphi_{c\ell}(x) + \frac{1}{2}\left\{[\partial_\mu \varphi_{c\ell}(x)]^2 - m^2\varphi^2_{c\ell}(x)\right\} - g\varphi^4_{c\ell}(x) \right) d^n x.$$

The associated equation of motion

$$(\Box + m^2)\varphi_{c\ell}(x) + 4g\varphi^3_{c\ell}(x) = h(x)$$

admits a solution we denote by $\varphi_{c\ell}(x; \varphi_{in}, \dot{\varphi}_{in}, h)$ that depends on the initial data $\varphi_{in}(\mathbf{x}), \dot{\varphi}_{in}(\mathbf{x})$ (say at time $t = 0$), and the source h. As a second choice we begin with a field $\Phi_{c\ell}(x, w)$ which also depends on an auxiliary, dimensionless variable $w \in (0, 1)$, and for this field we adopt the action functional

$$\int \left(h(x)\Phi_{c\ell}(x,w) + \frac{1}{2}\left\{ [\partial_\mu \Phi_{c\ell}(x,w)]^2 - m^2\Phi_{c\ell}^2(x,w) \right\} - g\Phi_{c\ell}^4(x,w) \right) d^n x dw;$$

note that the derivatives are still only with respect to x and none appear for w. The associated equation of motion

$$(\Box + m^2)\Phi_{c\ell}(x,w) + 4g\Phi_{c\ell}^3(x,w) = h(x),$$

coupled with the particular initial data $\Phi_{in}(\mathbf{x},w) = \varphi_{in}(\mathbf{x})$ and $\dot{\Phi}_{in}(\mathbf{x},w) = \dot{\varphi}_{in}(\mathbf{x})$ lead to a solution

$$\Phi_{c\ell}(x,w;\Phi_{in}\dot{\Phi}_{in},h) = \varphi_{c\ell}(x;\varphi_{in},\dot{\varphi}_{in},h),$$

with all relations holding for $w \in (0,1)$. In brief, if one cannot "initiate" or "test for" any w dependence, then the two action functionals lead to equations of motion with identical classical solutions. The present letter examines the quantum theory of the second ("diastrophic" [3]) theory as a quantum model for φ_n^4.

A formal expression for a path integral quantization may be readily obtained. If

$$\begin{aligned} Z\{h\} &\equiv \langle 0|Te^{i\int h(x)\varphi(x)d^n x}|0\rangle \\ &= \langle 0|Te^{i\int h(x)\Phi(x,w)d^n x\, dw}|0\rangle \end{aligned}$$

denotes the generating functional of time-ordered Green functions for the field $\varphi(x) \equiv \int \Phi(x,w)dw$, then Z admits a formal functional integral representation given by

$$\begin{aligned} Z\{h\} = \mathcal{N} \int \exp\Bigg(i \int h(x)\Phi(x,w)d^n x dw + i \int \Big\{ \frac{1}{2}\left[(\partial_\mu \Phi(x,w))^2 - m^2\Phi^2(x,w) \right] \\ - g\Phi^4(x,w) - P[\Phi(x,w)] \Big\} d^n x dw \Bigg) \Pi_{x,w} d\Phi(x,w)\,; \end{aligned}$$

the additional (nonpolynomial) potential $P[\Phi]$ ($= P[-\Phi]$) is discussed below. A corresponding expression for the Euclidean-space generating functional is formally given by

$$\begin{aligned} S\{h\} = \mathcal{N} \int \exp\Bigg(\int h(x)\Phi(x,w)d^n x dw - \int \Big\{ \frac{1}{2}[(\nabla\Phi(x,w))^2 + m^2\Phi^2(x,w)] \\ + g\Phi^4(x,w) + P[\Phi(x,w)] \Big\} d^n x dw \Bigg) \Pi_{x,w} d\Phi(x,w)\,, \end{aligned}$$

which, in turn, may be given an x *and* w, n-(hyper)cubic $\times$ linear lattice-space formulation as the continuum (and infinite space-time volume) limit implicit in the expression

$$S\{h\} = \lim_{\epsilon,a\to 0}\prod_{r=1}^{R} N\int \exp[\Sigma h_k\Phi_{kr}a^n\epsilon - \frac{1}{2}Y(a,\epsilon)\Sigma(\Phi_{k^*r}-\Phi_{kr})^2a^{n-2}\epsilon$$
$$-\frac{1}{2}m_0^2(a,\epsilon)\Sigma\Phi_{kr}^2a^n\epsilon - g_0(a,\epsilon)\Sigma\Phi_{kr}^4a^n\epsilon - \Sigma P(\Phi_{kr},a,\epsilon)a^n\epsilon]\Pi_k d\Phi_{kr}\,.$$

Here ϵ ($\equiv R^{-1}$) is the lattice spacing in w, $r \in \{1,2,\ldots\}$ labels a lattice site in w space, a is the lattice spacing in x, $k = (k_1,\ldots,k_n), k_j \in \{0,\pm1,\ldots\}$, labels a lattice site in x space, and k^*, and an implicit sum, includes $\frac{1}{2}$ the nearest neighbors of k, as usual. We have also anticipated and introduced cutoff-dependent coefficients Y, m_0, g_0, as well as for the as-yet-unspecified term P.

Clearly, any r-dependence of the integration variables Φ_{kr} is irrelevant, and in fact the expression prior to taking the limit is really the Rth power of a "base-theory" integral. Observe that R then enters in a manner similar to the number of "replicas" in statistical physics. Although that number originated from the limit of integration for w, we shall find it expedient to relax that condition (for $n \geq 4$) and allow for replica number renormalization by hereafter replacing that number by $\bar{R} \equiv [\bar{\epsilon}^{-1}] \equiv [\epsilon^{-1}Q(a)^{-1}]$ for some (n-dependent) $Q(a) \leq 1$, where $[A]$ denotes the integer part of A. As a consequence

$$S\{h\} = \lim_{\epsilon,a\to 0}[s(h)]^{\bar{R}}\,,$$
$$s(h) = N\int \exp[\Sigma h_k\Phi_k a^n\epsilon - \frac{1}{2}Y(a,\epsilon)\Sigma(\Phi_{k^*}-\Phi_k)^2a^{n-2}\epsilon$$
$$-\frac{1}{2}m_0^2(a,\epsilon)\Sigma\Phi_k^2a^n\epsilon - g_0(a,\epsilon)\Sigma\Phi_k^4a^n\epsilon - \Sigma P(\Phi_k,a,\epsilon)a^n\epsilon]\Pi d\Phi_k\,.$$

Apart from the term P and the appearance of the parameter ϵ, the expression for $s(h)$ is similar to a conventional lattice-space formulation for the model in question [4]. We can make that similarity even closer if we next assume that $Y(a,\epsilon) = Y(a)\epsilon$, $m_0^2(a,\epsilon) = m_0^2(a)\epsilon$, and $g_0(a,\epsilon) = g_0(a)\epsilon^3$; this dependence on ϵ reflects the multiplicative renormalization found necessary in previous operator treatments of such models [3]. A simple variable change then leads to our final integral representation for $s(h)$ given by

$$s(h) = N_0 \int \exp[\Sigma h_k \phi_k a^n - \frac{1}{2} Y(a) \Sigma(\phi_{k^*} - \phi_k)^2 a^{n-2} \\ - \frac{1}{2} m_0^2(a) \Sigma \phi_k^2 a^n - g_0(a) \Sigma \phi_k^4 a^n - \Sigma P_0(\phi_k, a, \epsilon) a^n] \Pi d\phi_k \,,$$

where we have introduced P_0 which now carries the only dependence on ϵ within $s(h)$. If it were not for P_0 this expression would exactly resemble the standard lattice formulation.

The factor N_0 is chosen so that $s(0) = 1$, and as a consequence the expression for $s(h)$ assumes the alternative form

$$s(h) = \exp\left[\sum_{\ell=1}^{\infty} (\ell!)^{-1} \sum h_{k_1} \cdots h_{k_\ell} \langle \phi_{k_1} \cdots \phi_{k_\ell} \rangle^T a^{\ell n}\right] ,$$

where averages $\langle \cdot \rangle$ are defined in the ϵ-dependent probability distribution implicit in the expression $s(h) \equiv \langle \exp(\Sigma h_k \phi_k a^n) \rangle$. Since the ultimate answer of interest involves $[s(h)]^{\bar{R}}$, and $\bar{R} \to \infty$ as $\bar{\epsilon} \to 0$, a nontrivial result emerges, when $\bar{\epsilon} \ll 1$, provided the truncated correlation functions satisfy

$$\langle \phi_{k_1} \phi_{k_2} \cdots \phi_{k_\ell} \rangle^T \propto \bar{\epsilon}$$

for all $\ell \geq 1$; with P_0 and the rest of the potential even, only the even-order (truncated) moments are nonzero in the present case. The advantage of arranging this proportionality to $\bar{\epsilon}$ will become clear shortly.

To this end, let us first recall the usual definition [4,5] of the dimensionless renormalized coupling constant $g_R \equiv \lim g(a)$ for the base theory given as a space-time continuum and infinite volume limit of the expression

$$g(a) = -\frac{\Sigma_{k\ell m} \langle \phi_0 \phi_k \phi_\ell \phi_m \rangle^T}{[\Sigma_k \langle \phi_0 \phi_k \rangle]^2 [\Sigma_\ell \ell^2 \langle \phi_0 \phi_\ell \rangle / \Sigma_m \langle \phi_0 \phi_m \rangle]^{n/2}} \,.$$

Let us initially ignore the amplitude dependence and present the usual argument for triviality. From the viewpoint of critical phenomena [5], it follows that $g(a) \propto a^{(\gamma + n\nu - 2\Delta)/\nu}$ for small a. For $n \geq 5$, when the critical exponents follow from mean field theory, $g(a) \propto$

$a^{(n-4)}$, an expression which tends to zero as $a \to 0$ leading to triviality. For $n = 4$, renormalization group arguments assert that $g(a) \propto 1/|\ln(\mu a)|$, for some mass parameter μ, which as $a \to 0$ also leads to triviality. For $n \leq 3$ hyperscaling ensures that $g(a) = O(1)$ and thus g_R is nonvanishing.

Next let us pursue the consequences of including the proper amplitude factors of the truncated correlation functions. For $n \geq 5$, $g(a) \propto \epsilon^{-1}Q(a)^{-1}a^{(n-4)}$, and so for $n \geq 5$ we choose $Q(a) = [\mu a/(1+\mu a)]^{(n-4)}$ with μ a suitable mass parameter. For $n = 4$ we choose $Q(a) = 1/(|\ln(\mu a)| + 1)$, while for $n \leq 3$ we set $Q(a) \equiv 1$. Therefore, for every $n \geq 1$, it follows that $g(a) \propto \epsilon^{-1}$ and consequently, as $a \to 0$, $g_R \propto \epsilon^{-1}$ and is nonvanishing as well.

The purpose of the term P_0 is to ensure the required dependence of the truncated correlation functions on $\bar{\epsilon}$. When $\bar{\epsilon}$ is small it follows that

$$\langle \phi_{k_1} \cdots \phi_{k_\ell} \rangle = \langle \phi_{k_1} \cdots \phi_{k_\ell} \rangle^T + O(\bar{\epsilon}^2) = O(\bar{\epsilon})\,,$$

and thus the role of P_0, as well as the remaining model parameters (Y, m_0, and g_0) in the single-lattice-site distribution is, roughly speaking, to divide the total probability distribution into the sum of two contributions: (i) one term, a highly concentrated distribution, largely determined by P_0, where $\phi_k \simeq 0$ with a total probability of $1-O(\bar{\epsilon})$; and (ii) the other term, a nonconcentrated distribution, largely determined by the remaining model parameters, where ϕ_k takes on general values but with a total probability $O(\bar{\epsilon})$. Such a division of probabilities is exactly how various Poisson distributions avoid the Gaussian vise-grip of the central limit theorem [6].

It is entirely reasonable that an extra term should appear in the renormalized lattice action the purpose of which is to reduce the probability of large field values. Recall, for classical functions, that the Sobolev inequality

$$[\int \phi^4(x) d^n x]^{\frac{1}{2}} \leq C \int [(\nabla\phi(x))^2 + m^2\phi^2(x)] d^n x$$

holds for finite C whenever $n \leq 4$, but fails to hold for any $C < \infty$ whenever $n \geq 5$ [7]. Elsewhere we have interpreted such inequalities to imply that any nonrenormalizable interaction (e.g., ϕ^4_n, $n \geq 5$) acts partially as a *hard core* in function space projecting out certain field histories otherwise allowed if the interaction had been entirely absent [8]. Ignoring fluctuations, the troublesome and so-projected fields have local singularities [e.g., for $|x| < 1$] of the form $\phi \sim |x|^{-\gamma}$, where $(n/4) < \gamma < (n-2)/2$; these large amplitude fields necessarily involve high momenta, but not all fields involving high momenta [e.g., $\phi \sim \cos(x^{-2})$] are troublesome in this sense. Thus a reasonable renormalization for such fields should reweight their distribution in the manner indicated.

An indication of the general form of P_0 that should accomplish our goal may be gleaned from the study of the explicitly soluble case for $n = 1$ (i.e., Euclidean quantum mechanics, and so we set $x = t$) [9]. In this case $Q(a) \equiv 1$, but even after $a \to 0$, we are still able to choose $P_0(\phi, \epsilon)$ so that $\langle \phi_{k_1} \cdots \phi_{k_\ell} \rangle^T \propto \epsilon$. In particular, for $n = 1$ and any γ, $\frac{1}{2} < \gamma < \frac{3}{2}$, we can, after letting $a \to 0$, choose [9]

$$\begin{aligned} P_0(\phi, \epsilon) &= \frac{1}{2}\frac{[\gamma(\gamma+1)\phi^2 - \gamma\delta^2(\epsilon)]}{[\phi^2 + \delta^2(\epsilon)]^2}\,, \\ \delta^2(\epsilon) &= (G\epsilon)^{2/(2\gamma-1)}\,, \\ G &= \sqrt{\pi}\,\Gamma(\gamma - \tfrac{1}{2})/\Gamma(\gamma)\,. \end{aligned}$$

Observe in this case that P_0 is not unique. Nevertheless each such P_0 may be interpreted as a regularized form of a formal interaction proportional to $\int \Phi^{-2}(t, w)dw$, which in turn should be viewed as a (decidedly unconventional) renormalization counterterm for $\int \dot{\Phi}^2(t, w)dw$ rather than for the quartic interaction. This interpretation is borne out by the fact that P_0 does not vanish when $g_0 \to 0$, meaning that the zero-coupling limit of the interacting theory, which retains the effects of the hard core, is not the free theory but a so-called pseudofree theory; it is expected that a perturbation theory in the quartic interaction exists about the pseudofree theory [8]. As a renormalization counterterm for the kinetic energy, P_0 contains an implicit multiplicative factor of $\hbar^2$, which in the classical limit, $\hbar \to 0$, implies that the expected classical theory emerges as desired. For $n = 1$ this expected behavior has been confirmed [10].

For $n \geq 2$, and for a suitable (n-dependent) choice of A, B, and C, it is suggestive that P_0 has the form

$$P_0(\phi, a, \epsilon) = \frac{A(a,\epsilon)\phi^2 - B(a,\epsilon)}{[\phi^2 + C(a,\epsilon)]^2}$$

based on what holds for $n = 1$. Although we have no real evidence for this hypothesis, it is clear that an expression of this form will accomplish the stated purposes, and it has the further advantage that it too may be interpreted as a renormalization counterterm for the kinetic energy. The coefficients must be chosen so that, like the case for $n = 1$, the distribution is a generalized Poisson process, rather than just a compound Poisson process [6]. It may even be possible to determine P_0 to a certain degree based on high-temperature series expansions that exist for a general, even, single-site field distribution [5]. One may try to determine the necessary $\bar{\epsilon}$-dependence of the moments of the single-site field distribution which ensures that the truncated correlation functions are proportional to $\bar{\epsilon}$ when $\bar{\epsilon} \ll 1$; for this purpose it is sufficient to focus on the second- and fourth-order truncated correlation functions. In this way one may be able to suggest a suitable $P_0(\phi, a, \epsilon)$ that leads to the required behavior of the moments. Given a candidate choice for $P_0(\phi, a, \epsilon)$, Monte-Carlo methods and/or renormalization-group techniques may then be introduced. Although high-temperature series exist only for $n \leq 4$, the principles set forth in this letter may be applied to study ϕ^4_4 and ϕ^6_4, or even ϕ^4_3, ϕ^6_3, and ϕ^8_3, which include super renormalizable, renormalizable and nonrenormalizable examples.

The resultant field will be infinitely divisible in the sense of probability theory, as indeed are all Poisson distributions or limits thereof [6]. A study of such fields has been made previously [11] where it was shown that under certain assumptions such fields had vanishing truncated four-point functions and an S-matrix of unity. Among those assumptions was asymptotic completeness, a property that is missing for the present models. Indeed, we have made the achievemant of a nonvanishing four-point function our central goal, and granting reasonable spectral properties of the theory, we expect to attain nontrivial scattering. This would be a necessary precondition to have a classical limit that agreed with the known proper nontrivial classical theory.

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