Theory of Local Observables and KMS–Condition

Dedicated to BERT SCHROER at the occasion of his 60th birthday

H.J. BORCHERS

Institut für Theoretische Physik
Universität Göttingen
Bunsenstrasse 9, D 37073 Göttingen

Abstract:
We will look at C*–dynamical systems with the translations of $\mathbb{R}^d$ as dynamical group and will define KMS–functionals for β taking values in $\mathbb{C}^d$. The properties of such functionals will be investigated. In particular if a KMS–functional exists for a complex β then it will be shown that there exists a KMS–state for the real part of β. If the system is a theory of local observables then it will be shown that locality condition implies that there do not exist β–KMS–functionals such that β is spacelike or has a spacelike real part.

1. Introduction:

It is generally believed that the thermodynamical functions should be analytic in the temperature except for points of phasetransitions. In most textbooks non–analyticity is taken as definition for phasetransition. In this setting the surface of coexistence becomes cuts in some analytic domain. The detailed structure of these cuts is used to classify phasetransitions.

In quantum statistical mechanics the temperature–states are defined by the KMS–condition [HHW67]. Therefore, it is not obvious that temperature–states should have analyticity properties in the temperature or rather in the inverse temperature β. If we allow arbitrary C*–dynamical systems then one cannot expect any analyticity in the inverse temperature β. This follows from the investigations of O. Bratteli and others [Bra82]. They have shown that to any given closed subset of the positive real axis there exists a C*–dynamical system which has β–KMS.–states only for β's belonging to this given set. Therefore, one can only hope for an acceptable behaviour in the temperature if we are dealing with reasonable dynamics. From the experience with quantum field theory we know that locality and spectrum condition seem to be good for defining acceptable dynamics.

If we use the theory of local observables then we have to ask for the direction of the time axis. Starting from the Gibbs approach to the temperature–states, the time direction is fixed because one considers boxes at rest. But in any other approach, starting from the infinite system, there is no convincing argument for a distinguished time–direction. This phenomenon is known from the kinetic theory of gasses. Therefore, one should allow any direction inside the forward light cone as a time–direction. Indeed, looking at the free field, P. Junglas [Jun85] was able to show that in this example KMS–states exist for all β belonging to the forward and backward light cone.

Unfortunately it is unknown whether or not every local net allows KMS–states for all temperatures. One only knows that the nuclearity condition of Buchholz and Wichmann [BuWi86] implies the existence of KMS–states for all temperatures [BuJu89]. These states are locally normal with respect to the vacuum state.

An investigation of complex temperatures will be started in this note. The first step is the generalization of β–KMS–states to complex values of the temperature. In the complex it is not reasonable anymore to stay with the concept of states, we rather have to introduce the notation of KMS–functionals. The problem of analyticity will be discussed in a later investigation. Having analyticity in mind might serve as a justification for looking at KMS–functionals (and not states) when dealing with complex temperatures.

The second aspect of this note is the introduction of C*–dynamical systems where the translation group of $\mathbb{R}^d$ is acting as a group of automorphisms. In such a situation we allow the temperature – rather the inverse temperature – to be elements of $\mathbb{C}^d$.

In the next section we shall define the KMS–functionals for arbitrary complex values of the inverse temperature and shall look at the properties of these functionals. In the third section we shall turn to the theory of local observables and shall derive some consequences of the locality condition. The main outcome of the third section is the result that the locality condition does not allow spacelike temperatures.

In reacent years there have been other investigations also using the locality condition especially for the so–called relativistic KMS–condition [BrBu92,93,94]. Our interest in this note is the functional dependence on the temperature and therefore this investigation is complementary to those of the relativistic KMS–condition. But hopefully the connection between the two subjects will be established one day.

For the rest of this section we collect some mathematical tools.

The theory of KMS–states is closely related to the Tomita–Takesaki theory [Tom67], [Tak70]. Since we need some results from this we collect its main aspects for the convenience of the reader.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{M}$ be a von Neumann algebra acting on this space with commutant $\mathcal{M}'$. A vector Ω is cyclic and separating for $\mathcal{M}$ if $\mathcal{M}\Omega$ and $\mathcal{M}'\Omega$ are dense in $\mathcal{H}$. If these conditions are fulfilled then a modular operator Δ and a modular conjugation J is associated to the pair $(\mathcal{M}, \Omega)$ such that

(i) Δ is self-adjoint, positive, invertible and fulfills

$$\Delta\Omega = \Omega, \qquad J\Omega = \Omega,$$

(ii) the unitary group $\Delta^{\imath t}$ defines a group of automorphisms of $\mathcal{M}$

$$\sigma_t(\mathcal{M}) := \mathrm{ad}\, \Delta^{-\imath t}\mathcal{M} = \mathcal{M} \qquad \forall t \in \mathbb{R},$$

(iii) the operator J is a conjugation i.e. J is antilinear and $J^2 = 1$ and commutes with $\Delta^{\imath t}$, i.e.

$$\mathrm{ad}\, J\Delta = \Delta^{-1},$$

J maps $\mathcal{M}$ onto its commutant

$$\mathrm{ad}\, J\mathcal{M} = \mathcal{M}',$$

(iv) for every $A \in \mathcal{M}$ the vector $A\Omega$ belongs to the domain of $\Delta^{\frac{1}{2}}$.
The operators $S := J\Delta^{\frac{1}{2}}$ and $S^* = J\Delta^{-\frac{1}{2}}$ have the property

$$SA\Omega = A^*\Omega \qquad \forall A \in \mathcal{M},$$
$$S^*A'\Omega = A^{'*}\Omega \qquad \forall A' \in \mathcal{M}'.$$

(v) The function

$$f(t) := (\Omega, B\sigma_t(A)\Omega), \qquad A, B \in \mathcal{M}$$

has an analytic continuation into the strip $S(0,1) = \{z; 0 < \Im m\, z < 1\}$. At the upper boundary it takes the values

$$f(x+\imath) = (\Omega, \sigma_t(A)B\Omega).$$

This relation is called the KMS–boundary–condition.
(vi) Let $\mathcal{M}$ be a von Neumann algebra with cyclic and separating vector Ω and let $\mathrm{ad}\, U(t)$ define a group of automosphisms of $\mathcal{M}$ and assume $U(t)\Omega = \Omega$ and $(\Omega, BU(t)A\Omega)$ fulfill the KMS–boundary condition for $\beta = 1$ then $U(t)$ is the modular group of $\mathcal{M}$.

For the proof see the exposition of Takesaki [Tak70] or textbooks as for instance Bratteli and Robinson [BrRo79] or Kadison and Ringrose [KaRi89].

We often have to use the polar decomposition of functionals. If $\phi \in \mathcal{A}^*$ then there exists a positive linear functional ω and a partial isometry $V \in \mathcal{A}^{**}$ with

$$\phi(A) = \omega(V^*A).$$

The partial isometry V is unique if we require that $V^*V = E$ is the support–projection of ω. The positive functional ω is faithful on the von Neumann algebra $\mathcal{A}^{**}_E$.

2. C*–dynamical systems and complex temperatures

We start the investigation by looking at a C*–dynamical system where the dynamics are given by the translation group of $\mathbb{R}^d$. We assume the following

2.1 Notation:

1. $\{\mathcal{A}, \alpha, \mathbb{R}^d\}$ denotes a C*–dynamical system where
   a) $\mathcal{A}$ is a C*–algebra,
   b) $\alpha : \mathbb{R}^d \longrightarrow Aut(\mathcal{A})$, i.e., α_a is a representation of the translations of $\mathbb{R}^d$ as automorphisms of $\mathcal{A}$,

c) the representation α_a is strongly continuous. This means for $A \in \mathcal{A}$ the function $a \to \alpha_a(A)$ is a continuous function with values in $\mathcal{A}$ equipped with the norm topology.

2. $\mathcal{A}_0$ denotes those elements in $\mathcal{A}$ such that $\alpha_a(A)$ has an extension as entire analytic function for $a \in \mathbb{C}^d$. $\mathcal{A}_0$ is norm-dense in $\mathcal{A}$.

3. For $A \in \mathcal{A}_0$ one has the relation $(\alpha_{\overline{z}}(A))^* = \alpha_z(A^*)$ for all $z \in \mathbb{C}^d$. This is due to the fact that both sides are analytic in $z \in \mathbb{C}^d$ and coincide for real z.

4. We denote by $\mathcal{A}_{00}$ those elements in $\mathcal{A}_0$ for which $\alpha_a(A)$ is an entire analytic function of exponential growth. $A \in \mathcal{A}_{00}$ if $\alpha_a(A)$ is the Fourier-transform of an operator-valued distribution with compact support. $\mathcal{A}_{00}$ is norm-dense in $\mathcal{A}_0$ and hence also in $\mathcal{A}$.

Next we want to introduce the temperature–functionals. Here we follow the definition of Haag, Hugenholtz, and Winnink [HHW67] by means of the KMS boundary condition. We will only use the subset of analytic elements in the algebra. Since we allow the temperature to be in $\mathbb{C}^d$ we look at these functionals as a vector–bundle over $\mathbb{C}^d$.

2.2 Definition:

Let $\{\mathcal{A}, \alpha, \mathbb{R}^d\}$ be a C^*–dynamical system fulfilling the assumptions 2.1. For $z \in \mathbb{C}^d$ we set

$$F_z = \{\varphi \in \mathcal{A}^*; \varphi(B\alpha_{\imath z}(A) - AB) = 0 \ \forall \ A, B \in \mathcal{A}_0\}.$$

We call F_z the KMS–fibre over z.

First we collect some obvious results, which holds for every KMS–fibre. Notice that every KMS–fibre contains at least the element zero.

2.3 Lemma:

Let $\{\mathcal{A}, \alpha, \mathbb{R}^d\}$ be a C^*–dynamical system fulfilling the assumptions 2.1, then every KMS–fibre has the properties:

(*i*) *The set F_z is weakly closed.*

(*ii*) *The sets F_z are invariant under all automorphisms commuting with the translation, in particular with the translations themselves. This means*

$$\varphi \in F_z \text{ implies } \alpha_a\varphi \in F_z, \quad \forall a \in \mathbb{R}^d.$$

(*iii*) *If $\varphi \in F_z$ then $\varphi^* \in F_{\overline{z}}$.*

(*iv*) *If $\varphi \in F_z$, $z = \xi + \imath\eta$ and $A, B \in \mathcal{A}$ then the function $f(x) = \varphi(B\alpha_{x\xi}(A))$ has an analytic continuation in the variable $\zeta = x + \imath y$ into the strip $S(0,1) := \{\zeta; 0 < \Im m\, \zeta < 1\}$. In this strip $f(\zeta)$ is bounded by*

$$|f(\zeta)| \leq \|A\|\|B\|\|\varphi\|,$$

and $f(\zeta)$ has continuous boundary values. We find the relation

$$f(x + \imath) = \varphi(\alpha_{x\xi+\eta}(A)B)$$

at the upper boundary.

Proof: (i) F_z is defined as the annihilator of family of elements. Hence it is the intersection of weakly closed linear subspaces.
(ii) is due to the fact that α_a is a commutative group. This implies that the set of elements, annihilated by F_z, is invariant under translations.
(iii) Recall the definition of φ^* by the formula $\varphi^*(A) = \varphi(A^*)$. This implies by the fact that α_x is a $*$–automorphism:

$$\varphi^*(B\alpha_{\imath\bar{z}}(A) - AB) = \varphi(\alpha_{-\imath z}(A^*)B^* - B^*A^*).$$

Replacing $A^* = \alpha_{\imath z}(A')$ we obtain

$$\varphi^*(B\alpha_{\imath\bar{z}}(A) - AB) = \varphi(A'B^* - B^*\alpha_{\imath z}(A')) = 0.$$

(iv) For $A, B \subset \mathcal{A}_0$ we know that $f_{A.B}(\zeta) := \varphi(B\alpha_{\imath\zeta\xi}(A))$ is analytic in ζ. Since $\varphi \in F_z$ we find

$$\varphi(B\alpha_{(x+\imath)\xi}(A)) = \varphi(B\alpha_{\imath z}(\alpha_{x\xi+\eta}(A)) = \varphi(\alpha_{x\xi+\eta}(A)B).$$

This implies that the function $f_{A,B}(\zeta)$ is bounded in the strip $S(0,1)$ with the estimate

$$|f_{A,B}(\zeta)| \leq \|\varphi\|\|A\|\|B\|, \quad \text{for} \quad \zeta \in S(0,1).$$

If we take $A, B \in \mathcal{A}$ and choose sequences $A_n,\ B_n \subset \mathcal{A}_0$ converging in norm to A and B respectively, then $f_{A_n,B_n}(\zeta)$ is a sequence of bounded analytic functions on $S(0,1)$ which converge on the boundary of the strip. The limit–function is continuous on the boundary because the translations act continuously. Hence we obtain the convergence of the analytic functions. □

Next we want to look at the invariance properties of KMS–functionals. It is known that β–KMS–states are invariant under "time" translations. For complex temperatures one finds the following results:

2.4 Proposition:
Assume $\varphi \in F_z$ and let $z = \xi + \imath\eta$ then we obtain:

$$\begin{aligned} \alpha_{\lambda\xi}\varphi &= \varphi \quad \forall\, \lambda \in \mathbb{R} \\ \alpha_{n\eta}\varphi &= \varphi \quad n = 0, \pm 1, \pm 2, ... \end{aligned}$$

The first equation shows that $\alpha_{\lambda\xi}\varphi$ is constant while $\alpha_{\lambda\eta}\varphi$ is periodic.

Proof: Choosing $B = 1$ we obtain $\varphi(\alpha_{\imath z}(A)) = \varphi(A)$ which implies $\varphi(\alpha_{n\imath z}(A)) = \varphi(A)$. Replacing A by $\alpha_{n\eta}A$ we obtain from the last equation

$$|\varphi(\alpha_{\imath n\xi}(A))| = |\varphi(\alpha_{n\eta}(A))| \leq \|\varphi\|\|A\|.$$

Now choosing $A \in \mathcal{A}_{00}$ and replacing A by $\alpha_{x\xi}(A)$ we see that the function $\varphi(\alpha_{\zeta\xi}(A))\quad \zeta \in \mathbb{C}$ is bounded and hence $\varphi(\alpha_{\zeta\xi}(A))$ does not depend on ζ. As $\mathcal{A}_{00}$ is norm dense in $\mathcal{A}$ we obtain the first equation. Using this result we obtain the second equation from the relation $\alpha_{\imath n z}\varphi = \varphi$. $\square$

The next Lemma collects results concerning the polar decomposition of states. The first part is of general nature and the second and third part is valid for KMS–functionals only.

2.5 Lemma

A. Let $\mathcal{A}$ be a C^*-algebra, $\alpha \in Aut(\mathcal{A})$ and $\varphi \in \mathcal{A}^*$. Assume $\omega = |\varphi|$ and $\alpha\varphi = \varphi$. Then

a. $\alpha\omega = \omega$.

b. If $\varphi(A) = \omega(V^*A)$ with V^*V the support–projection of ω then $\alpha(V^*V) = V^*V$.

c. If $\omega = \omega_\psi$ then one has $\alpha(V)\psi = V\psi$.

B. The KMS–boundary condition

$$\varphi(B\alpha_{\imath z}(A)) = \varphi(AB)$$

can be extended to all elements belonging to $\mathcal{A}^{**}$. This means for $A, B \in \mathcal{A}^{**}$ the function $\varphi(B\alpha_{x\xi}(A))$ in x is the boundary–value of an analytic function holomorphic in the strip $0 < \Im m\, x < 1$. It has continuous boundary–values at $\Im m\, x = 1$ and satisfies

$$\varphi(B\alpha_{(x+\imath)\xi}(A)) = \varphi(\alpha_{x\xi+\eta}(A)B).$$

C. Assume $\varphi \in F_z$ and $\varphi(A) = \omega(V^*A)$ where $E = V^*V$ is the support–projection of ω then one has:

d. $EV = VE = V \in \mathcal{A}_E^{**}$.

e. $\alpha_a(V) = V$ for $a = \lambda\xi$ or $a = \eta$.

Proof: If $\varphi = 0$ then the statement is trivial. Now assume $\|\varphi\| = 1$ and let $\varphi(A) = \omega(V^*A)$ be the polar decomposition of φ in $\mathcal{A}^{**}$ where V^*V is the support–projection of $\omega > 0$. With this decomposition we obtain $\omega(V^*\alpha(V)) = \varphi(\alpha(V)) = \varphi(V) = \omega(V^*V)$. This implies the following: If we write $\omega = \omega_\psi$ we conclude from the equation $(V\psi, V\psi) = (V\psi, \alpha(V)\psi)$ that $\alpha(V)\psi = V\psi$ because $\|V\psi\| = 1$ and $\|\alpha(V)\psi\| \leq 1$. This is statement A.c. Now we obtain $\|\alpha(V)\psi\|^2 = \omega(\alpha(V^*V)) = \|V\psi\|^2 = \omega(V^*V)$. If we denote $\alpha(V^*V) = F$ then we obtain $(\psi, F\psi) = (\psi, \psi)$ which implies $F\psi = \psi$ and we get $F \geq E$ since E is the support–projection of ψ. Replacing α by α^{-1} we obtain the invariance of $E = V^*V$. Hence we have shown A.b. Combining A.b.,c. we find $\omega(\alpha(A)) = \omega(V^*V\alpha(A)) = \omega(\alpha(V^*VA)) = (\alpha(V)\psi, \alpha(V\pi(A))\psi) = (V\psi, \alpha(V\pi(A))\psi) = \varphi(\alpha(VA)) = \varphi(VA) = (V\psi, V\pi(A)\psi) = \omega(A)$. This shows part A. of the lemma.

We know from Proposition 2.4 that ω is invariant under translations in the direction of ξ. Hence there exists a continuous representation $U(x)$ with $U(x)\pi(A)\psi = \pi(\alpha_{x\xi}(A))\psi$ in the representation space $\mathcal{H}_\omega$. This implies that the two equations

$$(\psi, V^*\pi(B)U(x)\pi(A)\psi) = \varphi(B\alpha_{x\xi}(A))$$
$$(\psi, V^*\pi(\alpha_{-\eta}(A))U(-x)\pi(B)\psi) = \varphi(\alpha_{-\eta}(A)\alpha_{-x\xi}(B))$$

can be extended to all of $\mathcal{A}^{**}$ as continuous functions. This follows from Kaplansky's density theorem and from the continuity of $U(x)$. We know that the approximating functions are connected by bounded analytic functions. This implies by Vitali's theorem the convergence of the analytic functions, so that also the limiting functions satisfy the KMS–boundary condition.

We know from A.b that $\alpha_z(E) = E$. This implies by the KMS–condition for $(\mathcal{A}_E^{**}, \psi)$ and the invariance of φ the relation $\varphi(EA) = \varphi(E\alpha_{\imath z}(A)) = \varphi(AE) = \varphi(A)$. Using the representation of φ we obtain: $\omega(V^*EA) = \omega(V^*A)$ or if $\omega = \omega_\psi$ we find the relation $(EV - V)\psi = 0$ by inserting $A = (EV - V)$. Applying $\pi(\mathcal{A})'$ to this equation we get $EVE - VE = 0$. By construction we also have $VE = V$ and therefore $V \in \mathcal{A}_E^{**}$. This is the first statement of C. From $\alpha(E) = E$ we obtain $\alpha(V) \in \mathcal{A}_E^{**}$. The invariance of V follows from $\alpha_a(V)\psi = V\psi$ since ψ is separating for $\mathcal{A}_E^{**}$. □

¿From this last lemma we can draw important conclusions for the representation defined by the positive functional ω.

2.6 Proposition:

Let $\mathcal{A}$ fulfill the conditions 2.1 and assume $\varphi \in F_z$ with $\|\varphi\| = 1$. Let $\omega = |\varphi|$ and $\omega = \omega_\psi$. Then the cyclic vector ψ is also separating for the representation π.

Proof: Let us denote the central support of E by F. We show that the assumption $E \neq F$ leads to a contradiction. Using the polar decomposition of φ and the KMS–boundary condition we obtain:

$$\begin{aligned}\omega(B\alpha_{\imath\xi}(A)) &= \omega(V^*VB\alpha_{\imath\xi}(A)) = \varphi(VB\alpha_{\imath\xi}(A)) \\ &= \varphi(\alpha_\eta(A)VB) = \omega(V^*\alpha_\eta(A)VB)\end{aligned}$$

¿From this equation we see the following: If $E \neq F$ then there exists an element $B \in \mathcal{A}^{**}$ with $0 \neq B = EB = B(F - E)$. Moreover, we know that $\pi(\alpha_{\imath\xi}(\mathcal{A}_0))\psi$ is dense in the representation space. Hence there exists an element $A \in \mathcal{A}_0$ with $\omega(B\alpha_{\imath\xi}(A)) \neq 0$. This is the contradiction because $\omega(.B) = 0$. □

Using the Lemma 2.5 and the last proposition we can say more about the structure of elements in the fibre F_z.

2.7 Proposition:

Assume $\varphi \in F_z$, $z = \xi + \imath\eta$ and $\varphi(A) = \omega(V^*A)$ with $E = V^*V$ the support–projection of ω. Define for $A \in \mathcal{A}_E^{**}$ the automorphism $\beta(A) = V^*AV$, then one has on $\mathcal{A}_E^{**} = \pi(\mathcal{A})''$:

(i) $\beta\omega = \omega$.
(ii) β *commutes with* α_η *and* $\alpha_{r\xi}$ *for* $r \in \mathbb{R}$.
(iii) $\beta^{-1} = \alpha_\eta$.

Proof: First we obtain by means of the polar decomposition of functionals:

$$\begin{aligned}\omega(B\alpha_{\imath\xi}(A)) &= \omega(V^*VB\alpha_{\imath\xi}(A)) = \varphi(VB\alpha_{\imath\xi}(A)) \\ &= \varphi(\alpha_\eta(A)VB) = \omega(V^*\alpha_\eta(A)VB).\end{aligned}$$

Next notice that Lemma 2.3 implies $\omega(.V) \in F_{\overline{z}}$ if $\omega(V^*.) \in F_z$ and hence we find together with 2.5.e

$$\begin{aligned}\omega(B\alpha_{\imath\xi}(A)) &= \omega(B\alpha_{\imath\xi}(A)V^*V) = \omega(B\alpha_{\imath\xi}(AV^*)V) \\ &= \omega(\alpha_{-\eta}(AV^*)BV) = \omega(\alpha_{-\eta}(A)V^*BV).\end{aligned}$$

Hence we obtain by comparing the two representations of $\omega(B\alpha_{\imath\xi}(A))$:

$$\omega(\beta(\alpha_\eta(A))B) = \omega(\alpha_{-\eta}(A)\beta(B)). \tag{$*$}$$

Choosing $A = 1$ we find

$$\omega(B) = \omega(\beta(B))$$

which is the first statement. In order to show the second statement we use the invariance property of ω and of V established in 2.5.a. and e. respectively. Together with $(*)$ we obtain

$$\omega(\beta(\alpha_\eta(A))B) = \omega(\alpha_{-\eta}(A)\beta(B)) = \omega(\beta^{-1}(\alpha_{-\eta}(A))B).$$

Since ω is separating for $\mathcal{A}_E^{**}$ we find $\beta\alpha_\eta = \beta^{-1}\alpha_{-\eta}$.

We know from 2.5.A.a. and 2.7.i. that ω is invariant under $\beta \circ \alpha_\eta$. Let $\omega = \omega_\psi$ and define the unitary operator U by

$$UA\psi = \beta(\alpha_\eta(A))\psi.$$

As in the proof of (i) we have

$$\omega(B\alpha_{\imath\xi}(A)) = \omega(\beta(\alpha_\eta(A))B).$$

This means (For simplicity we write A instead of $\pi(A)$.):

$$(B^*\psi, \alpha_{\imath\xi}(A)\psi) = (\beta(\alpha_\eta(A))^*\psi, B\psi),$$

and if we introduce the operator S for the pair $(\pi(\mathcal{A})'', \psi)$ we see from the last equation

$$\alpha_{\imath\xi}(A)\psi \in \mathcal{D}(S^*)$$

and hence we obtain

$$S^*\alpha_{\imath\xi}(A)\psi = \beta(\alpha_\eta(A))^*\psi = S\beta(\alpha_\eta(A))\psi$$

This implies that both sides belong to the domain of S and hence we get

$$\Delta^{-1}\alpha_{\imath\xi}(A)\psi = \beta \circ \alpha_\eta(A)\psi.$$

Here Δ is the modular operator of the pair $(\pi(\mathcal{A})'', \psi)$. Multiplying both sides with $\alpha_{\imath\xi}(A)\psi$ and using the commuting property of $\alpha_{\lambda\xi}$ with $\beta\alpha_\eta$ we find with $A \in \mathcal{A}_0$ and with Lemma 2.5.e.:

$$\begin{aligned}(\alpha_{\imath\xi}(A)\psi, \Delta^{-1}\alpha_{\imath\xi}(A)\psi) &= (\alpha_{\imath\xi}(A)\psi, \beta(\alpha_\eta(A))\psi) \\ &= (\alpha_{\imath\xi/2}(A)\psi, \beta(\alpha_\eta(\alpha_{\imath\xi/2}(A)))\psi) \\ &= (\alpha_{\imath\xi/2}(A)\psi, U\alpha_{\imath\xi/2}(A)\psi).\end{aligned}$$

Since the lefthand side is positive and since $\{\alpha_{\imath\xi/2}(A)\psi\}$ is dense in $\{\pi(A)\psi\}$ we obtain $U \geq 0$ which implies $U = 1$. □

¿From this proposition we obtain the following result concerning $F_{\xi+\imath\eta}$ and F_ξ:

2.8 Theorem:

Assume $\varphi \in F_z$ and $\varphi(A) = \omega(V^*A)$ with $z = \xi + \imath\eta$ then:

(i)

$$\omega \in F_\xi.$$

(ii) The fibre F_ξ is generated by its positive elements.

(iii) Assume $\omega \in F_\xi$ is a state and $\eta \in \mathbb{R}^d$. If

(α) $\alpha_\eta\omega = \omega$, and

(β) if E is the support of ω and if α_η is inner for the algebra $\mathcal{A}_E^{**}$.

Then with $z = \xi - \imath\eta$ there exists an element $\varphi \in F_z$ with $|\varphi| = \omega$.

Proof: We know that α_η is inner for the algebra $\pi(\mathcal{A})''$. Therefore, with the same calculation used in the proof of 2.7.(i) we obtain

$$\omega(B\alpha_{\imath\xi}(A)) = \omega(\beta(\alpha_\eta(A))B) = \omega(AB).$$

Hence $\omega \in F_\xi$.

(ii) If $\varphi \in F_\xi$ then by Lemma 2.3 also $\varphi^* \in F_\xi$. So F_ξ is generated by its self-adjoint elements. Next assume $\varphi(.) = \omega(U.)$ with U^2 the support–projection of ω. Since φ and ω both belong to F_ξ this holds also for $2\varphi^+ = \omega + \varphi$ and for $2\varphi^- = \omega - \varphi$.

(iii) Since α_η is inner in $\mathcal{A}_E^{**}$ exists $V \in \mathcal{A}_E^{**}$ with $\alpha_\eta(A) = V^*AV$ on $\mathcal{A}_F^{**}$. Hence we obtain

$$\omega(V^*B\alpha_\eta \circ \alpha_{\imath\xi}(A)) = \omega(\alpha_\eta(A)V^*B) = \omega(V^*AB).$$

Here we have used that $VV^* = E$ is the identity of the representation π_ω. This shows $\omega(V^*.)$ belongs to F_z. □

We end this section with a remark on possible intersection of two fibres.

2.9 Lemma:

Assume we have $\varphi \in F_{z_1} \cap F_{z_2}$ then $\alpha_{\imath(z_1-z_2)}$ restricted to $\mathcal{A}_E^{**}$ is trivial. In this expression E denotes the support–projection of $|\varphi|$.

Proof: We obtain

$$\varphi(B\alpha_{\imath(z_1-z_2)}(A)) = \varphi(\alpha_{-\imath z_2}(A)B) = \varphi(A\alpha_{\imath z_2}(B)) = \varphi(BA).$$

$B^*V\psi$ is dense in the representation space of ω since $\varphi(.) = \omega(V^*.)$ and since V is unitary in $\mathcal{A}_E^{**}$ This implies $U(z_1 - z_2)$ is trivial when restricted to $\mathcal{A}_E^{**}$, where U is defined by the equation $U(z_1 - z_2)\pi(A)\psi = \pi(\alpha_{(z_1-z_2)}A)\psi$. □

3. KMS-condition and local nets

In this section we will assume in addition to 2.1 that we are dealing with a theory of local observables. We introduce the following

3.1 Notation:

A theory of local observables will be denoted by $\{\mathcal{A}(O), \mathcal{A}, \alpha, \mathbb{R}^d\}$. The symbols have the following meaning:
(i) A C^*-algebra $\mathcal{A}(O)$ fulfilling isotony is associated with every bounded open region O in the Minkowski space $\mathbb{R}^d$.
(ii) $\mathcal{A}$ denotes the C^*-algebra generated by all $\mathcal{A}(O)$.
(iii) The local net fulfills locality, i.e. if O_1 and O_2 are spacelike separated then the two algebras $\mathcal{A}(O_1)$ and $\mathcal{A}(O_2)$ commute.
(iv) α is a representation of $\mathbb{R}^d$ in $\mathrm{Aut}(\mathcal{A})$ with the property

$$\alpha_a \mathcal{A}(O) = \mathcal{A}(O + a) \quad a \in \mathbb{R}^d.$$

(v) We use the following Minkowski scalar product: $x^2 = x_0^2 - (\vec{x})^2$.

The aim of this section is to show that there do not exist "spacelike" temperatures in the theory of local observables. By Theorem 2.8 this implies also that there is no complex temperature with spacelike real part. In other words: The result known from the free fields, that the possible values of $\imath\beta$ belong to the forward or backward tube, is a general feature of local quantum field theories. We will show

3.2 Theorem:

Let $\{\mathcal{A}(O), \mathcal{A}, \alpha, \mathbb{R}^d\}$ be as described above. Assume ξ is a spacelike vector. If $\omega \in F_\xi$ is a state and if $\{\pi, \mathcal{H}, \Omega\}$ is the G.N.S.-representation of ω then

$$\mathcal{A}/\ker\pi \quad \text{is an abelian algebra.}$$

This means in particular: If $\mathcal{A}$ is simple then $F_\xi = \{0\}$.

For the preparation of the proof of the theorem we remark first:

3.3 Lemma:

Let $\{\mathcal{A}(O), \mathcal{A}, \alpha, \mathbb{R}^d\}$ be as described above. Assume O is a bounded open region centered at the origin. Let ξ be a spacelike vector. If $\omega \in F_\xi$ is a state and $A, B \in \mathcal{A}(O)$ then

$$f(z) = \omega(B\alpha_{z\xi}(A))$$

is a periodic function with period $\imath$. If $a > 0$ is the smallest number such that

$$O + x\xi \subset O' \quad \text{for} \quad |x| > a$$

then $f(z)$ is analytic except for the cuts

$$[-a, a] + \imath n, \qquad n = 0, \pm 1, \pm 2, \ldots$$

and $f(z)$ is bounded everywhere:

$$|f(z)| \leq \max\{\sqrt{\omega(BB^*)\omega(A^*A)}, \sqrt{\omega(B^*B)\omega(AA^*)}\}.$$

Proof: Since $\omega \in F_\xi$ we obtain from the KMS-condition

$$\omega(B\alpha_{(x+\imath)\xi}(A)) = \omega(\alpha_{x\xi}(A)B).$$

If $|x| > a$ then $\alpha_{x\xi}(A)$ and B commute. For $|x| > a$ these two relations imply $f(x) = f(x+\imath)$ and hence $f(z)$ is periodic in the imaginary direction with period $\imath$. Since $f(z)$ is analytic in the strip $0 < \Im m\, z < 1$ we see that $f(z)$ is analytic except for the cuts. Since $f(z)$ is bounded everywhere it must take its maximum at the cuts. By the Cauchy-Schwarz inequality we obtain for $\Im m\, z = +0$:

$$|f(z)| \leq \sqrt{\omega(BB^*)\omega(A^*A)},$$

and for $\Im m\, z = 1-0$:

$$|f(z)| \leq \sqrt{\omega(B^*B)\omega(AA^*)}.$$

This shows the lemma. □

In order to obtain more properties of this state we will compute its Fourier-transform. We obtain

3.4 Lemma:

With

$$\mathcal{F}f(\epsilon) = \int \omega(B\alpha_{x\xi}(A))e^{-\imath\epsilon x}dx$$

we obtain

$$\mathcal{F}f(\epsilon) = \begin{cases} g(\epsilon)\frac{1}{1-e^{-\epsilon}} & \text{for} \quad \epsilon > 0 \\ -g(\epsilon)\frac{e^{\epsilon}}{1-e^{\epsilon}} & \text{for} \quad \epsilon < 0 \end{cases} = g(\epsilon)\frac{1}{1-e^{-\epsilon}}. \tag{1}$$

The function $g(\epsilon)$ is the Fourier-transform of the commutator

$$g(\epsilon) = \int_{-a}^{a} \omega([B\alpha_{x\xi}(A)])e^{-\imath\epsilon x}dx \tag{2}$$

and hence it is an entire analytic function. At the origin of $\mathcal{F}f(\epsilon)$ there will appear an additional δ-function contribution describing the cluster property of $f(x)$.

Proof: For $\epsilon > 0$ we can deform the path of integration into the lower half plane and remain with the integrals around the cuts. So we obtain in this situation

$$\begin{aligned}\mathcal{F}f(\epsilon) &= \sum_{n=0}^{-\infty} \int_{-a+n\imath}^{a+n\imath} \omega([B\alpha_{x\xi}(A)])e^{-\imath\epsilon z}dz \\ &= \int_{-a}^{a} \omega([B\alpha_{x\xi}(A)])e^{-\imath\epsilon x}dx \frac{1}{1-e^{-\epsilon}}.\end{aligned}$$

If $\epsilon < 0$ we can deform the contour of integration into the upper half plane. In this situation two things are changing, namely the sum goes only from 1 to ∞ and the integration around the cuts changes the direction, which accounts for the sign in the formula. Since the function $f(x)$ is the Fourier-transform of a measure, the only additional contribution at the origin can be a δ-function. But in the following its exact coefficient is not of interest.□

Next we want to compute the Fourier-transform of the commutator, i.e. we want to look at the function $g(\epsilon)$. But first let us introduce some notations. Let $\omega \in F_\xi$ be a state. Let $(\pi, U(x\xi), \psi)$ be the G.N.S.–representation given by ω, then we can write

$$U(x\xi) = \int \exp\{\imath\epsilon x\}\, dE(\epsilon)$$

where $E(\epsilon)$ is the spectral resolution of the one parametric group $U(x\xi)$. From the KMS–condition follows that $U(x\xi)$ is also the modular group for $\pi(\mathcal{A})''$. We obtain

3.5 Lemma:

In the situation described above we have with

$$(\psi, \pi(B)U(x\xi)\pi(A)\psi) = \int (\psi, \pi(B)dE(\epsilon)\pi(A)\psi)e^{\imath x\epsilon}$$

the equation

$$\omega([B, \alpha_{x\xi}(A)]) = \int (\psi, \pi(B)dE(\epsilon)\pi(A)\psi)(1 - e^{-\epsilon})e^{\imath x\epsilon} \tag{3}$$

where the expression on the right hand side is well defined because of the KMS–condition.

Proof: With $S = J\Delta^{1/2}$ we obtain

$$\begin{aligned}\omega([B, \alpha_{x\xi}(A)]) &= (\pi(B^*)\psi, U(x\xi)\pi(A)\psi) - (\pi(A^*)\psi, U(-x\xi)\pi(B)\psi)\\ &=(\pi(B^*)\psi, U(x\xi)\pi(A)\psi) - (S\pi(A)\psi, U(-x\xi)S\pi(B^*)\psi)\\ &=(\pi(B^*)\psi, U(x\xi)(1-\Delta)\pi(A)\psi)\end{aligned}$$

The last expression is well defined since $\pi(A)\psi$ and $\pi(B^*)\psi$ are both in the domain of definition of $\Delta^{1/2}$. Taking now the inverse Fourier-transform we obtain the stated result. □

Since $(1 - e^{-\epsilon})$ vanishes at the origin the contribution coming from the invariant state drops out. Details are described in the next result.

3.6 Lemma:

Let E_0 be the projection onto the invariant vectors of $U(x\xi)$. Define

$$E'(\epsilon) = \begin{cases} E(\epsilon) & \text{for} \quad \epsilon < 0 \\ E(\epsilon) - E_0 & \text{for} \quad \epsilon \geq 0 \end{cases}$$

then

$$g(\epsilon)\frac{d\epsilon}{1 - e^{-\epsilon}} = (\psi, \pi(B)dE'(\epsilon)\pi(A)\psi)$$

is an entire analytic function of exponential type a.

Proof: ¿From (2) and (3) we obtain

$$g(\epsilon)d\epsilon = (\psi, \pi(B)dE(\epsilon)\pi(A)\psi)(1 - e^{-\epsilon}). \tag{3'}$$

¿From (2) we know that $g(\epsilon)$ is an entire analytic function of exponential type a. Eqn. (3) implies $g(0) = 0$. Hence $g(\epsilon)(1 - e^{-\epsilon})^{-1}$ is again an entire analytic function of exponential type a. If we want to divide (3') by $(1 - e^{-\epsilon})$ we must remember that this factor has annihilated the $\delta(\epsilon)$ contribution, i.e. at zero we only obtain the continuous part of the spectral family. This is the statement of the lemma. □

Proof of the Theorem: The function $\{f(z) - (\psi, \pi(B)E_0\pi(A)\psi)\}d\epsilon$ is the Fourier-transform of

$$(\psi, \pi(B)dE'(\epsilon)\pi(A)\psi).$$

By Lemma 3.6 this expression is entire analytic and of exponential type a. Hence its Fourier-transform vanishes for $|x| > a$. Since $f(z)$ is the boundary value of an analytic function we get the vanishing everywhere. This implies that $f(z)$ is constant and coincides with $(\psi, \pi(B)E_0\pi(A)\psi)$. Next notice that $\cup\mathcal{A}(O)$ is dense in $\mathcal{A}$. This implies that $f(z)$ is constant for arbitrary $A, B \in \mathcal{A}$. Therefore, the translations in the direction ξ are represented trivially. Together with the locality–condition follows that $\mathcal{A}/\ker\pi$ is abelian.□

This result means that there exist no spacelike temperatures and by Theorem 2.8 also no complex temperatures with spacelike real part. Therefore, $\imath$ times temperatures can only belong to the forward– or to the backward tube.

The KMS–functionals in the forward tube are connected with the vacuum state and those in the backward tube are related to the plenum (ceiling) state.

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