Quantum Field Theory and Modular Structures

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Abstract:
A survey of the use of the Tomita–Takesaki modular theory in the theory of local observables will be given. This contains, in particular, the characterization of chiral field theories in terms of the concept of modular inclusions. Moreover, it contains the reconstruction of the translation–goup fulfilling spectrum condition out of the modular conjugations of the algebras associated with wedge–domains. Finally examples will be given wich show that neither duality nor wedge–duality are consequences of the standard axioms. These examples are constructed by using the modular theory.

Contents:

1. Historical remarks

At the Baton Rouge conference 1967 Tomita [To] distributed a preprint containing his theory on the standard form of von Neumann algebras. At the same time Haag, Hugenholtz und Winnink [HHW] published their paper on the description of thermodynamic equilibrium states using the KMS-condition. Probably N. Hugenholtz and M. Winnink have been the first realizing the similarity between certain aspects of their approach and Tomita's theory and hence the importance of this new mathematical theory for theoretical physics. (See e.g. the thesis of M. Winnink [Win].) But general knowledge became Tomita's theory only by Takesaki's [Ta] treatment published in the Lecture Notes in Mathematics. Since then this theory is usually called the Tomita-Takesaki theory.

A central role in this theory is played by faithful normal states of von Neumann algebras. As a consequence of the Reeh-Schlieder Theorems [RS] we know that the vacuum-state has this property for every local algebra in quantum field theory. Therefore, several people hoped that the Tomita-Takesaki theory could be made a useful tool also for quantum field theory. It seems as if these wishes will become true only nowadays. The long delay is due to the fact that the modular group has only an abstract definition and therefore its geometric meaning (if there is any) is not obvious.

The first application of the Tomita-Takesaki theory has been in thermodynamics because of the KMS boundary condition. In the usual quantum field theory the Tomita-Takesaki theory appeared first in connection with Wightman fields. In 1975 Bisognano and Wichmann [BW1,2] discovered that in a Lorentz–covariant Wightman theory the modular group of the algebra connected with a wedge domain coincides with the Lorentz boosts which leave this wedge invariant. This shows that in certain cases the modular group has a geometric meaning. A similar result has been obtained by Hislop and Longo [HL] in 1982 proving that in a conformal Wightman field theory the modular group of the algebra, associated to a double cone, is again a geometric transformation group. Knowing the geometric structure of the modular group usually leeds to the possibility of answering questions of physical interest. In the case of Bisognano and Wichmann it was the wedge duality which could be proved by using the new techniques.

Another attempt to use the Tomita-Takesaki theory is based on Connes' [Co] classification of type III factors using the essential spectrum of the modular operator. Informations about the types of different von Neumann algebras can be obtained if one succeeds in getting relations between the modular operators of the different von Neumann algebras. This has been used by Fredenhagen [Fre] in 1985 in order to obtain informations about the algebra of the double cone from the known structure of the wedge-algebra. That the same method can be used for obtaining relations between the types of von Neumann algebras in different sectors has been shown by Borchers and Wollenberg [BW].

The reason for hoping that the modular group acts geometrically is the following: In the theory of local observables there are only global dynamics. Unfortunately something is missing that can be used to define local dynamics. Probably one can take the modular groups for this purpose. These groups have to fulfill certain requirements in order to schow that this is correct. If we fix a small double cone and embed it into a larger domain then the action of the modular group of the large domain restricted to elements of the algebra of the double cone shall be close to the action of the time translation. This means if we do a proper rescaling of the modular group of the large domain and if we go with the large domain to infinite then the rescaled modular dynamics shall locally converge to the real dynamics. In the two explicitly known cases, the Bisognano-Wichmann and the Hislop-Longo situation, this is true, as one can easily check.

If one has local dynamics then it might be possible to define also local Gibbs states and to look at thermodynamical limits of such states. This would serve as the missing link between the abstractly defined KMS states and the local dynamics defined in the vacuum representation. This program has been carried through by Buchholz and Junglas [BJ] but with some extra condition and different local dynamics. The new input they need is the nuclearity condition which allows to define local dynamics in a different way. However, that the hamiltonian dynamics and the modular dynamics can not be too different is implied by the coincidence of the nuclearity condition for the hamiltonian and the local modular operator. This has been discovered by Buchholz, D'Antoni and Longo [BDL1,2].

My interest in the modular group originates in the observation that the modular group $\Delta^{\imath t}$, applied to expressions of the form $A\Omega$ where A is an element of the algebra (or its commutant), permits some analytic continuation. This together with the spectrum condition leeds to matrix elements of products of operators analytic not only in the translations but also in the modular action. It is known from many experiences in field theory that such analyticity properties can give rise to drastic restrictions. The first result was the observation that looking at functions in space and modular variables one finds expressions which have some periodicity property in a complex direction, [Bch1] 1990. Mostly this is of little use since in general the domain of holomorphy does not contain any line in the direction of periodicity. If there is enough analyticity then interesting conclusions can be drawn. Such a case has been found by the author in 1992 [Bch2]. It implies that in two-dimensional quantum field theory the modular group of the wedge can be interpreted as a Lorentz transformation. This means the translations together with the modular group of the wedge give rise to a representation of the two-dimensional Poincaré group. In addition there is a localization which implies that a Poincaré covariant local net has been constructed. One can start from the right or the left wedge. In general the answer will be different. Only in a theory satisfying wedge-duality the two group representations coincide. If the two representations are different then also the localizations are different. This means if an algebra is localized in one scheme it is not localized in the other scheme.

The use of the modular theory for two-dimensional field theory has been further developped in 1992 by Wiesbrock [Wie1,2,3]. He observed that there are subalgebras of the wedge-algebra which are mapped into itself by a part of the modular group of the wedge. Using this information he was able to reconstruct the translation group which necessarily fulfills the spectrum condition. Moreover, this information can be used to give an algebraic characterization of chiral field theories.

Together with the modular structure there is a modular conjugation. It acts like the CPT operator but only in two dimensions. In higher dimensions we only know this for the Bisognano–Wichmann situation. On the other hand every Poincaré transformation can be composed out of reflections. Buchholz and Summers [BSu] used this to reconstruct the translations out of these reflections. This can be done if certain additional requirements are fulfilled. These translations will fulfill the spectrum condition. The original requirements of Buchholz and Summers can be weakened as shown in a recent paper [Bch3].

2. Preliminaries

In this section we want to collect the notations and the background of this representation. All results will be cited without proofs.

A) *Tomita-Takesaki theory*

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a von Nueumann algebra acting on this space with commutant $\mathcal{M}'$. A vector Ω is cyclic and separating for $\mathcal{M}$ if $\mathcal{M}\Omega$ and $\mathcal{M}'\Omega$ are dense in $\mathcal{H}$. If these conditions are fulfilled then a modular operator Δ and a modular conjugation J is associated to the pair $(\mathcal{M}, \Omega)$ such that
(i) Δ is self-adjoint, positive and invertable

$$\Delta\Omega = \Omega, \qquad J\Omega = \Omega.$$

(ii) The unitary group $\Delta^{\imath t}$ defines a group of automorphisms of $\mathcal{M}$

$$\mathrm{ad}\,\Delta^{\imath t}\mathcal{M} = \mathcal{M} \qquad \forall t \in \mathbb{R}.$$

(iii) For every $A \in \mathcal{M}$ the vector $A\Omega$ belongs to the domain of $\Delta^{\frac{1}{2}}$.
(iv) The operator J is a conjugation, i.e. J is anti–linear and $J^2 = 1$, where J commutes with $\Delta^{\imath t}$. This implies the relation

$$\mathrm{ad}\,J\Delta = \Delta^{-1}.$$

(v) J maps $\mathcal{M}$ onto its commutant

$$\mathrm{ad}\,J\mathcal{M} = \mathcal{M}'.$$

(vi) The operators $S := J\Delta^{\frac{1}{2}}$ and $S^* = J\Delta^{-\frac{1}{2}}$ have the property

$$\begin{aligned} SA\Omega &= A^*\Omega \qquad \forall A \in \mathcal{M}, \\ S^*A'\Omega &= A^{'*}\Omega \qquad \forall A' \in \mathcal{M}'. \end{aligned}$$

For the proof see Takesaki [Ta] or textbooks as Bratteli and Robinson [BR] or Kadison and Ringrose [KR].

B) *The theory of local observables*

In the theory of local observables one associates to every bounded open region O in Minkowski space $\mathbb{R}^d$ a C^*-algebra $\mathcal{A}(O)$. For any unbounded open set G the C^*-algebra is defined as the C^* inductive limit of the $\mathcal{A}(O)$ with $O \subset G$. These algebras are subject to the following conditions:
(i) They fulfill isotony i.e., if $O_1 \subset O_2$ then $\mathcal{A}(O_1) \subset \mathcal{A}(O_2)$.
(ii) They fulfill locality, i.e. if O_1 and O_2 are spacelike separated regions then the corresponding algebras commute, i.e.

$$A \in \mathcal{A}(0_1),\ B \in \mathcal{A}(O_2) \quad \text{implies} \quad [A, B] = 0.$$

(iii) They fulfill translational covariance, i.e. the translation group of $\mathbb{R}^d$ acts as automorphisms on $\mathcal{M}(\mathbb{R}^d)$. For every $a \in \mathbb{R}^d$ there exists an automorphism $\alpha_a \in \mathrm{Aut}\,\mathcal{M}$ with

$$\alpha_a \mathcal{A}(O) = \mathcal{A}(O + a).$$

A representation π of $\mathcal{A}(\mathbb{R}^d)$ is called a particle representation if
(i) π is a non–degenerate representation on a Hilbert space $\mathcal{H}$.
(ii) $U(a)$ is a continuous representation of the translation group such that
(α) the spectrum of $U(a)$ is contained in the forward light cone,
(β) the representation $U(a)$ implements the automorphism α_a, which means for every $A \in \mathcal{A}(\mathbb{R}^d)$ one has

$$\mathrm{ad}\, U(a)\pi(A) = \pi(\alpha_a A).$$

(iii) A representation π is called a vacuum representation if
(α) π is a particle representation,
(β) in $\mathcal{H}$ exists a vector Ω with

$$U(a)\Omega = \Omega \qquad \forall a \in \mathbb{R}^d.$$

In the following we will always deal with vacuum representations and we set

$$\mathcal{M}(O) = \pi(\mathcal{A}(O))''.$$

For more details about the theory of local observales see the book of R. Haag [Ha].

C) *The result of Bisognano and Wichmann*

The first calculation of a modular group in quantum field theory is due to Bisognano and Wichmann [BW1,2]. This is done again in Wightman field theory. The assumptions and notations are the same as described by the CPT-theorem. The product and sums of field operators will be denoted by $\Phi(\underline{f})$, which includes summation over the indices. We ascertain that $\underline{f}$ has support in G if all components have their support in G with respect to every variable.

The domain which is of special importance is the *wedge*. Such a domain can be characterized in two ways:
(i) First characterization: Let t, s be two perpendicular vectors in $\mathbb{R}^d$. i.e. $(t, s) = 0$, such that $t^2 = 1$ and t belongs to the forward light cone and $s^2 = -1$ is spacelike. In this situation one defines

$$W(t, s) := \{a \in \mathbb{R}^d; |(a, t)| < (a, s)\}.$$

If, for instance, t is the time direction and s is the 1-direction then this becomes $W_R = \{a; |a_0| < a_1\}$.
(ii) Second characterization: Every two-plane containing a timelike direction must cut the

boundary of the forward light cone in two light rays. Let these light rays be described by the two lightlike vectors ℓ_1, ℓ_2. These vectors are different. Now define:

$$W(\ell_1, \ell_2) := \{\lambda_1 \ell_1 - \lambda_2 \ell_2 + \tilde{a}; \lambda_1, \lambda_i > 0, (\tilde{a}, \ell_i) = 0, i = 1, 2\}.$$

It is easy to see that the two definitions result in the same set of wedges. The two definitions coincide if $\{t, s\}$ and $\{\ell_1, \ell_2\}$ span the same two-plane and if $s = \lambda_1 \ell_1 - \lambda_2 \ell_2$ with positive coefficients.

The opposite wedge of a wedge W is the negative of W. It is obtained by replacing s by $-s$ in the first description and by interchanging the two lightlike vectors in the second description.

Given a wedge W there is exactly a one-parametric subgroup of the Lorentz group which maps this wedge onto itself. In the above example of the zero– and one–direction the Lorentz transformations are the boosts in the (0,1)-plane. We will write these transformations (in case the wedge is the right–wedge in the (0,1)-plane) as

$$\Lambda(t) = \begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t \\ -\sinh 2\pi t & \cosh 2\pi t \end{pmatrix}.$$

Bisognano and Wichmann showed the following results:
(i) If the support of $\underline{f}$ is contained in W_R then

$$U(\Lambda(t))\Phi(\underline{f})\Omega$$

allows in t an analytic continuation into the strip $-\frac{1}{2} < \Im m\, \tau < 0$.
(ii) Let R be the proper rotation such that RP is the reflection in the (0,1)-plane where P is the reflection defined by the CPT-operator. Define $J = \Theta U(R)$ then

$$JU(\Lambda(-\frac{\imath}{2}))\Phi(\underline{f})\Omega = \Phi(\underline{f})^*\Omega.$$

(iii) If the algebras generated by the field operators are affiliated with a local net of von Neumann algebras $\mathcal{M}(O)$ then one has wedge-duality which means

$$\mathcal{M}(W_R)' = \mathcal{M}(W_L).$$

We learn from these results that one can interprete $U(\Lambda(t))$ as the modular group of the wedge algebra and that J can be interpreted as its modular conjugation.

D) *The result of Hislop and Longo*

Because of the spectrum condition the forward tube

$$T^+ := \{z = x + \imath y \in \mathbb{C}^d; y \in V^+\},$$

where V^+ denotes the forward light cone, is of great importance in quantum field theory. The invariance group of T^+ consists of Lorentz transformations, dilatations and in addition the map

$$x \longrightarrow -\frac{x}{x^2}.$$

The group generated by these elements is the conformal group. This implies that there are field theories with this group as symmetry group.

The last transformation $-\frac{x}{x^2}$ has the property that it maps a shifted wedge onto a double cone. Therefore, in a theory with this invariance property one can calculate the action of the modular group from the known results about the wedge. In this case the modular group acts again geometric and one finds:

$$\Delta^{\imath t} = U(g(t))$$

where for the double cone

$$|x_0| + r < 1, \qquad \text{with} \qquad r^2 = \vec{x}^2$$

$g(t)$ is the following transformation:

$$g(t) : \xi_\pm \longrightarrow \frac{(1+\xi_\pm) - e^{2\pi t}(1-\xi_\pm)}{(1+\xi_\pm) + e^{2\pi t}(1-\xi_\pm)}.$$

In this formula $\xi_\pm$ means $x_0 \pm r$. For more details see Hislop and Longo [HL].

E) *Remarks on the edge of the wedge problem*

We will have to deal with the edge of the wedge the problem in two variables. Let us assume that we have the two tubes based on the first and third quadrant. If the coincidence domain is also the first quadrant then the envelope of holomorphy can easily be computed by mapping the upper half-plane of each variable biholomorphicly onto the strip $0 < \Im m\, w < \pi$. Now one can use the tube theorem.

This calculation implies the following property about real lines: If one has a linear manifold which is real for real values of z_1, z_2 and if this real line intersects the first quadrant then all its non–real points belong to the envelope of holomorphy computed before.

If the coincidence domain is the half–space $ax_1 + bx_2 > 0$, $a, b \geq 0$ then the envelope of holomorphy consists of $\mathbb{C}^2$, except for a cut in the variable $az_1 + bz_2$ i.e.$\mathbb{C}^2 \setminus \{az_1 + bz_2 \in \mathbb{R}^-\}$.

3. The fundamental relations

Large parts of modern investigations concerning modular groups are based on the following results.

Theorem A:

Let $\mathcal{M}, \mathcal{N}$ be two von Neumann algebras with the common cyclic and separating vector Ω and denote the modular operators and conjugations by $\Delta_{\mathcal{M}}, J_{\mathcal{M}}$ and $\Delta_{\mathcal{N}}, J_{\mathcal{N}}$, respectively. Let $V \in \mathcal{B}(\mathcal{H})$ be a unitary operator with
(i) $V\Omega = \Omega$ and
(ii) $\mathrm{ad}\, V\mathcal{N} \subset \mathcal{M}$
then the function $V(t) := \Delta_{\mathcal{M}}^{-\imath t} V \Delta_{\mathcal{N}}^{\imath t}$ has the properties
(a) $V(t)$ is $*$–strong continuous in $t \in \mathbb{R}$.
(b) $V(t)$ possesses an analytic extension into the strip $S(0, \frac{1}{2})$ as holomorphic function with values in the normed space $\mathcal{B}(\mathcal{H})$.
(c) In this strip we have the estimate

$$\|V(\tau)\| \leq 1$$

(d) $V(\tau)$ has boundary values at $\Im m\, \tau = 0$ and at $\Im m\, \tau = \frac{1}{2}$ in the $*$–strong topology.
(e) On the upper boundary the value is given by

$$V(t + \imath \frac{1}{2}) = J_{\mathcal{M}} V(t) J_{\mathcal{N}},$$

hence by (a) also this function is $*$–strong continuous in t.

3.1 Remark:

The functions $V(t)$ fulfill the following chain rule: If $U\mathcal{P}U^* \subset \mathcal{N}$ and $V\mathcal{N}V^* \subset \mathcal{M}$ then $W = VU$ maps $\mathcal{P}$ into $\mathcal{M}$ and one finds

$$W(t) = V(t)U(t).$$

Moreover, with $\mathcal{N}' \supset V^*\mathcal{M}'V$ one obtains

$$V^*(t) = V(-t)^*.$$

Notice that the function $V(\bar{z})^*$ is again an analytic function holomorphic in $S(-\frac{1}{2}, 0)$. Therefore, the last relation reads in the complex

$$V^*(z) = V(-\bar{z})^*.$$

Theorem B:

Let $\mathcal{M}, \mathcal{N}$ be two von Neumann algebras with the common cyclic and separating vector Ω. Let $W(s) \in \mathcal{B}(\mathcal{H})$ be an operator family fulfilling the following requirements with respect to the triple $(\mathcal{M}, \mathcal{N}, \Omega)$.
(i) For $s \in \mathbb{R}$ the operators $W(s)$ are unitary and strongly continuous and fulfill $W(s)\Omega = \Omega$.
(ii) The function $W(s)$ possesses an analytic continuation into the strip $S(0, \frac{1}{2})$ and has continuous boundary values.

(iii) The operators $W(\frac{\imath}{2}+t)$ *are again unitary.*
(iv) The function $W(\sigma)$ *is bounded, hence* $\|W(\sigma)\| \leq 1$.
(v) For $t \in \mathbb{R}$ *one has* $W(t)\mathcal{N}W(t)^* \subset \mathcal{M}$ *and* $W(\frac{\imath}{2}+t)\mathcal{N}'W(\frac{\imath}{2}+t)^* \subset \mathcal{M}'$.
In this situation the modular operator and the transformations $W(s)$ *fulfill the following transformation rules:*

$$\Delta^{\imath t}_{\mathcal{M}} W(s) \Delta^{-\imath t}_{\mathcal{N}} = W(s-t),$$
$$J_{\mathcal{M}} W(s) J_{\mathcal{N}} = W(\frac{\imath}{2}+s).$$

Special versions of Theorem A can be found in [Bch2] and in [BDL1] and of Theorem B in [Bch2] and [Wie2].

Proof of Theorem A: The continuity properties are shown by standard methods. The interesting parts are the analyticity properties. Take $A' \in \mathcal{N}'$. We consider the vector

$$\psi(t,s) := \Delta^{-\imath t}_{\mathcal{M}} V \Delta^{\imath s}_{\mathcal{N}} A'\Omega$$

and look at possible analytic continuations. Since $\Delta^{\imath s}_{\mathcal{N}}$ is the modular group for $\mathcal{N}$ we can analytically continue $\psi(t,s)$ in the variable s into the strip $S(0,\frac{1}{2})$. This continuation has continuous boundary values and we find

$$\psi(t, s+\imath\frac{1}{2}) = \Delta^{-\imath t}_{\mathcal{M}} V j_{\mathcal{N}}(\sigma^s_{\mathcal{N}}(A'^*))\Omega.$$

The operator $j_{\mathcal{N}}(\sigma^s_{\mathcal{N}}(A'^*))$ belongs to the algabra $\mathcal{N}$. Therefore $V j_{\mathcal{N}}(\sigma^s_{\mathcal{N}}(A'^*))V^*$ belongs to $\mathcal{M}$. Now we can continue the vector $\psi(t, s+\imath\frac{1}{2})$ in the variable t into the strip $S(0,\frac{1}{2})$. Again we obtain continuous boundary values and find

$$\psi(t+\imath\frac{1}{2}, s+\imath\frac{1}{2}) = j_{\mathcal{M}}(\sigma^{-t}_{\mathcal{M}}(V j_{\mathcal{N}}(\sigma^s_{\mathcal{N}}(A'))V^*))\Omega = \Delta^{-\imath t}_{\mathcal{M}} J_{\mathcal{M}} V J_{\mathcal{N}} \Delta^{\imath s}_{\mathcal{N}} A'\Omega.$$

Notice that the operator $\Delta^{-\imath t}_{\mathcal{M}} J_{\mathcal{M}} V J_{\mathcal{N}} \Delta^{\imath s}_{\mathcal{N}}$ is unitary, so that we obtain

$$\|\psi(t+\imath\frac{1}{2}, s+\imath\frac{1}{2})\| = \|A'\Omega\|.$$

Now using the Malgrange-Zerner-Theorem (see e.g Epstein [Ep]) we see that $\psi(t,s)$ has an analytic continuation into the tube based on the triangle defined by the points

$$\Im m\,(t,s) = \{(0,0), \quad (0,\frac{1}{2}), \quad (\frac{1}{2},\frac{1}{2})\}.$$

The strip $S(0,\frac{1}{2})$ of the complex manifold $t=s$ belongs to the boundary of the above tube. Therefore, we have to prove that the function stays analytic in the remaining variable. To this end we define $\phi(x,y) = \psi(x+y, x-y)$. In this variables we have analyticity in the tube based on the triangle with the corners

$$\Im m\,(x,y) = \{(0,0), \quad (\frac{1}{4},-\frac{1}{4}), \quad (\frac{1}{2},0)\}.$$

The functions $\phi(x,y)$ are for fixed y analytic in strip $S(-\Im m\, y, \frac{1}{2} + \Im m\, y)$. We choose $\Re e\, y = 0$ and with $\Im m\, y$ we want to reach 0. For fixed $\eta < 0$ the function $\phi(x, \imath\eta)$ has continuous boundary values. Since the strip can be transformed bi–holomorphically onto the unit circle it follows that $\phi(x, \imath\eta)$ can be expressed with a transformed Cauchy kernel as an integral over the boundary values. This kernel is $\mathcal{L}^1$ on the boundary and depends real–analytically on η in a neighbourhood of zero. On the boundary we are only dealing with one modular group so that we know that the boundary values are continuous also for $\eta \to 0$. Hence the "Cauchy" integral converges to a function which is analytic in the variable t. The function $\psi(\tau, \tau)$ is analytic in the strip $S(0, \frac{1}{2})$. On the lower and upper boundary we find $\|\psi(t,t)\| = \|\psi(t + \imath\frac{1}{2}, t + \imath\frac{1}{2})\| = \|A'\Omega\|$. This implies by the maximum modulus theorem $\|V(\tau)A'\Omega\| \leq \|A'\Omega\|$. This shows that $V(\tau)$ is bounded in norm by 1 on $\mathcal{M}'\Omega$. Since this set of vectors is dense in $\mathcal{H}$ it can be continuously extended to a bounded operator with the same norm. The value of $V(\tau)$ at the upper boundary follows from the above remark. If τ is an interior point of $S(0, \frac{1}{2})$ with distance d from the boundary and if we make a power series expansion around this point then we find that the n-th coefficient is a bounded operator with norm at most d^{-n}. Therfore, the power series is converging in the norm topology. □

The proof of Theorem B will be splitted into two parts. We start with the following result:

3.2 Proposition:

Let $W(s)$ fulfill the requirements listed in Theorem B. Then the operator function

$$W(s,t) := \Delta_{\mathcal{M}}^{-\imath t} W(s) \Delta_{\mathcal{N}}^{\imath t}$$

has an analytic continuation into the tube domain based on the quadrangle with the four corners

$$\Im m\,(\sigma, \tau) = \{(0,0), \quad (0, \frac{1}{2}), \quad (\frac{1}{2}, 0), \quad (-\frac{1}{2}, \frac{1}{2})\}.$$

In this domain we have the estimate

$$\|W(\sigma, \tau)\| \leq 1.$$

At the four corners it has the values

$$\begin{aligned} W(s,t) &= \Delta_{\mathcal{M}}^{-\imath t} W(s) \Delta_{\mathcal{N}}^{\imath t} \\ W(s, \frac{\imath}{2} + t) &= \Delta_{\mathcal{M}}^{-\imath t} J_{\mathcal{M}} W(s) J_{\mathcal{N}} \Delta_{\mathcal{N}}^{\imath t} \\ W(\frac{\imath}{2} + s, t) &= \Delta_{\mathcal{M}}^{-\imath t} W(\frac{\imath}{2} + s) \Delta_{\mathcal{N}}^{\imath t} \\ W(\frac{\imath}{2} + s, \frac{\imath}{2} + t) &= \Delta_{\mathcal{M}}^{-\imath t} J_{\mathcal{M}} W(\frac{\imath}{2} + s) J_{\mathcal{N}} \Delta_{\mathcal{N}}^{\imath t} \end{aligned}$$

Proof: This result is a consequence of the last theorem together with the Malgrange-Zerner theorem. First we can continue for real s in the variable t into the strip $S(0, \frac{1}{2})$

and for real t in the variable s into the strip $S(0, \frac{\imath}{2})$. For $\tau = \frac{\imath}{2} + t$ we have the expression $W(s, \frac{\imath}{2} + t) = \Delta_{\mathcal{M}}^{-\imath t} J_{\mathcal{M}} W(s) J_{\mathcal{N}} \Delta_{\mathcal{N}}^{\imath t}$ which has an analytic continuation in s into the strip $S(-\frac{\imath}{2}, 0)$. Since one has to take into consideration that J is an antilinear opertor it follows that the function $J_{\mathcal{M}} W(\overline{\sigma}) J_{\mathcal{N}}$ is analytic. The values at the corners are obtained by simple computations. □

Proof of Theorem B: We choose two operators $A \in \mathcal{N}$ and $B \in \mathcal{M}'$ and introduce two functions

$$\begin{aligned} F^+(s,t) &:= (\Omega, BW(s,t)A\Omega), \\ F^-(s,t) &:= (\Omega, AW(s,t)^* B\Omega). \end{aligned}$$

It is clear that the operator function $W(s,t)^*$ is analytic in the conjugate complex of the domain of analyticity for $W(s,t)$. Now we look at the points where the two functions coincide. We have the following identities:

$$\begin{aligned} F^+(s,t) &= F^-(s,t) \quad s,t \in \mathbb{R}, \\ F^+(-\frac{\imath}{2} + s, \frac{\imath}{2} + t) &= F^-(\frac{\imath}{2} + s, -\frac{\imath}{2} + t) \quad s,t \in \mathbb{R}. \end{aligned}$$

Since $A \in \mathcal{N}$ one has $W(s,t)AW(s,t)^* \in \mathcal{M}$. This implies the first statement. Next the operator $\Delta_{\mathcal{M}}^{-\imath t} J_{\mathcal{M}} W(\frac{\imath}{2} + s) J_{\mathcal{N}} \Delta_{\mathcal{N}}^{\imath t} A \{\Delta_{\mathcal{N}}^{-\imath t} J_{\mathcal{N}} W(\frac{\imath}{2} + s) J_{\mathcal{M}} \Delta_{\mathcal{M}}^{\imath t}\}^*$ belongs again to $\mathcal{M}$ since J interchanges the algebra and its commutant and $W(\frac{\imath}{2} + s)$ maps the commutant of $\mathcal{N}$ into the commutant of $\mathcal{M}$. From this follows the second statement. As a consequence of this result we see that the two functions are two different representations of one function $F(s,t)$ which is periodic, i.e.

$$F(s,t) = F(s + \imath, t - \imath).$$

Moreover, by the edge of the wedge theorem and the tube theorem we find that this function is analytic in the tube-domain

$$\{-\frac{1}{2} < \Im m\, \sigma + \Im m\, \tau < \frac{1}{2}\}.$$

Since the operator $W(s,t)$ is bounded in norm by 1 we see that the function $F(s,t)$ is bounded by $\|A\Omega\| \|B\Omega\|$. Therefore, the function F is entire and bounded in the direction of periodicity which implies that the function $F(\sigma, \tau)$ depends only on one variable, i.e.

$$F(\sigma, \tau) = F(\sigma + z, \tau - z), \qquad z \in \mathbb{C}.$$

In the above equation for $F(\sigma, \tau)$ we choose real arguments and for z the value of t. Then we get $F(s,t) = F(s+t, 0)$. Inserting the expression for F we find $(\Omega, B\Delta_{\mathcal{M}}^{-\imath t} W(s) \Delta_{\mathcal{N}}^{+\imath t} A\Omega)$ $= (\Omega, BW(s+t)A\Omega)$. This is the first statement in matrix elements. Since Ω is cyclic and separating the equation for the matrix elements becomes an equation for the operators. This yields the first relation. The second relation is obtained by choosing the value $\frac{\imath}{2}$ for τ. □

4. One-dimensional situation

In the two-dimensional theories the structure of quantum fields is much more transparent than in the general theory because of the product structure of the forward light cone. This simplification has its counterpart also in the structure of the modular groups. Therefore, we start with this subject. But we will simplify the situation even more by looking at theories depending only on one variable. Remember that in classical physics the solutions of the free wave equations in two dimensions split into the sum of two solutions depending only on one of the light cone coordinates $x_0 \pm x_1$. The same is true for certain quantum fields. First we shall look at theories depending only on one light cone variable. For the sake of definiteness we will deal with "right movers". First we want to show the following general result:

4.1 Theorem:

Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω, and Δ, J be the modular operator and conjugation of the pair $(\mathcal{M}, \Omega)$. Then
(a) the following statements are equivalent:
(i) There exists a unitary group $U(s)$ with positive generator fulfilling

$$U(s)\mathcal{M}U(-s) \subset \mathcal{M} \quad \text{for} \quad t \geq 0, \qquad \text{and} \quad U(t)\Omega = \Omega.$$

(ii) There exists a proper subalgebra $\mathcal{N} \subset \mathcal{M}$ with Ω as cyclic vector, such that

$$\Delta_{\mathcal{M}}^{\imath t}\mathcal{N}\Delta_{\mathcal{M}}^{-\imath t} \subset \mathcal{N} \quad \text{for} \quad t \leq 0.$$

(b) If the conditions (a) are fulfilled then the modular group of $\mathcal{M}$ and the translations $U(a)$ fulfill the relations

$$\Delta^{\imath t}U(\lambda)\Delta^{-\imath t} = U(e^{-2\pi t}\lambda),$$
$$JU(\lambda)J = U(-\lambda).$$

If we are dealing with the situation (a, ii) then one has (up to a scale transformation)

$$\mathcal{N} = U(1)\mathcal{M}U(-1).$$

(c) If (a) or (b) is fulfilled and if the generator of $U(a)$ is bounded then $U(a)$ is identical $\mathbb{1}$.

The implication (a,i) $\to$ (b) and hence $\to$ (a,ii) can be found in [Bch2]. The implication (a,ii) $\to$ (a,i) is due to Wiesbrock [Wie2]. The situation described in (ii) of Theorem 4.1 is called *half-sided modular inclusion.* If the sign is of some importance we speak about $\pm$ half-sided modular inclusion. The second result deals with the uniqueness of the interpolating family of von Neumann algebras. It is generally believed that all local von Neumann algebras are the same. Therefore, the theory is not defined by one algebra only but one needs at least two of them. In this situation we have

4.2 Theorem:

Let $\mathcal{M}_a$ and $\mathcal{N}_a$ be two families of von Neumann algebras on the Hilbert spaces $\mathcal{H}_m, \mathcal{H}_n$ with the cyclic and separating vector Ω_m, Ω_n, respectively. Assume that both fulfill isotony and covariance and that their translation groups fulfill spectrum condition. If we assume that there exists a unitary map W with $W\mathcal{H}_n = \mathcal{H}_m$ and $W\Omega_n = \Omega_m$ and in addition

$$\mathcal{M}_0 = W\mathcal{N}_0 W^*, \quad \text{and} \quad \mathcal{M}_1 = W\mathcal{N}_1 W^*,$$

then follows

$$\begin{aligned} \mathcal{M}_a &= W\mathcal{N}_a W^* \qquad \forall \quad a \in \mathbb{R} \\ U_m(a) &= W U_n(a) W^*. \end{aligned}$$

The same is true if we require that $\mathcal{M}_0$ and $\mathcal{M}_1$ as well as $\mathcal{N}_0$ and $\mathcal{N}_1$ both fulfill modular inclusion for negative arguments of the modular groups.

The uniqueness of the interpolating families is taken from [Bch3].

Proof of Theorem 4.1: (a,i) $\to$ (b). If $U(a)$ fulfills the assumptions of (a,i) then it has an analytic continuation into the upper half plane. For positive arguments $U(a)$ maps $\mathcal{M}$ into itself by assumption and hence $U(a)$ maps $\mathcal{M}'$ into itself for negative arguments. Hence we can apply Theorem B to the family $W(s) = U(e^{2\pi s})$ and obtain together with the analyticity of $U(a)$

$$\begin{aligned} \operatorname{ad} \Delta^{\imath t} U(e^{2\pi s}) &= U(e^{2\pi(s-t)}), \\ \operatorname{ad} \Delta^{\imath t} U(a) &= U(e^{-2\pi t} a), \\ \operatorname{ad} J U(a) &= U(-a). \end{aligned}$$

(b) $\to$ (a,ii). If we set $\mathcal{N} = U(1)\mathcal{M}U(-1)$ then one obtains $\Delta^{\imath t}\mathcal{N}\Delta^{-\imath t} = U(e^{-2\pi t})\mathcal{M}U(-e^{-2\pi t})$. This is contained in $\mathcal{N}$ for negative values of t.
(a,ii) $\to$ (a,i). Assume this to be true and assume $\mathcal{N} = U(1)\mathcal{M}U(-1)$ then one has $\Delta_{\mathcal{M}}^{-\imath t}\Delta_{\mathcal{N}}^{\imath t} = \Delta_{\mathcal{M}}^{-\imath t} U(1)\Delta_{\mathcal{M}}^{\imath t} U(-1) = U(e^{2\pi t} - 1)$. Therefore, one has to show that the product $\Delta_{\mathcal{M}}^{-\imath t}\Delta_{\mathcal{N}}^{\imath t} =: D(t)$ commutes for different values of the arguments. For this one uses Theorem B again. In the situation $\mathcal{N} \subset \mathcal{M}$ one can apply Theorem A with $V = \mathbb{1}$ and will find that $D(t)$ has an analytic continuation into the strip $S(0, \frac{1}{2})$. On both boundaries the expression is unitary. By assumption of the modular inclusion one obtains:

$$\begin{aligned} D(t)\mathcal{N}D(t)^* &\subset \mathcal{N}, \quad \text{for} \quad t \geq 0 \\ D(t)\mathcal{N}'D(t)^* &\subset \mathcal{N}', \quad \text{for} \quad t \leq 0 \\ D(\frac{\imath}{2} + t)\mathcal{N}'D(\frac{\imath}{2} + t)^* &\subset \mathcal{N}', \quad \text{for} \quad t \in \mathbb{R} \end{aligned}$$

The last statements follow from $D(\frac{\imath}{2} + t) = J_{\mathcal{M}} D(t) J_{\mathcal{N}}$. $J_{\mathcal{N}}$ maps $\mathcal{N}'$ onto $\mathcal{N}$, $D(t)$ maps this into $\mathcal{M}$ and finally $J_{\mathcal{M}}$ maps this into $\mathcal{M}' \subset \mathcal{N}'$. Consequently one can apply Theorem B to the expression

$$W(s) = D(\frac{1}{2\pi}\log(e^{2\pi s} + 1)),$$

which leeds to the relation

$$\Delta_{\mathcal{N}}^{\imath t} D(\frac{1}{2\pi}\log(e^{2\pi s}+1))\Delta_{\mathcal{N}}^{-\imath t} = D(\frac{1}{2\pi}\log(e^{2\pi(s-t)}+1)).$$

Multiplying this equation from the left with $\Delta_{\mathcal{M}}^{-\imath t}$ and from the right with $\Delta_{\mathcal{N}}^{\imath t}$ then we get with $e^{2\pi x} = e^{2\pi s}+1$

$$\begin{aligned}\Delta_{\mathcal{M}}^{-\imath t}\Delta_{\mathcal{N}}^{\imath t}\Delta_{\mathcal{M}}^{-\imath x}\Delta_{\mathcal{N}}^{\imath x} &= D(\frac{1}{2\pi}\log(e^{2\pi(s-t)}+1)+t)) = D(\frac{1}{2\pi}\log(e^{2\pi s}+e^{2\pi t}))\\ &= D(\frac{1}{2\pi}\log(e^{2\pi x}+e^{2\pi t}-1)).\end{aligned}$$

Since this expression is symmetric in x and t we obtain the commutativity of the operator family $D(t)$. If we set $U(e^{2\pi t}-1) = D(t)$ then the above equation reads

$$U(e^{2\pi x}-1)U(e^{2\pi t}-1) = U(e^{2\pi x}+e^{2\pi t}-2).$$

This shows that $U(a)$ is an additive unitary group with positive generator. It remains to show that $\mathcal{N}$ is of the form $U(1)\mathcal{M}U(-1)$. To this end we introduce:

4.3 Definition:

Let the modular inclusion stated in Theorem 4.1 be fulfilled we put

$$\begin{aligned}\mathcal{N}(e^{-2\pi t}) &= \Delta_{\mathcal{M}}^{\imath t}\mathcal{N}\Delta_{\mathcal{M}}^{-\imath t},\\ \mathcal{N}(-e^{-2\pi t}) &= \{\Delta_{\mathcal{M}}^{\imath t}J_{\mathcal{M}}\mathcal{N}J_{\mathcal{M}}\Delta_{\mathcal{M}}^{-\imath t}\}'.\\ \mathcal{N}(0) &= \{\bigcup_t \mathcal{N}(e^{-2\pi t})\}''.\end{aligned}$$

¿From the following will be shown that this is a good definition.

4.4 Lemma:

The von Neumann algebras $\mathcal{N}(t)$, defined above, fulfill the following relations:

$$\begin{aligned}t_1 < t_2 \quad &\text{implies} \quad \mathcal{N}(t_1) \supset \mathcal{N}(t_2),\\ &\mathcal{N}(0) = \mathcal{M},\\ t < 0 \quad &\text{implies} \quad \mathcal{N}(t) \supset \mathcal{M},\\ t > 0 \quad &\text{implies} \quad \mathcal{N}(t) \subset \mathcal{M}.\end{aligned}$$

Proof: Because of modular inclusion we have $\mathcal{N}(t) \subset \mathcal{N}(1)$ for $t < 1$. Since unitary transformations preserve order we obtain the first statement for positive arguments. Moreover, $\mathcal{N} \subset \mathcal{M}$ implies $\mathcal{N}(t) \subset \mathcal{M}$ for positive t. For negative t we obtain the corresponding statements by the properties of $J_{\mathcal{M}}$. Finally the algebra $\mathcal{N}(0)$ is a subalgebra of $\mathcal{M}$ which is invariant under the modular group of $\mathcal{M}$ and hence coincides with $\mathcal{M}$ by a result of Rigotti [Rig]. □

Proof of Theorem 4.1, continuation: ¿From the observation that $U(a)$ is a continuous group it follows that the family $\mathcal{N}(t)$ is also continuous at zero. Hence we obtain

$$\mathcal{M} = U(-1)\mathcal{N}U(1).$$

(c) If $U(a)$ has a bounded generator then $U(a)$ is entire analytic and hence the equation $[B, U(a)AU(-a)] = 0$ for $B \in \mathcal{M}'$ and $A \in \mathcal{M}$ is true for arbitrary a if it is true for positive a. This shows that $U(a)$ defines an automorphism of $\mathcal{M}$. Since the generator is bounded it defines an inner automorphism [Ka,Sak] and since the state defined by Ω is left fixed this automorphism is the identity. Hence also $U(a)$ is the identity.
This proves Theorem 4.1. □

It remains to prove the statements concerning the uniqueness. Also the proof of theorem 4.2 needs some preparation. We start with

4.5 Lemma:
Let $\mathcal{M}_a$ and $\mathcal{N}_a$ be two families with the same cyclic and separating vector Ω. Assume they fulfill the condition of Theorem 4.1. If we assume $\mathcal{M}_0 = \mathcal{N}_0$ and $\mathcal{M}_1 = \mathcal{N}_1$ then follows

$$\mathcal{M}_i = \mathcal{N}_i, \quad \textit{for all} \quad i \in \mathbb{Z}.$$

Proof: From Theorem 4.1 (b) we know that the modular conjugation of the algebra $\mathcal{M}_i$ maps the algebra $\mathcal{M}_{i+k}$ onto the commutant of the algebra $\mathcal{M}_{i-k}$. Therefore, if $\mathcal{M}_i = \mathcal{N}_i$ and $\mathcal{M}_{i+k} = \mathcal{N}_{i+k}$ then one has also $\mathcal{M}_{i-k} = \mathcal{N}_{i-k}$ by the identity of the modular conjugations. The smallest set containing 1 and 0, which is invariant under the above map, is the set of positive and negative integers. □

Proof of the Theorem 4.2: From Lemma 4.5 we know that the two families of algebras coincide for all entire values of a. Let $U(a)$ be the unitary group of the family $\mathcal{M}_a$ and $V(a)$ that of the family $\mathcal{N}_a$. Take $A \in \mathcal{M}_0$ and $B \in \mathcal{M}_0'$ and consider the two functions

$$\begin{aligned} F^+(a,b) &= (\Omega, BU(a)V(b)A\Omega) = (\Omega, BU(a)V(b)AV(-b)U(-a)\Omega) \\ F^-(a,b) &= (\Omega, AV(-b)U(-a)B\Omega) = (\Omega, U(a)V(b)AV(-b)U(-a)\Omega). \end{aligned}$$

F^+ is the boundary value of an analytic function holomorphic in the tube

$$T^+ = \{z_1, z_2 \in \mathbb{C}^2; \Im m\, z_1 > 0, \Im m\, z_2 > 0\}.$$

F^- has an analytic extension into the tube $T^- = -T^+$. By the assumption of isotony it follows that the two functions coincide for $a > 0, b > 0$. However, since we know that $\mathcal{M}_i$ and $\mathcal{N}_i$ coincide for $i \in \pm\mathbb{N}$ we obtain a larger coincidence-domain namely, all points which are larger than a point $(i, -i)$, $i \in \mathbb{Z}$ with respect to the order given by the first quadrant. The boundary of the coincidence domain is the sawtooth curve obtained by taking the boundary of the union of all the translated first quadrants. We want to enlarge this domain by computing the envelope of holomorphy. From the Remarks 2.E we know that the complex points of the line $a+b=C>0$ belong to the envelope of holomorphy. Using these straight lines we obtain by the continuity theorem that the coincidence domain consists of all points $a+b>0$. Again by the Remarks 2.E we obtain that the function is entire in the difference–variable $a-b$. Since it is bounded we obtain $U(a)V(-a)=1$. Hence $U(a)$ and $V(a)$ coincide. Since $\mathcal{M}_0$ and $\mathcal{N}_0$ are the same it follows that the two families are the same. If the two families of von Neumann algebras are in different Hilbert spaces then we apply the above result to the families $\mathcal{M}_a$ and $W\mathcal{N}_aW^*$ and obtain the Theorem 4.2.

We end this section by looking at the case that we are dealing with a + half sided modular inclusion.

4.6 Theorem:
Let $\mathcal{M}\supset\mathcal{N}$ be two von Neumann algebras with cyclic and separating vector Ω, and Δ, J be the modular operator and conjugation of the pair $(\mathcal{M},\Omega)$. Assume, moreover,

$$\Delta_{\mathcal{M}}^{\imath t}\mathcal{N}\Delta_{\mathcal{M}}^{-\imath t}\subset\mathcal{N}\quad\text{for}\quad t\geq 0,$$

then exists a unitary group $U(s)$ with negative generator fulfilling

$$U(s)\mathcal{M}U(-s)\subset\mathcal{M}\quad\text{for}\quad s\geq 0,\qquad\text{and}\quad U(t)\Omega=\Omega.$$

Between the given two algebras one has the relation

$$\mathcal{N}=U(1)\mathcal{M}U(-1).$$

Proof: Let us look at the operator function

$$D(t)=\Delta_0^{-\imath t}\Delta_{\mathcal{N}}^{\imath t}.$$

This function has the following transformation properties:

$$\begin{aligned}\operatorname{ad}D(t)\mathcal{N}&\subset\mathcal{N}\quad t<0,\\ \operatorname{ad}D(t)\mathcal{N}'&\subset\mathcal{N}'\quad t>0\\ \operatorname{ad}D(\frac{\imath}{2}+t)\mathcal{N}'&\subset\mathcal{N}'\quad\forall t\in\mathbb{R}.\end{aligned}\tag{$*$}$$

The first two equations follow by assumptions. The third equation is a consequence of the relation

$$\operatorname{ad}\{J_0\Delta_0^{-\imath t}\Delta_{\mathcal{N}}^{\imath t}J'_{\mathcal{N}}\subset\mathcal{M}'\subset\mathcal{N}'.$$

This implies that for $t\leq 0$ $D(t)$ maps $\mathcal{N}$ into itself and for all other cases it maps $\mathcal{M}'_{0,1}$ into itself. As in Theorem 4.1 this implies that $D(t)$ is a commuting family of unitaries. Precisely speaking, the operator $W(s)$ of Theorem B is

$$W(s)=D\Big(\frac{1}{2\pi}\log\frac{e^{2\pi s}}{e^{2\pi s}+1}\Big)$$

which implies after the same manipulation as in Theorem 4.1

$$D(t)D(\frac{1}{2\pi}\log\frac{e^{2\pi s}}{e^{2\pi s}+1}) = D(\frac{1}{2\pi}\log\frac{e^{2\pi(s-t)}}{e^{2\pi(s-t)}+1}) + t).$$

Incerting $\frac{e^{2\pi s}}{e^{2\pi s}+1} = e^{2\pi x}$ which implies $e^{2\pi s} = \frac{e^{2\pi x}}{1-e^{2\pi x}}$ we find

$$D(t)D(x) = D(\frac{1}{2\pi}\log\frac{e^{2\pi(x+t)}}{e^{2\pi x}+e^{2\pi t}-e^{2\pi t}e^{2\pi x}}).$$

Defining $D(t) =: U(e^{-2\pi t}-1)$ we obtain from $(*)$ and the definition of $\mathcal{M}_{a,b}$:

$$\begin{aligned} U(t)U(s) &= U(t+s), \\ U(t)\mathcal{M}_0 U(-t) &= \mathcal{M}_{0,\frac{1}{t}} \quad \text{for} \quad t>0, \\ U(t)\mathcal{M}_0' U(-t) &= \mathcal{M}_{-\frac{1}{t}} \quad \text{for} \quad t<0. \end{aligned}$$

The group $U(t)$ has by construction a negative generator. Moreover, $U(t)$ maps $\mathcal{M}_0$ into itself for positive arguments. Hence Theorem B can be applied to this algebra together with the group $U(t)$ and we obtain

$$\begin{aligned} \Delta_0^{\imath t} U(s) \Delta_0^{-\imath t} &= U(e^{2\pi t}s), \\ J_0 U(s) J_0 &= U(-s). \end{aligned}$$

□

5. Chiral quantum fields

We start again with the assumption that we are dealing with a half sided modular inclusion, i.e. with a triple $(\mathcal{M}\supset\mathcal{N},\Omega)$ such that $\Delta_{\mathcal{M}}^{\imath t}\mathcal{N}\Delta_{\mathcal{M}}^{-\imath t}\subset\mathcal{N}$ for $t\leq 0$. In addition we assume that the vector Ω is also cyclic for $\mathcal{M}\cap\mathcal{N}'$. First we have to introduce a notation.

5.1 Definition:

By a standard chiral field theory we understand an association of von Neumann algebras to the intervals I of the unit-circle S^1 fulfilling:
(i) Isotony, i.e. $I_1\subset I_2$ implies $\mathcal{M}(I_1)\subset\mathcal{M}(I_2)$,
(ii) duality, i.e, $\mathcal{M}(S^1\setminus I) = \mathcal{M}(I)'$,
(iii) there exists a vector Ω cyclic and separating for every algebra $\mathcal{M}(I)$ if $I\neq S^1$ and $I\neq\emptyset$,
(iv) the family of algebras is covariant under a unitary representation $W(g)$ of the Möbius group $Sl(2,\mathbb{R})/\mathbb{Z}_2$,
(v) the vector Ω is left fixed by the group representation $W(g)$,
(vi) the representation of the translations $W(a)$ has a positive generator.

The Möbius transformations are the maps of the unit–circle onto itself, which can be analytically continued into the interior of the circle. All of them are of the form:

$$w = e^{\imath\varphi}\frac{z-\alpha}{\overline{\alpha}z-1}.$$

Typical examples are:
(i) Rigid rotations: $z \longrightarrow e^{\imath\varphi}z$. These transformations appear for $\alpha = 0$.
If $\alpha \neq 0$ then the equation for fixed points is a quadratic equation. If the solutions are different we obtain:
(ii) Dilatations: If 1 and -1 are the fixed points then with $b > 0$ the transformation has the form:

$$w_b = \frac{w + \frac{1-b}{1+b}}{\frac{1-b}{1+b}w + 1}.$$

If the two fixed points become a double point then we speak about
(iii) Translations: If 1 is the fixed point then the transformation for $a \in \mathbb{R}$ is given by the formula:

$$w_a = -\frac{a-2\imath}{a+2\imath}\frac{w + \frac{a}{a-2\imath}}{\frac{a}{a+2\imath}w + 1}.$$

Every three different subgroups containing one dilatation generate the whole Möbius group.

Using the triple $(\mathcal{M}, \mathcal{N}, \Omega)$ we can construct a dilatation $\Delta^{\imath t}_{\mathcal{M}}$ and a translation $U(a)$, such that $\mathcal{N} = \operatorname{ad} U(1)\mathcal{M}$. With the help of these operators we introduce an algebra for every interval (a, b) by the

5.2 Definition:

For every pair $a, b \in \mathbb{R} \cup \infty$ and with $-\infty = \infty$ we define two algebras as follows:
(i) $\mathcal{M}_{a,\infty} = \operatorname{ad} U(a)\mathcal{M}$, $\quad \mathcal{M}_{\infty,a} = \operatorname{ad} U(a)\mathcal{M}'$.
(ii) If $-\infty < a < b < \infty$ we put $\mathcal{M}_{a,b} = \mathcal{M}_{\infty,b} \cap \mathcal{M}_{a,\infty}$.
(iii) If $-\infty < b < a < \infty$ we set $\mathcal{M}_{a,b} = \mathcal{M}'_{b,a}$.

¿From the assumption that Ω is cyclic for $\mathcal{M} \cap \mathcal{N}' = \mathcal{M}_{0,1}$ it follows that Ω is cyclic and separating for every $\mathcal{M}_{a,b}$ with $a \neq b$.

By means of the triple $(\mathcal{M}, \mathcal{N}' \cap \mathcal{M}, \Omega)$ we obtain the dilatation $\Delta^{\imath t}_{\mathcal{M}}$ and a second "translation" $V(t)$ such that one obtains $\mathcal{N}' \cap \mathcal{M} = \operatorname{ad} V(1)\mathcal{M}$. This transformation permits to give a second construction for the algebras of the intervals.

5.3 Definition:

With $1/\infty = 0$ we define two algebras for every pair $a, b \in \mathbb{R} \cup \infty$ as follows:
(i) $\tilde{\mathcal{M}}_{a,\infty} = \operatorname{ad} V(\frac{1}{a})\mathcal{M}$, $\quad \tilde{\mathcal{M}}_{\infty,a} = \operatorname{ad} V(\frac{1}{a})\mathcal{M}'$.
(ii) If $-\infty < \frac{1}{a} < \frac{1}{b} < \infty$ we put $\tilde{\mathcal{M}}_{a,b} = \tilde{\mathcal{M}}_{0,b} \cap \tilde{\mathcal{M}}'_{0,a}$.
(iii) If $-\infty < \frac{1}{b} < \frac{1}{a} < \infty$ we set $\tilde{\mathcal{M}}_{a,b} = \tilde{\mathcal{M}}'_{b,a}$.

With these notations we get the following characterzation of standard chiral field theories.

5.4 Proposition:

Let $\mathcal{N} \subset \mathcal{M}$ be a pair of von Neumann algebras with the following restrictions:
(i) There exists a vector Ω which is cyclic and separating for $\mathcal{M}, \mathcal{N}$, and $\mathcal{N}' \cap \mathcal{M}$.
(ii) The triple $(\mathcal{M}, \mathcal{N}, \Omega)$ fulfills the condition of $-$ half sided modular inclusion.
Then these data define a standard chiral field theory iff the algebras $\mathcal{M}_{a,b}$ and $\tilde{\mathcal{M}}_{a,b}$ coincide.

This result is essentially due to H.W. Wiesbrock [Wie4], although our formulation differs from his. Also our proof is not identical with his proof. Later we will give a condition implying the assumption of the theorem. This will coincide with the formulation of Wiesbrock.

Proof: We transform the real line onto the circle by the transformation

$$w = \frac{\imath - x}{\imath + x}.$$

The inverse transformation is ($w = e^{\imath\varphi}$):

$$x = \imath\frac{1-w}{1+w} = \frac{\sin\varphi}{1+\cos\varphi}.$$

By this map the algebra $\mathcal{M}$ becomes the algebra of the upper half circle, the algebra $\mathcal{N}$ is mapped onto the algebra of the second quadrant and $\mathcal{N}' \cap \mathcal{M}$ onto the algebra of the first quadrant.

The three groups, the modular automorphism of $\mathcal{M}$, the translations of the two triples $(\mathcal{M}, \mathcal{N}, \Omega)$ and $(\mathcal{M}, \mathcal{N}' \cap \mathcal{M}, \Omega)$ are transformed into geometric actions of the circle. The adjoint action of these groups operate by assumption in the correct geometric manner on the algebras of the intervals. Since these three groups generate the whole Möbius Group, also the adjoint representation of this group acts in the correct geometric manner on the algebras of the intervals. Hence we have constructed a standard chiral field theory. If we start from a standard chiral field then the conditions of the theorem are obviously fulfilled. □

It might be instructive to start only from the $-$ half sided modular inclusion $(\mathcal{M}, \mathcal{N}, \Omega)$ and to require that Ω is also cyclic for $\mathcal{M} \cap \mathcal{N}'$ and to look for different conditions in order to guarantee that we are dealing with a standard chiral field theory. Using Definition 5.2 we are able to construct every algebra $\mathcal{M}_{a,b}$. But instead of looking at the second translation $V(t)$ we look at the algebra $\mathcal{M}_{-1,1}$ and in particular at its modular conjugation $J_{-1,1}$.

5.5 Proposition:

Let Ω be cyclic and separating for $\mathcal{M}, \mathcal{N}, \mathcal{M} \cap \mathcal{N}'$ and assume that the triple $(\mathcal{M}, \mathcal{N}, \Omega)$ fulfills the condition of $-$ half sided modular inclusion. Define $\mathcal{M}_{a,b}$ as in Definition 5.2. Then this setting defines a standard chiral field theory iff the following two conditions are fulfilled:

(*i*)

$$\mathrm{ad}\, J_{-1,1}\mathcal{M} = \mathcal{M}$$

where $J_{-1,1}$ is the modular conjugation of $\mathcal{M}_{-1,1}$.
(ii) The algebra $\mathcal{N}$ coincides with the relative commutant of $\mathcal{M}\cap\mathcal{N}'$ in $\mathcal{M}$, i.e.:

$$\mathcal{N} = \mathcal{M} \cap \{\mathcal{M}\cap\mathcal{N}'\}'.$$

Proof: From the relation $\operatorname{ad} J_{-1,1}\mathcal{M} = \mathcal{M}$ and the fact that $J_{-1,1}$ is antiunitary we obtain the relations

$$J_{-1,1}J = JJ_{-1,1}$$
$$J_{-1,1}\Delta^{\imath t} = \Delta^{-\imath t}J_{-1,1}$$

where J and Δ are the modular conjugation and modular operator of the algebra $\mathcal{M}$. This implies in particular ($\lambda \in \mathbb{R}$):

$$\operatorname{ad} J_{-1,1}\mathcal{M}_{-\lambda,\lambda} = \mathcal{M}_{1/\lambda,-1/\lambda}.$$

Taking the intersection with $\mathcal{M}$ or with $\mathcal{M}'$ we obtain:

$$\operatorname{ad} J_{-1,1}\mathcal{M}_{0,a} = \mathcal{M}_{1/a,\infty}, \qquad a \in \mathbb{R}.$$

This is the relation where it is necessary that $\mathcal{N}$ coincides with its second relative commutant in $\mathcal{M}$. Since $J_{-1,1}$ maps $\mathcal{M}_{0,1}$ onto $\mathcal{M}_{1,\infty}$ we also obtain a relation between the corresponding translations.

$$\operatorname{ad} J_{-1,1}V(a) = U(1/a).$$

Together with the invariance of $\mathcal{M}$ under $\operatorname{ad} J_{-1,1}$ we find by this equation that the conditions of Proposition 5.4 are fulfilled. □

Sometimes the conditions for a standard chiral field theory are formulated in such a way that the difficulties are hidden. The result is the following:

5.6 Theorem:

Let $\mathcal{M}_1, \mathcal{M}_2$ be two von Neumann algebras and Ω be a vector, then these data define a standard chiral field theory iff the following conditions are fulfilled.
(i) Ω is cyclic for $\mathcal{M}_1 \cap \mathcal{M}_2$ and for $\mathcal{M}_1' \cap \mathcal{M}_2'$.
(ii) Let J_i denote the modular conjugation of $\mathcal{M}_i$ then one has

$$\operatorname{ad} J_1\mathcal{M}_2 = \mathcal{M}_2, \qquad \operatorname{ad} J_2\mathcal{M}_1 = \mathcal{M}_1.$$

(iii) Let Δ_i denote the modular operator of $\mathcal{M}_i$ then

$$\operatorname{ad} \Delta_1^{\imath t}\mathcal{M}_2 \subset \mathcal{M}_2 \quad \text{for} \quad t \geq 0$$
$$\operatorname{ad} \Delta_2^{\imath t}\mathcal{M}_1 \subset \mathcal{M}_1 \quad \text{for} \quad t \leq 0$$

Proof: This setting allows many different interpretations namely $\mathcal{M}_1$ corresponds to the algebra of the upper half circle and $\mathcal{M}_2$ to the algebra of the left half circle. If we apply a rigid rotation of the circle to this scheme then we obtain the other possible interpretation. We will use the first interpretation. It is clear from the assumptions that the triple $(\mathcal{M}_1, \mathcal{M}_1 \cap \mathcal{M}_2, \Omega)$ fulfills the condition of $+$ half sided modular inclusion. Therefore, the triple $(\mathcal{M}_1, \mathcal{M}_1 \cap \mathcal{M}_2', \Omega)$ fulfills the condition of $-$ half sided modular inclusion. Since J_2 maps $\mathcal{M}_1$ onto itself it follows that $\mathcal{M}_1 \cap \mathcal{M}_2'$ coincides with its second relative commutant with respect to $\mathcal{M}_1$ and hence the conditions of Proposition 5.5 are fulfilled. □

6. Two-dimensional theories

In this section we deal with a two-dimensional quantum field theory. But, most of the results will be valid also for higher dimensional theories in the situation where we fix one time– and one space–coordinate and assume that all sets are cylindrical in the other directions. As distinguished set we use the right wedge. The algebra associated to this set will be denoted by $\mathcal{M}(W)$. It has the property that the translations in the direction of this wedge map the algebra into itself. Two of these directions lie in the boundary of the forward or in the backward lightcone respectively. The translations in the directions of the lightcone coordinates fulfill the spectrum condition. This yields the connection with the investigations of section 4.

6.1 Theorem:

Assume $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$ with cyclic and separating vector Ω. Assume $U(a)$ is a representation of the two-dimensional translation group which fulfills the spectrum condition and which has Ω as fixed point.

If for every a in the closed right wedge one has

$$U(a)\mathcal{M}U(-a) \subset \mathcal{M}$$

then the following relations exist between the group representation and the modular group:

$$\Delta^{\imath t}U(a)\Delta^{-\imath t} = U(\Lambda(t)a),$$
$$JU(a)J = U(-a).$$

This means that we have a unitary representation of the two-dimensional Poincaré group. If we define

$$U(a)\mathcal{M}U(-a) = \mathcal{M}_a, \qquad U(a)\mathcal{M}'U(-a) = \mathcal{M}'_a$$
$$\mathcal{M}_{a,b} = \mathcal{M}_a \cap \mathcal{M}'_b,$$

provided one has $b - a \in W$. then this net transforms covariant under the Poincaré group.

This result is taken from [Bch2].

Proof: Introducing lightcone coordinates a^+, a^- we can use the results obtained in the last two section. One should notice that the group $U(a^-)$ used here and the corresponding group used in the proof of Theorem 5.2 differ by the sign. This leads to

$$\Delta^{\imath t}U(a^+)\Delta^{-\imath t} = U(e^{-2\pi t}a^+),$$
$$\Delta^{\imath t}U(a^-)\Delta^{-\imath t} = U(e^{+2\pi t}a^+),$$
$$JU(a)J = U(-a).$$

This shows the first statement. The second statement follows from this by the definition and the the commutation relations between the translations and the modular group. □

Also this situation can be characterized by using the modular group instead of the translations.

6.2 Theorem:

Let $\mathcal{N}_1 \subset \mathcal{M}$ and $\mathcal{N}_2 \subset \mathcal{M}$ be three von Neumann algebras with common cyclic and separating vector Ω and assume

$$\begin{aligned} \sigma_{\mathcal{M}}(t)\mathcal{N}_1 \subset \mathcal{N}_1 \quad &\text{for} \quad t \le 0, \\ \sigma_{\mathcal{M}}(t)\mathcal{N}_2 \subset \mathcal{N}_2 \quad &\text{for} \quad t \ge 0. \end{aligned}$$

Denote by $U_1(t)$, $U_2(t)$ the translations which exist according to Theorem 4.1 and fulfill the spectrum condition and the relations

$$\begin{aligned} \mathrm{ad}\, U_1(1)\mathcal{M} &= \mathcal{N}_1, \\ \mathrm{ad}\, U_2(-1)\mathcal{M} &= \mathcal{N}_2, \end{aligned}$$

then the following conditions are equivalent:
(a) $U_1(t)$ and $U_2(s)$ commute for arbitrary $t, s \in \mathbb{R}$.
(b) $\mathrm{ad}\, U_1(t)\mathcal{N}_2 \subset \mathcal{N}_2$ for $t \ge 0$.
(c) $\mathrm{ad}\, U_2(t)\mathcal{N}_1 \subset \mathcal{N}_1$ for $t \le 0$.
(d) One of the two products $J_1 J_{\mathcal{M}}, J_{\mathcal{M}} J_1$ commutes with one of the products $J_{\mathcal{M}} J_2, J_2 J_{\mathcal{M}}$. Here J_i stands for $J_{\mathcal{N}_i}$.
(e) Ω is cyclic for $\mathcal{N}_1 \cap \mathcal{N}_2$ and moreover:

$$\begin{aligned} \mathrm{ad}\, \Delta_1^{\imath t}(\mathcal{N}_1 \cap \mathcal{N}_2) \subset \mathcal{N}_1 \cap \mathcal{N}_2 \quad &\text{for} \quad t \le 0, \\ \mathrm{ad}\, \Delta_2^{\imath t}(\mathcal{N}_1 \cap \mathcal{N}_2) \subset \mathcal{N}_1 \cap \mathcal{N}_2 \quad &\text{for} \quad t \ge 0, \end{aligned}$$

where Δ_i, $i, k = 1, 2$ denotes the modular operators of the algebras $\mathcal{N}_i$.

If these equivalent conditions are fulfilled then there exists a two-parametric family $\mathcal{M}_a$, $a \in \mathbb{R}^2$ of von Neumann algebras. All of them have Ω as cyclic and separating vector. This family fulfills isotony, covariance under translations which filfill spectrum condition and hence also covariance under the Poincaré group. This family is connected to the given algebras by

$$\mathcal{M}_{(0,0)} = \mathcal{M}, \quad \mathcal{M}_{(1,0)} = \mathcal{N}_1, \quad \mathcal{M}_{(0,-1)} = \mathcal{N}_2.$$

The equivalence of (a) and (d) is due to H.-W. Wiesbrock [Wie3].

Proof: If (a) is equivalent to (b) then it is also equivalent to (c) by symmetry. Assume (a) then one finds for $t \ge 0$:

$$\mathrm{ad}\, U_1(t)\mathcal{N}_2 = \mathrm{ad}\, U_1(t)\mathrm{ad}\, U_2(-1)\mathcal{M} = \mathrm{ad}\, U_2(-1)\mathrm{ad}\, U_1(t)\mathcal{M} \subset \mathrm{ad}\, U_2(-1)\mathcal{M} = \mathcal{N}_2$$

and hence (b) is fulfilled. Next assume (b) then $U_1(e^{2\pi s})$ fulfills the condition of Theorem B with respect to the subalgebra $\mathcal{N}_2$. Hence we obtain

$$\Delta_2^{\imath t} U_1(a) \Delta_2^{-\imath t} = U_1(e^{-2\pi t} a) = \Delta_{\mathcal{M}}^{\imath t} U_1(a) \Delta_{\mathcal{M}}^{-\imath t}$$

and consequently

$$\Delta_{\mathcal{M}}^{-\imath t} \Delta_2^{\imath t} U_1(a) = U_1(a) \Delta_{\mathcal{M}}^{-\imath t} \Delta_2^{\imath t}.$$

¿From this follows (a) because of

$$\Delta_{\mathcal{M}}^{-\imath t} \Delta_2^{\imath t} = U_2(e^{2\pi t} - 1).$$

Since we have $J_1 = \mathrm{ad}\, U_1(1) J_{\mathcal{M}}$ it follows $J_1 J_{\mathcal{M}} = U_1(2)$. We get $J_{\mathcal{M}} J_1 = U_1(-2)$ and $J_2 J_{\mathcal{M}} = U_2(-2), J_{\mathcal{M}} J_2 = U_2(2)$ accordingly. Hence (a) implies (d). It is sufficient to show the converse for one of the combinations. For the other combination the result follows in the same manner. So we choose the commutator of the first products. ¿From $J_1 J_{\mathcal{M}} = U_1(2)$ and $J_2 J_{\mathcal{M}} = U_2(-2)$ we know that $U_1(2)$ and $U_2(-2)$ commute. Applying the modular automorphism of the algebra $\mathcal{M}$ to this commutator we obtain the commutation of $U_1(e^{-2\pi t} 2)$ and $U_2(-e^{2\pi t} 2)$. Since with two commuting operators also their powers commute we find that $U_1(e^{-2\pi t} 2m)$ and $U_2(-e^{-2\pi t} 2n)$ commute for $n, m \in \mathbb{Z}$. Let E_M^1 be the spectral projection of $U_1(a)$ for the interval $[0, M]$ then the expression

$$E_M^1 [U_1(z), U_2(-e^{-2\pi t} 2n)] E_M^1$$

is an entire analytic function of order M and bounded on the reals with zeros at $e^{-2\pi t} 2m$. But the density of zeros of an function of exponential type cannot be too high. (Carlsons Theorem, see Boas [Boa] 9.2.1.)Hence this function vanishes if $e^{-2\pi t} 2 < \{2\pi M\}^{-1}$. Now both unitaries are boundary values of analytic functions which implies that

$$E_M^1 [U_1(z), U_2(y)] E_M^1$$

vanishes. Since the value M was arbitrary we ascertain that $U_1(x)$ commutes with $U_2(y)$. If (a) is fulfilled then it is easy to derive (e). For the converse we will show that (e) implies (b) and (c). Since the proof needs some detailed calculations we will split off some part as a separate

6.3 Lemma:

Denote the translation group given by the modular inclusion of the pair $\{\mathcal{N}_1, \mathcal{N}_1 \cap \mathcal{N}_2\}$ by $U_2^{(1)}(s)$ and that of the pair $\{\mathcal{N}_2, \mathcal{N}_1 \cap \mathcal{N}_2\}$ by $U_1^{(2)}(s)$. First we have

$$\Delta_{\mathcal{M}}^{\imath t} U_1^{(2)}(s) \Delta_{\mathcal{M}}^{-\imath t} = U_2(1 - e^{2\pi t}) U_1^{(2)}(e^{-2\pi t} s) U_2(e^{2\pi t} - 1),$$
$$\Delta_{\mathcal{M}}^{\imath t} U_2^{(1)}(s) \Delta_{\mathcal{M}}^{-\imath t} = U_1(e^{-2\pi t} - 1) U_2^{(1)}(e^{2\pi t} s) U_2(1 - e^{-2\pi t}).$$

In addition we obtain the relation

$$U_1^{(2)}(1 - e^{-2\pi t}) U_2(e^{2\pi t} - 1) = U_2^{(1)}(e^{2\pi t} - 1) U_1(1 - e^{-2\pi t}).$$

Proof: The modular groups of $\mathcal{N}_1$ and $\mathcal{N}_2$ are given by the formulas

$$\Delta_1^{\imath t} = U_1(1)\Delta_{\mathcal{M}}^{\imath t} U_1(-1) = U_1(1 - e^{-2\pi t})\Delta_{\mathcal{M}}^{\imath t},$$
$$\Delta_2^{\imath t} = U_2(-1)\Delta_{\mathcal{M}}^{\imath t} U_2(1) = U_2(e^{2\pi t} - 1)\Delta_{\mathcal{M}}^{\imath t}.$$

¿From these formulas we easily obtain the first two relations by the assumed modular inclusions. The last relation follows from the fact that we can compute the modular group of the intersection in two different ways

$$\Delta_{\cap}^{\imath t} = \operatorname{ad} U_2^{(1)}(-1)\Delta_1^{\imath t} = \operatorname{ad} U_1^{(2)}(1)\Delta_2^{\imath t}.$$

This allows to express $\Delta_{\cap}^{\imath t}$ as a factor times $\Delta_{\mathcal{M}}^{\imath t}$. The identity of the factors leads to the second statement of the lemma. □

Proof of the theorem, continuation: We apply to the last relation of Lemma 6.3 the operator $\operatorname{ad} \Delta_{\mathcal{M}}^{\imath t}$ and use the first two relations of the lemma. This leads to:

$$U_2(1 - e^{2\pi s})U_1^{(2)}(e^{-2\pi s}(1 - e^{-2\pi t}))U_2(e^{2\pi s} - 1)U_2(e^{2\pi s}(e^{2\pi t} - 1)) =$$
$$U_1(e^{-2\pi s} - 1)U_2^{(1)}(e^{2\pi s}(e^{2\pi t} - 1))U_1(1 - e^{-2\pi s})U_1(e^{-2\pi s}(1 - e^{-2\pi t})).$$

Differentiating this equation with respect to t and then putting $t = 0$ yields

$$U_2(1 - e^{2\pi s})e^{-2\pi s}H_1^{(2)}U_2(e^{2\pi s} - 1) + e^{2\pi s}H_2 =$$
$$U_1(e^{-2\pi s} - 1)e^{2\pi s}H_2^{(1)}U_1(1 - e^{-2\pi s}) + e^{-2\pi s}H_1,$$

where $H_i^{(k)}$ denotes the generators of the unitary groups $U_i^{(k)}$. First we evaluate these relations by formal manipulation. Multiplying this equation with $e^{2\pi s}$ and taking the limit $s \to -\infty$ leads to

$$U_2(1)H_1^{(2)}U_2(-1) = H_1,$$

and multiplying with $e^{-2\pi s}$ and taking the limit $s \to \infty$ yields

$$U_1(-1)H_2^{(1)}U_1(1) = H_2.$$

But this is the same as condition (b). The correct calculation is the following: Put $e^{-2\pi s}(1 - e^{-2\pi t}) = \lambda$ and take the limit $s \to -\infty$. This leads to the first equation. If we set $e^{2\pi s}(e^{2\pi t} - 1) = \lambda$ and if we let s tend to ∞ we find the second relation. □

7. Higher dimensional theories

As we have seen in the last sections the modular groups gives some insight into the structure of quantum field theory in one or two dimensions. In higher dimensions the situation is more complicated. This is mainly due to the structure of the light cone.

However, there is one exception, i.e., the theory covariant under the conformal group. This result is due to Brunetti, Guido, and Longo [BGL]. Morover, we will discuss the procedure of Buchholz and Summers [BuSu] which gives a characterization of the vacuum state.

7.1 Theorem:

Let $\mathcal{K}$ be the family of sets consisting of double cones, wedges, and forward or backward translated light cones. Assume that for every $O \in \mathcal{K}$ we have a von Neumann algebra $\mathcal{M}(O)$ on a Hilbert space $\mathcal{H}$ fulfilling isotony and locality. Moreover, assume that there exists a vector $\Omega \in \mathcal{H}$ which is cyclic and separating for every $\mathcal{M}(O)$. Assume that this family is covariant under conformal transformations. Assume, in particular, that there exists a continuous unitary representation of the conformal group $U(g)$ with $U(g)\Omega = \Omega$ such that the translations fulfill the spectrum condition and $U(g)\mathcal{M}(O)U^*(g) = \mathcal{M}(O_g)$ whenever O and O_g belong to $\mathcal{K}$.

If this is fulfilled then the modular group acts as geometric transformation i.e:
As Lorentz boosts for wedge domains,
as dilatations for light cones,
as the transformations described in section 2 for double cones.
In particular one has

$$\Delta_O^{\imath t} = U_O(t)$$

when the geometric transformations are properly defined and where $U_O(t)$ is the image of the Lorentz transformations of the wedge under the map which sends the wedge onto O.

This is the result of Brunetti, Guido, and Longo [BGL]. Notice that we have absorbed the factor 2π which appears in the previous section when defining the geometric transformations. This simplifies the calculations.

Proof: For the dimension equal to one we have seen in section 5 that the result is true, so we have to consider the case $d \geq 2$. First we want to show that for any $O \in \mathcal{K}$ the one-parametric group

$$W(t) = \Delta_O^{\imath t} U_O(-t)$$

commutes with the representation of the conformal group and does not depend on the domain O. Since all these sets are the image of a wedge let us fix a wedge W and look at the equation

$$\Delta_W^{\imath t} U(g) \Delta_W^{-\imath t} = U_W(t) U(g) U_W(-t).$$

This is true for all transformations g which map the wedge onto itself as the Lorentz transformations of this wedge, the translations in the directions in the wedge, and the transformations

$$x \to -\frac{x}{x^2}, \quad \text{for} \quad d = 2$$

$$x \to -\frac{\{x_0, x_1, -x_2, \cdots\}}{x^2}, \quad \text{for} \quad d > 2.$$

These transformations belong to the connected component of the identity in the conformal group. Moreover, we know from the last section that this relation is also true for the translations in the two lightlike directions defining the wedge. But it is known [TMP] that these transformations together generate the connected component of the conformal group. Hence $W(t)$ commutes with the representation of the conformal group. Since $W(t)$ of one domain is mapped onto $W(t)$ of another domain by means of conformal transformations we get the independence of the domain. It remains to show that $W(t)$ is the identity. If $d > 2$ then we know that there is a rotation which maps the wedge onto the opposite wedge. This rotation commutes with the Lorentz boosts and the modular group onto their inverse. Hence we obtain

$$W(t) = U(R)W(t)U(R) = (U(R)\Delta_W U(R))^{\imath t} U_W(-t) = W(-t).$$

In two dimensions we have to use the modular conjugation J_W and obtain the same result. □

Next we turn to the result of Bucholz and Summers [BuSu] concerning a characterization of the vacuum state. The main idea is the following: It is well known that every translation can be decomposed into reflections. In vacuum sector of local quantum field theory the modular conjugations of the algebras belonging to wedges are reflections in the two-plane, spanned by the two light rays defining the wedge. Therefore, if these conjugations act locally on every local algebra it should be possible to construct a representation of the translation group. If this representation of the translation group is continuous then it will have the correct properties in configuration space and therefore, by a result of Wiesbrock [Wie1], this representation must necessarily fulfill the spectrum condition. So we obtain:

7.2 Theorem:

Assume that we are dealing with a representation of the theory of local observables on a Hilbert space $\mathcal{H}$ such that there is a vector $\Omega \in \mathcal{H}$ which is cyclic and separating for all wedge algebras $\mathcal{M}(W(\ell_1, \ell_2, a))$. Denote by $J(\ell_1, \ell_2, a)$ the corresponding modular conjugations and assume that these induce a geometric action on all wedge algebras. Suppose in addition that for every wedge $W(\ell_1, \ell_2, 0)$ the modular group of this wedge satisfies the condition of $\mp$ modular inclusion for the wedges obtained by translation in the direction ℓ_1 and $-\ell_2$ respectively then there exists a continuous unitary representation $U(x)$ of the translation group fulfilling

(a) $U(x)\mathcal{M}(W(\ell_1, \ell_2, a))U(-x) = \mathcal{M}(W(\ell_1, \ell_2, a) + x)$,

(b) $U(x)\Omega = \Omega$,

(c) $U(x)$ fulfills the spectrum condition.

This action of the group gives the right action on algebras of the double cones provided the algebras of the double cones are identical with the intersections of the algebra of all the wedges which contain this double cone.

Before proving this theorem we need some explanations:
Given the triple (ℓ_1, ℓ_2, a) then we can decompose the Minkowski space into the two-plane $E(\ell_1, \ell_2)$ spanned by the two lightlike vectors and the complement $E(\ell_1, \ell_2)^\perp$ where the orthogonal complement is computed with respect to the Minkowski scalar product. Now the vector a has a unique decomposition $a = a_1 + a^\perp$ where a_1 belongs to $E(\ell_1, \ell_2)$. If x is an arbitrary vector we can also decompose it: $x = x_1 + x^\perp$, where x_1 belongs again to

$E(\ell_1, \ell_2)$. Now we define the reflection $r(\ell_1, \ell_2, a)$ by the formula

$$r(\ell_1, \ell_2, a)x = -x_1 + 2a_1 + x^{\perp}.$$

The correct reflection, mentioned in the theorem, is given by the formula

$$J(\ell_1, \ell_2, a)\mathcal{M}(W)J(\ell_1, \ell_2, a) = \mathcal{M}(r(\ell_1, \ell_2, a)W).$$

By the $\mp$–modular inclusion we mean the following: Let $\Delta(\ell_1, \ell_2)$ be the modular operator of the algebra $\mathcal{M}(\ell_1, \ell_2, 0)$ and $p > 0$ then we require

$$\begin{aligned}\Delta^{\imath t}(\ell_1, \ell_2)\mathcal{M}(W(\ell_1, \ell_2, p\ell_1))\Delta^{-\imath t}(\ell_1, \ell_2) &\subset \mathcal{M}(W(\ell_1, \ell_2, p\ell_1)) \\ &\text{for} \qquad t \leq 0, \\ \Delta^{\imath t}(\ell_1, \ell_2)\mathcal{M}(W(\ell_1, \ell_2, -p\ell_2))\Delta^{-\imath t}(\ell_1, \ell_2) &\subset \mathcal{M}(W(\ell_1, \ell_2, -p\ell_2)) \\ &\text{for} \qquad t \geq 0,\end{aligned}$$

The original work of Buchholz and Summers contains much more assumptions. Here we follow the ideas of Borchers [Bch3] reducing the assumptions to the essential ones. The strategy consists in looking first in a fixed lightlike direction and constructing the translation group in this direction. Afterwards one has to show that all these one-dimensional groups fit together.

7.3 Proposition:

With the assumptions of Theorem 7.2 let ℓ_2, ℓ_2 be two fixed lightlike vectors in the boundary of the forward lightcone. Then there exists a one parametric continuous unitary group $U(a\ell_1)$ fulfilling the assumptions of the theorem. This group is given by the formula

$$U(2a\ell_1) = J(\ell_1, \ell_2, 0)J(\ell_1, \ell_2, -a\ell_1).$$

Proof: During this proof we denote the algebra $\mathcal{M}(W(\ell_1, \ell_2, a\ell_1))$ by $\mathcal{M}_a$ and its commutant by $\mathcal{M}'_a$. The corresponding modular conjugation will simply be denoted by J_a. The requirement concerning the action of the modular conjugations becomes

$$J_a\mathcal{M}_bJ_a = \mathcal{M}'_{(2a-b)}.$$

Since the algebra and its commutant have the same modular conjugation we obtain from this relation the equation

$$J_aJ_bJ_a = J_{(2a-b)}. \qquad (*)$$

As a consequence of this relation we show that the products $J_0J_{-\frac{1}{2}a}$ form a group for rational values of a.

7.4 Lemma:

Assume equation $(*)$ and let $c \in \mathbb{R}$ be fixed. If $a, b \in \mathbb{Q}c$ then

1) *the products J_aJ_b depend only on the difference $b - a$,*

2) *the unitary operators* $V_c(a) := J_0 J_a$ *define a unitary representation of the additive group of* $\mathbb{Q}$ *i.e.*

$$V_c(a)V_c(b) = V_c(a+b).$$

Proof: Equation $(*)$ leeds to the relation $J_0 J_a = J_{-a} J_0$. First we show that equation $(*)$ implies

$$(J_0 J_a)^n = J_{-ka} J_{(n-k)a} \qquad k = 0, \cdots, n. \qquad (**)$$

We prove this relation by induction with respect to n. The statement is obviously correct for $n = 1$. Assume we know the statement for $i = 1 \ldots n-1$. Then we want to show it for n. Let $0 < k < n$ and write $(J_0 J_a)^n = (J_0 J_a)^k (J_0 J_a)^{n-k}$. Incerting the induction hypothesis we obtain

$$(J_0 J_a)^n = J_{-la} J_{(k-l)a} J_{-ma} J_{(n-k-l)a}.$$

In order to use equation $(*)$ we must choose $l = m$. We obtain
$J_{-(l+k)a} J_{(n-l-k)a}$. This is the stated formula. We can also choose $(k-l) = (n-k-m)$. This again gives the representation $J_{-la} J_{(n-l)a}$. As restriction we have $l = 0, \ldots, k$ and $m = 0, \ldots n-k$. Since k does not take the values 0 and n only both constructions together cover the whole of formula $(**)$. Now note that the equation $J_0 J_{-a} J_0 J_a = J_a J_0 J_0 J_a = 1$, implies $J_0 J_{ma} J_0 J_{na} = J_0 J_{(m+n)a}$ for $n, m \in \mathbb{N}$. Let now q_1, q_2 be two rational numbers then by writing $q_1 = \frac{m}{r}, q_2 = \frac{n}{r}$ we obtain $J_0 J_{q_1 a} J_0 J_{q_2 a} = J_0 J_{q_2 a} J_0 J_{q_1 a} = J_0 J_{(q_1+q_2)a}$ for arbitrary $q_1, q_2 \in \mathbb{Q}$. □

If we would know the continuity of the expression $J_0 J_a$ in the variable a then we would obtain a continuous unitary representation of the additive group of the real line. In order to obtain the continuity we start from the requirement that the algebra $\mathcal{M}_1$ fulfills the condition of $-$ modular inclusion with respect to the algebra $\mathcal{M}_0$. Hence by Theorem 4.1 there is a unitary group $U_1(a)$ of the real line fulfilling the spectrum condition, leaving the vector Ω fixed and which maps $\mathcal{M}_0$ onto $\mathcal{M}_1$ i.e.

$$U_1(1)\mathcal{M}_0 U_1(-1) = \mathcal{M}_1.$$

We could have started with another subalgebra as e.g. $\mathcal{M}_{\frac{1}{2}}$. Then we would have obtained another group $U_{\frac{1}{2}}(a)$ and, after proper rescaling, the relation

$$U_{\frac{1}{2}}(1)\mathcal{M}_0 U_{\frac{1}{2}}(-1) = \mathcal{M}_1.$$

We know from Theorem 4.2 that these are the same groups.

For every $a > 0$ we can construct a unitary group $U_a(t)$ fulfilling spectrum condition and the equation

$$U_a(a)\mathcal{M}_0 U_a^*(a) = \mathcal{M}_a$$

by using Theorem 4.1. Moreover, the family

$$\mathcal{M}_t^a = U_a(t)\mathcal{M}_0 U_a(-t)$$

fulfills the condition of isotony. Moreover, we have $\mathcal{M}^a(2a) = \mathcal{M}^{2a}(2a)$. It follows from Theorem 4.2 that $U_a(t)$ and $U_{2a}(t)$ coincide. Repeating the argument for other values with a common multiple we find that $U_q(t) = U(t)$ does not depend on q for rational values of q. The isotony implies that the family $\mathcal{M}_a$ is continuous because for every irrational a $\mathcal{M}_a$ can be approximated from the inside and the outside by algebras with rational values of the index. On the algebras we have the action of $U(t)$ which is a continuous group. Hence we obtain continuity for $\mathcal{M}_a$. This implies the representation $\mathcal{M}_a = U(a)\mathcal{M}_0 U(-a)$ with $U(a)$ fulfilling the spectrum condition. Moreover, from Theorem 4.1 we obtain

$$J_0 J_{-\frac{1}{2}a} = J_0 U(-\frac{1}{2}a) J_0 U(\frac{1}{2}a) = U(a).$$

This shows the statements of Proposition 7.2. □

Next we look at a fixed wedge $W(\ell_1, \ell_2, 0)$ and we find two unitary groups $U(a^+)$ and $U(a^-)$ translating in the directions ℓ_1 and ℓ_2 both fulfilling spectrum condition. Now we have to show that these groups commute. For this we need a preparation.

7.5 Lemma:

Let $\mathcal{M}_a$ be a one-parametric family of von Neumann algebras with a common cyclic and separating vector Ω. Assume that this family has the properties of isotony, covariance and spectrum condition. Let V be a unitary operator having Ω as fixed point and which maps every $\mathcal{M}_a$ into itself. Then V commutes with the translations.

Proof: Let $U(a)$ be the unitary group defining the covariance and fulfilling the spectrum condition. Define a family of unitary operators $T(a) = U(a)VU(-a)$. Since V maps every $\mathcal{M}_b$ into itself, it follows from the definition that the same is true for $T(a)$. Now choose $A \in \mathcal{M}_0$ and $B \in \mathcal{M}_0'$. Then $U(b)BU(-b)$ and $T(a)AT^*(a)$ commute for $b \leq 0$. Therefore the two functions

$$F^+(b,a) = (\Omega, BU(-b)T(a)A\Omega) = (\Omega, U(b)BU(-b)T(a)AT^*(a)\Omega),$$
$$F^-(b,a) = (\Omega, AT^*(a)U(b)B\Omega) = (\Omega, T(a)AT^*(a)U(b)BU(-b)\Omega)$$

coincide for negative values of b. Because of $U(-b)T(a) = U(a-b)VU(-a)$ it follows that $F^+(b,a)$ is the boundary-value of a bounded analytic function holomorphic in the tube $\Im m\,(a-b) > 0, \Im m\, a < 0$. Since we get $T^*(a)U(b) = U(a)V^*U(b-a)$ it follows that $F^-(b,a)$ has an analytic extension into the opposite tube. Hence we have to deal with an edge of the wedge problem. Let us denote the common analytic extension of both functions by $F(b,a)$. For solving this edge of the wedge problem let us introduce the variables $z_1 = a,\ z_2 = a-b$. Then we obtain the two tubes based on the second and on the fourth quadrant. The coincidence domain becomes $x_1 - x_2 > 0$. According to III.3 $F(b,a)$ is holomorphic except for the cut in the b-variable along the negative axis. In particular F is entire analytic in the variable a. Since F is also bounded, it does not depend on this variable. Because the vector Ω is cyclic, as well for $\mathcal{M}_0$ as for $\mathcal{M}_0'$, $T(a)$ does not depend on a. The equation $T(a) = T(0)$ is equivalent to the statement of the lemma. □

Next we show the commuting of the two groups. Let a^+ be the positive lightcone coordinate. Then $U(a^+) = J_0 J_{-\frac{1}{2}a^+}$ fulfills spectrum condition. Now the operator $J_0 J_{-\frac{1}{2}a^-}$ fulfills the relation:

$$J_0 J_{-\frac{1}{2}a^-} \mathcal{M}_{a^+} J_{-\frac{1}{2}a^-} J_0 = J_0 \mathcal{M}'_{-a^- - a^+} J_0 = \mathcal{M}_{a^+ + a^-}.$$

¿From the isotony follows that for negative values of a^- the product $J_0 J_{-\frac{1}{2}a^-}$ maps every $\mathcal{M}_{a^+}$ into itself. Hence by 7.6 it commutes with $U(a^+)$. Since we have modular inclusion for both lightlike directions also the group $U(a^-) = J_0 J_{-\frac{1}{2}a^-}$ fulfills the spectrum condition. Hence by analytic continuation $U(a^+)$ and $U(a^-)$ commute for all values of their arguments.

Proof of the theorem: We start with two wedges which have a common lightlike vector ℓ i.e the two wedges $W(\ell, \ell_1, 0)$ and $W(\ell, \ell_2, 0)$. We can apply the two-dimensional situation to every of this wedges and obtain two representations of the translations along the direction ℓ. Both of these representations fulfill the spectrum condition. We have to show that these representations coincide. Define $U(\ell, \ell_1)(a^+) = J(\ell, \ell_2, 0) J(\ell, \ell_1, -\frac{a^+}{2}\ell)$, then the reflection symmetry yields:

$$U(\ell, \ell_1)(a^+) \mathcal{M}(\ell, \ell_2, b^+) U^*(\ell, \ell_1)(a^+) = \mathcal{M}(\ell, \ell_2, r(\ell, \ell_1, 0) r(\ell, \ell_1, \frac{a^+}{2}) b^+).$$

This implies that $U(\ell, \ell_1)(a^+)$ maps the family of wedge algebras $\mathcal{M}(\ell, \ell_2, b^+)$ onto itself. Hence, by Lemma 7.5, the two groups $U(\ell, \ell_1)(a^+)$ and $U(\ell, \ell_2)(a^+)$ coincide. Therefore, the groups in every two different lightlike directions commute. Since every vector can be decomposed into two lightlike vectors we see that the system is covariant and that it fulfills the spectrum condition, because it is true for every lightlike direction. □

8. Examples

In this section we want to present some examples showing that duality and wedge-duality do not result from the usual axioms without Lorentz covariance. From these models we construct examples in higher dimensions showing that the modular group does not always act locally in the direction of invariance of the wedge. These results are due to J.Yngvason [Yng].

Suppose Φ is a hermitian Wightman field that transforms covariantly under space-time translations, but not necessarily under Lorentz transformations, and depends only on one light cone coordinate, say x_+. Locality implies that the commutator $[\Phi(x_+), \Phi(y_+)]$ has support only for $x_+ = y_+$. Moreover, from the spectrum condition it follows that the generator for translations of Φ in the x_+-direction, $P^0 - P^1$, is positive semidefinite. This implies that the Fourier transform of the two point function, $\mathcal{W}_2$, defined by $(\Omega, \Phi(x_+)\Phi(y_+)\Omega) = (1/2\pi) \int \exp[ip(x_+ - y_+)] \tilde{\mathcal{W}}_2(p) dp$ has the form

$$\tilde{\mathcal{W}}_2(p) = \theta(p) p Q(p^2) + c\delta(p).$$

In this formula Ω is the vacuum vector, $Q(p^2)$ is a positive, even polynomial in $p \in \mathbb{R}$ and $\theta(s) = 1$ for $s \geq 0$ and zero else, and $c = (\Omega, \Phi(x_+)\Omega)^2 \geq 0$ is a constant. Subtracting $c^{1/2}$ from Φ if necessary, we may drop the $\delta(p)$-term. For simplicity of notation from now on we write x, y instead of x_+, y_+.

The models we consider are generalized free fields with the two point function given above (without the δ-term). They are characterized by the commutation relations

$$[\Phi(x), \Phi(y)] = DQ(D^2)\delta(x-y)\mathbb{1},$$

where we have for convenience denoted id/dx by D. Let $\mathcal{H}_{Q,1}$ be the Hilbert space of functions $f(p)$ such that $\int_0^\infty |f(p)|pQ(p)dp < \infty$. Define for $f \in \mathcal{H}_{Q,1}$ the unitary Weyl operators as usual by

$$W(f) = e^{i\Phi(f)}.$$

The Weyl relations are

$$W(f)W(g) = e^{-K(f,g)/2}W(f+g)$$

with

$$K(f,g) = (\Omega, [\Phi(f), \Phi(g)]\Omega) = \int_{-\infty}^{\infty} p\,Q(p^2)\tilde{f}(-p)\tilde{g}(p)dp.$$

It follows that $W(f)$ commutes with $W(g)$ if and only if $K(f,g) = 0$, in particular if f and g have disjoint supports. The Weyl operators are defined on the Fock space $\mathcal{H}_Q$. For our future investigations we can restrict our attention to the one–particle Hilbert space $\mathcal{H}_{Q,1}$.

We know that the modular group of the half line acts as a delatation by the factor $e^{-2\pi t}$. This amounts in momentum space to a dilatation by the factor $\lambda = e^{2\pi t}$. If we denote the restriction of the modular group of the positive half line $\Delta_+^{\imath t}$ to the one–particle Hilbert space $\mathcal{H}_{Q,1}$ by $V_+(\lambda)$ we must get

$$(V_+(\lambda)\psi)(p) = \lambda\sqrt{\frac{Q(\lambda p)}{Q(p)}}e^{\imath\Phi(p)}\psi(\lambda p),$$

where the phase–factor $e^{\imath\Phi(p)}$ has to be determined. If $\psi(p)$ is analytic in the upper half plane then the same must be true for $(V_+(\lambda)\psi)(p)$. This condition can be solved by remembering the structure of $Q(p)$ which permits us to write

$$Q(p) = L(p)L(-p), \qquad \text{with} \qquad L(-p) = L(p)^*.$$

The polynomial $L(p)$ is fixed up to a sign by the requirement that its zeros lie in the closed upper half plane. Hence we find:

$$(V_+(\lambda)\psi)(p) = \lambda\frac{L(-\lambda p)}{L(-p)}\psi(\lambda p).$$

That this is the correct expression for the modular group can be checked by showing that the KMS–condition is fulfilled. For this one uses the analyticity property as well in p as in λ.

In the same manner we obtain for the left half–line

$$(V_-(\lambda)\psi)(p) = \lambda \frac{L(\lambda p)}{L(p)} \psi(\lambda p).$$

Since the algebra and its commutant have the same modular group we see that wedge duality is fulfilled iff $L(p)$ has only real zeros.

The duality conditions for intervals is a little more difficult. Let $I \subset \mathbb{R}$ be a bounded interval. If $L(p)$ and hence $Q(p)$ is not a constant, we can obviously find a test function g such that $DQ(D^2)g(x) = 0$ for $x \in I$, while $DL(D)g(x) \neq 0$ for all x in some open subinterval of I. The first property implies that $W(g) \in \mathcal{M}(I)'$ because of the commutation relations. On the other hand, by the latter property of g one can find a test function h with support in I such that

$$\int \tilde{h}(-p) p L(p) \tilde{g}(p) dp = \int h(x) DL(D) g(x) dx \neq 0.$$

Let f be the tempered distribution whose Fourier transform is $\tilde{f}(p) = \lim_{\varepsilon \to 0^+} \tilde{h}(p)/L(p + i\varepsilon)$. Since $\int |p||L(p)\tilde{f}(p)|^2 dp < \infty$, $W(f)$ and $K(f,g)$ are well defined we have

$$K(f,g) = \int pL(-p)\tilde{f}(-p)L(p)\tilde{g}(p)dp = \int \tilde{h}(-p)pL(p)\tilde{g}(p)dp \neq 0.$$

Hence $W(f)$ and $W(g)$ do not commute. Since $W(g) \in \mathcal{M}(I)'$, it is enough to check that $W(f) \in \mathcal{M}(I')'$ in order to verify our assertion that duality is violated. But if $u(x)$ is a test function with support in I', then

$$K(f,u) = \int pL(p)L(-p)\tilde{f}(-p)\tilde{u}(p)dp = \int pL(p)\tilde{h}(-p)\tilde{u}(p)dp = K(-DL(-D)h, u) = 0$$

because $DL(-D)h$ has support in I. Hence $W(f) \in \mathcal{M}(I')'$, so duality does not hold.

Finally we consider fields in n–dimensional Minkowski space, $n > 2$. The most general two-point function consistent with positivity, translational covariance, spectrum condition and locality has the form

$$\mathcal{W}_2(p) = \sum_{i=1}^{N} M_i(p) d\mu_i(p)$$

in Fourier space, where $d\mu_i$ is a positive Lorentz-invariant measure with support in the forward light cone and M_i is a polynomial that is positive on the support of $d\mu_i$, $i = 1, \ldots, N$. Guided by the low-dimensional examples considered above we shall compute the modular groups of the wedge algebras for generalized free fields on $\mathbb{R}^n$ in the special case that the sum contains only one term, i.e.,

$$\mathcal{W}_2(p) = M(p) d\mu(p),$$

and the polynomial M allows a factorization,

$$M(p) = F(p)F(-p),$$

where $F(p)$ is a function (in general not a polynomial) with certain analyticity properties to be specified below.

To describe the properties of F we use the light cone coordinates $x_\pm = x^0 \pm x^1$ for $x = (x^0, \ldots, x^{(n-1)}) \in \mathbb{R}^n$ and denote $(x^2, \ldots, x^n)$ by $\hat{x}$. The Minkowski scalar product is

$$\langle x, y\rangle = \tfrac{1}{2}(x_+ y_- + x_- y_+) - \hat{x}\cdot\hat{y}.$$

The right wedge, W_R, is characterized by $x_+ > 0$, $x_- < 0$; hence the Fourier transform, $\tilde{f}(p) = \int \exp(-i\langle p, x\rangle) f(x) d^n x$ of a test function f with support in W_R has for fixed $\hat{p} \in \mathbb{R}^{n-2}$ an analytic continuation in p_+ and p_- into the half planes $\operatorname{Im} p_+ > 0$, $\operatorname{Im} p_- < 0$. We require for F that $F(\pm p)$ is analytic and that $F(-p)$ is *without zeros* in this domain, with $F(-p) = F(p)^*$ for $p \in \mathbb{R}^n$. There is no lack of polynomials M allowing such a factorization; one example is

$$M(p) = (p^1)^2 + \cdots + (p^n)^2 + m^2$$

with

$$F(p) = \sqrt{\hat{p}\cdot\hat{p} + m^2} + ip^1 = \sqrt{\hat{p}\cdot\hat{p} + m^2} + \tfrac{i}{2}(p_+ - p_-).$$

If $d\mu(p) = \theta(p^0)\delta(\langle p, p\rangle - m^2)$ we can replace the polynomial by $(p^0)^2$. Hence the corresponding generalized free field is nothing but the time derivative $(d/dx^0)\Phi_m(x)$, where Φ_m is the free field of mass m.

In analogy with the first example we define for $\lambda > 0$ the unitary operators $V_R(\lambda)$ on the Fock space $\mathcal{H}$ over the one-particle space $\mathcal{H}_1 = L^2(\mathbb{R}^n, M(p)d\mu(p))$ by

$$V_R(\lambda)\varphi(p) = \frac{F(-\lambda p_+, -\lambda^{-1} p_-, -\hat{p})}{F(-p_+, -p_-, -\hat{p})}\varphi(\lambda p_+, \lambda^{-1} p_-, \hat{p}) \qquad (*)$$

for $\varphi \in \mathcal{H}_1$ and canonical extension to $\mathcal{H}$. Then we define by means of $V_R(\lambda)$ a one parameter group of automorphisms of the von Neumann algebra $\mathcal{M}(W_R)$ on $\mathcal{H}$ generated by the Weyl operators $W(f)$ with $\operatorname{supp} f \subset W_R$. By essentially the same computation that verified the example of the half line one shows that $(*)$ satisfies the KMS condition and that it is therefore the modular group defined by the vacuum state on $\mathcal{M}(W_R)$.

For the left wedge $W_L = \{x \mid x_+ < 0, x_- > 0\}$ the corresponding operators are

$$V_L(\lambda)\varphi(p) = \frac{F(\lambda p_+, \lambda^{-1} p_-, \hat{p})}{F(p_+, p_-, \hat{p})}\varphi(\lambda p_+, \lambda^{-1} p_-, \hat{p}).$$

By comparing the two modular groups we see that the field satisfies the wedge duality condition $\mathcal{M}(W_R)' = \mathcal{M}(W_L)$ if and only if $F(p) = F(-p)$ on the support of $d\mu$. This condition is, e.g., violated in the above mentioned example.

This example demonstrates also that the modular group of $\mathcal{M}(W_R)$ may act nonlocally in the $\hat{x}$-directions. In fact, let f be a test function with compact support in W_R. Under the transformation (7.7) the Fourier transform $\tilde{f}$ is mapped into

$$\tilde{f}_\lambda(p) = \frac{\sqrt{\hat{p}\cdot\hat{p} + m^2} - \frac{i}{2}(\lambda p_+ - \lambda^{-1} p_-)}{\sqrt{\hat{p}\cdot\hat{p} + m^2} - \frac{i}{2}(p_+ - p_-)}\tilde{f}(\lambda p_+, \lambda^{-1} p_-, \hat{p}).$$

This is no longer the Fourier transform of a function of compact support in the $\hat{x}$-directions, because it is not analytic in $\hat{p}$. From this lack of analyticity it is not difficult to deduce that $W(f_\lambda)$ does not belong to any wedge algebra generated by the field unless the wedge is a translate of W_R or W_L, but we refrain from presenting a formal proof. The operator $W(f_\lambda)$ is still localized in the x^0, x^1-directions in the sense that it is contained in $\mathcal{M}(W_R + a) \cap \mathcal{M}(W_R + b)'$ for some $a, b \in W_R$, in accordance with the discussion of Borchers [Bch2].

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