# Uniqueness and Reconstruction Formulae for Inverse $N$-Particle Scattering

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**Abstract**

For $N$ multidimensional nonrelativistic quantum particles interacting by local pair potentials of short range the high energy behaviour of the scattering operator between the totally free channels determines the potentials uniquely and we derive a reconstruction formula. If, in addition, long-range potentials are given (e.g. Coulomb potentials) then the short-range parts can be reconstructed uniquely from the modified scattering operator.

## 1 Introduction, Uniqueness

For $N$-body quantum systems in $\nu \geq 2$ space dimensions with interaction by local short-range pair potentials (multiplication operators) let $S$ denote the scattering operator between the free channels, i.e. there are no bounded subsystems asymptotically. Then $S$ determines the pair potentials uniquely (Theorem 1.1). More precisely, it is sufficient to know $S$ between states where the relative velocity between any two particles tends to infinity and we give a formula to reconstruct the potentials from the high velocity limit of $(S - \mathbf{1})$ (Theorem 2.1).

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If, in addition, long-range pair potentials are present then one has to know already (the far out "tail" of) the long-range part in order to be able to define a scattering operator, e.g. the modified Dollard scattering operator $S^D$. Then $S^D$ determines the short-range parts (and thus all) of the pair potentials uniquely (Theorem 1.2). Again a reconstruction formula is given in terms of high velocity limits of $(S^D - \mathbf{1})$ (Theorem 2.1). The proofs are given in Sections 2 and 3. The question of how to obtain the long-range part of the potential will be addressed in a separate paper.

Finally, we discuss in Section 4 other sequences of states which yield reconstruction formulas. The scattering operator between two-cluster channels where bound states are admitted can be used to reconstruct an effective potential between the projectile and the target.

These and related results, in particular rates of convergence, will be proved under weaker assumptions in a forthcoming paper [3]. A pedagogical presentation of the analogous approach to the inverse problem in two-body potential scattering is given in [2]. For the general background of $N$-body scattering theory see e.g. [6].

Previous work on uniqueness for inverse scattering is described in [3]. The best results in the case $N = 2$ had been obtained by Saito [7] and for arbitrary $N$ by Wang [8] for regular short-range potentials using stationary methods. They are covered by our approach.

We consider a system of $N$ nonrelativistic quantum particles with masses $m_j$ and positions $\tilde{\mathbf{x}}_j \in \mathbf{R}^\nu$ with $\nu \geq 2$. The free time evolution is generated by the free Hamiltonian

$$\tilde{H}_0 = \sum_{j=1}^{N} (2m_j)^{-1}\, \tilde{\mathbf{p}}_j^2\,, \quad \tilde{\mathbf{p}}_j = -i\, \nabla_{\tilde{\mathbf{x}}_j}. \tag{1.1}$$

As usual we separate off the center of mass motion

$$H_{CM} = \left(2\sum_{j=1}^{N} m_j\right)^{-1} \left(\sum_{j=1}^{N} \tilde{\mathbf{p}}_j\right)^2$$

and obtain as a free Hamiltonian

$$H_0 := \tilde{H}_0 - H_{CM} \tag{1.2}$$

and as state space the Hilbert space $\mathcal{H}$ represented e.g. by configuration space wave functions $\psi$ in

$$L^2(X), \;\; X = \left\{ (\tilde{\mathbf{x}}_1, ..., \tilde{\mathbf{x}}_N) \,\middle|\, \sum_{j=1}^{N} m_j\, \tilde{\mathbf{x}}_j = 0 \right\} \cong \mathbf{R}^{\nu(N-1)}$$

with measure on $X$ induced by the norm $|||\mathbf{x}|||^2 = \sum_j m_j \mathbf{x}_j^2$ on $\mathbf{R}^{\nu N}$ which is equivalent to Lebesgue measure on $\mathbf{R}^{\nu(N-1)}$. Fourier transformation maps $L^2(X)$ unitarily to

$$L^2(\hat{X}), \;\; \hat{X} = \left\{ (\tilde{\mathbf{p}}_1, ..., \tilde{\mathbf{p}}_N) \,\middle|\, \sum_{j=1}^{N} \tilde{\mathbf{p}}_j = 0 \right\} \cong \mathbf{R}^{\nu(N-1)},$$

the set of momentum space wave functions $\hat{\psi}$, where $\hat{X}$ is equipped with the dual metric induced by $\sum_j (m_j)^{-1} \tilde{\mathbf{p}}_j^2$ on $\mathbf{R}^{\nu N}$. For the given (abstract) state $\Psi \in \mathcal{H}$ we use both its configuration or momentum-space wave functions $\psi$, $\hat{\psi}$, respectively, where appropriate. $H_0$ is self-adjoint on its domain $\mathcal{D}(H_0) = W^{2,2}(X)$. The potential is assumed to be a sum of pair potentials which are multiplication operators

$$V = \sum_{i<j} V_{ij}(\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_i). \tag{1.3}$$

Each pair potential may be split into parts of short range or long range depending on their differentiability and decay rate towards infinity:

$$V_{ij}(\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_i) = V_{ij}^s(\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_i) + V_{ij}^\ell(\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_i),$$

$$\mathcal{V}_{SR} = \left\{ V^s = \sum_{i<j} V_{ij}^s(\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_i) \,\middle|\, V_{ij}^s \in C(\mathbf{R}^\nu),\ \int_0^\infty dR \sup_{|\mathbf{y}|\geq R} |V_{ij}^s(\mathbf{y})| < \infty \right\}, \tag{1.4}$$

$$\mathcal{V}_{LR} = \left\{ V^\ell = \sum_{i<j} V_{ij}^\ell(\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_i) \,\middle|\, V_{ij}^\ell \in C_\infty^1(\mathbf{R}^\nu),\ |\nabla V_{ij}^\ell(\mathbf{y})| \leq C(1+|\mathbf{y}|)^{-\gamma}, \gamma > \frac{3}{2} \right\}. \tag{1.5}$$

The continuity of the short-range parts is assumed here only for convenience of presentation. The space $C_\infty^1$ consists of differentiable functions which tend to 0 towards infinity. The splitting is not unique. Without loss of generality [5] we may make it in such a way that in addition

$$V_{ij}^\ell \in C^2(\mathbf{R}^\nu), \quad |(\Delta V_{ij}^\ell)(\mathbf{y})| \leq \mathrm{const}(1+|\mathbf{y}|)^{-2-\varepsilon}, \quad \varepsilon > 0 \tag{1.6}$$

is satisfied. The interacting Hamiltonian is

$$H = H_0 + V = H_0 + V^s + V^\ell \tag{1.7}$$

which is self-adjoint on $\mathcal{D}(H) = \mathcal{D}(H_0)$. It generates the time-evolution $e^{-iHt}$. Consider first the short-range case, i.e. $V_{ij}^\ell = 0$ for all $i < j$. Then the wave operators for the free channel are

$$\Omega_\pm = s - \lim_{t\to\pm\infty} e^{iHt}\, e^{-iH_0t}. \tag{1.8}$$

There are other channel wave operators where some particles are bounded. We will not need them here, however, see Section 4. Existence of the limits is well known, it will follow on certain vectors as a byproduct of our estimates. In particular $(\Omega_-)^*\,\Omega_- = \mathbf{1}$, $e^{-iHt}\,\Omega_\pm = \Omega_\pm\, e^{-iH_0t}$, and

$$\Omega_+ - \Omega_- = \int_{-\infty}^{\infty} dt\; e^{iHt}\; i(H-H_0)\; e^{-iH_0t} \tag{1.9}$$

between any vectors for which the integral is defined, e.g. $\Phi$, $\Psi \in \mathcal{D}(H_0)$. The scattering operator $S$ from the free channel to the free channel is defined as

$$S = (\Omega_+)^*\,\Omega_- \tag{1.10}$$

which implies $(S - \mathbf{1}) = (\Omega_+ - \Omega_-)^*\,\Omega_-$. We call $S$ as a mapping from $\mathcal{V}_{SR}$ into the bounded operators $S = S(V)$ the *scattering map.*

**Theorem 1.1.** *The scattering map $S : \mathcal{V}_{SR} \to L(\mathcal{H})$ is injective.*

If long-range forces are present we assume that a splitting according to (1.3)-(1.6) has been made and is kept fixed. The Dollard modified free time evolution on the free channel is generated by

$$H^D(t) := H_0 + \sum_{i<j} V_{ij}^\ell \left(t\, \mathbf{p}_{ij}/\mu_{ij}\right) \tag{1.11}$$

where $\mathbf{p}_{ij}/\mu_{ij} = (\tilde{\mathbf{p}}_j/m_j) - (\tilde{\mathbf{p}}_i/m_i)$ is the relative velocity of particles $i$ and $j$. The propagator is again a unitary multiplication operator on momentum space, for initial time 0 it reads

$$U^D(t,0) := e^{-iH_0 t}\ \exp\left\{-i \sum_{i<j} \int_0^t ds\, V_{ij}^\ell (s\, \mathbf{p}_{ij}/\mu_{ij})\right\}. \tag{1.12}$$

Then the modified Dollard wave operators on the free channel exist:

$$\Omega_\pm^D = s - \lim_{t\to\pm\infty}\ e^{iHt}\ U^D(t,0). \tag{1.13}$$

They are isometric and satisfy

$$\Omega_+^D - \Omega_-^D = \int_{-\infty}^{\infty} dt\ e^{iHt}\ i\left(H - H^D(t)\right)\ U^D(t,0) \tag{1.14}$$

between suitable states. The Dollard scattering operator between the free channels is

$$S^D = (\Omega_+^D)^*\ \Omega_-^D = S^D(V^\ell; V^s) \tag{1.15}$$

with $(S^D - \mathbf{1}) = (\Omega_+^D - \Omega_-^D)^*\ \Omega_-^D$.

The only long-range potential in physics is the Coulomb (or gravitational) potential. If the charges $q_i$ of the particles are known then e.g. $V_{ij}^\ell(\mathbf{y}) = q_i\, q_j (1 + \mathbf{y}^2)^{-1/2}$ describes the long-range part and other effects are of short range. In general let $V = \bar{V}^s + \bar{V}^\ell$ , $\bar{V}^s \in \mathcal{V}_{SR}$, $\bar{V}^\ell \in \mathcal{V}_{LS}$ be given. Then we adapt the splitting if necessary such that $V = V^s + V^\ell$ where $V^\ell$ satisfies in addition (1.6). This $V^\ell$ is then kept fixed and is used in the definition of $H^D$ and $U^D$. The next theorem states that for given long-range tail of the potential the short-range part is determined uniquely by the scattering map.

**Theorem 1.2.** *Let a long-range potential $V^\ell \in \mathcal{V}_{LR}$ be given (satisfying without loss of generality also (1.6) ) then the scattering map $S^D(V^\ell; \cdot) : \mathcal{V}_{SR} \to L(\mathcal{H})$ is injective.*

We will show at the end of the next section that both theorems follow easily from the reconstruction formula which we state there as Theorem 2.1.

# 2 A Reconstruction Formula

For any distinguished pair out of the $N$ particles we will construct a sequence of states where all particles have high velocities relative to each other. In the limit, the properly rescaled deviation of the scattering operator from the identity gives the Radon- or $X$-ray-transform of the corresponding pair potential or of its short-range part. This implies the uniqueness theorems of the previous section.

The family of states is chosen here for mathematical convenience. Other choices which are closer to physical situations will be discussed in Section 4. Before specifying the states we have to introduce some notation. To determine the potential for a given pair we use a numbering of the particles such that the pair of interest consists of particles 1 and 2. After separating off the center of mass motion we use various coordinates to describe the relative positions etc. of the particles. One $\nu$-dimensional variable is the relative position $\mathbf{x}$ and momentum $\mathbf{p}$ in the distinguished pair as is usual for two body systems

$$\mathbf{x} := \tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1, \ \mathbf{p} := -i\,\nabla_{\mathbf{x}} = m[(-i\nabla_{\tilde{\mathbf{x}}_2}/m_2) - (-i\nabla_{\tilde{\mathbf{x}}_1}/m_1)],$$

$$m = m_1\, m_2\,/\,(m_1 + m_2). \tag{2.1}$$

In addition, we use the position $\mathbf{x}_\ell$ and the momentum $\mathbf{p}_\ell$ of the $\ell$-th particle, $\ell = 3, \ldots, N$, relative to the center of mass of the distinguished pair. We denote the corresponding reduced masses by

$$\mu_\ell = m_\ell(m_1 + m_2)/(m_\ell + m_1 + m_2) \tag{2.2}$$

and for two particles not in the pair

$$\mu_{k\,\ell} = m_k\, m_\ell\,/(m_k + m_\ell), \quad k, \ell \in \{3,\ \ldots, N\}. \tag{2.3}$$

Denoting as in (1.1) by $\tilde{\mathbf{p}}_j := -i\,\nabla_{\tilde{\mathbf{x}}_j}$ the momentum relative to some origin, then

$$\mathbf{x}_\ell := \tilde{\mathbf{x}}_\ell - (m_1\tilde{\mathbf{x}}_1 + m_2\tilde{\mathbf{x}}_2)/(m_1 + m_2), \quad \ell = 3, \ldots, N, \tag{2.4}$$

$$\mathbf{p}_\ell = \mu_\ell \left( \frac{\tilde{\mathbf{p}}_\ell}{m_\ell} - \frac{\tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2}{m_1 + m_2} \right), \quad \ell = 3,\ \ldots, N, \tag{2.5}$$

are the positions and momenta and $\mathbf{p}_i/\mu_i$ the velocities relative to the center of mass of the pair. $\{\mathbf{x}, \mathbf{x}_3,\ \ldots, \mathbf{x}_N\}$ and $\{\mathbf{p}, \mathbf{p}_3,\ \ldots, \mathbf{p}_N\}$ are sets of $N-1$ independent $\nu$-dimensional configuration and momentum coordinates in the total center of mass frame. The relative momentum of particles $k$ and $\ell$ is

$$\mathbf{p}_{k\ell} = -i\,\nabla_{(\tilde{\mathbf{x}}_\ell - \tilde{\mathbf{x}}_k)} \tag{2.6}$$

where in the derivative the positions of all other particles as well as of the total center of mass are kept fixed. Their relative velocity is

$$\frac{\mathbf{p}_{k\ell}}{\mu_{k\ell}} = \frac{\tilde{\mathbf{p}}_\ell}{m_\ell} - \frac{\tilde{\mathbf{p}}_k}{m_k} = \frac{\mathbf{p}_\ell}{\mu_\ell} - \frac{\mathbf{p}_k}{\mu_k}. \tag{2.7}$$

We choose vectors $\Psi_{\mathbf{0}} \in \mathcal{H}$, $\|\Psi_{\mathbf{0}}\| = 1$, with product wave functions of the following form in momentum space

$$\Psi_{\mathbf{0}} \sim \hat{\psi}(\mathbf{p})\ \hat{\psi}_3(\mathbf{p}_3,\ \ldots, \mathbf{p}_N). \tag{2.8}$$

Here $\hat{\psi} \in C_0^\infty(\mathbf{R}^\nu)$ varies while $\hat{\psi}_3 \in C_0^\infty(\mathbf{R}^{\nu(N-2)})$ is a fixed normalized function with support in $\{(\mathbf{p}_3,\ \ldots, \mathbf{p}_N)|\ |\mathbf{p}_\ell| < \mu_\ell\}$, i.e. particles 3 to $N$ have speed smaller than one relative to the distinguished pair. Let $\hat{R} \geq 1$ be a bound for the radius of the velocity support of $\hat{\psi}$, i.e. $\hat{\psi}(\mathbf{p}) = 0$ for $|\mathbf{p}| \geq m\,\hat{R}$.

$$\mathbf{v} = v\,\hat{\mathbf{v}} \;\;,\; |\hat{\mathbf{v}}| = 1\,, v \geq 0 \tag{2.9}$$

is a velocity vector ($\hat{\mathbf{v}}$ a unit velocity vector, $v$ a dimensionless constant). It will turn out to be sufficient to consider $\hat{\mathbf{v}}$ in a circle, the intersection of $S^{\nu-1}$ with an arbitrary given plane through the origin. $\mathbf{e}_1$ is the unit velocity vector in the direction of the first coordinate axis. Then define

$$\Psi_{\mathbf{v}} \sim \hat{\psi}(\mathbf{p} - m\,\mathbf{v})\;\hat{\psi}_3(\mathbf{p}_3 - 2\cdot 3\mu_3 v^2\mathbf{e}_1,\;\ldots,\mathbf{p}_\ell - 2\,\ell\mu_\ell v^2\mathbf{e}_1,\ldots). \tag{2.10}$$

For large $v$ the relative velocity of the particles in the chosen pair (1,2) is approximately $\mathbf{v}$ while all other pairs separate with a speed of at least $2(v^2-1)$ or $(6v^2 - v - 1 - \hat{R})$. Since the effect of the interaction decreases for high velocities the potential part $V_{12}$ will be dominant in the limit. Let analogous to $\Psi_{\mathbf{v}}$

$$\Phi_{\mathbf{v}} \sim \hat{\phi}(\mathbf{p} - m\,\mathbf{v})\,\hat{\psi}_3(\ldots,\mathbf{p}_\ell - 2\,\ell\mu_\ell v^2\mathbf{e}_1,\ldots) \tag{2.11}$$

with the *same* $\hat{\psi}_3$. The wave functions of $\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}$ belong to the Schwartz space $\mathcal{S}(\mathbf{R}^{\nu(N-1)})$ and have rapid decay as $|\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_i| \to \infty$ for all $i, j$ *uniformly* in $\mathbf{v}$ (because a shift in velocity space amounts to multiplication by a phase factor in configuration space).

Now we are ready to state the reconstruction theorem.

**Theorem 2.1** *Let $H = H_0 + V^\ell + V^s$ satisfy (1.2) - (1.5) [and without loss of generality also (1.6)] and be $\Phi_{\mathbf{v}}$, $\Psi_{\mathbf{v}}$ as given in (2.10),(2.11). Then*

$$\lim_{v\to\infty} v\,(\Phi_{\mathbf{v}},\; i(S^D - \mathbf{1})\;\Psi_{\mathbf{v}})$$
$$= \int d^\nu x\;\overline{\phi(\mathbf{x})}\;\psi(\mathbf{x})\quad\int d\tau\{V^s_{12}(\mathbf{x} + \hat{\mathbf{v}}\tau) + V^\ell_{12}(\mathbf{x} + \hat{\mathbf{v}}\tau) - V^\ell_{12}(\hat{\mathbf{v}}\tau)\}. \tag{2.12}$$

The integrals exist due to the integrable decay (1.4) of $V^s_{12}$ and (1.5) of $(\nabla V^\ell_{12})$ and the rapid decay of the wave functions. Since the chosen pair was arbitrary this permits to recover the pair potential $V^s_{ij}$ for any pair. In the short-range case where $V^\ell_{ij} = 0$ for all $i < j$ we have $S^D = S$, the usual scattering operator. Thus this case is covered as a special case in the above theorem. In $\nu = 1$ dimension one gets in our approach only the numbers $(\Phi_{\mathbf{0}}, \Psi_{\mathbf{0}})\;\int d\tau\;V^s_{12}(\pm\tau)$, etc., which is insufficient for uniqueness. For the one-dimensional two-body inverse scattering problem it is well known that uniqueness fails. See Section 4 for reconstruction from other channel scattering operators and other sequences of states.

If in addition to the pair potentials one has three-body or multiparticle forces then they are not found with this approach. However, one may apply the chain rule and go first from the free motion to an evolution which includes only the pair potentials and in a next step add three-body potentials etc. If the potentials have some more regularity then one can recover multiparticle forces as well. See [8] for a similar approach.

We will finish this section by proving the uniqueness theorems.

**Proof of Theorems 1.1 and 1.2.** The integrated functions

$$W^s_{12}(\mathbf{x};\hat{\mathbf{v}}) := \int d\tau\;V^s_{12}(\mathbf{x} + \hat{\mathbf{v}}\tau)$$

$$W^\ell_{12}(\mathbf{x};\hat{\mathbf{v}}) := \int d\tau [V^\ell_{12}(\mathbf{x}+\hat{\mathbf{v}}\tau) - V^\ell_{12}(\hat{\mathbf{v}}\tau)]$$

exist, are continuous, and independent of the component of $\mathbf{x}$ parallel to $\hat{\mathbf{v}}$. $W^\ell_{12}$ is known (or zero). The family of functions $\psi, \phi$ is rich enough to determine the function $W^s_{12}(\cdot;\hat{\mathbf{v}})$ from (2.12) for any parameter $\hat{\mathbf{v}}$. It is sufficient to choose the directions $\hat{\mathbf{v}}$ on a circle, e.g. as unit vectors in the $x^1 - x^2$-plane. In dimension $\nu \geq 3$ keep then $\bar{x} = (x^3, \ldots, x^\nu)$ arbitrarily fixed and denote as $\tilde{V}^s_{12}(x^1, x^2)$ , $\tilde{W}^s_{12}(x^1, x^2; \hat{\mathbf{v}})$, the same objects as functions depending on the first two variables. The decay rate (1.4) ensures that $\tilde{V}^s_{12} \in L^2(\mathbf{R}^2)$ for any $\bar{x}$. Moreover, in two dimensions the integral along a line (the "$X$-ray transform") coincides with the Radon transform (integration over hyperplanes). Thus for any $\bar{x}$ the function $\tilde{W}^s_{12}$ is the Radon transform of the square integrable $\tilde{V}^s_{12}$. This transformation is known to be invertible, see Theorem 2.17 in Chapter I of [4]. □

# 3 Proof of the Reconstruction Theorem

In this section we give the main part of the proof of Theorem 2.1. The scattering operator between the totally free channels satisfies

$$i(S^D - \mathbf{1}) = \int_{-\infty}^{\infty} dt\, U^D(t,0)^* \, \left\{H - H^D(t)\right\} e^{-iHt}\, \Omega^D_-$$

$$= \int dt\, U^D(t,0)^* \left\{H - H^D(t)\right\}\, U^D(t,0)$$

$$+ \int dt\, U^D(t,0)^* \left\{H - H^D(t)\right\}\, e^{-iHt} \left[\Omega^D_- - e^{iHt}\, U^D(t,0)\right]. \tag{3.1}$$

The term in square brackets equals

$$-i \sum_{i<\ell} \int_{-\infty}^{t} ds\, e^{iHs} \left\{H - H^D(t)\right\}\, U^D(s,0). \tag{3.2}$$

From the usual estimates [1] on propagation into the classically forbidden region $\{|\tilde{x}_j - \tilde{x}_i| < v/2,\ i < j = 2, \ldots, N\}$ for the free and modified free time evolutions it follows that uniformly for $v \geq 2\hat{R} \geq 2$

$$v \int_{-\infty}^{\infty} dt \left\| \left\{H - H^D(t)\right\}\, U^D(t,0)\, \Psi_{\mathbf{v}} \right\| \leq M < \infty \tag{3.3}$$

and the same for $\Phi_{\mathbf{v}}$. See [3] for the complete derivation. Then the second summand in (3.1) is by a factor $1/v$ smaller than the first and can be neglected in the high velocity limit:

$$v \left(\Phi_{\mathbf{v}},\, i\,(S^D - \mathbf{1})\, \Psi_{\mathbf{v}}\right)$$

$$= v \int dt\, \left(\Phi_{\mathbf{v}},\, U^D(t,0)^* \left\{H - H^D(t)\right\}\, U^D(t,0)\, \Psi_{\mathbf{v}}\right) + O(1/v). \tag{3.4}$$

The estimate (3.3) ensures in particular the existence of modified (or ordinary) wave operators. Similarly one shows [1, 3] for the classically forbidden region $\{|\tilde{x}_\ell - \tilde{x}_i| < v^2,\ i < \ell = 3, \ldots, N\}$ uniformly for large $v$

$$v^2 \int dt \left\| \Big\{ [H - V_{12}(\mathbf{x})] - [H^D(t) - V_{12}^\ell(t\,\mathbf{p}/m)] \Big\}\ U^D(t,0)\ \Psi_{\mathbf{v}} \right\| \leq M < \infty. \tag{3.5}$$

Consequently, the leading term in (3.4) is up to an error of order $O(1/v)$

$$\begin{aligned} & v \int_{-\infty}^{\infty} dt\ \Big(\Phi_{\mathbf{v}},\ U^D(t,0)^*\ \Big\{V_{12}(\mathbf{x}) - V_{12}^\ell(t\,\mathbf{p}/m)\Big\}\ U^D(t,0)\ \Psi_{\mathbf{v}}\Big) \\ &= \int_{-\infty}^{\infty} d\tau\ \Big(\Phi_{\mathbf{v}},\ U^D(\tau/v,0)^*\ \Big\{V_{12}(\mathbf{x}) - V_{12}^\ell(\tau\,\mathbf{p}/mv)\Big\}\ U^D(\tau/v,0)\ \Psi_{\mathbf{v}}\Big) \end{aligned} \tag{3.6}$$

with the substitution $\tau = t\,v$. Note that we had defined $\tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1 = \mathbf{x}$ and $\mathbf{p}$ its conjugate momentum. An easy consequence [3] of (3.3) and (3.5) is the estimate

$$\Big|\Big(\Phi_{\mathbf{v}},\ U^D(\tau/v,\ 0)^* \{V_{12}(\mathbf{x}) - V_{12}^\ell(\tau\,\mathbf{p}/m\,v)\}\ U^D(\tau/v,\ 0)\ \Psi_{\mathbf{v}}\Big)\Big| \leq h(\tau) \tag{3.7}$$

with some $h \in L^1(\mathbf{R})$ uniformly for $v \geq 2\hat{R} \geq 2$. Thus by the dominated convergence theorem we may take the limit $v \to \infty$ pointwise w.r.t. $\tau$ in the integrand of (3.6). Clearly,

$$\begin{aligned} & \Big(\Phi_{\mathbf{v}},\ U^D(\tau/v,\ 0)^*\ V_{12}^\ell(\tau\,\mathbf{p}/m\,v)\ U^D(\tau/v,\ 0)\ \Psi_{\mathbf{v}}\Big) \\ &= \Big(\Phi_{\mathbf{v}},\ V_{12}^\ell(\tau\,\mathbf{p}/m\,v)\ \Psi_{\mathbf{v}}\Big) = \Big(\Phi_{\mathbf{0}},\ V_{12}^\ell(\tau(\mathbf{p} + m\,v\,\hat{\mathbf{v}})/m\,v)\ \Psi_{\mathbf{0}}\Big) \\ &\longrightarrow \Big(\Phi_{\mathbf{0}},\ V_{12}^\ell(\tau\,\hat{\mathbf{v}})\ \Psi_{\mathbf{0}}\Big) \quad \text{as } v \to \infty \end{aligned} \tag{3.8}$$

because $V_{12}^\ell$ is continuous and the support of $\hat{\psi}$ is compact. Moreover,

$$\left\| U^D(\tau/v, 0) - e^{-iH_0\,\tau/v} \right\| \leq \frac{\tau}{v} \sum_{i<j} \|V_{ij}^\ell\| \ \to 0 \quad \text{as } v \to \infty \tag{3.9}$$

as can be seen from the explicit form (1.12) of the Dollard propagator.

$$\begin{aligned} & (\Phi_{\mathbf{v}},\ e^{iH_0\tau/v}\ \ V_{12}(\mathbf{x})\ e^{-iH_0\tau/v}\ \Psi_{\mathbf{v}}) \\ &= (\Phi_{\mathbf{v}},\ V_{12}(\mathbf{x} + \mathbf{p}\tau/m\,v)\ \Psi_{\mathbf{v}}) = (\Phi_{\mathbf{0}},\ V_{12}(\mathbf{x} + \hat{\mathbf{v}}\tau + \mathbf{p}\tau/m\,v)\ \Psi_{\mathbf{0}}) \\ &\longrightarrow (\Phi_{\mathbf{0}},\ V_{12}(\mathbf{x} + \hat{\mathbf{v}}\tau)\ \Psi_{\mathbf{0}}) \quad \text{as } v \to \infty. \end{aligned} \tag{3.10}$$

Thus we have shown Theorem 2.1:

$$\begin{aligned} & \lim_{v\to\infty} \Big(\Phi_{\mathbf{v}},\ i(S^D - \mathbf{1})\ \Psi_{\mathbf{v}}\Big) \\ &= \int d\tau\ \Big(\Phi_{\mathbf{0}},\ \{V_{12}^s(\mathbf{x} + \hat{\mathbf{v}}\tau) + V_{12}^\ell(\mathbf{x} + \hat{\mathbf{v}}\tau) - V_{12}^\ell(\hat{\mathbf{v}}\tau)\}\ \Psi_{\mathbf{0}}\Big) \\ &= \int d\mathbf{x}\ \overline{\phi(\mathbf{x})}\ \psi(\mathbf{x}) \int d\tau \{V_{12}^s(\mathbf{x} + \hat{\mathbf{v}}\tau) + V_{12}^\ell(\mathbf{x} + \hat{\mathbf{v}}\tau) - V_{12}^\ell(\hat{\mathbf{v}}\tau)\} \end{aligned} \tag{3.11}$$

provided (3.3), (3.5), and (3.7) have been verified. This finishes the essential parts of our proof.

**Remark**: Our choice of relative speed proportional to $v^2$ for all particles except those in the pair is not essential, it gave an error const$/v$ in (3.4)–(3.6) with a constant which is in principle explicit. Any other speed $w$ with $v/w \to 0$ would yield the same theorem.

# 4 Other Approximating Sequences

We briefly indicate two other possibilities to construct sequences of high velocity states which yield asymptotically information about the potentials. We illustrate it for short-range potentials only. Let the states $\Psi_{\mathbf{0}}$ and $\Phi_{\mathbf{0}}$ be as above in (2.8) and (2.10). Then we define the momentum space wave function of the family of translated states as

$$\Psi_{\mathbf{v},\,a} \sim \hat{\psi}(\mathbf{p} - m\mathbf{v})\, e^{i3a\hat{\mathbf{v}}\cdot\mathbf{p}_3}\ \cdots\ e^{i\ell a\hat{\mathbf{v}}\cdot\mathbf{p}_\ell}\ \cdots\ \hat{\psi}_3(\mathbf{p}_3 - 3\mu_3 v\tilde{\mathbf{v}},\ \ldots,\ \mathbf{p}_\ell - \ell\mu_\ell v\tilde{\mathbf{v}},\ \ldots).$$

Here $\tilde{\mathbf{v}}$ is orthogonal to $\hat{\mathbf{v}}$, thus all relative velocities are proportional to $v$ and, moreover, all particles except those in the distinguished pair have a separation proportional to $a$ in a direction which is not parallel to their relative velocity. Analogous to Theorem 2.1 we obtain the short-range reconstruction formula

$$\lim_{v,\,a\to\infty} v\,(\Phi_{\mathbf{v},\,a},\ i\,(S-\mathbf{1})\,\Psi_{\mathbf{v},\,a}) = \int d^\nu x\ \overline{\phi(\mathbf{x})}\,\psi(\mathbf{x}) \int d\tau\ V^s_{12}(\mathbf{x} + \hat{\mathbf{v}}\tau). \tag{4.1}$$

The simple proof follows from the fact that e.g.

$$\lim_{a\to\infty} \int d\tau\ V^s_{34}(\tilde{\mathbf{x}}_4 - \tilde{\mathbf{x}}_3 + a\hat{\mathbf{v}} + \tilde{\mathbf{v}}\tau)\ \Psi_{\mathbf{0}} = 0$$

and similarly for potentials like $V^s_{13}$. If long-range potentials are present then one has to use an additional correction because

$$\int d\tau \left\{ V^\ell_{34}(a\hat{\mathbf{v}} + \tilde{\mathbf{v}}\tau) - V^\ell_{34}(\tilde{\mathbf{v}}\tau) \right\} \Psi_{\mathbf{0}}$$

typically diverges as $a \to \infty$. With the corresponding adjustments one gets (2.12) as well.

In a physical scattering experiment it is unrealistic to give all particles high velocities relative to each other. A less unrealistic choice is a single particle which hits with high velocity a target consisting of $N-1$ particles in a bound state. Since the strength of the interaction decreases with increasing speed the incoming and outgoing channels are the same in the high velocity limit and the bound state is not excited. Certainly it is impossible to reconstruct the individual pair potentials but the high velocity limit of the scattering operator determines uniquely the effective potential (4.7) between the projectile and the bounded cluster. We restrict the description again to the short-range case. This means that we assume implicitly a sufficiently rapid decay of the bound state function. Typically they decay exponentially which is more than enough, but if the bound state energy happens to be a threshold value then this assumption may be violated.

Let $\Psi_b$ be a bound state of the $N-1$ particle subsystem with energy $E_b$ in its center of mass frame. Denoting here by $\mathbf{x}$ and $\mathbf{p}$ the position and momentum of the first particle relative to the center of mass of the cluster, then

$$\Psi^{(2)}_{\mathbf{0}} = \Psi \otimes \Psi_b \sim \psi(x) \otimes \psi_b \tag{4.2}$$

denotes for $\psi \in L^2(\mathbf{R}^\nu)$ the vectors in the scattering channel $\mathcal{H}^{(2)}$ and for $\hat{\psi} \in C^\infty_0(\mathbf{R}^\nu)$ the states of interest here. Then for large $\mathbf{v}$ we the high velocity states are

$$\Psi^{(2)}_{\mathbf{v}} \sim \hat{\psi}(\mathbf{p} - m\mathbf{v}) \otimes \hat{\psi}_b. \tag{4.3}$$

The channel time evolution for this two-cluster situation is generated on $\mathcal{H}^{(2)}$ by

$$H^{(2)} = T_0 + E_b = H - \sum_{j=2}^{N} V^s_{1j}(\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_1). \tag{4.4}$$

$$T_0 = \mathbf{p}^2/2m$$

is the kinetic energy and $m$ the reduced mass of the first particle relative to the cluster. The channel wave operators and scattering operator are

$$\Omega^{(2)}_{\pm} = s - \lim_{t\to\infty} e^{iHt}\, e^{-iH^{(2)}t}, \quad S^{(2)} = \left(\Omega^{(2)}_{+}\right)^* \,\Omega^{(2)}_{-} \tag{4.5}$$

on states of type (4.2). As in the two-body case one obtains

$$\begin{aligned} &\lim_{v\to\infty} v\left(\Phi^{(2)}_{\mathbf{v}},\, i\,(S^{(2)} - \mathbf{1})\; \Psi^{(2)}_{\mathbf{v}}\right) \\ &= \int d^\nu x\; \overline{\phi(\mathbf{x})}\; \psi(\mathbf{x}) \int d\tau\; V_{\text{eff}}(\mathbf{x} + \hat{\mathbf{v}}\tau) \end{aligned} \tag{4.6}$$

for short-range effective potentials

$$V_{\text{eff}}(\mathbf{x}) := \sum_{j=2}^{N} \left(\Psi_b,\; V^s_{1j}(\tilde{\mathbf{x}}_j - \tilde{\mathbf{x}}_1)\; \Psi_b\right)_b . \tag{4.7}$$

The partial inner product $(\cdot,\,\cdot)_b$ for the state space of the cluster involves integration over all relative positions of particles $2, \ldots, N$ relative to their center of mass. Therefore the effective potential depends only on $\mathbf{x}$. See [3] for further details.

# References


[1] V. Enss, *Propagation properties of quantum scattering states*, J. Funct. Anal. **52** (1983), 219 - 251.

[2] V. Enss and R. Weder, *Inverse potential scattering: a geometrical approach*, to appear in Proceedings Summer School in Mathematical Quantum Theory (UBC Vancouver, Aug. 1993), J.Feldman, R.Froese, and L.Rosen eds., CRM Proceedings and Lecture Notes, AMS, Providence 1994

[3] V. Enss and R. Weder, *The geometrical approach to multidimensional inverse scattering*, in preparation.

[4] S. Helgason, *Groups and Geometric Analysis*, Academic Press, Orlando,1984.

[5] L. Hörmander, *The existence of wave operators in scattering theory*, Math. Z. **146** (1976), 69 - 91.

[6] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, III. Scattering Theory*, Academic Press, New York, 1979.

[7] Y. Saito, *An asymptotic behavior of the $S$-matrix in the inverse scattering problem*, J. Math. Phys. **25** (1984), 3105 - 3111.

[8] X.P. Wang, *On the uniqueness of inverse scattering for $N$-body systems*, Preprint Univ. de Nantes 93/11-2.