CORRECTIONS TO THE CRITICAL TEMPERATURE IN 2D ISING SYSTEMS WITH KAC POTENTIALS

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Abstract. We consider a $d = 2$ Ising system with a Kac potential whose mean field critical temperature is 1. Calling $\gamma > 0$ the Kac parameter, we prove that there exists $c^\star > 0$ so that the true inverse critical temperature $\beta_{\rm cr}(\gamma) > 1 + b\gamma^2 \log\gamma^{-1}$, for any $b < c^\star$ and γ correspondingly small. We also show that if $\gamma \to 0$ and $b \to c^\star$, suitably, then the correlation functions (normalized and rescaled) converge to those of a non trivial Euclidean field theory.

Keywords: Kac potential, Ising model, critical fluctuations, euclidean field theory.

In this short communication we study the corrections to the critical temperature in a $d = 2$ Ising model with Kac potentials, (proofs will follow in an extended version of this note). Recall that the Kac hamiltonian, [?], is

$$-\frac{1}{2}\sum_{x\neq y} J_\gamma(x,y)\sigma(x)\sigma(y) \tag{1.1}$$

where the spins $\sigma(x) = \pm 1$, $x \in \mathbb{Z}^d$, and the coupling strength is

$$J_\gamma(x,y) = c_\gamma \gamma^d J(\gamma|x-y|), \quad \sum_{y\neq x} J_\gamma(x,y) = 1, \quad \int dr J(|r|) = 1, \quad D := \int_{\mathbb{R}^2} dr J(|r|) r^2 \tag{1.2}$$

We suppose $J(|r|) \geq 0$ and, to fix the ideas, we take J "smooth", $J(|r|) > 0$ for $|r| < 1$ and $J(|r|) = 0$ for $|r| \geq 1$. $\gamma > 0$ is the scaling parameter of the Kac potential and we want to

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study the behavior of the inverse critical temperature $\beta_{\rm cr}(\gamma)$, for the system with interaction J_γ, as $\gamma \to 0$. c_γ is a normalization constant that goes to 1 as $\gamma \to 0$. It has been introduced to avoid spurious fluctuations of $\beta_{\rm cr}(\gamma)$ as γ varies: one should in fact compare systems at different γ with the same total interaction strength $\sum J_\gamma(0,x)$. The deviations of $\beta_{\rm cr}(\gamma)$ in [?] are entirely due to the variations of this quantity.

The deviations we are talking about are from the Lebowitz-Penrose critical temperature $\beta_{\rm cr}^{\rm LP}$, that, with the convention (1.2), is equal to 1, see, [?]. We believe that $\beta_{\rm cr}(\gamma)$ goes to 1 as $\gamma \to 0$. A more precise statement involves lower and upper bounds. For the latter we only have a conjecture:

Conjecture. *For any $\beta > 1$ there is $\gamma(\beta) > 0$ so that for all $\gamma \le \gamma(\beta)$, $\beta_{\rm cr}(\gamma) < \beta$.*

(We are not aware of a proof of this conjecture, except in cases where reflection positivity may be applied, as in the original Kac potential, see [?]. In general the proof should follow from a Peierles argument complemented by some large deviation estimates and bounds on the surface tension. Some progresses on the last two issues have been recently obtained, [?], [?]).

The other side of the bound comes from mean field: it is known that $\beta_{\rm cr}(\gamma) \ge \beta_{\rm cr}^{\rm LP} = 1$, as we will discuss later. Thus, assuming the validity of the Conjecture, we conclude that $\beta_{\rm cr}(\gamma) \to 1$ as $\gamma \to 0$. The next question concerns the rate of convergence. Our results show that $\beta_{\rm cr}(\gamma) > 1$ (strictly, for $\gamma > 0$) and that the deviations from 1 are responsible for the Wick regularization and the convergence of the block spin variables to a non trivial Euclidean field theory. Let us be more precise, starting from the first statement:

1. Theorem.

For any $b < c^\star := 1/(\pi D)$, D as in (1.2), there is $\gamma(b) > 0$ so that for all $0 < \gamma \le \gamma(b)$

$$\beta_{\rm cr}(\gamma) \ge \beta_b(\gamma) := 1 + b\gamma^2 \log \gamma^{-1} \tag{1.3}$$

This Theorem suggests to study the critical behavior by letting $b \to c^\star$ as $\gamma \to 0$. The analysis of this limit is in the same spirit of the continuum limit considered for instance in the works of Aizenmann, [?], [?], Sokal, [?], Frölich, [?], and Brydges, Frölich and Sokal, [?], and references therein, where they prove convergence to ϕ^4 in $d = 2, 3$ and to a Gaussian field in higher dimensions, using, and exploiting, the relation with Ising systems and Statistical Mechanics. Our systems do not seem to be included, at least explicitely, in the class considered in the above papers and we have worked out a specific proof that has the advantage of making explicit the relation between the shift of the critical temperature, the origin of the Wick regularization term and the fluctuations strength.

We study the convergence of the block spin variables, by considering the normalized and scaled correlation functions. We denote by $< \cdot >_{b,\gamma}$ the expectation with respect to the Gibbs measure with interaction J_γ and inverse temperature $\beta_b(\gamma)$, see (1.3). By Theorem 1 for $\gamma < \gamma(b)$ there is only one Gibbs state and no ambiguity may arise. Sometimes later we will write $< \cdot >_{\beta,\gamma}$, for the expectation with respect to the Gibbs measure at the inverse temperature β.

¿From the proof of Theorem 1 it follows that the correlations decay on the scale

$$\ell_{b,\gamma} = \frac{\gamma^{-2}}{\sqrt{(c-b)\log\gamma^{-1}}} \tag{1.4}$$

Thus, in interaction-length units, the correlation length is

$$\gamma\ell_{b,\gamma} = \frac{1}{\sqrt{\beta_{c^\star}(\gamma) - \beta_b(\gamma)}} \tag{1.5}$$

which shows that the mean field critical exponent is "correct" when the temperature "is not too close" to critical, i.e. $b < c^\star$, as in (1.3). (This remark complements that of Aizenmann about the validity of mean field for computing critical exponents in $d > 4$, see the beginning of Section 4 in [?]). Since Theorem 1 does not tell us that $\beta_{c^\star}(\gamma)$ is the true inverse critical temperature, the r.h.s. of (1.5) is only an upper bound to the critical exponent. On the other hand, since we prove in Theorem 2 below that our system behaves as a massive local field theory when $b \to c^\star$ and $\gamma \to 0$ (suitably), we expect a change of behavior when the distance from $\beta_{c^\star}(\gamma)$ is of the order of γ^2 (which is the order of the term responsible for the appearence of the mass in the teory). Therefore, at these values of the inverse temperature, the n.n. Ising critical exponent should be recovered.

We introduce the normalized and rescaled correlations

$$S_{b,\gamma}(r_1, .., r_{2n}) = \gamma^{-2n}\langle\sigma(\ell_{b,\gamma}r_1)\cdots\sigma(\ell_{b,\gamma}r_{2n})\rangle_{b,\gamma} \tag{1.6}$$

where $\ell_{b,\gamma}r_i$ should be replaced by its integer part, and all sites are supposed distinct. We have the following result:

2. Theorem.

There is $\epsilon_0 > 0$ and for any $\epsilon \le \epsilon_0$ there is a sequence b_γ such that

$$\lim \sqrt{(c-b_\gamma)\log\gamma^{-1}} = \epsilon^{-1} \tag{1.7}$$

and for any n and any distinct $r_1, .., r_{2n}$,

$$\lim S_{b,\gamma}(r_1, .., r_{2n}) = S^{(\epsilon)}(r_1, .., r_{2n}) \tag{1.8}$$

The Schwinger functions $S^{(\epsilon)}$ are continuous in $\{r_i \neq r_j\}$, and, for any test function $\psi(r_1, .., r_{2n})$,

$$\lim \int dr_1 \cdots dr_{2n} S_{b,\gamma}(r_1, .., r_{2n}) \psi(r_1, .., r_{2n}) = \int dr_1 \cdots dr_{2n} S^{(\epsilon)}(r_1, .., r_{2n}) \psi(r_1, .., r_{2n}) \tag{1.9}$$

The functions $S^{(\epsilon)}$ satisfy a recursive relation that is also satisfied by the Schwinger functions of the ϕ^4 Euclidean theory with interaction strength $\lambda = 1$ and mass ϵ^{-1}. In particular the truncated correlation functions constructed from $S^{(\epsilon)}$ are not identically 0.

Without entering into the proofs of the two Theorems, we just want to outline some points that may be of interest. We recall that by using the Dobrushin's techniques for uniqueness of Gibbs measures, [?], we get for the Vaserstein distance $R_x(\sigma, \sigma')$ between the conditional probabilities at x given two different boundary conditions, σ and σ',

$$R_x(\sigma, \sigma') \leq \beta \sum_{y \neq x} J_\gamma(x, y) \big|\sigma(y) - \sigma(y')\big| \tag{1.10}$$

Recalling (1.2) the Dobrushin's uniqueness condition $\beta \sum J_\gamma(x, y) < 1$ is satisfied for all $\beta < 1$ and all γ, hence $\beta_{\rm cr}(\gamma) \geq 1$. The same analysis in [?] allows to derive bounds for the correlation functions, which, in the case of the 2 point correlations yields

$$\langle \sigma(0)\sigma(x)\rangle_{\beta,\gamma} \leq \beta \sum_{y \neq 0} J_\gamma(x, y) \langle \sigma(0)\sigma(x)\rangle_{\beta,\gamma} + 2\beta J_\gamma(x, 0) \tag{1.11}$$

hence

$$\langle \sigma(0)\sigma(x)\rangle_{\beta,\gamma} \leq 2\Big(\big(1 - \beta J_\gamma\big)^{-1}\Big)_{x,0} \tag{1.12}$$

On the other hand, by the DLR equations,

$$\langle \sigma(0)\sigma(x)\rangle_{\beta,\gamma} = \langle \sigma(0) \tanh \beta h_\gamma(x)\rangle_{\beta,\gamma}, \qquad h_\gamma(x) = \sum_{y \neq x} J_\gamma(x, y)\sigma(y) \tag{1.13}$$

By Taylor expanding the hyperbolic tangent and retaining only the first two terms (as it can be rigorously justified using Newmann's Gaussian inequalities, [?], and (1.12)) we get

$$\langle \sigma(0)\sigma(x)\rangle_{\beta,\gamma} \approx \beta \langle \sigma(0) h_\gamma(x)\rangle_{\beta,\gamma} - \frac{1}{3}\beta^3 \langle \sigma(0) h_\gamma(x)^3\rangle_{\beta,\gamma} \tag{1.14}$$

Given any $\beta < 1$, it is not difficult to see, using the previous arguments, that also the last term is of higher order, in the limit as $\gamma \to 0$. We can then solve (1.13) and, more generally we find that for any $n \ge 1$ and any distinct $r_1, .., r_{2n}$ (and in any dimension d)

$$\lim_{\gamma \to 0} \gamma^{-2n} \langle \sigma(\gamma^{-1} r_1) \sigma(\gamma^{-1} r_{2n}) \rangle_{\beta,\gamma} = \sum_{\text{pairings}} \prod_{\ell=1}^{n} G\big(r_{i_\ell} - r_{j_\ell}\big) \tag{1.15}$$

where $G = G_1$ and

$$G_t(r) = \frac{1}{2\pi D t} e^{-r^2/2Dt} \tag{1.16}$$

D as in (1.2). The bound (1.11) is quite accurate when $\beta < 1$ is kept fixed as $\gamma \to 0$, but it starts deviating from the correct one as soon as $\beta = 1 - \gamma^2$. We can rewrite (1.14) as

$$\langle \sigma(0)\sigma(x) \rangle_{\beta,\gamma} \approx (\beta - \beta^3 C_{\beta,\gamma}) \langle \sigma(0) h_\gamma(x) \rangle_{\beta,\gamma} - \frac{1}{3} \beta^3 \langle \sigma(0) h_\gamma(x)^3 \rangle^T_{\beta,\gamma} \tag{1.17}$$

where $\langle \cdot \rangle^T_{\beta,\gamma}$ is the truncated correlation function and

$$C_{\beta,\gamma} = \langle h_\gamma(x)^2 \rangle_{\beta,\gamma} = \gamma^2 \log \gamma^{-1} [\frac{1}{\pi D} + 0\big(\frac{1}{\log \gamma^{-1}}\big)], \quad \text{for } \beta = \beta_b(\gamma),\ b < c^\star \tag{1.18}$$

(An upper bound of the right order at $\beta = 1 - \gamma^2$ follows directly from (1.11)). Because of the Gaussian structure of the correlations, evidentiated by (1.15), we should have, and we actually do have that the truncated correlation function may be neglected as $\gamma \to 0$.

¿From (1.17), (1.18) and after some computations, we then see that the effective temperature is

$$\beta - \beta^3 C_{\beta,\gamma} = \beta - c^\star \gamma^2 \log \gamma^{-1} + 0\big(\gamma^2\big) \tag{1.19}$$

This establishes the relation between the temperature shift and the Wick regularization, (i.e. replacing a correleation function by the truncated one). The same mechanism was observed in [?] in a $d = 3$ lattice approximation of ϕ^4, see also [?].

The hard part in the proof of Theorems 1 and 2 is to extend the validity of the above considerations beyond $\beta = 1$. We achieve that by making an ansatz on bounds on the correlation functions. We then prove, (this part of the argument is relatively simple), that the ansatz is consistent with the equations that are obtained by applying the DLR equation to correlation functions of any order (similar to those in (1.13) for the two body terms). We then need to prove that the actual correlations satisfy the ansatz and this is done by a continuity argument starting from $\beta = 1 - \gamma^2$. In this part we use extensively Newmann's Gaussian inequalities and convexity properties of the pressure to prove uniqueness of the even correlation functions. With this information we can then modify the usual mean field argument for the magnetization proving that also at $\beta_b(\gamma)$, $b < c^\star$, it is equal to 0, thus obtaining Theorem 1. To prove Theorem 2 we use Aizenmann inequality on the 4 point truncated correlation function to have a closed inequality for the 2 points correlations, that can be solved till $b = c^\star$ if ϵ^{-1} in (1.7) is large enough. By using an argument by contradiction we finally prove the existence of a non trivial limit.

Concluding Remarks

Very schematically we present motivations and some of the main open problems we would like to study.

(1) Unsatisfactory: we would like to work at γ small but non zero and $\beta = \beta_{c^\star} - a\gamma^2$, $a > 0$ and $a \to 0$. Aim: crossover from mean field to n.n. Ising critical exponent, see the discussion following (1.5).

(2) Original motivation: Jona-Lasinio's proposal, [?], of stochastic quantization via particle dynamics, in our case the Glauber dynamics. Namely to derive stochastic PDE's whose invariant measure is the ϕ^4 Euclidean field theory. This is the dynamical analogue of the approach to constructive ϕ^4 theories via Ising models, with the belief that, like in equilibrium, this may lead to substantial simplifications. Results in $d = 1$ have been already obtained, [?] and [?].

(3) $d = 3$?

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