Adiabatic Quantum Transport: Quantization and Fluctuations

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Abstract

Quillen's local index theorem is used to study the charge transport coefficients (adiabatic curvature) associated to the ground state of a Schrödinger operator for a charged (spinless) particle on a closed, multiply connected surface. The formula splits the adiabatic curvature into an explicit integral part and a fluctuating part which has a natural interpretation in terms of quantum chaos.

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Some of the interesting recent developments in quantum transport are connected with two complementary phenomena: the precise quantization of certain transport coefficients (e.g. conductance) [1], and the fluctuation of others [2]. Normally the two are viewed as disparate phenomena that have little to do with each other: they pertain to different kinds of physical systems, to different notions of conductances and the theoretical frameworks used to describe the two seem to have little in common.

Here we shall study a class of quantum systems for which the transport coefficients simultaneously display quantization and fluctuation, in a way that, we believe, sheds light on both. A precise description of the kind of systems we consider shall be given below. For the moment, however, these may be thought of roughly as describing the dynamics of a Schrödinger particle on a two dimensional multiply connected surface which has no boundary but is of *finite area*. We consider the non-dissipative transport in a constant magnetic field and $2h$ Aharonov-Bohm flux tubes that thread the h handles of the surface (for the ordinary torus $h = 1$).

There are two related notions of transport coefficients that we consider: *conductance* and *charge transport*. Change of flux through the handles generates electromotive forces around the loops. In the limit of small emf's the currents and emf's are related by the antisymmetric conductance matrix. The charge transport connects the charges transported around the loops to adiabatic increase of fluxes by one quantum unit. They are given by integrals of the corresponding elements of the conductance matrix.

The tool we bring to bear is a local index theorem due to Quillen [3]. When applicable, it splits the conductances into two parts. The first is explicit and universal, i.e. it is - up to factor - the canonical symplectic form on the space of

Aharonov-Bohm fluxes. Its two-dimensional integrals yield quantized charge transport (in units of e^2/h) and therefore provide a connection to the Integral Quantum Hall effect [1, 4]. The second piece in the formula is a complete derivative, hence it does not affect charge transport. It affects however the conductance as a fluctuation term. In contrast to the first, it depends on spectral properties of the Hamiltonian; it is related to (an appropriate regularization of) its determinant.

In systems which are classically chaotic there is a relation between the eigenvalues and the lengths of closed classical orbits. This relation is exact in situations where Selberg's trace formula applies [5] and is a semi-classical approximation in cases where Gutzwiller's formula holds [6]. In some special cases (see below) the determinant of the Hamiltonian can be expressed in terms of Selberg's zeta function.This connects conductance fluctuations to quantum chaos.

In the general context of adiabatic curvature [7], Quillen's theorem, when applicable, solves a problem posed by Robbins and Berry [8], namely, what does the adiabatic curvature say about classically chaotic systems.

There are several applications to this point of view; all of them are consequences of formula (15) below:

1. All transport coefficients associated to the ground state for any compact Riemann surface with h handles and sufficiently large magnetic field B are explicitly determined: they are ± 1 if the pair of fluxes is associated with intersecting loops and 0 otherwise. This solves a problem posed in [9], also solved, by different means, in [10].

2. The contribution of quantum chaos to conductance fluctuations in the ground state, in the limit of large magnetic fields, is exponentially small. This result

has the same flavor as results on the role of the thermodynamic limit on the quantization of the Hall conductance [11].

3. There are no finite volume correction to *charge transport* in the ground state for quantum systems for which Quillen's theorem applies.

4. In special cases where we can apply Selberg's trace formula, conductance fluctuations in the ground state are related to the periodic orbits of the corresponding classical system.

Now we shall describe the model in more detail. Consider a charged, non-relativistic, spinless particle on a topologically closed surface Σ with h handles. Fix a complex structure on it with local coordinate z and a conformal metric $\rho(z,\bar{z})|dz|^2$. The area element is the 2-form $\sigma = \frac{i}{2}\rho dz \wedge d\bar{z}$.

The magnetic field is a real 2-form $B\sigma$. We allow only constant magnetic fields (relative to σ) with total magnetic flux $2\pi f = \int_\Sigma B\sigma$, where f is an integer.

For B fixed consider the space $\mathcal{A}$ of real gauge potentials. For any $A \in \mathcal{A}$ we have $dA = B\sigma$, where

$$dA = \frac{\partial A}{\partial z} dz + \frac{\partial A}{\partial \bar{z}} d\bar{z} \tag{1}$$

and locally on Σ

$$A = \bar{A}^{0,1}(z,\bar{z})dz + A^{0,1}(z,\bar{z})d\bar{z}. \tag{2}$$

To show explicitly the connection between gauge potentials and Aharonov-Bohm fluxes we introduce on Σ a canonical basis of 1-cycles, i.e. simple oriented loops $\gamma_1, ..., \gamma_{2h}$, which consists of h disjoint pairs γ_j, γ_{j+h} (one pair for each handle) such that γ_j intersects γ_{j+h} exactly at one point. Consider the dual basis $a_1, ..., a_{2h}$ of harmonic one forms on Σ normalized by

$$\int_{\gamma_j} a_k = \delta_{jk}, \qquad j,k = 1,...2h. \tag{3}$$

Fix some $A_0 \in \mathcal{A}$; then any gauge potential $A \in \mathcal{A}$ is uniquely represented in the form

$$A = A_0 + \sum_{j=1}^{2h} \phi^j a_j + i\, dg\, g^{-1}, \tag{4}$$

where $g : \Sigma \to U(1)$ is a gauge transformation. Factorizing the space $\mathcal{A}$ by the action of the gauge group, we obtain a 2h-dimensional torus Φ parametrized by the Aharonov-Bohm fluxes.

The flux torus Φ carries natural symplectic and complex structures. (Both of them are necessary to formulate Quillen's theorem). In terms of the Aharonov-Bohm fluxes the symplectic structure is given by

$$\Omega = \sum_{i=1}^{h} d\phi^j \wedge d\phi^{j+h} \tag{5}$$

.

To define the complex structure, consider the Hodge star operator on harmonic 1-forms on Σ,

$$*(\alpha\, dz + \bar{\alpha}\, d\bar{z}) = i(\bar{\alpha}\, d\bar{z} - \alpha\, dz) \tag{6}$$

Since $*^2 = -1$, we can regard $*$ as a multiplication by i; this gives rise to a complex structure on Φ which we denote by J. It is explicitly given by $J = \Omega\, g$, where g is the Riemannian metric on Φ

$$g_{ik} = \int_\Sigma a_i \wedge *a_k. \tag{7}$$

Now we return to the discussion of particles on Σ in the presence of a constant magnetic field B. The family of Schrödinger operators we consider is defined through the kinetic energy of a spinless particle on Σ including a gauge potential A and is given by

$$H(\phi) = (d + iA)^*(d + iA) = 4D^*D + B, \tag{8}$$

where $D = \frac{\partial}{\partial \bar{z}} + iA^{0,1}$ and $D^* = -\frac{1}{\rho}(\frac{\partial}{\partial z} + i\bar{A}^{0,1})$. The Schrödinger operator acts on the Hilbert space H of sections of a U(1) line bundle with the scalar product

$$(\psi, \psi) = \int_\Sigma |\psi|^2 \, \sigma. \tag{9}$$

Let us recall a few basic facts about adiabatic transport [1, 4]. Let $P(\phi)$ denote the spectral projection on the ground state of $H(\phi)$ [12]. Suppose that the fluxes $\phi^j(t)$ depend adiabatically on time. Then, the current around the k-th flux is given by

$$I_k(P, \phi) = i \sum_{j=1}^{2h} \dot{\phi}^j \, \omega_{jk}(P, \phi). \tag{10}$$

Here $\omega_{jk}(P, \phi)$ is the jk component of the adiabatic curvature 2-form $\omega\,(P, \phi)$ [7]:

$$\omega(P, \phi) = Tr\,(P\mathbf{d}P \wedge \mathbf{d}P) = Tr\left(P(\mathbf{d}H)\,\widehat{(H - E)}^{-2}(\mathbf{d}H)\right), \tag{11}$$

where $\mathbf{d}$ denotes exterior derivative with respect to flux and $\mathbf{d}P = \sum \frac{\partial P}{\partial \phi^i}\mathbf{d}\phi^i$. The hat in $\widehat{(H - E)}^{-1}$ excises the pole at E (the reduced resolvent). The expression on the right hand side of Eq. (11) is Kubo's formula in operator notation. In the case at hand, the adiabatic curvature is also given by the formula

$$\omega(P, \phi) = Tr\,(P\mathbf{d}D^* \frac{1}{DD^*}\mathbf{d}D). \tag{12}$$

The periods of the current one form have a direct physical interpretation. The total charge transported around the k-th flux driven by the adiabatic increase by one unit of quantum flux in the j-th flux is:

$$Q_{jk} = \int_{0 \le \phi^j \le 2\pi} \omega_{jk}(P, \phi) d\phi^j. \tag{13}$$

In the situation we consider Quillen's local index theorem gives an explicit formula for the curvature form of the so-called determinant bundle of the family of Cauchy-Riemann operators D, parametrised by Aharonov-Bohm fluxes ϕ. To apply Quillen's formula to the analysis of adiabatic curvature, notice first that by Riemann-Roch

$$dim\, ker\, D - dim\, ker\, D^* = f - h + 1. \tag{14}$$

By a standard positivity argument $ker\, D^* = 0$ when $f \geq 2h-1$. Then $dim\, ker\, D = f - h + 1$ is constant on the flux space and Quillen's theorem yields the following formula for conductance:

$$\frac{i}{2\pi}\omega(P,\phi) = -\frac{1}{(2\pi)^2}\Omega - \frac{1}{4\pi}\, d\, J\, d \log \det D^* D. \tag{15}$$

Here $\det D^* D$ is defined via the zeta function regularization,

$$\log\ \det\ D^* D = \lim_{\varepsilon\to 0} \int_\varepsilon^\infty \mathrm{Tr}\left(e^{-t\, D^* D}\right) \frac{dt}{t}. \tag{16}$$

The determinant term is an exact two form on Φ, hence it does not contribute to charge transport, and represents conductance fluctuations. The natural symplectic form Ω on the flux torus depends only on the topology of the surface and in this sense is universal. Furthermore it determines the charge transport. Thus, Quillen's theorem divides the conductance into a constant and a fluctuating term.

In addition to the consequences of formula (15) mentioned at the beginning we can draw the following conclusions:

1. Formula (15) suggests, that the behavior of conductance is qualitatively more complicated for small magnetic fields, i.e. $f = 0, 1, ..., 2(h-1)$.

2. It is known that for the special case of the flat two torus $\det D^*D$ is constant on the flux torus. Hence conductance has no fluctuating term. This relates well to the fact that classical dynamics on the flat torus is integrable.

3. More interesting is the case of surfaces of constant negative curvature -1. They are conveniently realized as the orbit spaces of discrete subgroups Γ of $SL(2,R)$ acting on the upper half plane H . In this case the determinant of the Hamiltonian can be expressed in terms of Selberg's zeta function,

$$Z(s,\phi) = \prod_{primitive\,\gamma} \prod_{k=0}^{\infty} \left(1 - \chi(\gamma,\phi) e^{-(s+k)\ell(\gamma)}\right). \tag{17}$$

Here $\gamma \in \Gamma$ are primitive hyperbolic elements of Γ representing conjugacy classes and may be thought of as closed geodesics on Σ; $\ell(\gamma)$ denotes the length of γ and $\chi(\gamma,\phi)$ are phase factors that carry information about the fluxes:

$$\chi(\gamma,\phi) = \exp i \left(\sum_1^{2h} \phi_j\, n_j(\gamma)\right) \tag{18}$$

where $n_j(\gamma)$ counts the number of times (including signs) the closed geodesic γ goes around the j-th fundamental loop. The determinant of D^*D is expressed in terms of Selberg zeta function by [13]:

$$\det D^*D = c_h\, Z(B,\chi), \tag{19}$$

c_h is a constant independent of the fluxes. When B becomes large, $Z(B,\chi)$ tends to one and its derivatives decrease exponentially. Therefore the chaotic contribution to conductance is exponentially small for large magnetic fields. Moreover the leading term in the asymptotics of the fluctuating term for large B is given by $\exp - (B\ell(\gamma_{min}))$, where γ_{min} is the shortest homologically non trivial closed geodesic.

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