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Inverse Potential Scattering: A Geometrical Approach

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1. Results

In short-range potential scattering the wave operators and the scattering operator S are naturally uniquely defined and have the expected properties. For space dimension $\nu \geq 2$ we show in this note by a simple time-dependent geometrical proof that the high velocity limit of the scattering operator determines the potential uniquely. In the long-range case one has to know (the tail part of) the long-range potential before one can even define modified wave operators and a corresponding scattering operator. Thus, given a long-range potential we show that the short-range part (and thus all) of the potential is uniquely determined by the high velocity behavior of the scattering operator. The result is known in the short-range case [**9**], but our approach seems to be simpler and it easily generalizes to include long-range potentials. Moreover, it extends to N-body systems interacting by pair potentials [**4**].

On the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^\nu), \nu \geq 2$ the self-adjoint free Hamiltonian is $H_0 = -(1/2\,m)\,\Delta$. For simplicity of presentation we give here the proof for bounded continuous potentials, the general case including singular potentials is treated in [**4**]. The class of short-range potentials consists here of multiplication operators V^s from the set

$$\mathcal{V}_{SR} := \{V \in C(\mathbb{R}^\nu) \mid \sup_{|\mathbf{x}| \geq R} |V(\mathbf{x})| \in L^1([0,\infty), dR)\}. \tag{1.1}$$

The interacting Hamiltonian $H = H_0 + V^s$ is self-adjoint on the same domain as H_0. It is well known that the wave operators

$$\Omega_\pm(H_0, H_0 + V^s) := \operatorname*{s-lim}_{t\to\pm\infty} e^{i(H_0+V^s)t}\; e^{-iH_0 t} \tag{1.2}$$

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exist and are complete and that the scattering operator

$$S(H_0, H_0 + V^s) := [\Omega_+(H_0, H_0 + V^s)]^* \, \Omega_-(H_0, H_0 + V^s) \tag{1.3}$$

is unitary on $\mathcal{H}$ etc., see e.g. [**8**] as a general reference for mathematical scattering theory. We will consider now the scattering operator as a mapping from the potentials into bounded (even unitary) operators and call it the "scattering map".

THEOREM 1.1. (short-range) *The scattering map*

$$\begin{aligned} S : \mathcal{V}_{SR} &\longrightarrow L(\mathcal{H}), \\ V^s &\longrightarrow S(H_0, H_0 + V^s) \end{aligned}$$

is injective.

Actually, the high energy behavior of S is enough to determine V^s, see Theorem 1.4 below for the explicit formula.

This result has a long history beginning with Faddeev [**5**] and Berezanskii [**1**]. Under essentially the same assumptions as above it was shown by Saito [**9**], see [**4**] for further references.

The differentiable long-range potential is assumed to satisfy

$$\lim_{|\mathbf{x}|\to\infty} V^\ell(\mathbf{x}) = 0, \quad |(\nabla V^\ell)(\mathbf{x})| \leq \mathrm{const}(1 + |\mathbf{x}|)^{-\gamma}, \quad \gamma > 3/2. \tag{1.4}$$

The splitting $V = V^\ell + V^s$ into long- and short-range parts is not unique. By [**7**, Lemma 3.3] one can choose it *without loss of generality* such that V^ℓ is smooth and in addition

$$|(\Delta V^\ell)(\mathbf{x})| \leq \mathrm{const}(1 + |\mathbf{x}|)^{-(2+\varepsilon)}, \quad \varepsilon > 0 \tag{1.5}$$

is satisfied. We will use this convention in the sequel. Under these assumptions the ordinary wave operators between $H^\ell := H_0 + V^\ell$ and $H = H^\ell + V^s$ exist on $P^{\mathrm{cont}}(H^\ell)\mathcal{H}$, the continuous spectral subspace of H^ℓ, and are complete.

THEOREM 1.2. (long-range) *The scattering map*

$$\begin{aligned} S^\ell : \mathcal{V}_{SR} &\longrightarrow L(\mathcal{H}), \\ V^s &\longrightarrow S(H^\ell, H^\ell + V^s) \end{aligned}$$

is injective.

This result is simple to state but it has the disadvantage that we cannot control the evolution generated by H^ℓ effectively and it is of little use in physics. We will only sketch its proof. However, an explicitly known time evolution like the Dollard evolution requires the prior knowledge of the long-range part of the potential. Without the latter it is impossible to even define modified wave operators and the scattering operator. Therefore, from now on a long-range part V^ℓ of the potential which satisfies (1.4) and (1.5) is given. The question how V^ℓ can be obtained from information about scattering of wave packets will be addressed in a later paper.

As a modified Dollard time evolution we use

$$U^D(t, \tau) := e^{-iH_0(t-\tau)} \, \exp\{-i \int_\tau^t ds \, V^\ell(s\,\mathbf{p}/m)\} \tag{1.6}$$

which is generated by $H^D(t) := H_0 + V^\ell(t\,\mathbf{p}/m)$. The corresponding modified Dollard wave operators are

$$\Omega^D_\pm(H_0, H_0 + V^\ell + V^s) := \operatorname*{s-lim}_{t\to\pm\infty} e^{i(H_0+V^\ell+V^s)t}\, U^D(t,0). \tag{1.7}$$

They are well known to exist and to be complete for any $V^s \in \mathcal{V}_{SR}$. The unitary modified Dollard scattering operator is

$$S^D(H_0, H_0 + V^\ell + V^s) = [\Omega^D_+(H_0, H_0 + V^\ell + V^s)]^*\, \Omega^D_-(H_0, H_0 + V^\ell + V^s). \tag{1.8}$$

The non-uniqueness of the splitting $V = V^\ell + V^s$ and of the definition (1.6) for U^D does not cause problems. The wave operators establish a connection between scattering states (given by their initial condition at $t = 0$) and asymptotic configurations. The latter can be represented by vectors in a Hilbert space in many different but equivalent ways. The scattering operator is a mapping from incoming to outgoing asymptotic configurations. Its concrete form clearly depends on the chosen representation. (1.6) - (1.8) are a convenient and common choice.

THEOREM 1.3. (long-range) *The Dollard modified scattering map*

$$\begin{aligned} S^D : \mathcal{V}_{SR} &\longrightarrow L(\mathcal{H}) \\ V^s &\longrightarrow S^D(H_0, H_0 + V^\ell + V^s) \end{aligned}$$

is injective.

Again the high velocity behavior is enough to reconstruct the potential. To make this precise we need some notation. Let $\Psi_\mathbf{0}$ be a normalized asymptotic configuration with compact momentum support in the open ball of radius $m\hat{R}$ around the origin

$$\hat{\psi}_\mathbf{0} \in C_0^\infty(B_{m\hat{R}}(0)), \quad \|\Psi_\mathbf{0}\| = 1. \tag{1.9}$$

For later use we mention that there is a function $g \in C_0^\infty(B_{m\hat{R}}(0))$ such that $g(\mathbf{p})\,\Psi_\mathbf{0} = \Psi_\mathbf{0}$ (i.e. $g = 1$ on the momentum support of $\Psi_\mathbf{0}$). The boosted configuration ($\mathbf{x}$ the position operator)

$$\Psi_\mathbf{v} := e^{i\,m\,\mathbf{v}\cdot\mathbf{x}}\,\Psi_\mathbf{0} \quad \Leftrightarrow \quad \hat{\psi}_\mathbf{v} \in C_0^\infty(B_{m\hat{R}}(m\,\mathbf{v})) \tag{1.10}$$

has velocity support of radius $\hat{R}$ around $\mathbf{v}$. We will use the high velocity limit in an arbitrary fixed direction $\hat{\mathbf{v}} = \mathbf{v}/|\mathbf{v}|$, $v := |\mathbf{v}| \to \infty$.

THEOREM 1.4. (reconstruction formula) *For* $\Psi_\mathbf{v}$, $\Phi_\mathbf{v}$ *as defined in* (1.9), (1.10) *the following high velocity limit exists for any* $\hat{R}, \hat{\mathbf{v}}$

$$\lim_{v\to\infty} i\,v\,(\Phi_\mathbf{v}, (S^D - \mathbf{1})\,\Psi_\mathbf{v})$$

$$= (\Phi_\mathbf{0}, \int_{-\infty}^{\infty} d\tau\, \left[V^s(\mathbf{x}+\hat{\mathbf{v}}\tau) + V^\ell(\mathbf{x}+\hat{\mathbf{v}}\tau) - V^\ell(\hat{\mathbf{v}}\tau)\right]\, \Psi_\mathbf{0}). \tag{1.11}$$

This contains short-range scattering as a special case with $V^\ell = 0$ and $S^D = S$. With the notation $\mathbf{x}^\perp := \mathbf{x} - (\mathbf{x}\cdot\hat{\mathbf{v}})\,\hat{\mathbf{v}} \equiv \mathbf{x} - x^\parallel \hat{\mathbf{v}}$ the integrals

$$W^s(\mathbf{x}^\perp;\hat{\mathbf{v}}) := \int d\tau\, V^s(\mathbf{x}+\hat{\mathbf{v}}\tau),$$
$$W^\ell(\mathbf{x}^\perp;\hat{\mathbf{v}}) := \int d\tau\, [V^\ell(\mathbf{x}+\hat{\mathbf{v}}\tau) - V^\ell(\hat{\mathbf{v}}\tau)]$$

exist and are continuous by (1.1) and (1.4), respectively, and W^ℓ is given (possibly zero).

Varying $\hat{R}$ in (1.9) the set of $\Phi_{\mathbf{0}}, \Psi_{\mathbf{0}}$ is rich enough to determine for any $\hat{\mathbf{v}}$ the continuous function $W^s(\cdot;\hat{\mathbf{v}})$ uniquely from the r.h.s. of (1.11):

$$\int dx\, \overline{\phi_{\mathbf{0}}(\mathbf{x})}\, \psi_{\mathbf{0}}(\mathbf{x})\, W^s(\mathbf{x}^\perp;\hat{\mathbf{v}}). \tag{1.12}$$

In $\nu = 2$ dimensions $W^s(\mathbf{x}^\perp;\hat{\mathbf{v}})$ is the Radon transform of $V^s(\mathbf{x})$, the latter is square integrable. It is known that it determines V^s uniquely [**6**, p. 115f]. In $\nu \geq 3$ dimensions it is the X-ray transform. One can fix arbitrarily $(x^3, \cdots, x^\nu)$ and apply the same to the resulting two-dimensional function. Thus, varying $\hat{\mathbf{v}}$ in a plane is actually sufficient to reconstruct $V^s(\mathbf{x})$ from $W^s(\mathbf{x}^\perp;\hat{\mathbf{v}})$. Alternatively, one can see this intuitively from looking at (1.12) in momentum space:

$$\int dp^\parallel\, dp^\perp\, d\tilde{p}^\perp\, \overline{\hat{\phi}_0(p^\parallel,\mathbf{p}^\perp)}\, \hat{V}^s(0,\mathbf{p}^\perp - \tilde{\mathbf{p}}^\perp)\, \hat{\psi}_0(p^\parallel,\tilde{\mathbf{p}}^\perp). \tag{1.13}$$

To obtain $\hat{V}^s(\mathbf{q})$ choose $\hat{\mathbf{v}}$ orthogonal to $\mathbf{q}$ such that $\mathbf{q} = \mathbf{q}^\perp$. Then with a normalized one-dimensional χ let $\hat{\psi}_0(p^\parallel,\tilde{\mathbf{p}}^\perp) = \chi(p^\parallel)\,\delta_n(\tilde{\mathbf{p}}^\perp)$, $\hat{\phi}_0(p^\parallel,\mathbf{p}^\perp) = \chi(p^\parallel)\,\delta_n(\mathbf{p}^\perp - \mathbf{q}^\perp)$ for a "δ-sequence" of square integrable functions δ_n of $\nu-1$-dimensional argument. Clearly, $V^s(\mathbf{x})$ or $\hat{V}^s(\mathbf{q})$ need not be determined pointwise but it is enough to know their action on a (form) core for H_0. This fixes $H = H_0 + V^s$ uniquely. See [**4**] for a complete proof.

Thus Theorems 1.1 and 1.3 follow from the more detailed Theorem 1.4. One can prove Theorem 1.2 directly but here it is easier to use the chain rule for wave operators.

Remark. For applications of our method to multiparticle systems we mention that completeness of the wave operators and unitarity of the scattering operator have not been used in the proofs. All we need is existence of (modified) wave operators on states whose Fourier transform has compact support away from the origin in momentum space. Actually, this follows as a byproduct of our estimates.

In the next section we give the simple and intuitive proof of Theorem 1.4 in the short-range case, the slightly more technical modifications to include long-range potentials follow in Section 3.

2. Proof in the Short-Range Case

We use the shorthands $\Omega_\pm = \Omega_\pm(H_0, H_0 + V^s)$ and $S = S(H_0, H_0+V^s)$. The wave operators are well known to be isometric : $\Omega_\pm^*\ \Omega_\pm = \mathbf{1}$ and to satisfy the intertwining relations $\Omega_\pm\, H_0 = H\,\Omega_\pm$. Moreover, the following identity holds

$$\Omega_+ - \Omega_- = i\int_{-\infty}^{\infty} dt\; e^{iHt}\; V^s\; e^{-iH_0 t}, \tag{2.1}$$

see e.g. [**8**]. Thus

$$
\begin{aligned}
i(S-\mathbf{1}) &= i(\Omega_+^*\,\Omega_- - \Omega_-^*\,\Omega_-) = i(\Omega_+ - \Omega_-)^*\Omega_- \\
&= \int dt\; e^{iH_0 t}\; V^s\; e^{-iHt}\;\Omega_- = \int dt\; e^{iH_0 t}\; V^s\Omega_-\; e^{-iH_0 t} \\
&= \int dt\; e^{iH_0 t}\; V^s\; e^{-iH_0 t} \;+\; \int dt\; e^{iH_0 t}\; V^s(\Omega_- - \mathbf{1})\; e^{-iH_0 t}. \qquad (2.2)
\end{aligned}
$$

The first term on the r.h.s. of (2.2) – the first Born approximation – is the leading term which decays as v^{-1} if the velocity $\mathbf{v}$ goes to infinity. It will be used to recover the potential. The second summand – representing "multiple scattering" – is of order $o(v^{-1}) = O(v^{-2})$ and can therefore be neglected asymptotically.

Intuitively it is clear that each instance of scattering by a short-range potential yields a factor v^{-1} : the strength of the interaction as measured by $(S-\mathbf{1})$ or $(\Omega_\pm - \mathbf{1})$ is proportional to the time which the particle spends in the interaction region where the potential is strong. This time is inverse proportional to the speed v.

Under translations in momentum or configuration space, generated by $\mathbf{x}$ or $\mathbf{p}$, respectively, we get

$$
e^{-im\mathbf{v}\cdot\mathbf{x}}\; f(\mathbf{p})\; e^{im\mathbf{v}\cdot\mathbf{x}} = f(\mathbf{p} + m\mathbf{v}), \qquad (2.3)
$$

in particular,

$$
e^{-im\mathbf{v}\cdot\mathbf{x}}\; e^{-iH_0 t}\; e^{im\mathbf{v}\cdot\mathbf{x}} = e^{-i\mathbf{p}\cdot\mathbf{v}t}\; e^{-iH_0 t}\; e^{-imv^2 t/2}, \qquad (2.4)
$$

$$
e^{i\mathbf{p}\cdot\mathbf{v}t}\; f(\mathbf{x})\; e^{-i\mathbf{p}\cdot\mathbf{v}t} = f(\mathbf{x} + \mathbf{v}\,t). \qquad (2.5)
$$

With $\Phi_{\mathbf{v}}, \Psi_{\mathbf{v}}$ as defined in (1.9), (1.10)

$$
\begin{aligned}
& v\int dt\; (\Phi_{\mathbf{v}}, e^{iH_0 t}\; V^s(\mathbf{x})\; e^{-iH_0 t}\; \Psi_{\mathbf{v}}) \\
&= \int d(v\,t)\; (\Phi_{\mathbf{0}}, e^{iH_0 t}\; V^s(\mathbf{x} + \mathbf{v}\,t)\; e^{-iH_0 t}\; \Psi_{\mathbf{0}}) \\
&= \int d\tau (\Phi_{\mathbf{0}}, e^{iH_0 \tau/v}\; V^s(\mathbf{x} + \hat{\mathbf{v}}\tau)\; e^{-iH_0\tau/v}\; \Psi_{\mathbf{0}}) \qquad (2.6)
\end{aligned}
$$

with the substitution $\tau := v\,t$ and $\hat{\mathbf{v}} = \mathbf{v}/v$ the arbitrary fixed unit vector in the direction of $\mathbf{v}$. For any $\tau \in \mathbb{R}$ we have $\lim_{v\to\infty} \exp\{-iH_0\tau/v\}\Psi_{\mathbf{0}} = \Psi_{\mathbf{0}}$. Thus the pointwise limit of the integral in (2.6) yields the desired limit (1.11) in Theorem 1.4 (without V^ℓ). To justify it we need a bound for all large enough v

$$
|(\Phi_{\mathbf{0}}, e^{iH_0\tau/v}\; V^s(\mathbf{x} + \hat{\mathbf{v}}\tau)\; e^{-iH_0\tau/v}\; \Psi_{\mathbf{0}})| \;\le\; h(\tau) \qquad (2.7)
$$

where h is integrable. The left hand side is dominated by

$$
\|V^s(\mathbf{x} + \hat{\mathbf{v}}\tau)\; F(|\mathbf{x}| \le |\tau|/2)\| + \|V^s\|\; \|F(|\mathbf{x}| \ge |\tau|/2)\; e^{-iH_0\tau/v}\; \Psi_{\mathbf{0}}\|. \qquad (2.8)
$$

The first summand is bounded by $h_1(\tau) := \sup_{|\mathbf{y}|\ge\tau/2}\; |V(\mathbf{y})|$ which is independent of v and integrable by (1.1). If $v > 2\,\hat{R}$ ($\hat{R}$ the radius of the velocity support of $\Psi_{\mathbf{0}}$) the second term describes free propagation into the "classically forbidden region" which is rapidly decaying: With g as introduced following (1.9), i.e. $g(\mathbf{p})\,\Psi_\mathbf{0} = \Psi_\mathbf{0}$, $g \in C_0^\infty(B_{m\hat{R}}(0))$, we get for $v \geq 4\,\hat{R}$

$$\begin{aligned}
&\|F(|x| \geq |\tau|/2)\, e^{-iH_0\tau/v}\ \Psi_\mathbf{0}\| \\
&\leq \|F(|x| \geq |\tau|/8)\Psi_\mathbf{0}\| \\
&\quad + \|F(|x| \geq |\tau|/8 + \hat{R}|\tau|/v + |\tau|/8)\, e^{-iH_0\tau/v}\ g(p)\, F(|x| \leq |\tau|/8)\| \\
&= h_2(\tau) + h_3(\tau).
\end{aligned} \tag{2.9}$$

Both terms decay rapidly in $(1+|\tau|)$ uniformly for large v: For the first it is obvious because $\psi_\mathbf{0} \in \mathcal{S}(\mathbb{R}^\nu)$; for the second it is a standard non-stationary phase estimate, see e.g. [**2**, Proposition 2.10] where $r = |\tau|/8$ is the distance from the classically allowed region. We have with $h = h_1 + h_2 + h_3$ an integrable bound for all $v \geq 4\,\hat{R}$. By the dominated convergence theorem we may take pointwise limits and obtain

$$\begin{aligned}
&\lim_{v\to\infty}\ v \int dt\ (\Phi_\mathbf{v},\, e^{iH_0t}\ V^s\ e^{-iH_0t}\ \Psi_\mathbf{v}) \\
&= (\Phi_\mathbf{0},\, \int d\tau\ V^s(\mathbf{x} + \hat{\mathbf{v}}\tau)\ \Psi_\mathbf{0}).
\end{aligned} \tag{2.10}$$

Remark. The free time evolution of a wave packet can be considered as consisting of two parts: a purely kinematical part of translation with the average velocity and an additional evolutionary part which changes the shape of the packet (spreading etc.). Locally in the interaction region the translational part is dominant in the high velocity limit yielding the simple formula. The evolutionary part which is absent for improper states with sharp energy can be neglected asymptotically likewise as in the stationary approach. This remains true for interacting time evolutions as well.

To finish the proof we have to show that the second "multiple scattering" term in (2.2) is small compared to the leading first one.

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty} dt\ \Big(\Phi_\mathbf{v},\ e^{iH_0t}\ V^s(\Omega_- - \mathbf{1})\ e^{-iH_0t}\ \Psi_\mathbf{v}\Big)\right| \\
&= \left|\int_{-\infty}^{\infty} dt\ \left(V^s\ e^{-iH_0t}\ \Phi_\mathbf{v},\ \int_{-\infty}^{0} dT\ e^{iHT}\ V^s\ e^{-iH_0(T+t)}\ \Psi_\mathbf{v}\right)\right| \\
&\leq \int_{-\infty}^{\infty} dt\ \|V^s\ e^{-iH_0t}\ \Phi_\mathbf{v}\|\ \int_{-\infty}^{\infty} dT\ \|V^s\ e^{-iH_0T}\ \Psi_\mathbf{v}\|.
\end{aligned} \tag{2.11}$$

With the same calculations (2.3) - (2.5) as above

$$\begin{aligned}
&v \int dt\ \|V^s(\mathbf{x})\, e^{-iH_0t}\ \Phi_\mathbf{v}\| = v \int dt\ \|V^s(\mathbf{x} + \mathbf{v}t)\, e^{-iH_0t}\ \Phi_\mathbf{0}\| \\
&\leq \int d(vt) \Big\{\|V^s(\mathbf{x} + \mathbf{v}t)\, F(|\mathbf{x} + \mathbf{v}t| > |t\,v|/2)\| \\
&\qquad\qquad + \|F(|x| > |t\,v|/2)\, e^{-iH_0t}\ \Phi_\mathbf{0}\|\Big\}.
\end{aligned} \tag{2.12}$$

The first integral is finite uniformly in v by (1.1) and the same applies to the second by (2.9). This shows that (2.11) is $O(v^{-2})$ at infinity and thus negligible compared to the leading term (2.10) in the high velocity limit. This finishes the proof of Theorem 1.4 in the case $V^\ell = 0$. It implies also Theorem 1.1. $\square$

Remark. In the situation of Theorem 1.2 one has $[\Omega^\ell_\pm]^* \, \Omega^\ell_\pm = P^{\rm cont}(H^\ell)$, $H^\ell = H_0 + V^\ell$, and one obtains analogously

$$\lim_{v\to\infty} i\, v(\Phi_{\mathbf{v}},\, (S^\ell - P^{\rm cont}(H^\ell))\, \Psi_{\mathbf{v}}) = (\Phi_{\mathbf{0}}, \int d\tau\; V^s(\mathbf{x} + \hat{\mathbf{v}}\tau)\; \Psi_{\mathbf{0}}). \tag{2.13}$$

Asymptotically, $\Psi_{\mathbf{v}}$ is orthogonal to all bound states of H^ℓ. It is easy to verify that for fixed τ

$$\lim_{v\to\infty} \; e^{imv\tau/2}\; e^{i\mathbf{p}\cdot\hat{\mathbf{v}}\tau}\; e^{-im\mathbf{v}\cdot\mathbf{x}}\; e^{-i(H^\ell)\tau/v}\; e^{im\mathbf{v}\cdot\mathbf{x}}\; \Psi_{\mathbf{0}}$$

$$= \lim_{v\to\infty}\; e^{i\mathbf{p}\cdot\hat{\mathbf{v}}\tau}\; e^{-i\mathbf{p}\cdot\hat{\mathbf{v}}\tau + i(H^\ell)\tau/v}\; \Psi_{\mathbf{0}} = \Psi_{\mathbf{0}}.$$

However, the uniform bound corresponding to (2.7) is harder in this case. See the end of the next section for a different approach which relies on results which we prove anyway for Theorems 1.3. and 1.4.

3. Proof in the Long-Range Case

With the shorthands $\Omega^D_\pm$ and S^D we obtain in this case instead of (2.2)

$$\begin{aligned}
i(S^D - \mathbf{1}) &= i(\Omega^D_+ - \Omega^D_-)^*\; \Omega^D_- \\
&= \int dt\; U^D(t,0)^* \; \left\{V^s(\mathbf{x}) + V^\ell(\mathbf{x}) - V^\ell(\mathbf{p}t/m)\right\} e^{-iHt}\; \Omega^D_- \\
&= \int dt\; U^D(t,0)^* \; \left\{V^s(\mathbf{x}) + V^\ell(\mathbf{x}) - V^\ell(\mathbf{p}t/m)\right\} U^D(t,0) \\
&\quad + \int dt\; U^D(t,0)^* \; \left\{V^s(\mathbf{x}) + V^\ell(\mathbf{x}) - V^\ell(\mathbf{p}t/m)\right\} \left(e^{-iHt}\; \Omega^D_- - U^D(t,0)\right).
\end{aligned} \tag{3.1}$$

Again the first summand will give the leading contribution while the second is asymptotically negligible. Analogous to (2.6) we obtain

$$v \int dt \left(\Phi_{\mathbf{v}},\, U^D(t,0)^* \; \left\{V^s(\mathbf{x}) + V^\ell(\mathbf{x}) - V^\ell(\mathbf{p}t/m)\right\} U^D(t,0)\; \Psi_{\mathbf{v}}\right)$$

$$\begin{aligned}
&= \int d(vt) \Big(\Phi_{\mathbf{0}},\, \tilde{U}(\mathbf{v};t,0)^*\; e^{iH_0 t} \\
&\qquad \times \left\{V^s(\mathbf{v}t + \mathbf{x}) + V^\ell(\mathbf{v}t + \mathbf{x}) - V^\ell(\mathbf{v}t + \mathbf{p}t/m)\right\} e^{-iH_0 t}\; \tilde{U}(\mathbf{v};t,0)\; \Psi_{\mathbf{0}}\Big)
\end{aligned}$$

$$\begin{aligned}
&= \int d\tau \Big(\Phi_{\mathbf{0}},\, \tilde{U}(\mathbf{v};\tau/v,0)^*\; e^{iH_0\tau/v} \\
&\times \left\{V^s(\hat{\mathbf{v}}\tau + \mathbf{x}) + V^\ell(\hat{\mathbf{v}}\tau + \mathbf{x}) - V^\ell(\hat{\mathbf{v}}\tau + \mathbf{p}\tau/mv)\right\} e^{-iH_0\tau/v}\; \tilde{U}(\mathbf{v};\tau/v,0)\; \Psi_{\mathbf{0}}\Big).
\end{aligned} \tag{3.2}$$

Here the translated correction term in the Dollard evolution is

$$\tilde{U} \equiv \tilde{U}(\mathbf{v}; t, 0) = \exp\{-i \int_0^t ds\, V^\ell(\mathbf{v}\, s + \mathbf{p}\, s/m)\} \tag{3.3}$$

and its rescaled version $\tilde{U}(\mathbf{v}; \tau/v, 0)$ tends for fixed τ strongly to $\mathbf{1}$ in the limit $v \to \infty$, similarly as $\exp\{\pm iH_0\tau/v\}$ do. Moreover, $V^\ell(\hat{\mathbf{v}}\tau + \mathbf{p}\tau/m\, v)$ converges strongly to $V^\ell(\hat{\mathbf{v}}\,\tau)$.

Thus, taking a pointwise limit of the integrand of (3.2) we obtain as $v \to \infty$ the limit

$$\left(\Phi_\mathbf{0}, \int d\tau \left\{V^s(\hat{\mathbf{v}}\,\tau + \mathbf{x}) + V^\ell(\hat{\mathbf{v}}\,\tau + \mathbf{x}) - V^\ell(\hat{\mathbf{v}}\,\tau)\right\}\, \Psi_\mathbf{0}\right) \tag{3.4}$$

as claimed in Theorem 1.4. To justify this we show several estimates. We include the details for the convenience of the reader to make the article self-contained although some of them are standard in long-range scattering theory [**3, 8**].

Proposition 3.1. *For $\Phi_\mathbf{v}$ as given in* (1.9), (1.10) *there is a function $h \in L^1(\mathbb{R})$ such that for $v \geq 4\,\hat{R}$*

$$\left\|\left\{V^s(\mathbf{x}) + V^\ell(\mathbf{x}) - V^\ell(\mathbf{p}t/m)\right\} U^D(t, 0)\, \Phi_\mathbf{v}\right\| \leq h(vt). \tag{3.5}$$

Proof. It is convenient to split the long-range potential V^ℓ into an inner part and an outer part with easily controllable decay properties. Let $\chi \in C^\infty$ satisfy $\chi(\lambda) = 1$ for $\lambda \geq 1/2$ and $\chi(\lambda) = 0$ for $\lambda \leq 1/4$ and define

$$V^\ell_{vt}(\mathbf{y}) := V^\ell(\mathbf{y})\, \chi(|\mathbf{y}/vt|). \tag{3.6}$$

Then $\operatorname{supp}(V^\ell - V^\ell_{vt}) \subset B_{|vt|/2}(0)$, $\|V^\ell - V^\ell_{vt}\| \leq \|V^\ell\|$, and the decay of V^ℓ for large arguments translates into decay w.r.t. $|vt|$ independent of the argument: from (1.4) and (1.5) follows

$$\|\nabla V^\ell_{vt}\| = \sup_\mathbf{y} |(\nabla V^\ell_{vt})(\mathbf{y})| \leq \text{const } |vt|^{-\gamma}, \quad \gamma > 3/2, \tag{3.7}$$

$$\|\Delta V^\ell_{vt}\| \leq \text{const } |v\,t|^{-(2+\varepsilon)}, \quad \varepsilon > 0. \tag{3.8}$$

The norm in (3.5) equals

$$\left\|\left\{V^s(\mathbf{x}) + V^\ell(\mathbf{x}) - V^\ell(t\,\mathbf{p}/m)\right\} e^{-iH_0 t}\, \exp\left\{-i \int_0^t ds\, V^\ell(s\,\mathbf{p}/m)\right\}\, e^{im\mathbf{v}\cdot\mathbf{x}}\, \Phi_\mathbf{0}\right\|$$

$$\begin{aligned} \leq\ & \left\|V^s(t\,\mathbf{v} + \mathbf{x})\, e^{-iH_0 t}\, \tilde{U}(\mathbf{v}; t, 0)\, \Phi_\mathbf{0}\right\| \\ &+ \left\|(V^\ell - V^\ell_{vt})(t\,\mathbf{v} + \mathbf{x})\, e^{-iH_0 t}\, \tilde{U}(\mathbf{v}; t, 0)\, \Phi_\mathbf{0}\right\| \\ &+ \left\|\left\{V^\ell_{vt}(t\,\mathbf{v} + \mathbf{x}) - V^\ell_{vt}(t\,\mathbf{v} + t\,\mathbf{p}/m)\right\} e^{-iH_0 t}\, \tilde{U}(\mathbf{v}; t, 0)\, \Phi_\mathbf{0}\right\|. \end{aligned} \tag{3.9}$$

We have used (2.3)-(2.5), the shorthand (3.3), and the fact that $V^\ell(t\,\mathbf{v} + t\,\mathbf{p}/m) = V^\ell_{vt}(t\,\mathbf{v} + t\,\mathbf{p}/m)$ on the momentum support of $\Phi_\mathbf{0}$ for $v > 2\,\hat{R}$.

Note that $|t\,\mathbf{v}+\mathbf{x}| < |tv|/2$ implies $|x| > |tv|/2$, thus the summand with V^s is estimated by

(3.10)
$$\|V^s(t\,\mathbf{v}+\mathbf{x})\;F(|t\,\mathbf{v}+\mathbf{x}| > |tv|/2)\| + \|V^s\|\;\|F(|x| > |tv|/2)\;e^{-iH_0 t}\;\tilde{U}\;\Phi_{\mathbf{0}}\|.$$

The first term decays integrably in vt by (1.1), the integrable decay of the second will be shown later in Lemma 3.2. The second line in (3.9) is bounded as well by

$$\|V^\ell\|\;\|F(|x| > |tv|/2)\;e^{-iH_0 t}\;\tilde{U}\;\Phi_{\mathbf{0}}\| \tag{3.11}$$

due to the support of $(V^\ell - V^\ell_{vt})$. For the tail part of the long-range potential in (3.9) we use $\exp(iH_0 t)\;f(\mathbf{x})\;\exp(-iH_0 t) = f(\mathbf{x}+t\,\mathbf{p}/m)$ and obtain

$$\begin{aligned}
&\left\|\left\{V^\ell_{vt}(t\,\mathbf{v}+t\,\mathbf{p}/m+\mathbf{x}) - V^\ell_{vt}(t\,\mathbf{v}+t\,\mathbf{p}/m)\right\}\;\tilde{U}\;\Phi_{\mathbf{0}}\right\| \\
&\leq \int_0^1 ds\;\left\|\frac{d}{ds}\;V^\ell_{vt}\;(t\,\mathbf{v}+t\,\mathbf{p}/m+s\,\mathbf{x})\;\tilde{U}\;\Phi_{\mathbf{0}}\right\| \\
&= \int_0^1 ds\,\|(\nabla\,V^\ell_{vt})\;(t\,\mathbf{v}+t\,\mathbf{p}/m+s\,\mathbf{x})\cdot\mathbf{x}\;\tilde{U}\;\Phi_{\mathbf{0}}\;-[it/(2m)]\;(\Delta V^\ell_{vt}\;(\cdots)\;\tilde{U}\;\Phi_{\mathbf{0}}\| \\
&\leq \text{const}\;\left[|vt|^{-\gamma}\;\|\mathbf{x}\;\tilde{U}\;\Phi_{\mathbf{0}}\| + |t|\;|vt|^{-(2+\varepsilon)}\right].
\end{aligned} \tag{3.12}$$

The gradient-term is the same as for commuting arguments of the potential. The correction with the Laplacian of the potential (at the same argument) is obtained e.g. using the Fourier representation and the Baker-Campbell-Hausdorff formula: $\exp(A+B) = \exp(A)\;\exp(B)\;\exp([A,B]/2)$ if $[A,B]$ commutes with A and B. The formula is used to isolate the terms depending on s in the exponent.

$$|t|\;|vt|^{-(2+\varepsilon)} = v^{-1}\;|vt|^{-(1+\varepsilon)} \tag{3.13}$$

is integrable in vt uniformly for large v. We show in Lemma 3.2 that

$$\|\mathbf{x}\,\tilde{U}\;\Phi_{\mathbf{0}}\| \leq \text{const}\;\;v^{-2}\;|vt|^{2-\gamma} + \|\mathbf{x}\,\Psi_{\mathbf{0}}\|. \tag{3.14}$$

Thus we get as a bound for (3.12) for large $|vt|$

$$\text{const}\;\left[v^{-2}\;|vt|^{2-2\gamma} + |vt|^{-\gamma}\|\mathbf{x}\;\Psi_{\mathbf{0}}\| + v^{-1}|vt|^{-(1+\varepsilon)}\right] \tag{3.15}$$

and uniform boundedness for all v, t. Since $\gamma > 3/2$ we have shown integrability in vt of this contribution uniformly for $v \geq 2\,\hat{R}$. □

LEMMA 3.2. *For $\Phi_{\mathbf{0}}$ as given in (1.9), $\tilde{U}(\mathbf{v};t,0)$ as defined in (3.3) and $v > 4\hat{R}$*

(3.16)
$$\|\mathbf{x}\;\tilde{U}(\mathbf{v};t,0)\;\Phi_{\mathbf{0}}\| \leq \|\mathbf{x}\,\Phi_{\mathbf{0}}\| + \text{const}\;\,v^{-2}|v\,t|^{2-\gamma},$$

(3.17)
$$\begin{aligned}
&\|F(|\mathbf{x}| > |tv|/2)\;e^{-iH_0 t}\;\tilde{U}(\mathbf{v};t,0)\;\Phi_{\mathbf{0}}\| \\
&\leq \text{const}\;\left[1 + v^{-2}\;|vt|^{2-\gamma} + v^{-3}\;|vt|^{1-\varepsilon} + v^{-4}\;|vt|^{4-2\gamma}\right]\;(1+|vt|^2)^{-1}.
\end{aligned}$$

Both constants depend on V^ℓ, the one in (3.17) depends in addition on g, $\|\mathbf{x}\,\Phi_{\mathbf{0}}\|$, and $\|\mathbf{x}^2\;\Phi_{\mathbf{0}}\|$. In particular (3.17) is integrable in vt uniformly for large v.

Remark. With a longer proof one shows decay faster than any inverse power of vt in (3.17), see [**2**, Corollary 2.5].

PROOF. We can use that V^ℓ may be replaced by V^ℓ_{vs} in the exponent of $\tilde U$ on states with momentum support in $B_{m\hat R}(0)$, i.e. on the range of $g(\mathbf{p})$ as defined following (1.9), if $v > 2\hat R$.

$$[\mathbf{x},\, \tilde U(\mathbf{v};t,0)]\, g(\mathbf{p}) = \int_0^t ds\, \frac{s}{m}\, (\nabla\, V^\ell_{vs})(\mathbf{v}\, s + \mathbf{p}\, s/m)\, \tilde U(\mathbf{v};t,0)\, g(\mathbf{p}), \tag{3.18}$$

$$\|\, [\mathbf{x},\ \tilde U(\mathbf{v};t,0)]\, g(\mathbf{p})\| \le \text{const}\ \ v^{-\gamma}\, \left|\int_0^t ds\ s^{1-\gamma}\right| \le const\ v^{-2}|vt|^{2-\gamma}, \tag{3.19}$$

$$\begin{aligned}\|\, [\mathbf{x}, [\mathbf{x},\ \tilde U(\mathbf{v};t,0)]]\, g(\mathbf{p})\| &\le \left|\int \cdots\right|^2 + \left|\int_0^t ds\ (\frac{s}{m})^2\ (\Delta\, V^\ell_{vs})(\mathbf{v}\ s + \mathbf{p}\ s/m)\right| \\ &\le \text{const}\ \left[v^{-4}\ |vt|^{4-2\gamma} + v^{-3}\ |vt|^{1-\varepsilon}\right].\end{aligned} \tag{3.20}$$

The constants depend only on V^ℓ. We get (3.16) from

$$\|\mathbf{x}\ \tilde U(\mathbf{v};t,0)\ \Phi_\mathbf{0}\| \le \|\mathbf{x}\,\Phi_\mathbf{0}\| + \|\, [\mathbf{x},\ \tilde U(\mathbf{v};t,0)]\ g(\mathbf{p})\|.$$

For (3.17) observe that

$$\begin{aligned}&\|F(|\mathbf{x}| > |tv|/2)\ e^{-iH_0 t}\ \tilde U\ \Phi_\mathbf{0}\| \\ &\le \|F(|\mathbf{x}| > |tv|/2)\, e^{-iH_0 t}\, g(\mathbf{p})\, F(|\mathbf{x}| < |tv|/8)\| + \|F(|\mathbf{x}| > |tv|/8)\, \tilde U\, \Phi_\mathbf{0}\|.\end{aligned} \tag{3.21}$$

Again, the first summand has rapid decay in $|vt|$, e.g.

$$\|F(|\mathbf{x}| > |tv|/2)\ e^{-iH_0 t}\ g(p)\ F(|\mathbf{x}| < |tv|/8)\| \le \text{const}(1 + |vt|^2)^{-1} \tag{3.22}$$

for all $v \ge 4\hat R$ by non-stationary phase estimates for propagation into classicaly forbidden regions, cf. (2.9).

$$\begin{aligned}&(|tv|/8)^2\ \|F(|\mathbf{x}| > |tv|/8)\ \tilde U\ \Phi_\mathbf{0}\| \\ &\le \|F(|\mathbf{x}| > |tv|/8)\ \mathbf{x}^2\ \tilde U\ \Phi_\mathbf{0}\| \\ &\le \|\, [\mathbf{x}, [\mathbf{x}, \tilde U]]\ g(\mathbf{p})\| + 2\|\, [\mathbf{x},\ \tilde U]\ g(\mathbf{p})\|\ \|\,\mathbf{x}\ \Phi_\mathbf{0}\| + \|\mathbf{x}^2\ \Phi_\mathbf{0}\| \\ &\le \text{const}\ \left[v^{-4}\, |vt|^{4-2\gamma} + v^{-3}\, |vt|^{1-\varepsilon} + v^{-2}\, |vt|^{2-\gamma} + 1\right]\end{aligned} \tag{3.23}$$

for a constant which depends on g, V^ℓ, $\|\mathbf{x}\,\Phi_\mathbf{0}\|$, and $\|\mathbf{x}^2\,\Phi_\mathbf{0}\|$. □

PROPOSITION 3.3. *For $\Psi_\mathbf{v}$ as given in (1.9), (1.10) there is a bound $M < \infty$ such that*

$$\sup_{t\in\mathbb{R}}\ v\, \|\, \left(e^{-iHt}\, \Omega^D_\pm - U^D(t,0)\right)\ \Psi_\mathbf{v}\| \le M. \tag{3.24}$$

PROOF.

$$\begin{aligned}
&\left(e^{-iHt}\,\Omega^D_\pm - U^D(t,0)\right)\\
&= \lim_{t'\to\pm\infty} e^{-iHt}\, e^{iHt'}\, U^D(t',0) - U^D(t,0)\\
&= \lim_{T\to\pm\infty} e^{iHT}\, U^D(T+t,0) - U^D(t,0)\\
&= \int_0^{\pm\infty} ds\; e^{iHs}\; i\,\{H - H^D(s+t)\} U^D(s+t,0).
\end{aligned}$$

This yields the uniform bound

$$\begin{aligned}
&\sup_{t\in\mathbb{R}} v\, \|\left(e^{-iHt}\,\Omega^D_\pm - U^D(t,0)\right)\;\Psi_{\mathbf{v}}\|\\
\le v &\int_{-\infty}^{\infty} ds\, \left\|\left\{V^s(\mathbf{x}) + V^\ell(\mathbf{x}) - V^\ell(\mathbf{p}\,s/m)\right\}\; U^D(s,0)\;\Psi_{\mathbf{v}}\right\|\\
&\le \int_{-\infty}^{\infty} d(vs)\; h(vs)\; = M < \infty
\end{aligned}$$

by Proposition 3.1. $\square$

Now we are ready to finish the proof of Theorem 1.4. The integrable bound for (3.2) uniform for large v is a direct consequence of Proposition 3.1. By the dominated convergence theorem pointwise limits as $v \to \infty$ yield the correct result (3.4). To show that the second term in (3.1) is $O(v^{-2})$ – and thus small compared to the first one – we estimate it with $vt = \tau$ and Propositions 3.1, 3.3

$$\begin{aligned}
v^2 \int dt\; &\left\|\left\{V^s(\mathbf{x}) + V^\ell(\mathbf{x}) - V^\ell(\mathbf{p}\,t/m)\right\}\; U^D(t,0)\,\Phi_{\mathbf{v}}\right\|\\
&\times \quad \sup_{t\in\mathbb{R}} \|\left(e^{-iHt}\,\Omega^D_\pm - U^D(t,0)\right)\;\Psi_{\mathbf{v}}\|
\end{aligned}$$

$$\le \int d\tau\; h(\tau)\; M = M^2. \tag{3.25}$$

This finishes the proof of Theorem 1.4 and as a consequence of Theorem 1.3. $\square$

It remains to show Theorem 1.2. For shorter writing let $H^\ell := H_0 + V^\ell$, $H := H_0 + V^\ell + V^s$. Since all wave operators exist and are complete we may use the chain rule (see e.g. [**8**]) and obtain

$$\Omega_\pm(H^\ell, H) = \Omega^D_\pm(H_0, H)\; [\Omega^D_\pm(H_0, H^\ell)]^*.$$

This yields for the scattering operator

$$\begin{aligned}
S(H^\ell, H) &= \Omega^D_+(H_0, H^\ell)\; [\Omega^D_+(H_0, H)]^*\; \Omega^D_-(H_0, H)\; [\Omega^D_-(H_0, H^\ell)]^*\\
&= \Omega^D_+(H_0, H^\ell)\;\; S^D(H_0, H)\;\; [\Omega^D_-(H_0, H^\ell)]^*.
\end{aligned}$$

Since the ranges of $[\Omega^D_\pm(H_0, H^\ell)]^*$ are all of $\mathcal{H}$ we conclude that the knowledge of $S(H^\ell, H)$ determines $S^D(H_0, H)$ uniquely. By Theorem 1.3 this fixes V^s as well. For a proof of formula (2.13) see [**4**]. $\square$

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Abstract. In multidimensional short-range potential scattering the high energy limit of the scattering operator determines the potential uniquely. For given long-range potential the high energy behavior of the Dollard modified scattering operator determines the potential uniquely.

We use geometrical, time-dependent methods for the proof and obtain a reconstruction formula as well.

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