EQUIVARIANT INSTANTONS AND THE ADHM CONSTRUCTION

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Abstract. We develop a version with a certain $SU(2)$-symmetry of the Atiyah-Drinfeld-Hitchin-Manin (ADHM) "monad" construction of Yang-Mills instantons. We construct all such "$SU(2)$-equivariant ADHM monads" and thus obtain the following: for each second Chern number of the form $m(m+1)/2$ (where m is a non-negative integer) a unique $SU(2)$-equivariant instanton exists, and is given explicitly.

1. Introduction

An *instanton* is a connection with anti-self-dual curvature on a complex vector bundle over the four-sphere S^4. We consider here the simplest case, that of a rank-2 vector bundle with structure group $SU(2)$. Associated with such an instanton is an integer invariant called the second Chern number C_2, determining the topological type of the bundle. Instantons exist only on bundles with non-negative second Chern number.

We consider here instantons that admit a certain $SU(2)$-symmetry, described as follows. The $SU(2)$-action on the base space S^4 is the action induced on the unit sphere in $\mathbb{R}^5$ by the 5-dimensional irreducible representaion of $SU(2)$. By an "$SU(2)$-equivariant instanton" we mean an instanton invariant under an $SU(2)$-action by bundle automorphisms inducing the above $SU(2)$-action on the base space S^4. In previous work we have studied this $SU(2)$-symmetry in relation to the problem of finding non-self-dual solutions to the Yang-Mills equations. It was found [ASSS,BM] that such $SU(2)$-actions on a bundle with a second Chern number C_2 are naturally classified by a pair of positive odd integers n_+ and n_- such that $C_2 = (n_+^2+n_-^2)/8$, and furthermore [SS1,SS2,Bo], that $SU(2)$-equivariant instantons cannot exist if $n_- \neq 1$. More precisely, for each pair (n_+, n_-), the $SU(2)$-action on the bundle induces an action on the moduli space of instantons, so that $SU(2)$-equivariant instantons form the fixed point set, which is empty if $n_- \neq 1$. It was then shown in [Bo, sect. 3], via an equivariant index calculation, that for $n_- = 1$ this fixed point set is zero-dimensional. Here we establish that the fixed point set is not empty when $n_- = 1$. There is in fact a unique $SU(2)$-equivariant instanton and we can even describe it explicitly. Setting $n_+ = 2m+1$ our result is formulated as follows.

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Theorem 1.1. *For every bundle with second Chern number of the form $C_2 = m(m+1)/2$ (m non-negative integer), there is a unique $SU(2)$-equivariant instanton. Furthermore, an explicit formula for the instanton can be given (see proposition 4.4).*

Our proof of theorem 1.1 is based on the work of Atiyah, Drinfeld, Hitchin and Manin [ADHM], which gives a bijective correspondence between instantons and the so-called "ADHM monads". A *monad*, in this paper, consists of three complex vector spaces W, Z, and V of dimensions C_2, 4, and $2\,C_2 + 2$ respectively, together with a bilinear map

$$A : W \times Z \to V. \tag{1.1}$$

An *ADHM monad* further satisfies certain "ADHM conditions" (see theorem 3.3). Given an ADHM monad, one can easily and explicitly construct the corresponding instanton.

When an instanton is $SU(2)$-equivariant, the above vector spaces W, Z, and V are naturaly $SU(2)$ representation spaces, and the map A is $SU(2)$-equivariant. The $SU(2)$-action on Z is determined by the $SU(2)$-action on S^4 alone (independently of $SU(2)$-action on the bundle given by m) and is quite easily seen to be the four-dimensional irreducible $SU(2)$-representation (proposition 3.1). W and V are decomposed into irreducible $SU(2)$-representations (proposition 3.2) using their interpretation as the kernels of certain elliptic linear differential operators (Dirac operators) associated with a given instanton, and an equivariant index theorem. Using these decompositions, we write in theorem 3.4 the ADHM conditions for an $SU(2)$-equivariant ADHM monad as a set of reality conditions, inequalities, and quadratic equations for the complex coefficients of the bilinear map A, and solve these equations explicitly in proposition 4.4. We use a number of identities, apparently new, concerning four-fold tensor products of $SU(2)$-representations. For certain Abelian group actions, equivariant monads have appeared previously in [At2,BA,Au].

Finally, let us remark that some of our result here can be arrived at from at least two additional points of view. The ADHM monads construct certain holomorphic bundles [ADHM,At1] on the complex projective space $\mathbb{CP}^3$, which correspond via the Penrose "twistor" transform to instantons [Wa,AW]. The holomorphic bundles with $SU(2)$-symmetry admit a complexification of the symmetry group to $SL(2,\mathbb{C})$. It is also possible to study the holomorphic bundles with $SL(2,\mathbb{C})$-symmetry directly (without using ADHM monads) yielding essentially the same results as this paper. Details will be published elsewhere. ¿From a different point of view, a recent result of Sadun [Sa, Thm. 1.1] concerning the solutions of a certain ordinary differential equation system is equivalent to the existence and uniqueness statement of our theorem 1.1, but does not yield explicit solutions. This ODE system comes from the dimensional reduction of the anti-self-duality equation on S^4 by the $SU(2)$-symmetry.

2. The structure of $SU(2)$ representations

Let $\mathcal{V} \equiv \mathbb{C}[x,y]$ be the polynomial ring with complex coefficients generated by the two indeterminates x and y. $SL(2,\mathbb{C})$ acts on $\mathcal{V}$ in the usual way by linear substitutions in x and y. The subspace $\mathcal{V}_d \subset \mathcal{V}$ of degree-d homogeneous polynomials is an irreducible $SL(2,\mathbb{C})$ representation. Every irreducible $SL(2,\mathbb{C})$ representation is isomorphic to $\mathcal{V}_d$ for some non-negative integer d. The representation theory of the real form $SU(2)$ is identical to that of $SL(2,\mathbb{C})$. The action of the Lie algebra $su(2)$ on $\mathcal{V}$ is generated by the linear differential operators

$$J_1 \equiv -\frac{i}{2}\left(x\frac{\partial}{\partial y}+y\frac{\partial}{\partial x}\right), \quad J_2 \equiv \frac{1}{2}\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right), \quad J_3 \equiv \frac{i}{2}\left(x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y}\right), \tag{2.1}$$

with $[J_1, J_2] = J_3$, etc. Complexification yields the Lie algebra $sl(2,\mathbb{C}) \simeq su(2)\otimes_{\mathbb{R}}\mathbb{C}$.

Tensor products of irreducible representations decompose according to the Clebsch-Gordan formula

$$\mathcal{V}_l \otimes \mathcal{V}_j \simeq \mathcal{V}_{|l-j|} \oplus \mathcal{V}_{|l-j|+2} \oplus \cdots \oplus \mathcal{V}_{l+j-2} \oplus \mathcal{V}_{l+j}. \tag{2.2}$$

The projections implicit in this formula are described in terms of the "transvectants" (Überschiebungen) of classical invariant theory. We adopt the notation of Bryant [Br], and refer to [GY] or [KR] for additional details. For each non-negative integer p, the p-th transvectant is the $\mathbb{C}$-bilinear map $\langle\ ,\ \rangle_p : \mathcal{V}\times\mathcal{V}\to\mathcal{V}$ defined by

$$\langle u, v\rangle_p \equiv \frac{1}{p!}\sum_{k=0}^{p}(-1)^k\binom{p}{k}\frac{\partial^p u}{\partial x^{p-k}\partial y^k}\frac{\partial^p v}{\partial x^k \partial y^{p-k}}. \tag{2.3}$$

The $SU(2)$-equivariance of the transvectants follows from the property

$$J_n\langle u, v\rangle_p = \langle J_n u, v\rangle_p + \langle u, J_n v\rangle_p, \qquad u, v \in \mathcal{V}, \quad n = 1, 2, 3. \tag{2.4}$$

The $\mathbb{C}$-bilinearity immediately implies that the transvectants are in fact $SL(2,\mathbb{C})$-equivariant. The restriction $\langle \mathcal{V}_l, \mathcal{V}_j\rangle_p \to \mathcal{V}_{l+j-2p}$ is a nonzero map for $0 \le p \le \min(l,j)$, and the corresponding linear map $\mathcal{V}_l \otimes \mathcal{V}_j \to \mathcal{V}_{l+j-2p}$ is the $SL(2,\mathbb{C})$-equivariant projection implicit in the Clebsch-Gordan formula eq.(2.2).

$\mathcal{V}$ has a natural mod-2 grading

$$\mathcal{V} = \mathcal{V}_{even} \oplus \mathcal{V}_{odd}, \tag{2.5}$$

where $\mathcal{V}_{even}$ is is the $\mathbb{C}$-linear span of monomials $x^a y^b$ with $a - b = 0 \pmod 2$, and $\mathcal{V}_{odd}$ of those with $a - b = 1 \pmod 2$. The notation reflects the fact that $\mathcal{V}_d$ is a subspace of $\mathcal{V}_{even}$ if d is even, and of $\mathcal{V}_{odd}$ if d is odd. The following symmetry property will be important:

$$\langle u, v\rangle_p = (-1)^p\langle v, u\rangle_p, \qquad u, v \in \mathcal{V}. \tag{2.6}$$

The restriction $\langle \mathcal{V}_l, \mathcal{V}_l\rangle_l \to \mathcal{V}_0 \simeq \mathbb{C}$ is a nondegenerate $SL(2,\mathbb{C})$-equivariant bilinear form, symmetric if l is even, antisymmetric if l is odd. The bilinear form $\langle \mathcal{V}, \mathcal{V}\rangle \to \mathbb{C}$ defined by

$$\langle \mathcal{V}_l, \mathcal{V}_l\rangle \equiv \langle \mathcal{V}_l, \mathcal{V}_l\rangle_l, \qquad \langle \mathcal{V}_l, \mathcal{V}_j\rangle \equiv 0 \quad \text{if } l \neq j, \tag{2.7}$$

restricts to a symmetric bilinear form $\langle \mathcal{V}_{even}, \mathcal{V}_{even}\rangle \to \mathbb{C}$ on $\mathcal{V}_{even}$, and to an antisymmetric bilinear form $\langle \mathcal{V}_{odd}, \mathcal{V}_{odd}\rangle \to \mathbb{C}$ on $\mathcal{V}_{odd}$.

Real or quaternionic structures on $\mathcal{V}_d$ are defined via the $\mathbb{C}$-antilinear map $\sigma : \mathcal{V} \to \mathcal{V}$, which acts on monomials by

$$\sigma(\xi x^a y^b) \equiv (-1)^a \bar{\xi} x^b y^a, \qquad \xi \in \mathbb{C}, \tag{2.8}$$

where $\bar{\xi}$ denotes the complex conjugate of ξ. The restriction $\sigma_{even} : \mathcal{V}_{even} \to \mathcal{V}_{even}$ is a real structure map, that is, a $\mathbb{C}$-antilinear map with $(\sigma_{even})^2 = 1$. For even d, we denote by $\mathcal{V}_d^{\mathbb{R}} \subset \mathbb{R}[x, y]$ the corresponding real representation. The restriction $\sigma_{odd} : \mathcal{V}_{odd} \to \mathcal{V}_{odd}$ is a quaternionic structure map, that is, a $\mathbb{C}$-antilinear map with $(\sigma_{odd})^2 = -1$. The equation

$$\sigma J_n = J_n \sigma, \qquad n = 1, 2, 3 \tag{2.9}$$

shows that σ commutes with the action of $SU(2)$, but *not* with the action of $SL(2, \mathbb{C})$ since σ is not $\mathbb{C}$-linear. The transvectants are compatible with σ in the sense

$$\sigma \langle u, v \rangle_p = \langle \sigma u, \sigma v \rangle_p. \tag{2.10}$$

Finally, a hermitian inner product $(\mathcal{V}, \mathcal{V}) \to \mathcal{V}_0 \simeq \mathbb{C}$ is defined by

$$(u, v) \equiv \langle \sigma u, v \rangle. \tag{2.11}$$

The inner product is $SU(2)$-equivariant (but *not* $SL(2, \mathbb{C})$-equivariant) because the linear maps $J_n : \mathcal{V}_d \to \mathcal{V}_d$ are anti-hermitian,

$$(J_n u, v) = -(u, J_n v), \qquad n = 1, 2, 3. \tag{2.12}$$

The inner product is by construction compatible with σ in the sense

$$\sigma (u, v) = (\sigma u, \sigma v). \tag{2.13}$$

The nondegeneracy of the inner product $(\ ,\ )$, and consequently of the bilinear form $\langle\ ,\ \rangle$, follows easily from

$$\|x^a y^b\|^2 \equiv (x^a y^b, x^a y^b) = a!\, b! \neq 0. \tag{2.14}$$

3. The $SU(2)$-equivariant ADHM conditions

In the ADHM construction, the four-sphere S^4 appears as the quaternionic projectivization $\mathbb{HP}^1$ of the quaternionic vector space $Z \simeq \mathbb{H}^2$. We consider the $SU(2)$ action $Z \simeq \mathcal{V}_3$, with quaternionic structure map $\sigma : \mathcal{V}_3 \to \mathcal{V}_3$.

Proposition 3.1. *([ASSS2]) The $SU(2)$ action on the quaternionic projectivization $\mathbb{HP}^1$ of $\mathcal{V}_3$ is isomorphic to the action on the unit sphere $S^4 \subset \mathcal{V}_4^{\mathbb{R}} \simeq \mathbb{R}^5$.*

Proof. Let $X \simeq \mathbb{R}^5$ be the space of σ-invariant hermitian $\mathbb{C}$-linear endomorphisms $Z \to Z$ with zero trace. S^4 is identified with the unit sphere in X relative to the operator norm. The $+1$ eigenspace of $A \in S^4 \subset X$ is a quaternionic subspace of $Z \simeq \mathbb{H}^2$, or equivalently an element $q(A) \in \mathbb{HP}^1$. This defines the standard identification $q : S^4 \xrightarrow{\sim} \mathbb{HP}^1$.

Now consider the $SU(2)$ action via $Z \simeq \mathcal{V}_3$. The possible representations on $X \simeq \mathbb{R}^5$ are $\mathcal{V}_4^{\mathbb{R}}$, $\mathcal{V}_2^{\mathbb{R}} \oplus 2\,\mathcal{V}_0^{\mathbb{R}}$, and the trivial reperesentation $5\,\mathcal{V}_0^{\mathbb{R}}$. Schur's lemma tells us that the $SU(2)$ action on X cannot contain $\mathcal{V}_0^{\mathbb{R}}$, since nonzero scalar multiples of the identity endomorphism have nonzero trace. The only possibility is $X \simeq \mathcal{V}_4^{\mathbb{R}}$. □

We will not consider here instantons with the other $SU(2)$ symmetry $X \simeq \mathcal{V}_2^{\mathbb{R}} \oplus 2\,\mathcal{V}_0^{\mathbb{R}}$, corresponding to $Z \simeq 2\,\mathcal{V}_1$, which were studied in [Ta].

The vector spaces W and V of the ADHM monad corresponding to an instanton connection on the bundle $E \to S^4$ are naturally identified with kernels of elliptic differential operators [ADHM, Hi]:

$$\begin{aligned} W &= \operatorname{Ker}\left[D_1 : \Gamma(E \otimes S^-) \to \Gamma(E \otimes S^+)\right] \\ V &= \operatorname{Ker}\left[D_2 : \Gamma((E \otimes S^-) \otimes S^-) \to \Gamma((E \otimes S^-) \otimes S^+)\right], \end{aligned} \tag{3.1}$$

where S^- and S^+ denote the bundles of negative and positive spinors respectively, D_1 and D_2 are the Dirac operators coupled to the connection on E and $E \otimes S^-$ respectively. The cokernels of D_1 and D_2 are trivial by a Weitzenbock formula involving the anti-self-duality of the curvature [AHS].

Proposition 3.2. *If there exists an $SU(2)$-equivariant instanton of symmetry type $n_+ = 2m+1$ and $n_- = 1$, then the vector spaces W and V of the corresponding ADHM monad contain at most one summand of any irreducible representation $\mathcal{V}_d$. Specifically,*

a) *$\mathcal{V}_d \subset W$ iff $d = 2l$, where $l = m-1 \pmod 2$ and $0 \leq l \leq m-1$.*
b) *$\mathcal{V}_d \subset V$ iff d is odd, $d \leq 2m+1$, and $d \neq 2m-1$.*

Proof. The representation on W is computed in [Pa, Sect. 4]. We adapt here a similar computation in [Bo, Sect. 3]. Consider a $U(1)$ subgroup of the $SU(2)$ symmetry group, parametrized by complex numbers u with $|u| = 1$. The action of $U(1)$ on S^4 has exactly two fixed points. We use the Atiyah-Bott-Lefschetz (ABL) fixed-point formula [AB] to compute the virtual character $\operatorname{Tr}(u, \operatorname{Ker} D_i) - \operatorname{Tr}(u, \operatorname{Coker} D_i)$, which equals the character $\operatorname{Tr}(u, \operatorname{Ker} D_i)$ since $\operatorname{Coker} D_i$ is trivial. Setting $n_+ = n$ for notational simplicity, the ABL formula yields

$$\begin{aligned} \chi_W \equiv \operatorname{Tr}(u, \operatorname{Ker} D_1) &= \frac{(u^n + u^{-n}) - (u + u^{-1})}{(u - u^{-1})(u^2 - u^{-2})} \\ \chi_V \equiv \operatorname{Tr}(u, \operatorname{Ker} D_2) &= \frac{(u^n + u^{-n})(u^3 + u^{-3}) - (u + u^{-1})^2}{(u - u^{-1})(u^2 - u^{-2})}, \end{aligned} \tag{3.2}$$

which can be simplified to

$$\begin{aligned} \chi_W &= \chi_{n-3} + \chi_{n-7} + \cdots \\ \chi_V &= \chi_n + \chi_{n-4} + \chi_{n-6} + \cdots, \end{aligned} \tag{3.3}$$

where

$$\chi_d = \frac{u^{d+1} - u^{-(d+1)}}{u - u^{-1}} = u^d + u^{d-2} + \cdots + u^{-d}$$

is the character of $\mathcal{V}_d$. □

We have now proved that for all $SU(2)$-equivariant ADHM monads with $Z \simeq \mathcal{V}_3$, we can identify $W \subset \mathcal{V}_{even}$ and $V \subset \mathcal{V}_{odd}$ as subspaces of a certain special form. We illustrate the first few cases:

$m = 0,\ n_+ = 1;$	$W \simeq \{0\},$	$V \simeq \mathcal{V}_1,$
$m = 1,\ n_+ = 3;$	$W \simeq \mathcal{V}_0,$	$V \simeq \mathcal{V}_3,$
$m = 2,\ n_+ = 5;$	$W \simeq \mathcal{V}_2,$	$V \simeq \mathcal{V}_5 \oplus \mathcal{V}_1,$
$m = 3,\ n_+ = 7;$	$W \simeq \mathcal{V}_4 \oplus \mathcal{V}_0,$	$V \simeq \mathcal{V}_7 \oplus \mathcal{V}_3 \oplus \mathcal{V}_1,$
$m = 4,\ n_+ = 9;$	$W \simeq \mathcal{V}_6 \oplus \mathcal{V}_2,$	$V \simeq \mathcal{V}_9 \oplus \mathcal{V}_5 \oplus \mathcal{V}_3 \oplus \mathcal{V}_1,$
$m = 5,\ n_+ = 11;$	$W \simeq \mathcal{V}_8 \oplus \mathcal{V}_4 \oplus \mathcal{V}_0,$	$V \simeq \mathcal{V}_{11} \oplus \mathcal{V}_7 \oplus \mathcal{V}_5 \oplus \mathcal{V}_3 \oplus \mathcal{V}_1.$

We have already used the identification $Z \simeq \mathcal{V}_3 \subset \mathcal{V}_{odd}$ to induce the quaternionic structure map $\sigma : Z \to Z$. Similarly, we induce the real structure map $\sigma : W \to W$, the quaternionic structure map $\sigma : V \to V$, the $\mathbb{C}$-bilinear antisymmetric form $\langle V, V\rangle \to \mathbb{C}$, and the hermitian inner product $(V, V) \to \mathbb{C}$.

We now state the ADHM conditions. In this paragraph we are *not* assuming $SU(2)$-equivariance of monads. The monad $A : W \times Z \to V$ is said to be *linear-equivalent* to the monad $\tilde{A} : W \times Z \to V$ iff

$$\tilde{A}(w, z) = h\, A(gw, z), \qquad w \in W, \quad z \in Z \tag{3.4}$$

for some $\mathbb{C}$-linear isomorphism $g : W \to W$ that is σ-preserving (i.e. real and invertible), and for some $\mathbb{C}$-linear isomorphism $h : V \to V$ that is σ-preserving and unitary (i.e. symplectic). Associate to a monad $A : W \times Z \to V$ the quadrilinear map

$$\begin{aligned} I_A : W \times W \times Z \times Z &\to \mathbb{C} \\ (v, w, r, s) &\mapsto \langle A(v, r), A(w, s)\rangle, \end{aligned} \tag{3.5}$$

and denote the symmetrization on the last two arguments by

$$I_A^+(v, w, r, s) \equiv I_A(v, w, r, s) + I_A(v, w, s, r). \tag{3.6}$$

An *ADHM monad* is a monad that satisfies the "ADHM conditions" of the fundamental classification theorem, stated below in a slightly nonstandard form.

Theorem 3.3. *(*[ADHM,At1]*) There is a bijective correspondence between gauge-equivalence classes of instantons and and linear-equivalence classes of monads $A : W \times Z \to V$ that satisfy the follwing "ADHM conditions":*

a) *The "reality condition": $\sigma\, A(w, z)) = A(\sigma w, \sigma z)$.*
b) *The "injectivity condition": $A(w, z) = 0$ only if $w = 0$ or $z = 0$.*
c) *The "isotropy condition": $I_A^+ = 0$.*

In the sequel we consider only $SU(2)$-equivariant monads with W and V as in proposition 3.2. The Clebsch-Gordan formula eq.(2.2) and Schur's lemma reduce such an $SU(2)$-equivariant monad to a set of complex-valued coefficients, defined as follows. The restriction of an $SU(2)$-equivariant bilinear map $A : W \times \mathcal{V}_3 \to V$ to $\mathcal{V}_{2l} \subset W$ is a linear combination of transvectants of the form

$$\begin{aligned} A_l : \mathcal{V}_{2l} \times \mathcal{V}_3 &\to V \\ (w, s) &\mapsto \sum_{p=0}^{\min(2l,3)} a_{l,p} \langle w, s\rangle_p, \end{aligned} \tag{3.7}$$

with complex coefficients $a_{l,p}$. We shall always define $a_{m-1,1} \equiv 0$ since $\mathcal{V}_{2m-1} \not\subset V$.

Theorem 3.4. *For an $SU(2)$-equivariant monad A with $C_2 = m(m+1)/2$, the ADHM conditions are equivalent to the following conditions on the coefficients $a_{l,p}$, where $l = m-1 \pmod 2$ and $0 \le l \le m-1$, and $0 \le p \le \min(2l,3)$.*

a) *The ADHM reality condition is equivalent to $a_{l,p} \in \mathbb{R}$ for all l and all p.*
b) *The ADHM injectivity condition is equivalent to $a_{l,0} \neq 0$ for all l.*
c) *The ADHM isotropy condition is equivalent to the following two conditions:*
   i) The "diagonal isotropy condition":

$$a_{l,2}^2 = \frac{(2l+1)}{(2l-1)^2} a_{l,0}^2 + \frac{2l(2l-3)}{(2l-1)^2} a_{l,1}^2 \qquad \text{for all } l \ge 1, \tag{3.8}$$

$$a_{l,3}^2 = \frac{(2l+2)(2l+5)}{(2l-1)^2} a_{l,0}^2 - \frac{9(2l+1)}{(2l-1)^2} a_{l,1}^2 \qquad \text{for all } l \ge 2, \tag{3.9}$$

   ii) The "off-diagonal isotropy condition":

$$a_{l,0} a_{l+2,2} = a_{l,1} a_{l+2,3} \qquad \text{for all } 1 \le l \le m-3. \tag{3.10}$$

Proof of 3.4 a) and b).

a) Immediate from the $\mathbb{C}$-antilinearity of σ and from the compatibility eq.(2.10) with transvectants .
b) The injectivity condition holds for $A : W \times \mathcal{V}_3 \to V$ iff it holds for each of the restrictions $A_l : \mathcal{V}_{2l} \times \mathcal{V}_3 \to V$. The injectivity condition holds for $A_l : \mathcal{V}_{2l} \times \mathcal{V}_3 \to V$ iff it holds for $a_{l,p}\langle\ ,\ \rangle_p : \mathcal{V}_{2l} \times \mathcal{V}_3 \to V$ for at least one $0 \le p \le \min(2l,3)$. The injectivity condition holds for $\langle\ ,\ \rangle_p : \mathcal{V}_{2l} \times \mathcal{V}_3 \to V$ iff $p = 0$, so we conclude $a_{l,0} \neq 0$ for each l. $\square$

To prove part c), we define quadrilinear maps

$$\begin{aligned} F_{l,k,p} : \mathcal{V}_{2l} \times \mathcal{V}_{2k} \times \mathcal{V}_3 \times \mathcal{V}_3 &\to \mathcal{V}_0 \\ (v,w,r,s) &\mapsto \langle \langle v,r\rangle_{p+l-\min(l,k)}, \langle w,s\rangle_{p+k-\min(l,k)} \rangle, \end{aligned} \tag{3.11}$$

and their symmetrization on the last two arguments

$$F^+_{l,k,p}(v,w,r,s) \equiv F_{l,k,p}(v,w,r,s) + F_{l,k,p}(v,w,s,r). \tag{3.12}$$

Lemma 3.5.

a) *The vector space $\mathcal{Y}_{l,k}$ of $SL(2,\mathbb{C})$-equivariant $\mathbb{C}$-quadrilinear maps*

$$\mathcal{V}_{2l} \times \mathcal{V}_{2k} \times \mathcal{V}_3 \times \mathcal{V}_3 \to \mathcal{V}_0 \tag{3.13}$$

   has basis $F_{l,k,p}$ with $0 \le p \le \min(l+k,3) - |l-k|$.
b) *If $l = k \pmod 2$, then the subspace $\mathcal{Y}^+_{l,k} \subset \mathcal{Y}_{l,k}$ of maps that are symmetric in the last two arguments has dimension:*
   i) $\min(l,2)$ *if $k = l$,*
   ii) 1 *if $|k-l| = 2$ and* $\min(k,l) \ge 1$,
   iii) 0 *in all other cases.* $\square$

Proof.

a) Immediate from the Clebsch-Gordan formula eq.(2.2).

b) By the symmetry property eq.(2.6), the space of symmetric equivariant bilinear maps $\mathcal{V}_3 \times \mathcal{V}_3 \to \mathcal{V}_0$ is isomorphic to $\mathcal{V}_4 \oplus \mathcal{V}_0$. The results now follow from the Clebsch-Gordan formula. $\square$

We see that the set of maps $F^+_{l,k,p}$ spans $\mathcal{Y}^+_{l,k}$, but that this set is not linearly independent in general.

Proposition 3.6. *If* $l = k \pmod 2$, *then the linear dependence of the* $F^+_{l,k,p}$ *in* $\mathcal{Y}^+_{l,k}$ *is generated by:*

i) *The "diagonal relations" if* $k = l$*:*

$$F^+_{l,l,0} = \begin{cases} -\dfrac{(2l+1)}{(2l-1)^2} F^+_{l,l,2} - \dfrac{(2l+2)(2l+5)}{(2l-1)^2} F^+_{l,l,3} & \text{if } l \geq 2; \\ -3\, F^+_{1,1,2} & \text{if } l = 1; \\ 0 & \text{if } l = 0, \end{cases} \tag{3.14}$$

$$F^+_{l,l,1} = \begin{cases} -\dfrac{2l(2l-3)}{(2l-1)^2} F^+_{l,l,2} + \dfrac{9(2l+1)}{(2l-1)^2} F^+_{l,l,3} & \text{if } l \geq 2; \\ 2\, F^+_{1,1,2} & \text{if } l = 1; \\ 0 & \text{if } l = 0. \end{cases} \tag{3.15}$$

ii) *The "off-diagonal relations" if* $|k-l| = 2$ *and* $\min(k,l) \geq 1$*:*

$$F^+_{l,k,0} = -F^+_{l,k,1}. \tag{3.16}$$

iii) *No nontrivial relations in all other cases.*

Proof. The number of linear dependence relations is the difference between the dimension of $\mathcal{Y}_{l,k}$ and the dimension of $\mathcal{Y}^+_{l,k}$, both of which are computed in lemma 3.5. To find a basis for the relations, we evaluate $F^+_{l,k,p}(v,w,r,s)$ for appropriate choices of arguments.

i) Substitute $w = x^a y^{2l-a}$ and $s = y^3$ into

$$\begin{aligned} F^+_{l,l,p}(\sigma\, w, w, \sigma\, s, s) &= \langle\langle \sigma\, w, \sigma\, s\rangle_p, \langle w, s\rangle_p\rangle + \langle\langle \sigma\, w, s\rangle_p, \langle w, \sigma\, s\rangle_p\rangle \\ &= \langle \sigma\, \langle w, s\rangle_p, \langle w, s\rangle_p\rangle - \langle \sigma \langle w, \sigma\, s\rangle_p, \langle w, \sigma\, s\rangle_p\rangle \\ &= \|\langle w, s\rangle_p\|^2 - \|\langle w, \sigma\, s\rangle_p\|^2, \\ &= \left\| \frac{1}{p!} \frac{\partial^p x^a}{\partial x^p} y^{2l-a} \frac{\partial^p y^3}{\partial y^p} \right\|^2 - \left\| \frac{1}{p!} x^a \frac{\partial^p y^{2l-a}}{\partial y^p} \frac{\partial^p x^3}{\partial x^p} \right\|^2. \end{aligned} \tag{3.17}$$

Evaluating this expression with eq.(2.14) for sufficiently many distinct values of a gives a linear system of equations. The diagonal relations form a basis for the space of solutions.

ii) The proof is similar, but even simpler. Since the dimension of the subspace is at most 1, it suffices to choose any argument for which $F^+_{l,l+2,0}$ does not vanish.

iii) $F^+_{l,k,p} = 0$ in all other cases since $\mathcal{Y}^+_{l,k}$ is zero-dimensional. $\square$

Proof of 3.4 c). The ADHM isotropy condition $I_A^+ = 0$ restricted to $\mathcal{V}_{2l} \times \mathcal{V}_{2k} \times \mathcal{V}_3 \times \mathcal{V}_3$ is equivalent to

$$\sum_{p=0}^{\min(l+k,3)-|l-k|} a_{l,p+l-\min(l,k)}\, a_{k,p+k-\min(l,k)}\, F^+_{l,k,p} = 0. \tag{3.18}$$

By the dimension count of lemma 3.5 b), there are only two nontrivial cases:
i) When $k = l$, eq.(3.18) reduces to

$$\sum_{p=0}^{\min(2l,3)} a^2_{l,p}\, F^+_{l,l,p} = 0. \tag{3.19}$$

This equation is equivalent, by proposition 3.6 i), to the diagonal isotropy condition of theorem 3.4 c)i).
i) When $|k - l| = 2$ and $\min(k, l) \geq 1$, then either $k = l + 2 \geq 3$ and eq.(3.18) reduces to

$$\sum_{p=0}^{1} a_{l,p} a_{l+2,p+2} F^+_{l,l+2,p} = 0, \tag{3.20}$$

or $l = k + 2 \geq 3$ and eq.(3.18) reduces to a similar equation. In either case the equation is equivalent, by proposition 3.6 ii), to the off-diagonal isotropy condition of theorem 3.4 c)ii). □

4. Solutions of the $SU(2)$-equivariant ADHM conditions

We now prove for each m the existence and uniqueness (modulo linear-equivalence) of $SU(2)$-equivariant ADHM monads. If k is an integer, we say that an $SU(2)$-equivariant monad is *k-canonical* if its coefficients $a_{l,p}$ satisfy the following condition:

$$l \geq k \Rightarrow\ a_{l,p} \text{ is real and non-negative for each } 1 \leq p \leq \min(2l, 3), \text{ and } a_{l,0} = 1. \tag{4.1}$$

Lemma 4.1. *For every C_2 of the form $m(m + 1)/2$, there is at most one $SU(2)$-equivariant ADHM monad that is 0-canonical.*

Proof. For a given m, suppse that $a_{l,p}$ and $\tilde{a}_{l,p}$ both describe equivariant 0-canonical ADHM monads. We shall show that $\tilde{a}_{l,p} = a_{l,p}$ for all l and p. If $m = 0$, the statement is trivial. Suppose $m \geq 1$. Because both are 0-canonical, $\tilde{a}_{m-1,0} = a_{m-1,0} \equiv 1$. If $m = 1$, we are done. Suppose $m \geq 2$. Since $\mathcal{V}_{2m-1} \not\in V$, we always have $\tilde{a}_{m-1,1} = \tilde{a}_{m-1,1} \equiv 0$. By the diagonal isotropy condition of theorem 3.4 c)i), $\tilde{a}^2_{m-1,2} = a^2_{m-1,2}$, so $\tilde{a}_{m-1,p} = a_{m-1,p}$ by non-negativity. If $m = 2$, we are done. Suppose $m \geq 3$. Again by the diagonal isotropy condition and non-negativity, $\tilde{a}_{m-1,3} = a_{m-1,3}$. If $m = 3$, $\tilde{a}_{0,0} = a_{0,0} \equiv 1$ and we are done.

Suppose $m \geq 4$. Let $0 \leq k \leq m - 3$ with $k = m - 1 \pmod 2$. We shall show that if $\tilde{a}_{k+2,p} = a_{k+2,p}$ for all p, then $\tilde{a}_{k,p} = a_{k,p}$ for all p. Induction on k then completes the proof of the lemma. If $k = 0$, only $p = 0$ occurs, and the statement is trivial because $\tilde{a}_{0,0} = a_{0,0} \equiv 1$. Assume $k \geq 1$. Here the key property of an equivariant ADHM monad is

$$a_{k,1} = \frac{a_{k,0}\, a_{k+2,2}}{a_{k+2,3}} \neq 0, \tag{4.2}$$

which follows immediately from the off-diagonal isotropy condition if $a_{k,1}a_{k+2,3} \neq 0$, or equivalently if $a_{k,0}a_{k+2,2} \neq 0$. By the injectivity condition, $a_{k,0} \neq 0$ and $a_{k+2,0} \neq 0$, so $a^2_{k+2,0} > 0$. Substituting the last inequality into the diagonal isotropy condition, we obtain $a^2_{k+2,2} > 0$, which implies $a_{k+2,2} \neq 0$, completing the verification of eq.(4.2). Since both monads are 0-canonical, eq.(4.2) implies $a_{k,1} = a_{k+2,2}/a_{k+2,3}$ and $\tilde{a}_{k,1} = \tilde{a}_{k+2,2}/\tilde{a}_{k+2,3}$. By assumption $\tilde{a}_{k+2,p} = a_{k+2,p}$ for all p, so $\tilde{a}_{k,1} = a_{k,1}$. Using this and $\tilde{a}_{k,0} = a_{k,0} \equiv 1$ in the the diagonal isotropy condition gives $\tilde{a}^2_{k,2} = a^2_{k,2}$, and also $\tilde{a}^2_{k,3} = a^2_{k,3}$ if $k \geq 2$. ¿From non-negativity we conclude that $\tilde{a}_{k,p} = a_{k,p}$ for all p. □

Suppose A and $\tilde{A}$ are linear-equivalent equivariant monads as in eq.(3.4). By Schur's lemma, the restriction of an equivariant real invertible map $g : W \to W$ to $\mathcal{V}_{2l} \subset W$ is the identity map multiplied by a nonzero real number γ_l. Similarly, the restriction of a symplectic map $h : V \to V$ to $\mathcal{V}_j \subset V$ is the identity map multiplied by $\kappa_j = \pm 1$. The coefficients $\tilde{a}_{l,p}$ of the equivariant monad $\tilde{A}$ are then related to the coefficients $a_{l,p}$ of the equivariant monad A by

$$\tilde{a}_{l,p} = \kappa_{2l+3-2p}\, \gamma_l\, a_{l,p}. \tag{4.3}$$

Lemma 4.2. *Every $SU(2)$-equivariant ADHM monad is linear-equivalent to an $SU(2)$-equivariant ADHM monad that is 0-canonical.*

Proof. Suppose $a_{l,p}$ describe an equivariant ADHM monad. If $m = 0$, the statement is trivial. Suppose $m \geq 1$. Letting $\kappa_{2m+1} = 1$ and $\gamma_{m-1} = 1/a_{m-1,0}$, we obtain $\tilde{a}_{m-1,0} = 1$. If $m = 1$ we are done. Suppose $m \geq 2$. As always, $\tilde{a}_{m-1,1} \equiv 0$. Choose κ_{2m-3} so that $\tilde{a}_{m-1,2} = \kappa_{2m-3}\gamma_{m-1}a_{m-1,2} \geq 0$. If $m = 2$, we are done. Suppose that $m \geq 3$. Choose κ_{2m-5} so that $\tilde{a}_{m-1,3} = \kappa_{2m-5}\gamma_{m-1}a_{m-1,3} \geq 0$. We have now constructed an $(m-1)$-canonical equivariant monad $\tilde{A}$ that is linear-equivalent to A. Letting $\gamma_{m-3} = \kappa_{2m-3}/a_{m-3,0}$, we obtain $\tilde{a}_{m-3,0} = 1$. If $m = 3$, we are done.

Suppose $m \geq 4$. Let $0 \leq k \leq m-3$ with $k = m-1 \pmod 2$. Given a $(k+2)$-canonical equivariant ADHM monad A, we shall construct a linear-equivalent k-canonical monad $\tilde{A}$. Induction on k then completes the proof of the lemma. Let $\kappa_j = 1$ for $j \geq 2k+1$. Let $\gamma_k = 1/a_{k,0}$, and $\gamma_l = 1$ for $l > k$. Then $\tilde{a}_{k,0} = 1$. If $k = 0$, we are done. Assume that $k \geq 1$. Since A is $(k+2)$-canonical $a_{k+2,2}$ and $a_{k+2,3}$ are positive, and eq.(4.2) implies $a_{k,1}/a_{k,0} > 0$, proving the positivity of $\tilde{a}_{k,1} = \gamma_k a_{k,1} = a_{k,1}/a_{k,0}$. Choose κ_{2k-1} so that $\tilde{a}_{k,2} \geq 0$. If $k = 1$, we are done. Assume $k \geq 2$. Choose κ_{2k-3} so that $\tilde{a}_{k,3} \geq 0$. $\tilde{A}$ is k-canonical and linear-equivalent to A. □

The uniqueness statement of theorem 1.1 follows:

Proposition 4.3. *For every bundle with second Chern number of the form $C_2 = m(m+1)/2$, there is at most one $SU(2)$-equivariant instanton.*

Proof. For a given m, suppose A and B are equivariant ADHM monads. By lemma 4.2, there exists a 0-canonical equivariant ADHM monad $\tilde{A}$ that is linear-equivalent to A, and a 0-canonical equivariant ADHM monad $\tilde{B}$ that is linear-equivalent to B. By lemma 4.1, $\tilde{A} = \tilde{B}$. Therefore A and B are linear-equivalent, and correspond to the same instanton by theorem 3.3. □

The existence statement of theorem 1.1 follows from the following explicit non-recursive formula.

Proposition 4.4. *For every C_2 of the form $m(m+1)/2$, the $SU(2)$-equivariant monad A with the coefficients*

$$a_{l,p} = \sqrt{(2m+1)^2\left((2\,l+1)\;p^2 - (6\,l-1)\;p + 6\,l - 3\right)^2 - (2\,l-1)^2\,(2\,l+3)^2}, \tag{4.4}$$

(where $0 \le l \le m-1$ with $l = m-1 \pmod 2$, and $0 \le p \le \min(2l, 3)$) satisfies the ADHM conditions.

Proof. The proof consits of substituting the coefficients $a_{l,p}$ into the equivariant ADHM conditions of theorem 3.4. One may check that $a_{m-1,1} = 0$. The inequality $l \le m-1$ is equivalent to $(2l+3)^2 \le (2m+1)^2$, and the reality and non-negativity of the $a_{l,p}$ are immediate from:

$$\begin{aligned} a_{l,0}^2 &= (2l-1)^2\left(9(2m+1)^2 - (2l+3)^2\right) > 0 \\ a_{l,1}^2 &= (2l-1)^2\left((2m+1)^2 - (2l+3)^2\right) \ge 0 \\ a_{l,2}^2 &= (2l+3)^2\left((2m+1)^2 - (2l-1)^2\right) > 0 \\ a_{l,3}^2 &= (2l+3)^2\left(9(2m+1)^2 - (2l-1)^2\right) > 0 \end{aligned} \tag{4.5}$$

The first inequality of eq.(4.5) implies the injectivity condition $a_{l,0} \neq 0$. Verification of the diagonal and-off diagonal isotropy conditions requires a simple calculation. □

Note that the equivariant ADHM monad of proposition 4.4 is *not* 0-canonical, but lemma 4.2 lets us find the unique 0-canonical ADHM monad for each m. We exhibit the first few examples.

The first equivariant monad, with $m = 0$ and $(n_+, n_-) = (1, 1)$, is trivial, corresponding to the flat connection on the trivial bundle $C_2 = 0$. $W = \{0\}$, so A is the zero map, and there are no coefficients $a_{l,p}$ to determine.

The equivariant monad with $m = 1$, and $(n_+, n_-) = (3, 1)$, corresponds to the standard BPST (anti)-instanton [BPST] on the bundle with $C_2 = 1$. The monad is described by the single coefficient $a_{0,0} = 1$. Neither the diagonal nor the off-diagonal isotropy condition enters.

The equivariant monad with $m = 2$, and $(n_+, n_-) = (5, 1)$, has $C_2 = 3$, and has coefficients:
$a_{1,0} = 1, \quad a_{1,1} = 0, \qquad a_{1,2} = \sqrt{3}.$
The diagonal isotropy condition enters, but the off-diagonal does not.

The equivariant monad with $m = 3$, and $(n_+, n_-) = (7, 1)$, has $C_2 = 6$, and has coefficients:
$a_{2,0} = 1, \quad a_{2,1} = 0, \qquad a_{2,2} = \sqrt{5/9}, \qquad a_{2,3} = \sqrt{6},$
$a_{0,0} = 1.$

Both the diagonal and off-diagonal isotropy conditions enter in this, and all following monads.

The equivariant monad with $m=4$, and $(n_+,n_-)=(9,1)$ has $C_2=10$, and has coefficients:
$a_{3,0}=1$, $a_{3,1}=0$, $a_{3,2}=\sqrt{7/25}$, $a_{3,3}=\sqrt{88/25}$,
$a_{1,0}=1$, $a_{1,1}=\sqrt{7/88}$, $a_{1,2}=\sqrt{125/44}$.

The equivariant monad with $m=5$, and $(n_+,n_-)=(11,1)$, has $C_2=15$, and has coefficients:
$a_{4,0}=1$, $a_{4,1}=0$, $a_{4,2}=\sqrt{9/49}$, $a_{4,3}=\sqrt{130/49}$,
$a_{2,0}=1$, $a_{2,1}=\sqrt{9/130}$, $a_{2,2}=\sqrt{343/585}$, $a_{2,3}=\sqrt{147/26}$,
$a_{0,0}=1$.

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