Quantum Plasma Model with Hydrodynamical Phase Transition

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Abstract. We derive the electro-hydrodynamics of the Jellium plasma model from its many-particle Schrödinger equation, subject to certain general initial and regularity conditions, and prove that it undergoes a transition from deterministic to stochastic flow when a certain parameter, representing the non-uniformity in the initial density and drift velocity profiles, reaches a certain critical value. Thus, the model exhibits a phase transition far from equilibrium.

1 Introduction

The quantum Jellium model is a system of electrons, interacting via Coulomb forces both with one another and with a uniform, positively charged, neutralising background. It is thus a model of a many-particle system with realistic interactions. At the level of mathematical physics, it has been proved to enjoy 'good' thermodynamic [1,2] and hydrodynamic [3] properties. In fact, apart from Davies's [4] derivation of Fourier's law of heat conduction for a certain model of interacting atoms, the passage from quantum mechanics to Eulerian hydrodynamics in [3] represents, to the best of our knowledge, the only rigorous quantum statistical derivation of a macroscopic continuum mechanics. It is, however, based on the assumption of regularity conditions, which exclude the possibility of hydrodynamical phase transitions.

The object of the present article is to provide a further quantum mechanical treatment of

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the hydrodynamics of the Jellium model, in which certain regularity assumptions of [3] are weakened in such a way as to admit non-equilibrium phase transitions. In fact, we show that, under the new assumptions, the model exhibits a transition from a deterministic to a stochastic hydrodynamics when a certain parameter, representing the non-uniformity of the initial density and velocity profiles, attains a critical value. The method by which we obtain this result is based on two main steps. Firstly, we derive a Vlasov equation for the large scale dynamics of the model from its many-particle Schrödinger equation, subject to specified initial and regularity conditions. We then show that the Vlasov dynamics reduces to a deterministic Eulerian hydrodynamics if the initial density and velocity profiles lie below a certain non-uniformity threshold, and that otherwise the flow becomes stochastic. Specifically, in the latter case, the flow corresponds to a *statistical mixture* of different streams, and thus the local density and drift velocity have macroscopic dispersions.

We present our treatment as follows. In §2, we extend the scheme of Refs. [3,5] so as to derive a Vlasov equation, governing the large-scale dynamics of the Jellium model, from its many-particle Schrödinger equation, subject to rather general initial conditions. To be more precise, we provide a treatment here of both the Jellium model itself and a regularised version of it, obtained by introducing a short distance cut-off in the Coulomb potential. For the regular model, our derivation of the Vlasov dynamics is based exclusively on the Schrödinger equation and the initial conditions. For the true Jellium model, on the other hand, certain supplementary regularity assumptions are also required. The essential idea behind these is that the repulsive character of the inter-electronic forces keeps the electrons apart and thereby tames the singularity in the Coulomb potential. The precise form of the assumptions is specified by the conditions (R.1-4). We remark here that we have not avoided repeating much of the formalism of Ref. [3] in this Section, because it is essential for our present purposes to reset it in the context of our new (weakened) regularity conditions

In §3, we note that the Vlasov equation is just the Liouville equation for a certain *Lagrangian* hydrodynamics, governing the evolution of the time (t)-dependent position, $X_t(x)$, of a 'fluid particle' located initially at the point x. We then show that the Vlasov dynamics reduces to a deterministic Eulerian hydrodynamics if and only if the function X_t is invertible: otherwise it corresponds a stochastic flow, in the sense described above.

In §4, we analyse the conditions for deterministic versus stochastic flow in both the true Jellium model and the regularised one, in the situation where the initial density and velocity profiles depend on only one spatial coordinate and so are essentially one-dimensional. For this case, we are able to show explicitly that both models undergo transitions from deterministic to stochastic flow when the initial conditions attain certain non-uniformity thresholds.

We conclude, in §5, with some brief further comments on the results obtained here and on their possible relevance to the theory of turbulence.

2. Basis of the Vlasov Dynamics

The Jellium model, $\Sigma^{(N,L)}$, consists of N electrons in a cube, $K^{(L)}$, of side L, with uniform neutralising positive charge background. We assume periodic boundary conditions. Our objective will be to obtain a quantum theoretical derivation of the hydrodynamics of $\Sigma^{(N,L)}$ in a limit where N and L tend to infinity and the mean particle density,

$$\overline{n} = N/L^3 \tag{2.1}$$

remains fixed and finite.

We denote the position vectors and momenta of the electrons by $X_1, \ldots, X_N$ and $P_1, \ldots, P_N$, respectively. Thus, $P_j = -i\hbar\nabla_j^{(L)}$, where $\nabla^{(L)}$ is the gradient operator in $K^{(L)}$. At the microscopic level, the pure states of the system are given by the normalised, antisymmetric wave-functions $\Psi^{(N)}(X_1, \ldots, X_N)$, and the Hamiltonian takes the form

$$H^{(N,L)} = \frac{-\hbar^2}{2m}\sum_{j=1}^{N}\Delta_j^{(L)} + e^2\sum_{j,k(>j)=1}^{N}U^{(L)}(X_j - X_k) \tag{2.2}$$

where $-e, m$ are the electronic charge and mass, respectively, $\Delta^{(L)}$ is the Laplacian for $K^{(L)}$, and $U^{(L)}(X)$ is the difference between $|X|^{-1}$, periodicised w.r.t. $K^{(L)}$, and its space average over that cube, i.e.

$$U^{(L)}(X) = \frac{4\pi}{L^3}\sum^{(L)}\frac{\exp(iQ.X)}{Q^2} \tag{2.3}$$

the superscript (L) over Σ signifying that summation is taken over the non-zero vectors $Q = (2\pi/L)(n_1, n_2, n_3)$, with the n's integers. The time-dependent Schrödinger equation for $\Sigma^{(N,L)}$, with T the time variable, is

$$i\hbar\frac{\partial\Psi_T^{(N)}}{\partial T} = H^{(N,L)}\Psi_T^{(N)} \tag{2.4}$$

We shall assume the following initial kinetic and potential energy bounds for $\Sigma^{(N,L)}$-more precisely, for the family of systems $\{\Sigma^{(N,L)}\}$, with N, L satisfying (2.1).

$(I.1)^{(L)}$ The expectation value of the total kinetic energy per particle, for the initial state $\Psi_0^{(N)}$, is less than some finite $N-$independent constant $B/2m$, i.e.

$$(\Psi_0^{(N)}, P_1^2\Psi_0^{(N)}) < B \tag{2.5}$$

$(I.2)^{(L)}$ The expectation value of the total potential energy, for the the initial state $\Psi_0^{(N)}$, is less than some finite $N-$independent constant, $e^2C/2$, times $N^{5/3}$. This bound corresponds to the electrostatic energy of a continuous distribution of charge, whose density is a smooth function of X/L, and signifies that

$$(\Psi_0^{(N)}, U^{(L)}(X_1 - X_2)\Psi_0^{(N)}) < CN^{2/3} \tag{2.6}$$

We base our macroscopic description of the model on scales of length, time and particle momentum given by $L,\ \omega^{-1}$ and $mL\omega$, where ω is the classical plasma frequency, i.e.

$$\omega = (4\pi\overline{n}e^2/m)^{1/2} \tag{2.7}$$

For this description, we employ the rescaled space and time coordinates, $x = X/L,\ t = \omega T$, respectively. Under this rescaling, $\Sigma^{(N,L)}$ is mapped onto a system $\Sigma^{(N)}$ of particles in a unit cube, K, with periodic boundaries. Correspondingly, the state, $\Psi_T^{(N)}$, of $\Sigma^{(N,L)}$ is transformed to that of $\Sigma^{(N)}$ given by

$$\psi_t^{(N)}(x_1,..,x_N) = L^{3N/2}\Psi_{\omega^{-1}t}^{(N)}(Lx_1,..,Lx_N) \tag{2.8}$$

The Schrödinger equation (2.4) thus transforms to

$$i\hbar_N\frac{\partial\psi_t^{(N)}}{\partial t} = H^{(N)}\psi_t^{(N)} \tag{2.9}$$

where

$$H^{(N)} = \frac{1}{2}\sum_{j=1}^{N}p_j^2 \ + \ N^{-1}\sum_{j,k(>j)=1}^{N}U(x_j - x_k) \tag{2.10}$$

$$p_j = -i\hbar_N\nabla_j \tag{2.11}$$

$$\hbar_N = \frac{\hbar}{mL^2\omega} = \frac{\hbar}{m\omega}(\frac{\overline{n}}{N})^{2/3} \tag{2.12}$$

is a dimensionless effective 'Planck constant', ∇ is the gradient operator for K, and

$$U(x) = U_c(x) := \sum_q^{(1)}\exp(iq.x)/q^2 \tag{2.13}$$

the superscript (1) over Σ signifying that summation is taken over the non-zero vectors $2\pi(n_1, n_2, n_3)$, with the n's integers. Our reason for introducing the symbol U_c here is that we want to employ the Hamiltonian given by (2.10) both for the model $\Sigma^{(N)}$, with $U = U_c$, and for a modified version of this, where U is a 'smoothed out' Coulomb potential. We note that it follows from (2.13) that

$$\Delta U_c(x) = 1 - \delta(x) \tag{2.14}$$

where Δ is the Laplacian and δ the Dirac distribution for K.

Note. Two key features of the rescaled description, as given by (2.9)-(2.14), are that

(a) the effective, dimensionless Planck constant, $\hbar_N$, governing the quantum behaviour of the model, tends to zero as $N\to\infty$; and

(b) the pair interaction potential scales as N^{-1}.

The properties (a) and (b) are generally the hallmarks of a classical and of a mean field theory, respectively, in the limit $N\to\infty$. In fact, as we shall presently show, the model does indeed reduce to a classical mean field theoretic one, governed by Vlasov dynamics, in this limit.

We formulate the dynamics of $\Sigma^{(N)}$ in terms of its characteristic functions,

$$\mu_t^{(N,n)}(\xi_1,..,\xi_n;\eta_1,..,\eta_n) = (\psi_t^{(N)}, \Pi_{j=1}^n(\exp(i\xi_j.p_j/2)\exp(i\eta_j.x_j)\exp(i\xi_j.p_j/2))\psi_t^{(N)}) \tag{2.15}$$

where the ξ's and η's run over the ranges $\mathbf{R}^3$ and $(2\pi\mathbf{Z})^3$, respectively. These functions are in one-to-one correspondence with the reduced density matrices for the state $\psi_t^{(N)}$; and, in particular, the n-particle spatial density function

$$\rho_t^{(N,n)}(x_1,..,x_n) = \int dx_{n+1}..\,.dx_N|\psi_t(x_1,x_2,..,x_N)|^2 \tag{2.16}$$

is the Fourier transform of $\mu_t^{(N,n)}$ w.r.t. the η's, when the ξ's are held at the value zero.

The initial condition for $\Sigma^{(N)}$, corresponding to $(I.1)^{(L)}$ for $\Sigma^{(N,L)}$, is

$$(\psi_0^{(N)}, p_1^2\psi_0^{(N)}) < N^{-2/3}b \to 0 \ as \ N\to\infty \tag{2.17}$$

with b a finite constant. We shall find it useful to generalise this condition by imposing an initial position-dependent drift velocity, $u_0 = \nabla\phi$, on the system. This is achieved by rephasing ψ_0 by the factor $\exp(i\sum_{j=1}^N\phi(x_j)/\hbar_N)$, and results in the replacement of (2.17) by

$(I.1)$

$$(\psi_0^{(N)}, (p_1 - u_0(x_1))^2\psi_0^{(N)}) < N^{-2/3}b\to 0 \ as \ N\to\infty \tag{2.18}$$

We shall assume that the function u_0 is continuously differentiable.

The initial condition $(II.2)^{(L)}$ transforms to the following form for $\Sigma^{(N)}$, which is unaffected by the above rephasing of the initial state.

$(I.2)$

$$(\psi_0^{(N)}, U(x_1 - x_2)\psi_0^{(N)}) < c \tag{2.19}$$

where c is a finite constant. We further assume that

$(I.3)$

$$\lim_{N\to\infty}\rho_0^{(N,1)}(x) = \sigma_0(x) \ \forall x\in K \tag{2.20}$$

where the function σ_0 is continuous; and that

$(I.4)$

$$\lim_{N\to\infty}(\mu_0^{(N,n)}(\xi_1,..,\xi_n;\eta_1,..,\eta_n) - \Pi_1^n\mu_0^{(N,1)}(\xi_j,\eta_j))\equiv 0 \tag{2.21}$$

This last condition signifies that, in the limit $N\to\infty$, the initial correlations of $\Sigma^{(N)}$ are of zero range: the assumption behind this is that the initial state of the microscopic model $\Sigma^{(N,L)}$, carries only short range correlations, as in a pure phase [6,7].

We note here that it was shown by explicit construction, in the Appendix of Ref. [3], that the initial conditions (I.1)-(I.4) are perfectly viable.

The macroscopic dynamics of the Jellium model, then, is represented by the time-dependence of the characteristic functions $\mu_t^{(N,n)}$, in the limit $N \to \infty$, subject to the initial conditions (I.1-4). Before attempting to extract this dynamics from the Schrödinger equation (2.9), we shall first summarise results on the corresponding problem for the simpler model, $\Sigma_g^{(N)}$, obtained by replacing the (singular) Coulomb potential, U_c, by a suitably regular one, U_g, given by

$$U_g(x) = \int_K dy g(x-y) U_c(y) \tag{2.22}$$

where the 'smoothing function' g, and hence U_g, is twice continuously differentiable.*

The Regularised Model, $\Sigma_g^{(N)}$. The Hamiltonian for this model is still given by (2.10), but now with $U = U_g$. We note that this Hamiltonian has the following simplifying features.

(a) The effective Planck constant, $\hbar_N$, vanishes in the limit $N \to \infty$; and

(b) the two-body potential is a regular one, scaled by a factor N^{-1}.

It follows immediately from Ref. [5] that (a) and (b) lead to a classical Vlasov dynamics. Specifically, under the initial conditions (I.1-4), we have the following results.

(A) *The functions* $\mu_t^{(N,n)}$ *converge pointwise, as* $N \to \infty$, *to the characteristic functions,* $\mu_t^{(n)}$, *of classical probability measures* $m_t^{(n)}$ *on* $(K \times \mathbf{R}^3)^n$, *i.e.*

$$\mu_t^{(n)}(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n) = \int \exp\left(\sum_{j=1}^N i(x_j.\eta_j + v_j.\xi_j)\right) dm_t^{(n)}(\xi_1, \ldots, \xi_n; \eta_1, \ldots, \eta_n) \tag{2.23}$$

Moreover, $m_t^{(n)}$ *is the restriction to* $(K \times \mathbf{R}^3)^{(n)}$ *of a unique probability measure* m_t *on* $(K \times \mathbf{R}^3)^{\mathbf{N}}$.

(B) *The initial form of* m *is given by*

$$dm_0^{(n)}(x_1, \ldots, x_n; v_1, \ldots, v_n) = \Pi_{j=1}^n dm_0^{(1)}(x_j, v_j) \tag{2.24}$$

where, as a consequence [3] of $(I.1)$ *and* $(I.3)$, $m_0^{(1)}$ *is given formally by*

$$dm_0(x, v) = \sigma_0(x) \delta(v - u_0(x))$$

i.e.

$$\int dm_0^{(1)}(x, v) f(x, v) = \int dx \sigma_0(x) f(x, u_0(x)) \tag{2.25}$$

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* This last condition ensures that the model meets the requirements of Refs. [5,8] for the derivation of the results given by (A)-(D) below.

for test functions, f, on $K \times \mathbf{R}^3$, that are continuous and have compact support.

(C) *The probability measure m_t evolves according to the Vlasov hierarchy, in its weak form, i.e.*

$$\frac{d}{dt}\int f^{(n)}(x_1,..,x_n;v_1,..,v_n)dm_t^{(n)}(x_1,..,x_n;v_1,..,v_n) =$$

$$\sum_{j=1}^n \int v_j.\frac{\partial f^{(n)}}{\partial x_j}(x_1,..,x_n;v_1,..,v_n)dm_t^{(n)}(x_1,..,x_n;v_1,..,v_n)$$

$$-\sum_{j=1}^n \int \nabla U(x_j - x_{n+1}).\frac{\partial f^{(n)}}{\partial v_j}(x_1,..,x_n;v_1,..,v_n)dm_t^{(n+1)}(x_1,..,x_{n+1};v_1,..,v_{n+1}) \tag{2.26}$$

for test-functions $f^{(n)}$ that are continuously differentiable and have compact support.

(D) [3,5] *The decorrelation property (B) is preserved at all times, i.e.*

$$dm_t^{(n)}(x_1,..,x_n;v_1,..,v_n) = \Pi_{j=1}^n dm_t^{(1)}(x_j,v_j) \tag{2.27}$$

and consequently, by (2.26), $m_t^{(1)}$, evolves according to the weak form of the classical Vlasov equation, i.e.

$$\frac{d}{dt}\int f(x,v)dm_t^{(1)}(x,v) = \int v.\frac{\partial}{\partial x}f(x,v)dm_t^{(1)}(x,v)$$

$$-\int \nabla U(x-y).\frac{\partial}{\partial v}f(x,v)dm_t^{(1)}(x,v)dm_t^{(1)}(y,w) \tag{2.28}$$

(E) [9] *This last equation has a unique solution, for the given initial conditions. Further, it is related to the (unique) solution of the Newtonian mean field theoretic problem*

$$\frac{d\chi_t(x,v)}{dt} = \mathcal{V}_t(x,v);\ \frac{d\mathcal{V}_t(x,v)}{dt} = -\int dm_0(y,w))\nabla U(\chi_t(x,v) - \chi_t(y,w)); \tag{2.29}$$

with

$$\chi_0(x,v) = x;\ \mathcal{V}_0(x,v) = v \tag{2.30}$$

by the formula

$$\int dm_t^{(1)}(x,v)f(x,v) = \int dm_0^{(1)}(x,v)f(\chi_t(x,v),\mathcal{V}_t(x,v)) \tag{2.31}$$

for test functions f, that are continuous and have compact support. Thus, defining

$$X_t(x) = \chi_t(x,u_0(x));\ V_t(x) = \mathcal{V}_t(x,u_0(x)) \tag{2.32}$$

it follows from (2.25) that (2.29-31) may be re-expressed as

$$\frac{dX_t(x)}{dt} = V_t(x);\ \frac{dV_t(x)}{dt} = -\int dy\sigma_0(y)\nabla U(X_t(x) - X_t(y));\ X_0(x) = x;\ V_0(x) = u_0(x) \tag{2.33}$$

and

$$\int dm_t^{(1)}(x,v)f(x,v) = \int dx\sigma_0(x)f(X_t(x),V_t(x)) \tag{2.34}$$

The Coulomb Model $\Sigma^{(N)}$. The problems posed by this model stem from the singularity in the Coulomb potential, U_c. Our treatment of the model will be based on certain regularity assumptions, designed to represent the idea that the repulsive character of the inter-electronic forces tends to keep the particles apart and thus tames the Coulomb singularity.

We express the first of these assumptions in terms of the one- and two-particle spatial densities, $\rho^{(N,1)}$, $\rho^{(N,2)}$, specified by (2.16). We employ these densities to define the conditional expectation, $\mathcal{E}_t^{(N)}(f(x,y)|x)$, of a two-point function $f(x,y)$, given x, by the standard formula

$$\int dx\rho_t^{(N,1)}(x)\mathcal{E}_t^{(N)}(f(x,y)|x)g(x) = \int dxdy\rho_t^{(N,2)}(x,y)f(x,y)g(x) \tag{2.35}$$

for all continuous functions g on K. We then introduce the following regularity assumption, to the effect that the Coulomb repulsion keeps the electrons apart sufficiently to ensure that the magnitude of the internal electric field remains bounded.

(R.1) For any finite $\tau(>0)$, there is a constant, $B_\tau(<\infty)$, such that

$$\int dy\rho_t^{(N,1)}(x)|\nabla U(x-y)| < B_\tau \tag{2.36}$$

and

$$\mathcal{E}_t^{(N)}(|\nabla U(x-y)|\ |x) < B_\tau\ \forall t\in[0,\tau],\ x\in K \tag{2.37}$$

Comments. (1) This is *implicitly* a condition on the initial state $\psi_0^{(N)}$. It is indeed restrictive since wave-functions can be constructed in such a way that their evolution leads, in the limit $N\to\infty$, to a catastrophic collapse, in which the microscopic dynamics breaks down within a finite time [10].

(2) Assumption (R.1)) is weaker than the corresponding one of Ref. [3], which required that $\rho_t^{(N,2)}$ itself was uniformly bounded over finite time intervals and led to a smooth hydrodynamics from which dynamical phase transitions were excluded. By contrast, (R.1) admits the possibility of singularities in both $\rho_t^{(N,1)}$ and $\rho_t^{(N,2)}$, in the limit $N\to\infty$, and thus, as we shall see, of hydrodynamical phase transitions.

The following Proposition is an extension of results obtained in Ref. [3] to the situation where the assumption of the uniform boundedness of $\rho_t^{(N,2)}$ is replaced by (R.1). It is a straightforward matter to check that the proofs given there of these results prevail under the above weaker assumption.

Proposition 2.1. *Assuming the conditions (I.1-4) and (R.1), the above results (A)-(C) are valid for the Jellium model, with the modification that here the convergence of* $\mu_t^{(N,n)}$ *to* $\mu_t^{(n)}$ *is subsequential. Thus, the system evolves according to the classical Vlasov hierarchy (2.26), subject to the initial conditions (2.24) and (2.25). Furthermore, the magnitude of the electric field, in the limit* $N \to \infty$*, satisfies the estimate*

$$\int dm_t^{(1)}(y,w)|U(x-y)| < B_\tau \ \forall x \in K, \ t \in [0,\tau] \tag{2.38}$$

We now assume that, as in $\Sigma_g^{(N)}$, the *macroscopic* decorrelation property (I.4) persists in time, i.e. that the (assumedly tamed) Coulomb singularity does not lead to correlations of long range on the microscopic scale.

(R.2) The factorisation property (2.27) prevails at all times.

As an immediate consequence of Prop. 2.1 and (R.2), we have

Proposition 2.2. *Under the further assumption (R.2), the single particle probability measure,* $m_t^{(1)}$*, evolves according to the weak form (2.28) of the classical Vlasov equation, subject to the initial condition (2.25).*

Our next regularity assumption is that as the (assumedly tamed) Coulomb singularity does not affect the uniqueness property (D), that was operative for the regularised model.

(R.3) The Vlasov equation (2.28), subject to the regularity condition (2.38) and the given initial conditions, has a unique solution.

Our last regularity condition is the counterpart to (R.3) for the Newtonian mean field dynamics, as given by (2.33).

(R.4) The Newtonian mean field problem (2.33), subject to the regularity condition

$$\int dy \sigma_0(y)|\nabla U(X_t(x) - X_t(y))| < B_\tau, \ \forall x \in K, \ t \in [0,\tau] \tag{2.39}$$

has a unique solution.

The following Proposition is an immediate consequence of Prop.2.2 and assumptions (R.3,4), since equations (2.33) and (2.34) imply that $m_t^{(1)}$ satisfies the Vlasov equation (2.28).

Proposition 2.3. *Under the further assumptions (R.3,4), the time-dependent macroscopic probability measure is given by (2.34), with* (X_t, V_t) *the unique solution of the Newtonian problem (2.33) for the Jellium model.*

3. Eulerian Versus Stochastic Hydrodynamics.

The Newtonian mean field theory, given by (2.33), corresponds to a *Lagrangian* hydrodynamics, in which $X_t(x)$ and $V_t(x)$ are the position and velocity, respectively, of a 'fluid particle'; and the Vlasov equation (2.28) is just the Liouville equation representing its probabilistic description. Our aim now is to investigate the conditions, both for the Jellium model and its regularised version, under which the Vlasov dynamics reduces to a deterministic Eulerian hydrodynamics. In fact, we shall show that it does so, provided that X_t is an invertible function of position; and that otherwise it is stochastic. Note here that the invertibility of the canonical transformation $(x,v)\to(\chi_t(x,v),\mathcal{V}_t(x,v))$ does *not* imply that of the mapping $x\to X_t(x)\equiv\chi_t(x,u_0(x))$.

Case (a): X_t Invertible. In this case, the Jacobean

$$J_t(x)=\frac{\partial(X_{1,t},X_{2,t},X_{3,t})}{\partial(x_1,x_2,x_3)} \tag{3.1}$$

with x_j (resp. $X_{j,t}$) the j'th component x (resp. X_t), is strictly positive, and so we can define

$$\sigma_t(x)=\sigma_0(X_t^{-1}(x))/J_t(x) \tag{3.2}$$

and

$$u_t(x)=V_t(X_t^{-1}(x)) \tag{3.3}$$

Thus, since, by (2.34), (3.2) and (3.3),

$$\int dm_t^{(1)}(x,v)f(x,v)=\int dx\sigma_t(x)f(x,u_t(x)) \tag{3.4}$$

for continuous functions f on K, i.e., formally,

$$dm_t(x,v)=\sigma_t(x)\delta(v-u_t(x))dxdv$$

it follows that $u_t(x)$ and $\sigma_t(x)$ are the drift velocity and normalised particle density, respectively, at position x and time t.

It follows now from (2.14), (2.22), (2.33), (3.2) and (3.3) that u_t,σ_t evolve according to the following Euler-Maxwell hydrodynamical equations, previously obtained in Ref. [3].

$$\frac{\partial\sigma_t}{\partial t}+\nabla.(\sigma_t u_t)=0 \tag{3.5}$$

$$\frac{\partial u_t}{\partial t}+(u_t.\nabla)u_t=E_t \tag{3.6}$$

where, for the Jellium model,

$$\nabla.E_t=(\sigma_t-1) \tag{3.7}$$

and, for the regularised one,

$$\nabla.E_t=(\sigma_t^{(g)}-1) \tag*{(3.7)'}$$

where

$$\sigma_t^{(g)}(x) = \int dy g(x-y)\sigma_t(y) \tag{3.8}$$

Case (b): X_t Non-Invertible. In this case, we cannot employ the formulae (3.2) and (3.3) to define the time-dependent density and drift velocity. Instead, we have to consider the situation where the equation

$$X_t(y) = x \tag{3.9}$$

has several solutions, labelled by an index set J, for y as a function of x and t, i.e.

$$y = \{Y_t^{(j)}(x) | j \in J\} \tag{3.10}$$

In this case, equation (2.34) implies that

$$\int dm_t^{(1)}(x,v) f(x,v) = \sum_{j\in J} \int dx \sigma_t^{(j)}(x) f(x, u_t^{(j)}(x)) \tag{3.11}$$

where

$$\sigma_t^{(j)}(x) = \sigma_0(Y_t^{(j)}(x))|K_t^{(j)}(x)|; \quad u_t^{(j)}(x) = V_t(Y_t^{(j)}(x)) \tag{3.12}$$

and

$$K_t^{(j)}(x) = \frac{\partial(Y_{1,t}^{(j)}, Y_{2,t}^{(j)}), Y_{3,t}^{(j)})}{\partial(x_1, x_2, x_3)} \tag{3.13}$$

Thus, (3.11) signifies that, formally,

$$dm_t(x,v) = \sum_{j\in J} \sigma_t^{(j)} \delta(v - u_t^{(j)}(x)) dx dv$$

i.e. that the macroscopic state of the system at time t corresponds to a statistical mixture of different streams, the j'th of which has density $\sigma_t^{(j)}$ and drift velocity $u_t^{(j)}$. This implies that the local density and drift velocity are now *stochastic* variables of a hydrodynamics still governed by the Vlasov equation.

We may summarise the above observations in the following form.

Proposition 3.1. *If X_t is invertible, then the macroscopic dynamics of the system corresponds to a hydrodynamics given by the Euler-cum-Maxwell equations (3.5)-(3.7) (or (3.7)$'$). Otherwise, it corresponds to the flow of a mixture of streams, and its evolution is of a stochastic type, governed by the Vlasov equation (2.28).*

Comment. Unless the domain of non-invertibility of X_t is confined to a surface, the resultant mixture of streams does not correspond to a shock wave. On this basis, it will be seen that the example of Eulerian hydrodynamic breakdown in §4 is not that of a shock front.

4. Example of Hydrodynamic Phase Transition.

We shall now provide an example of initial conditions, which lead to a transition from a deterministic to a stochastic flow, in both the regularised and the Coulomb models. These are conditions where both σ_0 and u_0 are functions of just a single coordinate, say x_1, and u_0 is directed along Ox_1. In this case, the flow becomes effectively one- dimensional, for the following reasons. If

$$X_t^{(b)}(x) := X_t(x+b) - b$$

for arbitrary vectors b in the plane Ox_2x_3, then, in view of the periodicity of K, if X_t is a solution of the Newtonian problem (2.33), so too is $X_t^{(b)}$. Hence, by the uniqueness* of the solution of (2.33), $X_t {\equiv} X_t^{(b)}$, which implies that the component $X_{1,t}$ of X_t depends on the coordinate x_1 only, and that $X_{2,t}(x), X_{3,t}(x)$ reduce to $x_2 + \xi_2(t),\ x_3 + \xi_3(t)$, where the ξ's are functions of t only. Furthermore, it follows from the x_2- and x_3-components of (2.33) that these functions are both zero. In other words, the component $X_{1,t}$ of X_t depends only on x_1, while $X_{2,t}, X_{3,t}$ remain fixed at x_2, x_3, respectively. Hence, the macroscopic dynamics reduces to a one-dimensional flow. For notational convenience, we shall henceforth drop the suffix 1 from $X_{1,t}$ and x_1.

Thus, by (2.33),

$$X_t(x) = x + u_0(x)t + \int_0^t ds(t-s)\int_0^1 dy\sigma_0(y)F(X_s(x) - X_s(y)) \tag{4.1}$$

where

$$F(x) = -\int_0^1 dx_2 \int_0^1 dx_3 \frac{\partial U}{\partial x}(x, x_2, x_3) \tag{4.2}$$

Let

$$J_t(x) = \frac{\partial X_t(x)}{\partial x} \tag{4.3}$$

Then, by Prop. 3.1 and the implicit function theorem, the neccessary and sufficient condition for deterministic hydrodynamics is that J_t has no zeroes. We note also that the definition (4.3) permits us to re-express (4.1) in the form

$$X_t(x) = x + u_0(x)t + \int_0^t ds(t-s)\int_0^1 dy\sigma_0(y)F(\int_y^x dzJ_s(z))) \tag*{(4.1)$'$}$$

The Regularised Model. Here, $U = U_g$ and thus, by (4.2), F is a continuously differentiable function. Hence, by (4.1) and (4.3),

$$J_t(x)(1 - \int_0^t ds(t-s)\int_0^1 dy\sigma_0(y)F'(x-y)) = 1 + u_0'(x)t \tag{4.4}$$

* In the case of the Coulomb model, this is a consequence of assumption (R.4).

where the primes denote differentiation w.r.t. x. Thus, defining

$$\|F'\| = sup\{|F'(x)| \ |x\in[0,1]\},$$

and t_0, t_1 to be the times given by

$$t_0 = 2/\|F'\|^{1/2} \tag{4.5}$$

and

$$t_1 = min\{t(>0)| \ (1 + u_0'(x)t) = 0 \ for \ some \ x\in[0,1]\} \tag{4.6}$$

it follows immediately from (4.4)-(4.6) that J_t is strictly positive, and hence that X_t is invertible, if $0\leq t < min(t_0, t_1)$. Therefore, by Prop. 3.1, the system evolves, in this regime, according to a deterministic hydrodynamics, given by equations (3.5), (3.6), (3.7)$'$ and (3.8).

On the other hand, if the initial conditions are such that $t_1 < t_0$, then it follows from (4.4)-(4.6) that J_t changes sign during the interval $t\in(t_1, t_0)$ over some spatial domain $D(\subset[0,1])$. Hence, by Prop. 3.1, the hydrodynamics of the model becomes stochastic, and is (still) governed by the Vlasov equation (2.28).

We may summarise these results as follows.

Proposition 4.1. *The regularised model exhibits a deterministic hydrodynamics, given by equations (3.5), (3.6) and (3.7)$'$, over the time interval $0\leq t < min(t_0, t_1)$. However, if the initial velocity profile is such that $t_1 < t_0$, then the flow undergoes a transition to stochasticity at time t_1.*

Comment. It will be seen from the derivations of this result that, in the stochastic phase, the domain of non-invertibility of X_t is, in general, not confined to a single value of x, i.e. to a surface in K. Thus, in view of the comment following Prop. 3.1, the hydrodynamic phase transition described here does not correspond to the formation of a shock wave.

The Coulomb Model. We shall prove the following Proposition for this model.

Proposition 4.2. *Under the specified assumptions, the hydrodynamics of the Coulomb model takes the deterministic form (3.5)-(3.7) at all times, provided that the initial density and velocity profiles satisfy the condition*

$$(\sigma_0(x) - 1)^2 + (u_0'(x))^2 < (\sigma_0(x))^2 \ \forall x\in[0,1] \tag{4.7}$$

Otherwise there is a transition to a stochastic flow at a certain time τ, given by the least positive value of t for which

$$\sigma_0(x) + (1 - \sigma_0(x))\cos(t) + u_0'(x)\sin(t) = 0 \tag{4.8}$$

for some $x\in[0,1]$.

Comment. Again, the domain of non-invertibility of X_t in the stochastic phase is not confined to a single value of x, i.e. to a surface in K; and thus the hydrodynamical phase transition does not correspond to the formation of a shock wave.

To prove this Proposition, we shall first establish the following lemma.

Lemma 4.3. *If X_t is invertible for $t\in[0,\tau]$, where $\tau > 0$, then the regularity condition (2.39) is satisfied.*

Proof. We note first that, by (2.13) and (4.2),

$$F(x) = \sum_{n=1}^{\infty} i\exp(2\pi inx)/(2\pi n)$$

which implies that F is square integrable, hence absolutely integrable, over $[0,1]$. Thus, since, by (3.2), (4.2) and (4.3), the l.h.s. of (2.39) is equal to

$$\int_0^1 dy\sigma_0(y)J_t(y)|F(X_t(x)-X_t(y))|$$

$$\equiv\int_0^1 dy\sigma_0(X_t^{-1}(y))|F(X_t(x)-y)|$$

by the invertibility of X_t, it follows that the condition (2.39) is satisfied.

Proof of Prop. 4.2. Since $U = U_c$ here, it follows from (2.13), (2.14), (4.1) and (4.3) that

$$J_t(x) = 1+u_0'(x)t+\int_0^t ds(t-s)J_s(x)(-1+\int_0^1 dy\sigma_0(y)\delta(X_s(x)-X_s(y))) \tag{4.9}$$

where now δ is the Dirac distribution on $[0,1]$, subject to periodic boundary conditions.

Let us first suppose that X_t is invertible, i.e. that J_t is strictly positive, over a time interval $0\le t<\tau_0$, for some positive τ_0. In this case,

$$J_s(x)\delta(X_s(x)-X_s(y))\equiv\delta(x-y)\ \forall s\in[0,\tau_0)$$

and therefore (4.9) reduces to

$$J_t(x) = 1+u_0'(x)t+\int_0^t ds(t-s)(\sigma_0(x)-J_s(x)) \tag{4.10}$$

i.e.

$$(\frac{d^2}{dt^2}+1)J_t(x)=\sigma_0(x);\ \textit{with}\ J_0(x)=1;\ \dot{J}_0(x)=u_0'(x) \tag{4.11}$$

where $\dot{J}_t = dJ_t/dt$. Hence,

$$J_t(x) = \sigma_0(x) + (1 - \sigma_0(x))\cos(t) + u_0'(x)\sin(t) \tag{4.12}$$

In view of the non-negativity of σ_0, this equation implies that J_t is strictly positive for all $t \geq 0$ if and only if the condition (4.7) is fulfilled. Otherwise, J_t changes sign at some point $x \in [0, 1]$ when t reaches the value τ specified in the statement of the Proposition.

We may thus summarise these results as follows.

(a) If (4.7) is satisfied, then the function X_t given by substituting the formula (4.12) for J_t into the r.h.s. of $(4.1)'$ is invertible and satisfies both the Newtonian mean field equation (4.10) and, by Lemma (4.3), the regularity condition (2.39), for all $t \geq 0$. Hence, by (R.4), it is the unique solution of the Newtonian mean field equation, and persists for all positive t. Hence, by Prop. 3.1, the model exhibits the Eulerian hydrodynamics given by (3.5)-(3.7) at all times.

(b) If (4.7) is violated, then, by the same argument, the system exhibits this deterministic hydrodynamics for times $t \in [0, \tau)$, with τ as specified in Prop. 4.2.

(c) If (4.7) is violated, then there must be a transition to stochastic flow at $t = \tau$, since an assumption to the contrary becomes invalid when t passes through that value.

The results (a)-(c) establish the Proposition.

5. Concluding Remarks.

We have shown here that the quantum dynamics of the Jellium model leads to a hydrodynamics, which supports both deterministic and stochastic flows, and exhibits phase transitions between them. This hydrodynamics is therefore richer than that of the deterministic flow given by the Euler-cum-Maxwell equations. Furthermore, since the flow in the stochastic phase corresponds to a statistical mixture of different streams, one might envisage that this carries a germ of turbulence.

As regards possible ramifications of the present work, we note that the above hydrodynamical properties of the model stemmed from its Vlasov dynamics, which is simply the Liouville probabilistic version of its Lagrangian hydrodynamics (cf. §3). This suggests that, more generally, a natural way of formulating the theory of stochastic flows, even of turbulence, might be via a probabilistic treatment of Lagrangian hydrodynamics. That should presumably have some connection with Foias's [11] formulation of stochastic hydrodynamics on the basis of a Liouville equation governing Navier-Stokes flows. It need not, however, be equivalent to it, since, as we have seen in $\S's$ 3 and 4, the Eulerian and Lagrangian pictures of the present plasma model are not equivalent.

Finally, we remark here that the hydrodynamics obtained here is completely inviscid. The reason for this, as in Ref. [3] (cf. discussion there at the end of §1), is that our macroscopic description is effected on the largest available length scale, L, and that, consequently, the viscous forces are 'scaled away'. Thus, the hydrodynamic picture we have obtained should be regarded as no more than a skeletal version of that of a real plasma.

References.

1. E. H. Lieb and H. Narnhofer: J. Stat. Phys. **12**, 291 (1975)

2. Ph. Martin and Ch. Oguey: I. Phys. A **18**, 1995 (1985)

3. G. L. Sewell: J. Math. Phys. **26**, 2324 (1985)

4. E. B. Davies: J. Stat. Phys. **18**, 161 (1978)

5. H. Narnhofer and G. L. Sewell: Commun. Math. Phys. **79**, 9 (1981)

6. D. Ruelle: "Statistical Mechanics", Benjamin, New York (1969)

7. G. L. Sewell: "Quantum Theory of Collective Phenomena", Clarendon Press, Oxford (1991)

8. H. Spohn: Math. Meth. Appl. Sci. **3**, 445 (1981)

9. H. Neunzert: Fluid Dyn. Trans. **9**, 229 (1978)

10. C. Radin: Commun. Math. Phys. **54**, 69 (1974)

11. C. Foias: Russian Math. Surveys **29**, 293 (1974)