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Strong Clustering in Type III Entropic K-Systems

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Abstract

It is shown that automorphisms of some factors of type III_λ, with $0 < \lambda \leq 1$, corresponding to Kolmogorov quantum dynamical systems of entropic type are strongly clustering.

1 Introduction

In [1] entropic K-systems are introduced to extend the notion of Kolmogorov systems to quantum theory. We consider quantum systems that evolve under some automorphism and we say that they undergo a *complete memory loss* if observations become completely independent, the degree of independence being measured by the dynamical entropy of Connes, Narnhofer and Thirring (CNT-entropy) [2].

In [3] it is proved that, as a consequence of complete memory loss, automorphisms of type II_1 von Neumann algebras are strongly asymptotically Abelian.

The restriction on the type of the algebra provided better control over the dynamical entropy. In this paper we reduce the gap by showing that the result remains true for a class of type III_λ factors, with $0 < \lambda \leq 1$, including quantum mechanical systems in quasifree states.

2 Classical and Quantum K-Systems

We shall consider quantum dynamical systems $(\mathcal{M}, \Theta, \omega)$ where $\mathcal{M}$ is a von Neumann algebra, Θ an automorphism of $\mathcal{M}$ and ω a faithful, Θ-invariant state.

In [2] the dynamical entropy $h_\omega(\Theta, \mathbf{M})$ of Θ with respect to a finite dimensional subalgebra $\mathbf{M} \subseteq \mathcal{M}$ was introduced. Our arguments will be based on a slightly different, not equivalent, entropic functional, denoted by $H_\omega(\Theta, \mathbf{M})$, which is introduced in [4] and leads to the same dynamical entropy $h_\omega(\Theta)$ of Θ as in the CNT theory once the dependence on the finite dimensional subalgebras is eliminated (see also [5]):

$$h_\omega(\Theta) = \sup_{\mathbf{M}} h_\omega(\Theta, \mathbf{M}) = \sup_{\mathbf{M}} H_\omega(\Theta, \mathbf{M}) \, . \tag{1}$$

We establish some notations and recall a few results.

Let $\mathcal{B}$ be an Abelian von Neumann algebra, σ an automorphism of $\mathcal{B}$ and μ a σ-invariant state on $\mathcal{B}$. By means of the Gelfand transform, the triple $(\mathcal{B}, \sigma, \mu)$ can be identified with an Abelian dynamical system typical of the measure-theoretic approach to classical ergodic theory (see [6]), and its main concepts translated accordingly. In particular, finite partitions $P = \{p_i\}_{i=1}^p$ of a measure space $\mathcal{X}$ into p disjoint atoms p_i are replaced by finite dimensional subalgebras $P = \{\hat{p}_i\}_{i=1}^p$, their minimal projections $\hat{p}_i$ corresponding to the characteristic functions $\chi_i(\mathbf{x})$ of the atoms p_i.

Given a measure μ on $\mathcal{X}$, the characteristic functions can be represented as multiplication operators on the Hilbert space $L^2(\mathcal{X}, \mu)$. The strong-operator closure of their linear span is the von Neumann algebra $\mathcal{B}$ of essentially bounded functions on $\mathcal{X}$. The measure μ will in turn provide, by integration, a state on $\mathcal{B}$ (which we shall denote by the same symbol): $\mu(\hat{p}_i) = \int_{\mathcal{X}} \chi_i(\mathbf{x}) \mathrm{d}\mathbf{x}$.

Definition 1 *1. Let $\mathcal{B}$ be an Abelian von Neumann algebra acting on some Hilbert space and $P = \{\hat{p}_i\}_{i=1}^p$ a finite dimensional subalgebra. Given a state μ on $\mathcal{B}$, the entropy $S_\mu(P)$ of $P \subseteq \mathcal{B}$ with respect to the state μ is given by:*

$$S_\mu(P) = -\sum_{i=1}^{p} \mu(\hat{p}_i) \log \mu(\hat{p}_i) \,. \tag{2}$$

It increases under inclusion, namely $P_1 \subseteq P_2 \Rightarrow S_\mu(P_1) \le S_\mu(P_2)$.

2. Let $Q = \{\hat{q}_j\}_{j=1}^q \subseteq \mathcal{B}$ and $P \vee Q$ denote the subalgebra of $\mathcal{B}$ generated by the minimal projections $\{\hat{p}_i \hat{q}_j\}_{i,j}$ (the refinement of P and Q). The conditional entropy of P given Q,

$$S_\mu(P|Q) = S_\mu(P \vee Q) - S_\mu(Q) \,, \tag{3}$$

is positive and continuous with respect to both arguments. Moreover, under inclusion, it increases in the first argument and decreases in the second one.

3. Let $\sigma : \mathcal{B} \to \mathcal{B}$ be an automorphism of $\mathcal{B}$ and $\sigma^j(P)$, $j \in \mathbb{Z}$, the image of P under the j-th iteration of the dynamical mapping σ, namely the subalgebra with minimal projections $\{\sigma^j(\hat{p}_i)\}_{i=1}^p$. If $\mu \circ \sigma = \mu$, then $S_\mu(\sigma^j(P)) = S_\mu(P)$.

4. Let $P_{n-1}^0 = \bigvee_{j=0}^{n-1} \sigma^{-j}(P)$ be the refinement of n consecutive iterations (the limit $n \to +\infty$ in this and similar expressions is to be understood with respect to the strong-operator topology). The Kolmogorov-Sinai entropy of σ (with respect to P):

$$h_\mu(\sigma, P) = \lim_{n \to +\infty} \frac{1}{n} S_\mu(P_{n-1}^0) \,, \tag{4}$$

is well defined and corresponds to the average information gain about P provided by the dynamics with respect to the state μ.

5. Let $P_k^l = \bigvee_{j=l}^{k} \sigma^{-j}(P)$. Then, the remote past of $P \subseteq \mathcal{B}$, also called the tail *of P, is the von Neumann subalgebra arising from the (strong-operator) limit*

$$\mathrm{Tail}(P) = \lim_{l \to +\infty} \lim_{k \to +\infty} P_k^l \,. \tag{5}$$

6. Those triples $(\mathcal{B}, \sigma, \mu)$ where all finite dimensional subalgebras P have trivial tail, *i.e.* $\mathrm{Tail}(P) = \{\mathbb{C}\hat{1}\}$, *are called* K(olmogorov)-systems.

Remarks 1 1. Let $\{\mathbb{C}\hat{1}\}$ the trivial subalgebra of $\mathcal{B}$, then $S_\mu(P|\{\mathbb{C}\hat{1}\}) = S_\mu(P)$.

2. The past orbit of $P \subseteq \mathcal{B}$

$$P_- = P_\infty^1 = \bigvee_{j=1}^{\infty} \sigma^{-j}(P) \,, \tag{6}$$

together with the properties of the conditional entropy permits to write:

$$h_\mu(\sigma,P) = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n-1} S_\mu(P|P_k^1) = S_\mu(P|P_-)\,. \tag{7}$$

3. From subadditivity, $S_\mu(P_{k-1}^0)\leq k\,S_\mu(P)$, where $P\subseteq\mathcal{B}$ is any finite dimensional algebra and σ any automorphism of $\mathcal{B}$, it follows that

$$h_\mu(\sigma,P)\leq S_\mu(P)\,. \tag{8}$$

Let now $P_k^l(n) = \bigvee_{j=l}^k \sigma^{-jn}(P)$, $n>0$, and notice that, when $n\to+\infty$, $P_-(n)$ is eventually contained in Tail(P). Therefore, if $(\mathcal{B},\sigma,\mu)$ is a K-system, then (see [1]):

$$\lim_{n\to\infty} h_\mu(\sigma^n,P) = \lim_{n\to\infty} S_\mu(P|P_-(n)) = S_\mu(P)\,.$$

This means that $S_\mu(P_{k-1}^0(n))$ becomes additive asymptotically:

$$\lim_{n\to+\infty} S_\mu(P\vee\sigma^{-n}(P)\vee\ldots\vee\sigma^{-n(k-1)}(P)) = k\,S_\mu(P)\quad \forall k\in\mathbb{N}\,. \tag{9}$$

4. Classical K-systems $(\mathcal{B},\sigma,\mu)$ are algebraically distinguished by the existence of a (K-) subalgebra $\mathcal{B}_0\subseteq\mathcal{B}$ such that

$$p\leq q\Rightarrow \sigma^p(\mathcal{B}_0)\subseteq\sigma^q(\mathcal{B}_0)\,,\quad \forall p,q\in\mathbb{Z}$$
$$\bigvee_{j=-\infty}^{+\infty}\sigma^j(\mathcal{B}_0) = \mathcal{B}$$
$$\bigwedge_{j=-\infty}^{+\infty}\sigma^j(\mathcal{B}_0) = \{\mathbb{C}\hat{1\!\!1}\}\,,$$

where $\bigwedge$ denotes the smallest von Neumann algebra contained in all $\sigma^j(\mathcal{B}_0)$.

5. K-system are among the most random classical dynamical systems, randomness showing up in the rate of decorrelation. For instance, they are all kind of mixing [6], the so called strong mixing of classical ergodic theory (we warn the reader that there is a formally similar expression in quantum statistical mechanics which is termed weak mixing) corresponding to:

$$\lim_{k\to\pm\infty}\mu(\hat{b}_1\sigma^k(\hat{b}_2)) = \mu(\hat{b}_1)\mu(\hat{b}_2)\quad \forall\hat{b}_1\,,\hat{b}_2\in\mathcal{B}\,. \tag{10}$$

Definition 2 *We say that a quantum dynamical system* $(\mathcal{M},\Theta,\omega)$ *is* stationarily coupled *with an Abelian dynamical system* $(\mathcal{B},\sigma,\mu)$ *when there exists a state* λ *on the von Neumann algebra* $\mathcal{M}\otimes\mathcal{B}$ *which is invariant under the automorphism* $\Theta\otimes\sigma$ *of* $\mathcal{M}\otimes\mathcal{B}$ *and, when restricted to* $\mathcal{M}$*, respectively to* $\mathcal{B}$*, reduces to* ω *, respectively to* μ *, namely* $\lambda|_{\mathcal{M}}=\omega$*,* $\lambda|_{\mathcal{B}}=\mu$*.*

Any triple $(\mathcal{B},\sigma,\mu)$ *stationarily coupled to* $(\mathcal{M},\Theta,\omega)$ *will be called an* Abelian model.

Remarks 2 1. We suppose $\mathcal{M}$ to be represented on some Hilbert space $\mathcal{H}$. The restrictions of any state on $\mathcal{M}$ to a finite dimensional subalgebra $\mathbf{M} \subseteq \mathcal{M}$ are normal states and thus correspond to trace class operators (density matrices) [7]. Thus, it makes sense writing the *relative entropy* [9] of two states ω, ρ on $\mathcal{M}$ restricted to such an $\mathbf{M}$ as:

$$S(\omega|_{\mathbf{M}}, \rho|_{\mathbf{M}}) = \mathrm{Tr}\left\{\rho|_{\mathbf{M}}\Big(\log\rho|_{\mathbf{M}} - \log\omega|_{\mathbf{M}}\Big)\right\}. \tag{11}$$

2. Let $(\mathcal{M}, \Theta, \omega)$ be stationarily coupled to $(\mathcal{B}, \sigma, \mu)$ through the state λ and consider the product state $\omega \otimes \mu$ on $\mathcal{M} \otimes \mathcal{B}$:

$$\omega \otimes \mu(\hat{x} \otimes \hat{b}) = \omega(\hat{x})\mu(\hat{b}) \quad \forall \hat{x} \in \mathcal{M}, \hat{b} \in \mathcal{B}.$$

Let $\mathbf{M}$ and P be finite dimensional subalgebras of $\mathcal{M}$, respectively $\mathcal{B}$, with $\{\hat{p}_i\}_{i=1}^{p}$ the minimal projections of P. Given the states

$$\hat{\lambda}_i(\hat{m} \otimes \hat{p}) = \frac{\lambda(\hat{m} \otimes \hat{p}\hat{p}_i)}{\lambda(\hat{p}_i)} \quad \forall \hat{m} \in \mathcal{M}, \hat{p} \in \mathcal{B}, \tag{12}$$

let $\hat{\omega}_i$ denote the restrictions of $\hat{\lambda}_i$ to $\mathcal{M}$, π_i the point measures $\pi_i(\hat{p}_j) = \delta_{i,j}$ on $\mathcal{B}$ and set $\mu_i = \lambda(\hat{p}_i) = \mu(\hat{p}_i)$. Then, from (11):

$$\lambda = \sum_{i=1}^{p} \mu_i\, \hat{\omega}_i \otimes \pi_i, \quad \omega = \sum_{i=1}^{p} \mu_i \hat{\omega}_i \tag{13}$$

$$S(\lambda|_{\mathbf{M}} \otimes \lambda|_P, \lambda|_{\mathbf{M}\otimes P}) = \sum_{i=1}^{p} \mu_i\, S(\omega|_{\mathbf{M}}, \hat{\omega}_i|_{\mathbf{M}}). \tag{14}$$

3. By using the modular automorphism τ_ω of ω (assumed to be faithful), the states $\hat{\omega}_i$ contributing to the decompositions in (13) can be expressed by means of positive operators $\hat{x}_i \in \mathcal{M}$ such that $\sum_i \hat{x}_i = \hat{1\!\!1}$. Thus [2]:

$$\hat{\omega}_i(\hat{m}) = \frac{\lambda(\hat{m} \otimes \hat{p}_i)}{\mu_i} = \frac{\omega(\tau_\omega^{i/2}(\hat{x}_i)\hat{m})}{\omega(\hat{x}_i)}, \quad \hat{m} \in \mathcal{M}. \tag{15}$$

Definition 3 *The entropy of every finite dimensional subalgebra* $\mathbf{M} \subseteq \mathcal{M}$ *with respect to* ω *is defined to be [2, 10]:*

$$\begin{aligned} H_\omega(\mathbf{M}) &= \sup_{\omega=\sum_i \omega_i} \sum_i \omega_i(1\!\!1)\, S(\omega|_{\mathbf{M}}, \hat{\omega}_i|_{\mathbf{M}}) \\ &= \sup_{\omega=\sum_i \omega_i} \Big\{ S(\omega|_{\mathbf{M}}) - \sum_i \omega_i(\hat{1\!\!1}) S(\hat{\omega}_i|_{\mathbf{M}}) \Big\}, \end{aligned} \tag{16}$$

where $S(\omega|_{\mathbf{M}}) = -\mathrm{Tr}\,\{\omega|_{\mathbf{M}} \log \omega|_{\mathbf{M}}\}$ *is the* von Neumann entropy *of the state* $\omega|_{\mathbf{M}}$ *and the supremum is computed over all possible decompositions of the state* ω.

According to Remark 2.2, by considering triples $(\lambda, \mathcal{B}, P)$ where $(\lambda, \mathcal{B})$ correspond to all possible stationary couplings $(\mathcal{M} \otimes \mathcal{B}, \Theta \otimes \sigma, \lambda)$ and P to all possible finite dimensional subalgebras of $\mathcal{B}$, we can rewrite:

$$H_\omega(\mathbf{M}) = \sup_{\lambda,\mathcal{B},P} S(\omega|_{\mathbf{M}} \otimes \mu|_P, \lambda|_{\mathbf{M}\otimes P}) \,. \tag{17}$$

Let $H_\lambda(P|\mathbf{M})$ be (see [5] for the second equality below):

$$\begin{aligned} H_\lambda(P|\mathbf{M}) &= \sup_{\lambda=\sum_i \lambda_i} \sum_i \Big[S(\lambda|_P, \lambda_i|_P) - S(\lambda|_{\mathbf{M}}, \lambda_i|_{\mathbf{M}})\Big] &(18)\\ &= S_\mu(P) - S(\lambda|_{\mathbf{M}} \otimes \lambda|_P, \lambda|_{\mathbf{M}\otimes P}) \,, &(19)\end{aligned}$$

the supremum being taken over all possible decompositions of the global state λ. Then, because of (19), (16) reads

$$H_\omega(\mathbf{M}) = \sup_{\lambda,\mathcal{B},P} \Big\{ S_\mu(P) - H_\lambda(P|\mathbf{M}) \Big\} \,. \tag{20}$$

In [2] the 2-subalgebra entropy functional $H_\omega(\mathbf{M}_1, \mathbf{M}_2)$, $\mathbf{M}_1$ and $\mathbf{M}_2$ two finite dimensional subalgebras of $\mathcal{M}$, is defined by maximizing the quantity

$$\begin{aligned} H_{\{\omega_{ij}\}}(\mathbf{M}_1, \mathbf{M}_2) &= \sum_{ij} \eta(\omega_{ij}(\hat{1\!\!1})) - \sum_i \eta(\omega^1_i(\hat{1\!\!1})) - \sum_j \eta(\omega^2_j(\hat{1\!\!1})) &(21)\\ &+ \sum_i \omega^1_i(\hat{1\!\!1}) S(\omega|_{\mathbf{M}_1}, \omega^1_i|_{\mathbf{M}_1}) &(22)\\ &+ \sum_j \omega^2_j(\hat{1\!\!1}) S(\omega|_{\mathbf{M}_2}, \omega^2_j|_{\mathbf{M}_2}) \,, &(23)\end{aligned}$$

over all possible decompositions of ω in terms of doubly indexed states $\hat{\omega}_{ij}$ over $\mathcal{M}$: $\omega = \sum_{ij} \omega_{ij}(\hat{1\!\!1})\hat{\omega}_{ij}$, where $\eta(x) = \begin{cases} -x\log x \ldots 0 < x \leq 1 \\ 0 \qquad\quad \ldots x = 0 \end{cases}$, and, by means of positive operators $\hat{x}_{ij} \in \mathcal{M}$ such that $\sum_{ij} \hat{x}_{ij} = \hat{1\!\!1}$ as in (15), [2]

$$\begin{aligned} \omega_{ij}(\hat{m}) &= \omega(\tau^{i/2}_\omega(\hat{x}_{ij})\hat{m}) \,, \quad \omega^1_i = \sum_j \omega_{ij} \,, \quad \omega^2_j = \sum_i \omega_{ij} &(24)\\ \hat{\omega}_{ij}(\hat{m}) &= \frac{\omega_{ij}(\hat{m})}{\omega_{ij}(\hat{1\!\!1})} \,, \quad \hat{\omega}^1_i(\hat{m}) = \frac{\omega^1_i(\hat{m})}{\omega^1_i(\hat{1\!\!1})} \,, \quad \hat{\omega}^2_j(\hat{m}) = \frac{\omega^2_j(\hat{m})}{\omega^2_j(\hat{1\!\!1})} &(25)\\ \omega_{ij}(\hat{1\!\!1}) &= \omega_{ij}(\hat{x}_{ij}) \,, \, \omega^1_i(\hat{1\!\!1}) = \sum_j \omega_{ij}(\hat{x}_{ij}) \,, \, \omega^2_j(\hat{1\!\!1}) = \sum_i \omega_{ij}(\hat{x}_{ij}) \,. &(26)\end{aligned}$$

The extension to more subalgebras is straightforward and leads, in the CNT theory, to the following definition of the *average information gain* about $\mathbf{M} \subseteq \mathcal{M}$ provided by the dynamics Θ:

$$h_\omega(\Theta, \mathbf{M}) = \lim_{k\to+\infty} \frac{1}{k} H_\omega(\mathbf{M}, \ldots, \Theta^{k-1}(\mathbf{M})) \,, \tag{27}$$

and, subsequently to define the dynamical entropy (CNT-entropy) of Θ with respect to the invariant state ω [2] as:

$$h_\omega(\Theta) = \sup_{\mathbf{M}} h_\omega(\Theta, \mathbf{M}) \,, \tag{28}$$

whence the independence on the finite dimensional subalgebras.

A formulation different in spirit makes use of the following notion [4, 5]:

Definition 4 *Let $(\mathcal{B}, \sigma, \mu)$ be stationarily coupled with $(\mathcal{M}, \Theta, \omega)$ through λ and define the average information $H_\omega(\Theta, \mathbf{M})$ provided by the automorphism Θ about the finite dimensional subalgebra $\mathbf{M} \subseteq \mathcal{M}$ as*

$$H_\omega(\Theta, \mathbf{M}) = \sup_{\lambda, \mathcal{B}, P} \Big[h_\mu(\sigma, P) - H_\lambda(P|\mathbf{M}) \Big] \,, \tag{29}$$

where the supremum is taken over all finite dimensional subalgebras $P \subseteq \mathcal{B}$ of all possible stationarily coupled Abelian models.

Remark 3 In [4] it was shown that the supremum of $H_\omega(\Theta, \mathbf{M})$ over all finite dimensional subalgebras $\mathbf{M} \subseteq \mathcal{M}$ coincides with the dynamical entropy $h_\omega(\Theta)$ in (28) (see (1). The proof of the equivalence requires that the supremum be taken. Indeed (compare [4, Prop. 4.1]), in general, we have:

$$0 \leq H_\omega(\Theta, \mathbf{M}) \leq h_\omega(\Theta, \mathbf{M}) \leq H_\omega(\mathbf{M}) \,. \tag{30}$$

The essential observation in [4] is that it is sufficient to consider mixing Abelian models $(\mathcal{B}, \mu, \sigma)$.

Definition 5 *The dynamical triple $(\mathcal{M}, \omega, \Theta)$ is an* entropic K-system *if*

$$H_\omega(\mathbf{M}) > 0 \,, \tag{31}$$

$$\lim_{n \to \infty} H_\omega(\Theta^n, \mathbf{M}) = H_\omega(\mathbf{M}) \,. \tag{32}$$

for all finite dimensional subalgebras $\mathbf{M} \subseteq \mathcal{M}$.

Remarks 4 1. Because of (30), entropic K-systems according to Definition 5 might ask for more than in [1], where they were defined by requiring

$$\lim_{n \to +\infty} h_\omega(\Theta^n, \mathbf{M}) = H_\omega(\mathbf{M}) \,. \tag{33}$$

for all finite dimensional subalgebras $\mathbf{M} \subseteq \mathcal{M}$. Like the old one, the new definition entails that the entropy functionals $H_\omega(\mathbf{M}, \Theta^n(\mathbf{M}), \ldots)$, usually subadditive

$$H_\omega(\mathbf{M}, \Theta(\mathbf{M}), \ldots, \Theta^{k-1}(\mathbf{M})) \leq k H_\omega(\mathbf{M}) \,, \tag{34}$$

become additive asymptotically (use (27)and (34)):

$$H_\omega(\mathbf{M}) = \lim_{n\to+\infty} H_\omega(\Theta^n, \mathbf{M}) \leq \lim_{n\to+\infty} h_\omega(\Theta^n, \mathbf{M}) \leq H_\omega(\mathbf{M})\,. \tag{35}$$

2. The entropy functionals $H_\omega(\mathbf{M}, \Theta(\mathbf{M})\ldots)$ are defined for positive times, unlike the classical analogous in Remark 1.3. This makes no difference, for Θ is invertible and we can equally well use Θ^{-1}.

3. We also added condition (31 (ever true in type II_1 factors) (compare [3]), to make (32) not trivially fulfilled. It is satisfied if the GNS vector $|\Omega>$ corresponding to ω is separating for $\mathcal{M}$.

We conclude this section with a few particular cases where the computation of $H_\omega(\mathbf{C})$ can be explicitly carried through.

Lemma 1 *Let $\hat\sigma_x$, $\hat\sigma_y$, $\hat\sigma_z$ be the Pauli matrices, $\vec\sigma = (\hat\sigma_x, \hat\sigma_y, \hat\sigma_z)$. Let*

$$\begin{aligned}\hat\rho_a &= \frac{\hat{1\!\!1}}{2} + \frac{a}{1+a^2}\hat\sigma_z\,, \quad 0 \leq a \leq 1 \\ \hat p_c^\pm &= \frac{\hat{1\!\!1} \pm \vec c\cdot\vec\sigma}{2}\,, \quad |\vec c| = 1\end{aligned}$$

be a density matrix in $\mathbf{M}_2(\mathbb{C})$, respectively a minimal projection. Let $\mathbf{C}$ be the Abelian subalgebra generated by $\hat p_c^\pm$. Then,

1. *If $a = 1$, then $H_{\hat\rho_1}(\mathbf{C}) = 0$.*
2. *If $a = 0$, then $H_{\hat\rho_0}(\mathbf{C}) = \log 2$.*
3. *If $0 < a < 1$ and $\vec c = (0,0,1)$, then*

$$H_{\hat\rho_a}(\mathbf{C}_{\pi/2}) = -\frac{(1+a)^2}{2(1+a^2)}\log\frac{(1+a)^2}{2(1+a^2)} - \frac{(1-a)^2}{2(1+a^2)}\log\frac{(1-a)^2}{2(1+a^2)}\,.$$

4. *If $0 < a < 1$ and $\vec c = (\cos\theta, 0, \sin\theta)$, then*

$$\begin{aligned}H_{\hat\rho_a}(\mathbf{C}_\theta) = &- \frac{1+a^2+2a\,c_3}{2(1+a^2)}\log\frac{1+a^2+2a\,c_3}{2(1+a^2)} \\ &- \frac{1+a^2-2a\,c_3}{2(1+a^2)}\log\frac{1+a^2-2a\,c_3}{2(1+a^2)} \\ &+ \frac{1+a^2+R}{2(1+a^2)}\log\frac{1+a^2+R}{2(1+a^2)} + \frac{1+a^2-R}{2(1+a^2)}\log\frac{1+a^2-R}{2(1+a^2)}\,,\end{aligned}$$

where $R = \sqrt{(1+a^2)^2c_3^2 + (1-a^2)^2c_1^2} \leq 1 + a^2$.
The optimal values uniquely correspond to the decompositions

1. *$\hat\rho_1 = \hat\rho_1$.*
2. *$\hat\rho_0 = \frac{1}{2}\hat p_c^+ + \frac{1}{2}\hat p_c^-$.*
3. *$\hat\rho_a = \sqrt{\hat\rho_a}\hat p_c^+\sqrt{\hat\rho_a} + \sqrt{\hat\rho_a}\hat p_c^-\sqrt{\hat\rho_a}$.*
4. *$\hat\rho_a = \sqrt{\hat\rho_a}\hat p_\alpha^+\sqrt{\hat\rho_a} + \sqrt{\hat\rho_a}\hat p_\alpha^-\sqrt{\hat\rho_a}$, where*

$$\hat p_\alpha^\pm = \frac{\hat{1\!\!1} \pm \cos\alpha\sigma_x \pm \sin\alpha\sigma_z}{2}\,, \quad \tan\alpha = \frac{(1+a^2)c_3}{(1-a^2)c_1}\,.$$

Proof: see [12].

Remark 5 In the last case the result extends to the Abelian algebra $\mathbf{C}_{\theta,\phi}$ generated by the projections corresponding to the unit vector $\vec{c}_\phi = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$.

In fact, it is a general fact that if a finite dimensional subalgebra is rotated, $\mathbf{M} \to U\mathbf{M}U^{-1}$, it has the same entropy with respect to the rotated state $U\omega U^{-1}$ and that it is given by the rotated optimal decomposition:

$$\begin{aligned} H_\omega(\mathbf{M}) &= \sum_i \lambda_i S(\omega_I|_{\mathbf{M}}) \\ &= \sum_i \lambda_i S(U\omega_i U^{-1}|_{U\mathbf{M}U^{-1}}) = H_{U\omega U^{-1}}(U\mathbf{M}U^{-1}) \,. \end{aligned} \tag{36}$$

By choosing U to correspond to the rotation about the z-axis which turns $\vec{c}$ into $\vec{c}_\phi$, the state ρ_a remains invariant and $H_{\hat{\rho}_a}(\mathbf{C}_{\theta,\phi})$ is thus given by the projections

$$\hat{p}^{\pm}_{\alpha,\phi} = \frac{\hat{1\!\!1} \pm \cos\alpha\cos\phi\,\sigma_x \pm \cos\alpha\sin\phi\,\sigma_y \pm \sin\alpha\sigma_z}{2}\,, \quad \tan\alpha = \frac{1+a^2}{1-a^2}\tan\theta\,.$$

3 Clustering in Optimal Abelian Models

We say that the triple $(\mathcal{B}, \lambda, P \subseteq \mathcal{B})$ associated with an Abelian model $(\mathcal{B}, \sigma, \mu)$ and a stationary coupling λ is *optimal up to* $\epsilon > 0$ for $H_\omega(\Theta, \mathbf{M})$ if

$$S_\mu(P|P_-) - H_\lambda(P|\mathbf{M}) \geq H_\omega(\Theta, \mathbf{M}) - \epsilon\,. \tag{37}$$

Then, we prove:

Proposition 1 *Let $(\mathcal{M}, \Theta, \omega)$ be an entropic K-system according to Definition 5 and $\mathbf{M} \subseteq \mathcal{M}$ a finite dimensional subalgebra. For any $\epsilon_{1,2} > 0$ there exists an $n_0 \in \mathbb{N}$ such that for every dynamical system $(\mathcal{M}, \Theta^n, \omega)$ and $n > n_0$ we can find a stationarily coupled Abelian model $(\mathcal{Q}, \sigma, \mu)$ and a finite dimensional subalgebra $P \subseteq \mathcal{Q}$ such that:*

$$\left|\mu(\hat{p}\hat{r}) - \mu(\hat{p})\mu(\hat{r})\right| \leq \delta_1(\epsilon_1, \epsilon_2)\|\hat{p}\hat{r}\| \quad \forall \hat{p} \in P\,,\, \forall \hat{r} \in \mathcal{R}\,,$$

where $\mathcal{R} = \mathrm{Tail}(P)$ and $\delta_1(\epsilon_1, \epsilon_2)$ is a positive function that $\to 0$ with ϵ_1 and $\epsilon_2 \to 0$.

Proof: Fix $\epsilon_1 > 0$ and choose $N_1 \in \mathbb{N}$ such that for all $n > N_1$

$$H_\omega(\Theta^n, \mathbf{M}) \geq H_\omega(\mathbf{M}) - \epsilon_1\,.$$

Let $(\mathcal{B}, \lambda, P \subseteq \mathcal{B})$ be a triple from a stationarily coupled Abelian model $(\mathcal{B}, \sigma, \mu)$ that is optimal up to $\epsilon_2 > 0$ for $H_\omega(\Theta^n, \mathbf{M})$. Then,

$$S_\mu(P|P_-) - H_\lambda(P|\mathbf{M}) \geq H_\omega(\mathbf{M}) - \epsilon_1 - \epsilon_2\,, \tag{38}$$

where (7) and (29) have been used.

Let $\mathcal{R} = \mathrm{Tail}(P)$. Since $\{\mathbb{C}\hat{\mathbb{1}}\} \subseteq \mathcal{R} \subseteq P_-$ and the conditional entropy decreases as the second argument becomes larger, using Remark 1.1, (38) and (20) we get

$$\begin{aligned} S_\mu(P) - H_\lambda(P|\mathbf{M}) &\geq S_\mu(P|\mathcal{R}) - H_\lambda(P|\mathbf{M}) \\ &\geq S_\mu(P|P_-) - H_\lambda(P|\mathbf{M}) \\ &\stackrel{(38)}{\geq} H_\omega(\mathbf{M}) - \epsilon_1 - \epsilon_2 \\ &\stackrel{(20)}{\geq} S_\mu(P) - H_\lambda(P|\mathbf{M}) - \epsilon_1 - \epsilon_2 \,, \end{aligned} \tag{39}$$

whence

$$0 \geq S_\mu(P|\mathcal{R}) - S_\mu(P) \geq -\epsilon_1 - \epsilon_2 \,. \tag{40}$$

Let $\mathcal{Q} = \bigvee_{n\in\mathbb{Z}} \sigma^n(P)$ be the von Neumann algebra generated by the orbit of P and denote the restrictions of μ and σ to $\mathcal{Q}$ by the same symbols. The new dynamical triple $(\mathcal{Q}, \sigma, \mu)$ together with the generating subalgebra $P(\subseteq \mathcal{Q})$ and the same stationary coupling λ, can be used to get close to $H_\omega(\Theta^n, \mathbf{M})$ up to $\epsilon_2 > 0$.

Now, consider the subalgebra $P \vee \mathcal{R}$ (compare Definition 1.2) and the product measure $\tilde{\mu}$ on $P\vee\mathcal{R}$ defined by $\tilde{\mu}(\hat{p}\hat{r}) = \mu(\hat{p})\mu(\hat{r})$, $\hat{p} \in P$, $\hat{r} \in \mathcal{R}$. By using (3) and (11) the inequality (40) reads

$$S(\tilde{\mu}|_{P\vee\mathcal{R}}, \mu|_{P\vee\mathcal{R}}) = S_\mu(P) + S_\mu(\mathcal{R}) - S_\mu(P \vee \mathcal{R}) \leq \epsilon_1 + \epsilon_2 \,.$$

The relative entropy of two states $\omega_{1,2}$ on a C^* algebra $\mathcal{A}$ enjoys the following lower bound [9]:

$$S(\omega_1, \omega_2) \geq \|\omega_1 - \omega_2\|^2/2 \,, \quad \|\omega_1 - \omega_2\| = \sup_{\hat{a}\in\mathcal{A}} \frac{|\omega_1(\hat{a}) - \omega_2(\hat{a})|}{\|\hat{a}\|} \,.$$

Therefore, we can estimate:

$$\left|\mu(\hat{p}\hat{r}) - \mu(\hat{p})\mu(\hat{r})\right| \leq \underbrace{\sqrt{2(\epsilon_1 + \epsilon_2)}}_{\delta_1(\epsilon_1,\epsilon_2)} \|\hat{p}\| \, \|\hat{r}\| \,.$$

□

Corollary 1 *Let $(\mathcal{Q}, \sigma, \mu)$ be the optimal triple constructed in the previous lemma for $H_\omega(\Theta^n, \mathbf{M})$. Take $\hat{p} \in P$ and $\hat{q} \in \mathcal{Q}$, then*

$$\limsup_{k\to+\infty} \left|\mu(\sigma^k(\hat{p})\hat{q}) - \mu(\hat{p})\mu(\hat{q})\right| \leq \delta_2(\epsilon_1, \epsilon_2) \,,$$

where $\delta_2(\epsilon_1, \epsilon_2)$ depends on $\hat{p}$, $\hat{q}$ and goes to zero with ϵ_1 , ϵ_2.

Proof: Notice that both $\mathcal{Q}$ and $\mathcal{R} = \mathrm{Tail}(P)$ are strongly closed as von Neumann (sub)algebras. Therefore, given $\hat{q} \in \mathcal{Q}$, choose $n \in \mathbb{N}$ and $\hat{q}_n \in \bigvee_{j=-n}^{n} \sigma^j(P)$ such that $\mu((\hat{q} - \hat{q}_n)^*(\hat{q} - \hat{q}_n)) \leq \delta_1^2(\epsilon_1, \epsilon_2)$.

For large enough k, $\sigma^{-k}(\hat{q}_n)$ is nearly contained in $\mathrm{Tail}(P)$. For fixed ϵ_1, ϵ_2, pick up then $k \in \mathbb{N}$ and $\hat{r}_k$ in $\mathrm{Tail}(P)$ so that

$$\mu((\sigma^{-k}(\hat{q}_n) - \hat{r}_k)^*(\sigma^{-k}(\hat{q}_n) - \hat{r}_k)) \leq \delta_1^2(\epsilon_1, \epsilon_2) .$$

Applying the previous result and using $\left|\mu(\hat{p}\hat{q})\right|^2 \leq \mu(\hat{p}^*\hat{p})\mu(\hat{q}^*\hat{q})$, we estimate:

$$\begin{aligned}
\left|\mu(\sigma^k(\hat{p})\hat{q}) - \mu(\hat{p})\mu(\hat{q})\right| &\leq \left|\mu(\hat{p}\sigma^{-k}(\hat{q}_n)) - \mu(\hat{p})\mu(\hat{q}_n)\right| + 2\|\hat{p}\|\delta_1(\epsilon_1, \epsilon_2) \\
&\leq \left|\mu(\hat{p}\hat{r}_k) - \mu(\hat{p})\mu(\hat{r}_k)\right| + 4\|\hat{p}\|\delta_1(\epsilon_1, \epsilon_2) \\
&\leq \underbrace{\|\hat{p}\|(4 + \|\hat{r}_k\|)\delta_1(\epsilon_1, \epsilon_2)}_{\delta_2(\epsilon_1, \epsilon_2)} .
\end{aligned}$$

□

4 The Space of Weak-Clustering

Let $\mathcal{M}_1$, $\mathcal{M}_2$ be two von Neumann algebras with faithful states ϕ_1, respectively ϕ_2, τ_1, τ_2 their modular automorphisms and $\gamma : \mathcal{M}_1 \to \mathcal{M}_2$ a *completely positive map* such that $\phi_2 = \phi_1 \circ \gamma$. Then (see [2], Section VIII), there exists a *canonical adjoint* $\gamma^\dagger : \mathcal{M}_2 \to \mathcal{M}_1$ such that $\phi_1 = \phi_2 \circ \gamma^\dagger$, and uniquely defined by

$$\phi_2(\tau_2^{i/2}(\hat{x}_2)\gamma(\hat{x}_1)) = \phi_1(\tau_1^{i/2}(\gamma^\dagger(\hat{x}_2))\hat{x}_1) , \; \forall \hat{x}_1 \in \mathcal{M}_1 , \; \hat{x}_2 \in \mathcal{M}_2 . \tag{41}$$

Let $(\mathcal{M}, \Theta, \omega)$ and $(\mathcal{B}, \sigma, \mu)$ be stationarily coupled through λ and τ_λ, τ_ω be the corresponding modular automorphisms ($\tau_\mu = \hat{1\!\!1}$ since $\mathcal{B}$ is Abelian). Given the embedding $i : \mathcal{M} \to \mathcal{M} \otimes \mathcal{B}$ of $\mathcal{M}$ into $\mathcal{M} \otimes \mathcal{B}$,

$$i(\hat{m}) = \hat{m} \otimes \hat{1\!\!1} \quad \forall \hat{m} \in \mathcal{M} ,$$

its adjoint map $i^\dagger : \mathcal{M} \otimes \mathcal{B} \to \mathcal{M}$ is defined by:

$$\lambda(\tau_\lambda^{i/2}(\hat{m}_1 \otimes \hat{p})i(\hat{m}_2)) = \omega(\tau_\omega^{i/2}(i^\dagger(\hat{m}_1 \otimes \hat{p}))\hat{m}_2) \quad \forall \hat{p} \in \mathcal{B}, \hat{m}_{1,2} \in \mathcal{M} . \tag{42}$$

Since $\hat{1\!\!1} \otimes \hat{p}$ is in the centre of $\mathcal{M} \otimes \mathcal{B}$, hence invariant under τ_λ, we can construct a positive linear map $\gamma : \mathcal{B} \to \mathcal{M}$ and its adjoint $\gamma^\dagger : \mathcal{M} \to \mathcal{B}$, as follows:

$$\begin{aligned}
\lambda(\hat{m} \otimes \hat{p}) &= \lambda(\tau_\lambda^{i/2}(\hat{1\!\!1} \otimes \hat{p})i(\hat{m})) \\
&= \omega(\tau_\omega^{i/2}(i^\dagger(\hat{1\!\!1} \otimes \hat{p}))\hat{m}) && (43) \\
\gamma : \hat{p} &\mapsto \gamma(\hat{p}) = i^\dagger(\hat{1\!\!1} \otimes \hat{p}) && (44) \\
\omega(\tau_\omega^{i/2}(\gamma(\hat{p}))\hat{m}) &= \mu(\hat{p}\gamma^\dagger(\hat{m})) . && (45)
\end{aligned}$$

Notice that, given $P = \{\hat{p}_i\} \subseteq \mathcal{B}$, the decomposition $\omega(\cdot) = \sum_i \mu_i \hat{\omega}_i(\cdot)$ in (13) has $\omega_i(\hat{m}) = \omega(\tau_\omega^{i/2}(\gamma(\hat{p}_i))\hat{m})$.

Lemma 2 *Let $(\mathcal{M}\otimes\mathcal{B},\Theta\otimes\sigma,\lambda)$ be a stationary coupling between the noncommutative dynamical system $(\mathcal{M},\Theta,\omega)$ and the classical dynamical system $(\mathcal{B},\sigma,\mu)$.*

Let $\gamma:\mathcal{B}\to\mathcal{M}$ be the mapping defined in (44). Then, $\gamma\circ\sigma=\Theta\circ\gamma$.

Proof: Since $\lambda=\lambda\circ\Theta\otimes\sigma$ (stationary coupling) and $\omega\circ\Theta=\omega$, it follows that $\tau_\lambda\circ\Theta\otimes\sigma=\Theta\otimes\sigma\circ\tau_\lambda$ and $\Theta\circ\tau_\omega=\tau_\omega\circ\Theta$.

Next, take $\hat{p}\in\mathcal{B}$, $\hat{m}\in\mathcal{M}$ observe that the set $\tau_\omega^{-i/2}(\mathcal{M})$ is (norm) dense in $\mathcal{M}$. The result then follows from

$$\begin{aligned}
\omega(\gamma\circ\sigma(\hat{p})\tau_\omega^{-i/2}(\hat{m})) &= \lambda(\hat{m}\otimes\sigma(\hat{p}))=\lambda(\Theta^{-1}(\hat{m})\otimes\hat{p})\\
&= \omega(\tau_\omega^{i/2}(\gamma(\hat{p}))\Theta^{-1}(\hat{m}))\\
&= \omega(\Theta\circ\gamma(\hat{p})\tau_\omega^{-i/2}(\hat{m}))\,.
\end{aligned}$$

□

Lemma 3 *Let $(\mathcal{M},\Theta,\omega)$ be an entropic K-system, $\mathbf{M}\subseteq\mathcal{M}$ any finite dimensional subalgebra and $(\mathcal{Q},\lambda,P)$ the optimal triple for $H_\omega(\Theta^n,\mathbf{M})$ constructed in Proposition 1. Let $\hat{x}_\gamma=\gamma(\hat{p})$ for some $\hat{p}\in\mathcal{Q}$, then*

$$\limsup_{k\to+\infty}\left|\omega(\Theta^k(\hat{x}_\gamma)\hat{x})-\omega(\hat{x}_\gamma)\omega(\hat{x})\right|\leq\delta_3(\epsilon_1,\epsilon_2)\quad\forall\hat{x}\in\mathcal{M}\,, \tag{46}$$

with $\delta_3(\epsilon_1,\epsilon_2)\overset{\epsilon_{1,2}\to 0}{\longrightarrow}0$.

Proof: Because of (45) and Lemma 2, we have for all $\hat{y}\in\mathcal{M}$:

$$\omega(\Theta^k(\hat{x}_\gamma)\tau_\omega^{-i/2}(\hat{y}))-\omega(\hat{x}_\gamma)\omega(\hat{y})=\mu(\sigma^k(\hat{p})\gamma^\dagger(\hat{y}))-\mu(\hat{p})\mu(\gamma^\dagger(\hat{y}))\,.$$

Corollary 1 ensures us that, by choosing k large enough,

$$\left|\mu(\sigma^k(\hat{p})\gamma^\dagger(\hat{y}))-\mu(\hat{p})\mu(\gamma^\dagger(\hat{y}))\right|\leq\delta_2(\epsilon_1,\epsilon_2)\,.$$

Further, varying $\hat{y}$, the set spanned by $\tau_\omega^{-i/2}(\hat{y})$ is dense in $\mathcal{M}$, then, for all $\hat{x}\in\mathcal{M}$, an $\hat{y}$ can be found such that (use $|\omega(\hat{x}\hat{y})|^2\leq\omega(\hat{x}\hat{x}^*)\omega(\hat{y}^*\hat{y})$):

$$\omega((\hat{x}-\tau_\omega^{-i/2}(\hat{y}))^*(\hat{x}-\tau_\omega^{-i/2}(\hat{y}))\leq\delta_2^2(\epsilon_1,\epsilon_2)$$

$$\left|\omega(\Theta^k(\hat{x}_\gamma)\hat{x})-\omega(\hat{x}_\gamma)\omega(\hat{x})\right|\leq\underbrace{(1+2\|\hat{x}_\gamma\|)\delta_2(\epsilon_1,\epsilon_2)}_{\delta_3(\epsilon_1,\epsilon_2)}\,.$$

□

Definition 6 *The dynamical system $(\mathcal{M},\Theta,\omega)$ is* weakly clustering *if*

$$\lim_{k\to\pm\infty}\omega(\hat{x}_1\Theta^k(\hat{x}_2)\hat{x}_3)=\omega(\hat{x}_2)\omega(\hat{x}_1\hat{x}_3)\quad\forall\,\hat{x}_1,\hat{x}_2,\hat{x}_3\in\mathcal{M}\,.$$

Remarks 6 1. Since we assumed ω to be faithful, we can use the modular relation satisfied by ω with respect to its own modular automorphism τ_ω, to show that

$$\lim_{k\to\pm\infty} \omega(\Theta^k(\hat{x})\hat{y}) = \omega(\hat{x})\omega(\hat{y}) \quad \forall\, \hat{x}, \hat{y} \in \mathcal{M}$$

is equivalent to weak clustering.

2. If we know that the correlation functions $\omega(\Theta^k(\hat{x})\hat{y})$ cluster for $k \to +\infty$, the Θ-invariance of ω guarantees that $\omega(\Theta^k(\hat{x})\hat{y}) = \omega(\hat{x}\Theta^{-k}(\hat{y}))$ cluster for $k \to -\infty$.

3. Weak clustering implies weak asymptotic Abelianness [13]:

$$\lim_{n\to\pm\infty} \omega(\hat{a}^*\left[\hat{b}, \Theta^n(\hat{c})\right]\hat{d}) = 0 \quad \left(\mathrm{w} - \lim_{n\to\pm\infty} \left[\hat{b}, \Theta^n(\hat{c})\right] = 0\right),$$

for all $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$ in $\mathcal{M}$.

4. By going to the GNS-construction for $(\mathcal{M}, \Theta, \omega)$, weak clustering amounts to the request that $U_\omega \stackrel{k\to\pm\infty}{\longrightarrow} |\Omega >< \Omega|$ weakly, where U_ω is the unitary operator that implements Θ.

Let $\mathbf{M}$ be a finite dimensional subalgebra of an entropic K-system $(\mathcal{M}, \Theta, \omega)$. ¿From Definition 3 we can approximate $H_\omega(\mathbf{M})$ within any $\epsilon > 0$ by means of a *finite* set of positive operators $\hat{x}_i \in \mathcal{M}$, such that $\sum_{i=1}^N \hat{x}_i = \hat{1\!\!1}$ (see Remark 2.3). According to Proposition 1, we expect them to be better and better approximated with increasing n by the minimal projections $\{\hat{p}_j\}_{j=1}^M$ of the generator P of the classical system $(\mathcal{Q}_n, \sigma_n, \mu_n)$ which nearly gives $H_\omega(\Theta^n, \mathbf{M})$. The clue is given by the map $\gamma_n : \mathcal{Q}_n \to \mathcal{M}$ and its adjoint, together with the next lemma where the dependence on the time step n is put into evidence (in the following, we shall not indicate the dependence of $\mathcal{Q}$ and μ on n).

Lemma 4 *Let $(\mathcal{M}, \Theta, \omega)$ be an entropic K-system, $\mathbf{M}$ a finite dimensional subalgebra of $\mathcal{M}$ and $(\mathcal{Q}, \sigma, \mu)$ an optimal Abelian model for $H_\omega(\Theta^n, \mathbf{M})$ as in Proposition 1. The triple $(\lambda, \mathcal{Q}, P)$ is optimal for $H_\omega(\mathbf{M})$.*

Proof: It suffices to recall (20), inequality (39) and to let $\epsilon_{1,2} \to 0$ by first improving the Abelian model for $H_\omega(\Theta^n, \mathbf{M})$, n fixed, and then increasing the time-step.

□

Proposition 2 *Let $(\mathcal{M}, \Theta, \omega)$ (ω a faithful state) be an entropic K-system.*

Let $\mathbf{A} \subseteq \mathbf{M}$ be finite dimensional subalgebras of $\mathcal{M}$, $\mathbf{A}$ Abelian with minimal projections $\{\hat{a}_k\}_{k=1}^N$.

Assume $\mathbf{M}$ invariant under τ_ω and let $E : \mathcal{M} \to \mathbf{M}$, $\omega \circ E = \omega$, be the corresponding state-preserving conditional expectation [11].

Let n be such that $H_\omega(\Theta^n, \mathbf{A}) \geq H_\omega(\mathbf{A}) - \epsilon_1$, $\epsilon_1 > 0$.

Let $(\mathcal{Q},\sigma,\mu)$, $\mathcal{Q}=\bigvee_{j\in\mathbb{Z}}\sigma^j(P)$ *be an Abelian model with stationary coupling* λ *and* $P\subseteq\mathcal{Q}$ *such that* $(\mathcal{Q},\lambda,P)$ *gives* $H_\omega(\Theta^n,\mathbf{A})$ *up to* $\epsilon_2>0$.

Let $\{\hat{p}_j\}_{j=1}^M$ *be the minimal projections of* P *and* $\omega(\,\cdot\,)=\sum_{j=1}^M\omega(\tau_\omega^{i/2}(\gamma_n(\hat{p}_j))\,\cdot\,)$ *the corresponding decomposition of* ω *constructed by means of the map* $\gamma_n:\mathcal{Q}\to\mathcal{M}$ *introduced in (45).*

Assume that the decomposition $\omega(\,\cdot\,)=\sum_{i=1}^L\omega(\tau_\omega^{i/2}(\hat{x}_i)\,\cdot\,)$, *with projections* $\hat{x}_i\in\mathcal{M}$, $\sum_{I=1}^L\hat{x}_i=1\!\!1$, *gives* $H_\omega(\mathbf{A})$ *and has a minimal number of contributions (see the Remark 7 below).*

Then, $M\geq L$ *and the* $\hat{p}_j$*'s can be labelled such that*

$$\omega\Big((\gamma_n(\hat{p}_i)-\hat{x}_i)^*(\gamma_n(\hat{p}_i)-\hat{x}_i)\Big)\leq\delta_4^2(\epsilon_1,\epsilon_2)\,,\qquad i=1,\ldots,L \tag{47}$$

$$\omega(\gamma_n(\hat{p}_i))\leq\delta_5(\epsilon_1,\epsilon_2)\,,\qquad i=L+1,\ldots,M\,,$$

where $\delta_{4,5}(\epsilon_{1,2})$ *are functions that vanish with* $\epsilon_{1,2}\to 0$.

Proof: According to Lemma 4, if we first let ϵ_2 and then $\epsilon_1\to 0$ (by considering $n\to+\infty$ in the latter case), then the minimality of the optimal set $\{\hat{x}_i\}_{i=1}^L$ guarantees that for each i there must exist some $j=1,\ldots,M$ for which

$$\left|\omega\Big(\tau_\omega^{i/2}(\gamma_n(\hat{p}_j))\hat{a}_k\Big)-\omega(\tau_\omega^{i/2}(\hat{x}_i)\hat{a}_k)\right|\ \stackrel{\epsilon_{1,2}}{\longrightarrow}\ 0\,,$$

for all $k=1,\ldots,N$.

Then, we use the conditional expectation $E:\mathcal{M}\to\mathbf{M}$ to deduce

$$\left|\omega\Big(\Big(E\circ\tau_\omega^{i/2}(\gamma_n(\hat{p}_j))-\tau_\omega^{i/2}(\hat{x}_i)\Big)\hat{a}_k\Big)\right|\ \stackrel{\epsilon_{1,2}}{\longrightarrow}\ 0\,.$$

In fact, according to the assumptions, $E(\hat{x}_i)=\hat{x}_i$, as well as $E\circ\tau_\omega^{i/2}(\hat{x}_i)=\tau_\omega^{i/2}(\hat{x}_i)$, and $E(\hat{m}\hat{a})=\hat{m}\,E(\hat{a})$, for all $\hat{m}\in\mathcal{M}$, $\hat{a}\in\mathbf{A}$. Thus, the finite dimensionality of $\mathbf{A}$ gives

$$\|E\circ\tau_\omega^{i/2}(\gamma_n(\hat{p}_j))-\tau_\omega^{i/2}(\hat{x}_i)\|\ \stackrel{\epsilon_{1,2}\to 0}{\longrightarrow}\ 0\,. \tag{48}$$

Since $\sum_{j=1}^M\gamma_n(\hat{p}_j)=\sum_{i=1}^L\hat{x}_i=\hat{1\!\!1}$, $M-N$ contributions $\omega(\gamma_n(\hat{p}_j))$ must vanish with $\epsilon_{1,2}\to 0$, whereas the remaining $\hat{p}_j$'s are in one-to-one correspondence with the L operators $\hat{x}_i$, $1\leq i\leq L$.

We now follow [3, Lemma 4.1.] and use the Schwartz positivity

$$\begin{aligned}
\gamma_n(\hat{p}_j)=\gamma_n(\hat{p}_j^2)\geq\gamma_n(\hat{p}_j)\gamma_n(\hat{p}_j)\quad&\Rightarrow\\
\omega\Big((\gamma_n(\hat{p}_j)-\hat{x}_i)^*(\gamma_n(\hat{p}_j)-\hat{x}_i)\Big)\quad&\leq\quad\omega(\gamma_n(\hat{p}_j))+\omega(\hat{x}_i)\\
&-\quad\omega(E\circ\tau_\omega^{i/2}(\gamma_n(\hat{p}_j))\tau_\omega^{i/2}(\hat{x}_i))\\
&-\quad\omega(\tau_\omega^{i/2}(\hat{x}_i)E\circ\tau_\omega^{i/2}(\gamma_n(\hat{p}_j)))\,,
\end{aligned}$$

from which the result follows by applying (48) and by relabelling the $\hat{p}_j$'s.

$\square$

Corollary 2 *Let $(\mathcal{M},\Theta,\omega)$ be an entropic K-system and $\mathbf{A}\subseteq\mathcal{M}$ a finite dimensional Abelian subalgebra. With the assumptions of the previous proposition:*

$$\lim_{k\to\pm\infty}\omega(\Theta^k(\hat{x}_i)(\hat{m}))=\omega(\hat{x}_i)\omega(\hat{m})\quad i=1,\ldots,L\,,\forall\hat{m}\in\mathcal{M}\,.$$

Proof: With $k>0$ large enough, we can use Lemma 3 with $\hat{x}_\gamma=\gamma_n(\hat{p}_i)$, then the Cauchy-Schwartz inequality and Proposition 2 to estimate

$$\begin{aligned}\Big|\omega(\Theta^k(\hat{x}_i)\hat{m})-\omega(\hat{x}_i)\omega(\hat{m})\Big| &\leq \Big|\omega\Big(\Theta^k(\hat{x}_i-\gamma_n(\hat{p}_i))\hat{m}\Big)\Big|\\ &+ \Big|\omega(\hat{x}_i-\hat{x}_\gamma)\omega(\hat{m})\Big|\\ &+ \Big|\omega(\Theta^k(\hat{x}_\gamma)\hat{m})-\omega(\hat{x}_\gamma)\omega(\hat{m})\Big|\\ &\leq 2\|\hat{m}\|\delta_4(\epsilon_1,\epsilon_2)+\delta_3(\epsilon_1,\epsilon_2)\,.\end{aligned}$$

Clustering when $k\to+\infty$ follows by letting $\epsilon_{1,2}\to 0$, when $k\to-\infty$ from Remark 6.2.
$\square$

Remark 7 The point behind the request that the best decomposition be minimal is that we cannot exclude that further refinements of a decomposition can be performed without affecting the contribution to $H_\omega(\mathbf{A})$.

However, any decomposition can be coarse grained until it is minimal. In the following sense: let $H_\omega(\mathbf{A})$ be approximated within ϵ by a decomposition $\{\hat{x}_i\}_{i=1}^L$. Then, there can be found a minimal number $N<L$ of positive operators $\{\hat{y}_j\}_{J=1}^N$ such that

$$\omega\Big((\sum_{i\in J(j)}\hat{x}_i-\hat{y}_j)^*(\sum_{i\in J(j)}\hat{x}_i-\hat{y}_j)\Big)\leq\epsilon\,,$$

where $\bigcup_{j=1}^N J(j)=\{1,\ldots,L\}$ and the contribution to $H_\omega(\mathbf{A})$ from the $\hat{y}_j$'s is the same as that of the $\hat{x}_i$'s up to order ϵ (see for instance [3]).

5 A Class of Models

In order to arrive at a result holding throughout the von Neumann algebra $\mathcal{M}$, it is of great importance to know how large is the set of operators $\hat{x}_i$ when we vary the finite dimensional Abelian subalgebras $\mathbf{A}$. In [3] a von Neumann algebra $\mathcal{M}$ of type II_1 equipped with the tracial state ω was considered with the result that every projection $\hat{a}$ of $\mathcal{M}$ contributed optimally to $H_\omega(\mathbf{A})$, where $\mathbf{A}=\{\hat{a},\hat{1\!\!1}-\hat{a}\}$, the unique best decomposition being given by $\omega(\,\cdot\,)=\omega(\hat{a}\,\cdot\,)+\omega((\hat{1\!\!1}-\hat{a})\,\cdot\,)$. In spite of the fact that the general situation is fairly uncontrolled, we can make good use of Lemma 1 in the following case.

In this section we shall consider a concrete class of models, based on the algebra

$$\mathcal{M}_\infty = \bigotimes_{j=1}^{\infty} (\mathbf{M}_n(\mathbb{C}))_j \tag{49}$$

be the *algebra spanned by tensor products of the form

$$\begin{aligned} \hat{m}_{[k,l]} &= \hat{\mathbb{1}}_{k-1]} \otimes \bigotimes_{j=k}^{l} \hat{m}_j \otimes \hat{\mathbb{1}}_{[l+1}\,, \quad \hat{m}_j \in \mathbf{M}_n(\mathbb{C}) \\ \hat{\mathbb{1}}_{k-1]} &= \otimes_{j=1}^{k-1} (\hat{\mathbb{1}})_j\,, \quad \hat{\mathbb{1}}_{[k+1} = \otimes_{j=k+1}^{+\infty} (\hat{\mathbb{1}})_j\,,\ \hat{\mathbb{1}} \in \mathbf{M}_n(\mathbb{C})\,. \end{aligned}$$

Let $\omega(\cdot) = \bigotimes_{j=1}^{+\infty} \mathrm{Tr}\,\hat{\lambda}_j(\cdot)$ be the product state on $\mathcal{M}_\infty$ defined by

$$\omega(\hat{m}_{[k,l]}) = \prod_{j=k}^{l} \mathrm{Tr}\,\hat{\lambda}_j \hat{m}_j\,, \quad \hat{\lambda}_j = \begin{pmatrix} e^{-\lambda_{j1}} & \dots \\ \dots & \dots \\ \dots & e^{-\lambda_{jn}} \end{pmatrix}. \tag{50}$$

The state ω restricted to the finite dimensional subalgebra $\mathbf{M}_N \subseteq \mathcal{M}$ generated by operators of the form $\hat{m}_{[-N+1,N]}$ corresponds to a density matrix $\hat{\rho}_N = \otimes_{j=-N+1}^{+N} \hat{\lambda}_j$. Let $\{\hat{e}_i\}_{i=1}^{2N^n}$ be the set of eigenvectors of $\hat{\lambda}_N$ and $\hat{e}_{ij}$ the corresponding system of matrix units:

$$\hat{e}_{ii} = \hat{e}_i\,, \quad \sum_i \hat{e}_i = \hat{\mathbb{1}} \in \mathbf{M}_N \tag{51}$$

$$\hat{e}^*_{ij} = \hat{e}_{ji}\,, \quad \hat{e}_{ij}\hat{e}_{kl} = \delta_{jk}\hat{e}_{il}\,. \tag{52}$$

Let us consider the GNS-representation π_ω based on ω with GNS-vector Ω and GNS-Hilbert space $\mathcal{H}_\omega$ and call $\mathcal{M}$ the strong closure $\pi_\omega(\mathcal{M}_\infty)''$.

Remark 8 Since the state ω is not pure, the commutant $\mathcal{M}'$ acting on $\mathcal{H}_\omega$ is not trivial. Depending on the choice of the eigenvalues of the density matrices $\hat{\rho}_j$, $\mathcal{M}$ is a hyperfinite factor of type III_λ, $0 < \lambda < 1$, or $\lambda = 1$. Varying $\mathbf{M}_N$, the operators $\pi_\omega(\hat{e}_{ij})$ acting on $|\Omega>$ span a dense subspace of $\mathcal{H}_\omega$.

Lemma 5 *Let $\mathbf{M}_{ij} \subseteq \mathbf{M}_N$ be the subalgebra of $\mathcal{M}$ spanned by the matrix units $\hat{e}_i$, $\hat{e}_j$, $\hat{e}_{ij}$ and $\hat{\mathbb{1}} - \hat{e}_i - \hat{e}_j$ corresponding to the eigenvectors of the density matrix $\omega|_{\mathbf{M}_N}$.*

Let $\hat{f}$, $\hat{p}^+_\phi$ and $\hat{p}^-_\phi$ be the projections

$$\begin{aligned} \hat{f} &= \hat{\mathbb{1}} - \hat{e}_i - \hat{e}_j \\ \hat{p}^\pm_\phi &= \frac{\hat{\mathbb{1}} - \hat{f} \pm \mathrm{e}^{-i\phi}\hat{e}_{ij} \pm \mathrm{e}^{i\phi}\hat{e}_{ji}}{2} \end{aligned}$$

and $\mathbf{A}_\phi$ the Abelian subalgebra of $\mathbf{M}_{ij}$ spanned by them.

The entropy $H_\omega(\mathbf{A}_\phi)$ of $\mathbf{A}_\phi$ is given by a unique decomposition determined by the minimal projections of $\mathbf{A}_\phi$ itself:

$$\omega(\,\cdot\,) = \omega(\tau_\omega^{i/2}(\hat{p}^+_\phi)\,\cdot\,) + \omega(\tau_\omega^{i/2}(\hat{p}^-_\phi)\,\cdot\,) + \omega(\tau_\omega^{i/2}(\hat{f})\,\cdot\,)\,. \tag{53}$$

Proof: $\mathbf{M}_{ij}$ is isomorphic to the matrix algebras:

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & e \end{pmatrix}, \qquad a,b,c,d,e \in \mathbb{C}\,, \tag{54}$$

and $\mathbf{A}_\phi$ to the subalgebra:

$$\begin{pmatrix} (a+b)/2 & (a-b)\mathrm{e}^{-i\phi}/2 & 0 \\ (a-b)\mathrm{e}^{i\phi}/2 & (a+b)/2 & 0 \\ 0 & 0 & e \end{pmatrix}.$$

We shall look for an optimal decomposition $\omega = \sum_i \omega_i(\hat{1\!\!1})\hat{\omega}_i$ where, according to (45),

$$\omega_i(\,\cdot\,) = \omega_i(\hat{1\!\!1})\hat{\omega}_i(\,\cdot\,) = \omega(\tau_\omega^{i/2}(\hat{x}_i)\,\cdot\,)\,,$$

for $0 < \hat{x}_i \in \mathcal{M}$ such that $\sum_i \hat{x}_i = \hat{1\!\!1}$. In order to calculate $H_\omega(\mathbf{A}_\phi)$ we can work within the algebra $\mathbf{M}_3(\mathbb{C})$ by considering the 3×3 density matrix

$$\hat{\rho} = \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix},$$

corresponding to the restriction $\omega|_{\mathbf{M}_{ij}}$. The decompositions of ω when restricted to $\mathbf{M}_{ij}$ can then be represented as (compare Lemma 1)

$$\begin{aligned} \hat{\rho} &= \sum_{i=1}^N \sqrt{\hat{\rho}}\hat{d}_i\sqrt{\hat{\rho}} = \sum_{i=1}^N \sigma_i(\hat{1\!\!1})\hat{\sigma}_i \\ \sigma_i(\hat{1\!\!1}) &= \mathrm{Tr}\,\hat{\rho}\hat{d}_i\,, \quad \hat{\sigma}_i = \frac{\sqrt{\hat{\rho}}\hat{d}_i\sqrt{\hat{\rho}}}{\mathrm{Tr}\,\hat{\rho}\hat{d}_i}\,, \end{aligned}$$

where $\hat{d}_i$ are positive 3×3 positive matrices with $\sum_{i=1}^n \hat{d}_i = \hat{1\!\!1}$. Let

$$\hat{d}_f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We now make the following observation which applies to more general cases.

Let $\mathcal{M}$ be any algebra and $\mathbf{A}$ any finite dimensional subalgebra (not necessarily Abelian) which splits into an orthogonal sum such that

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_1 & 0 \\ 0 & \mathbf{A}_2 \end{pmatrix} \subset \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix} \subseteq \mathcal{M}\,,$$

where $\mathcal{M}_{ij}$ are subalgebras of $\mathcal{M}$. ¿From explicit calculations it turns out that

$$H_\omega(\mathbf{A}_1 \oplus \mathbf{A}_2) = H_\omega(\mathbf{A}_1 \oplus 1\!\!1) + H_\omega(1\!\!1 \oplus \mathbf{A}_2)\,.$$

If, moreover, $\omega = \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}$, where ω_i is the restriction of ω to $\mathcal{M}_{ii}$, then

$$H_\omega(\mathbf{A}_1 \oplus 1\!\!1) = H_{\omega_1}(\mathbf{A}_1)\,.$$

Since $\sqrt{\hat\rho}\hat d_f\sqrt{\hat\rho} = \hat\rho\hat d_f = \hat d_f\hat\rho$, the argument used in the third case of the same lemma indicates the choice $\hat d_N = \hat d_f$ as optimal, for then $S(\hat\sigma_N|_{\mathbf{A}_\phi}) = 0$. Next, we observe

$$\sum_{i=1}^{N-1}\hat d_i = \hat{1\!\!1} - \hat d_f \Rightarrow \mathrm{Tr}\,\sqrt{\hat\rho}\hat d_i\sqrt{\hat\rho}\hat a = \mathrm{Tr}\,\sqrt{\hat\rho}(\hat{1\!\!1} - \hat d_f)\hat d_i(\hat{1\!\!1} - \hat d_f)\sqrt{\hat\rho}\hat a$$

for all $\hat a \in \mathbf{A}_\phi$, when $1 \le i \le N-1$. The corresponding states $\hat\sigma_i$ act nontrivially only on the Abelian algebra (with identity $\hat{1\!\!1} - \hat f$) spanned by

$$\hat d^{\pm}_\phi = \frac{1}{2}\begin{pmatrix} 1 & \mathrm{e}^{-i\phi} & 0 \\ \mathrm{e}^{i\phi} & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Because of the above observation, the choice how to decompose indicated by Lemma 1, case 4, and the subsequent Remark 5 with $\theta = 0$, is optimal also in this special three dimensional case, namely $H_\omega(\mathbf{A}_\phi)$ is uniquely given by the decomposition which uses $\hat d_1 = \hat d^+_\phi$, $\hat d_2 = \hat d^-_\phi$, $\hat d_3 = \hat d_f$, or, with the notation of Proposition 2,

$$\hat a_1 = \hat p^+_\phi\,, \quad \hat a_2 = \hat p^-_\phi\,, \quad \hat a_3 = \hat f\,. \tag{55}$$

$\square$

We are now in the conditions to resort to Proposition 2 to get the following preliminary result.

Lemma 6 *Let $\mathcal{M}$ be equipped with an automorphism Θ which respects the product state ω and makes $(\mathcal{M}, \Theta, \omega)$ an entropic K-system.*

Let $\hat p^{\pm}_\phi$ be the projections of $\mathcal{M}$ given in the previous proposition. Then, it follows that, for all $\hat x \in \mathcal{M}$, with $\hat a_1 = \hat p^+_\phi$, $\hat a_2 = \hat p^-_\phi$ and $\hat a_3 = \hat f$,

$$\lim_{k\to\pm\infty}\omega(\Theta^k(\hat a_i)\hat x) = \omega(\hat a_i)\omega(\hat x) \qquad i = 1, \ldots, 3\,.$$

Proof: $\mathbf{M}_{ij}$ is invariant under the modular automorphism of ω, and thus the result is a consequence of Lemma 5 and of Corollary 2.

$\square$

According to Remark 6.1, the dynamical system $(\mathcal{M}, \Theta, \omega)$ exhibits weak clustering with respect to the set of all possible projections $\hat f$ and $\hat p^{\pm}_\phi$. We shall express this fact, avoiding explicit reference to the GNS-representation π_ω, as

$$\mathrm{w} - \lim_{k\to\pm\infty}\Theta^k(\hat p^{\pm}_\phi)|\Omega> = \omega(\hat p^{\pm}_\phi)|\Omega>\ .$$

We use the freedom in the choice of the parameter θ to extend this result to all matrix units $\hat e_{ij}$, hence to all of $\mathcal{M}$.

Theorem 1 *The entropic-K-systems* $(\mathcal{M}, \Theta, \omega)$ *(compare (49)-(50)) considered in this section are weakly clustering quantum dynamical systems.*

Proof: By varying ϕ, $\hat{e}_{ij}$ can be written as a linear combination of $\hat{f}$ and $\hat{p}_\phi^{\pm}$.
$\square$

5.1 Strong Clustering

We now strengthen the previous result proving that entropic K-systems enjoy a stronger type of clustering by considering invariant states in $(\mathcal{M}, \Theta, \omega)$ that are cyclic and separating for $\mathcal{M}$.

Definition 7 *A quantum dynamical system* $(\mathcal{M}, \Theta, \omega)$ *is* strongly clustering *if for all* $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{e}$ *in* $\mathcal{M}$

$$\lim_{k\to\pm\infty} \omega(\hat{a}\Theta^k(\hat{b})\hat{c}\Theta^k(\hat{d})\hat{e}) = \omega(\hat{a}\hat{c}\hat{e})\omega(\hat{b}\hat{d}) \,.$$

Remarks 9 1. For classical dynamical systems there is no difference between weak and strong clustering.

2. Strong clustering implies strong asymptotic Abelianness [13]:

$$\lim_{k\to\pm\infty} \omega(\hat{a}^*\left[\hat{b}, \Theta^k(\hat{c})\right]^*\left[\hat{b}, \Theta^k(\hat{c})\right]\hat{a}) = 0 \quad \left(\mathrm{st} - \lim_{k\to\pm\infty}\left[\hat{b}, \Theta^k(\hat{c})\right] = 0\right),$$

for all $\hat{a}$, $\hat{b}$ and $\hat{c}$ in $\mathcal{M}$.

3. Weak clustering and strong asymptotic Abeliannes together imply strong clustering [3, Lemma 3.1.4].

We proceed by adapting Lemma 3.1.5 of [3] to the case where there are no tracial properties to benefit from. We only ask that the state ω be faithful so that we can use its modular automorphism τ_ω.

Lemma 7 *Let* $(\mathcal{M}, \Theta, \omega)$ *be weakly asymptotically Abelian. If for all projections* $\hat{p}$ *and* $\hat{q}$ *in* $\mathcal{M}$ *it holds that:*

$$\mathrm{st} - \lim_{k\to\pm\infty}\left[\hat{p}, \Theta^k(\hat{p})\right] = 0 \tag{56}$$

$$\mathrm{st} - \lim_{k\to\pm\infty}\left[\hat{q}, \Theta^k(\hat{q})\right] = 0 \tag{57}$$

$$\mathrm{st} - \lim_{k\to\pm\infty}\left[\hat{p}+\hat{q}, \Theta^k(\hat{p}+\hat{q})\right] = 0\,, \tag{58}$$

then the system is strongly clustering.

Proof: Since $\mathcal{M}$, as a von Neumann algebra, is the strong closure of the linear span of its projections, we shall prove that

$$\mathrm{w} - \lim_{k\to\pm\infty} \left[\hat{p}, \Theta^k(\hat{q})\right] = 0 \Rightarrow \mathrm{st} - \lim_{k\to\pm\infty} \left[\hat{p}, \Theta^k(\hat{q})\right] = 0 \,.$$

In fact, the l.h.s. holds due to the assumption and Remark 6.3, and if the implication holds we can conclude the proof by means of Remark 9.3.

Let $\hat{m}_k$ denote $\Theta^k(\hat{m})$ for all $\hat{m} \in \mathcal{M}$ and $k \in \mathbb{Z}$. Upon using the modular relation $\omega(\hat{a}\hat{b}) = \omega(\tau_\omega^{-i/2}(\hat{b})\hat{a}) = \omega(\hat{b}\tau_\omega^{i/2}(\hat{a}))$, $\hat{a}$, $\hat{b}$ in $\mathcal{M}$, and the Cauchy-Schwartz inequality, the result follows by proving that, when $k \to \pm\infty$,

$$\omega(\left[\hat{p}, \hat{q}_k\right]\left[\hat{p}, \hat{q}_k\right]) = \omega(\hat{p}\hat{q}_k\hat{p}\hat{q}_k) + \omega(\hat{q}_k\hat{p}\hat{q}_k\hat{p}) - \omega(\hat{p}\hat{q}_k\hat{p}) - \omega(\hat{q}_k\hat{p}\hat{q}_k) \to 0 \,.$$

We rewrite the first term of the r.h.s. of the above equality. Thus:

$$\begin{aligned}
\omega(\hat{p}\hat{q}_k\hat{p}\hat{q}_k) &= \omega((\hat{p}+\hat{q})\hat{q}_k\hat{p}\hat{q}_k) - \omega(\hat{q}\hat{q}_k\hat{p}\hat{q}_k) \\
&= \omega((\hat{p}+\hat{q})(\hat{p}+\hat{q})_k\hat{p}\hat{q}_k) - \omega((\hat{p}+\hat{q})\hat{p}_k\hat{p}\hat{q}_k) - \omega(\hat{q}\hat{q}_k\hat{p}\hat{q}_k) \\
&= \omega((\tau_\omega^{-i}(\hat{q})(\hat{p}+\hat{q}))_k(\hat{p}+\hat{q})\hat{p}) - \omega((\hat{p}+\hat{q})\hat{p}(\hat{p}\hat{q})_k) \\
&- \omega((\tau_\omega^{-i}(\hat{q})\hat{q})_k\hat{q}\hat{p}) + \omega(\left[(\hat{p}+\hat{q}), (\hat{p}+\hat{q})_k\right]\hat{p}\hat{q}_k) \\
&- \omega(\tau_\omega^{-i}(\hat{q}_k)(\hat{p}+\hat{q})\left[\hat{p}_k, \hat{p}\right]) - \omega(\tau_\omega^{-i}(\hat{p}\hat{q}_k)\left[\hat{q}, \hat{q}_k\right]) \,.
\end{aligned}$$

By using the Cauchy-Schwartz inequality and (56)- (58) the last two terms are seen to vanish when $k \to \pm\infty$, as well as $\omega(\left[(\hat{p}+\hat{q}), (\hat{p}+\hat{q})_k\right]\hat{p}\hat{q}_k)$. Moreover, we assumed weak clustering to hold, whence $\omega(\hat{p}\hat{q}_k\hat{p})$ and $\omega(\hat{q}_k\hat{p}\hat{q}_k)$ tend to $\omega(\hat{p})\omega(\hat{q})$ when $k \to \pm\infty$ and

$$\begin{aligned}
\lim_{k\to\pm\infty} \omega(\hat{p}\hat{q}_k\hat{p}\hat{q}_k) &= \omega(\hat{p}\hat{q}+\hat{q})\omega(\hat{p}+\hat{q}\hat{p}) \\
&- \omega(\hat{p}+\hat{q}\hat{p})\omega(\hat{p}\hat{q}) - \omega(\hat{q})\omega(\hat{q}\hat{p}) \\
&= \omega(\hat{p})\omega(\hat{q}) \,.
\end{aligned}$$

A similar argument can be used to show that

$$\lim_{k\to\pm\infty} \omega(\hat{q}_k\hat{p}\hat{q}_k\hat{p}) = \omega(\hat{p})\omega(\hat{q}) \,.$$

□

We now consider the case of the previous section $\mathcal{M} = \pi_\omega(\bigcup_N \mathbf{M}_N)''$ when the dynamical system $(\mathcal{M}, \Theta, \omega)$ is an entropic dynamical system and therefore weakly asymptotically Abelian. In order to prove that $(\mathcal{M}, \Theta, \omega)$ is also strongly clustering, the first step is proving that the correlation functions of the set $\{\hat{p}^\pm_\phi, \hat{f}\}$ of projections of $\mathcal{M}$ factorize strongly (we already know that they do so weakly).

Lemma 8 *Let $(\mathcal{M}, \Theta, \omega)$ be an entropic K-system in the class defined by (49) and (50). The projections of $\mathcal{M}$ considered in Lemma 5, $\{\hat{a}_1 = \hat{p}^+_\phi, \hat{a}_2 = \hat{p}^-_\phi, \hat{a}_3 = \hat{f}\}$ are such that:*

$$\mathrm{st} - \lim_{k\to\pm\infty} \left[\hat{a}_i, \Theta^k(\hat{a}_j)\right] = 0 \qquad i, j = 1, 2, 3 \,.$$

Proof: Given the Abelian algebra spanned by the $\hat{a}_i$'s, use Proposition 2 and the operators $\gamma_n(\hat{p}_i) \in \mathcal{M}$ that satisfy (47) (for n large enough), namely:

$$\omega\Big((\gamma_n(\hat{p}_i) - \hat{a}_i)^*(\gamma_n(\hat{p}_i) - \hat{a}_i)\Big) \leq \delta_4^2(\epsilon_1, \epsilon_2)\,. \tag{59}$$

Because of Lemma 2 and of $\omega \circ \Theta = \omega$, we also have:

$$\omega\Big((\gamma_n(\sigma^k(\hat{p}_i)) - \Theta^k(\hat{a}_i))^*(\gamma_n(\sigma^k(\hat{p}_i)) - \Theta^k(\hat{a}_i))\Big) \leq \delta_4^2(\epsilon_1, \epsilon_2)\,. \tag{60}$$

The operators $\gamma_n(\hat{p}_i\sigma^k(\hat{p}_j))$, $i,j = 1,2,3$, are positive and such that

$$\gamma_n(\hat{p}_i) = \sum_{j=1}^{3} \gamma_n(\hat{p}_i\sigma^k(\hat{p}_j))\,, \quad \gamma_n(\sigma^k(\hat{p}_j)) = \sum_{i=1}^{3} \gamma_n(\hat{p}_i\sigma^k(\hat{p}_j)) \tag{61}$$

$$\sum_i \gamma_n(\hat{p}_i) = \sum_j \gamma_n(\sigma^k\hat{p}_j)) = \hat{1\!\!1} \qquad \forall n \in \mathbb{N}\,, \forall k \in \mathbb{Z}\,. \tag{62}$$

Let $\hat{x}_{ij}^{n,k} = \gamma_n(\hat{p}_i\sigma^k(\hat{p}_j))$ and suppose that:

$$\omega\Big((\hat{x}_{ij}^{n,k} - \hat{a}_i\hat{x}_{ij}^{n,k}\hat{a}_i)^*(\hat{x}_{ij}^{n,k} - \hat{a}_i\hat{x}_{ij}^{n,k}\hat{a}_i)\Big) \leq \delta_6^2(\epsilon_1, \epsilon_2) \tag{63}$$

$$\omega\Big((\hat{x}_{ij}^{n,k} - \Theta^k(\hat{a}_j)\hat{x}_{ij}^{n,k}\Theta^k(\hat{a}_j))^*(\hat{x}_{ij}^{n,k} - \Theta^k(\hat{a}_j)\hat{x}_{ij}^{n,k}\Theta^k(\hat{a}_j))\Big) \leq \delta_6^2(\epsilon_1, \epsilon_2)\,, \tag{64}$$

where $\delta_6(\epsilon_1, \epsilon_2) \stackrel{\epsilon_{1,2}\to 0}{\longrightarrow} 0$.

We choose k large enough in order to use the result of Corollary 1. Then, we observe that

$$\mu(\hat{p}_i) \stackrel{(45)}{=} \omega(\gamma_n(\hat{p}_i)) \Rightarrow \Big|\omega(\hat{a}_i) - \mu(\hat{p}_i)\Big| \stackrel{(59)}{\leq} \delta_4(\epsilon_1, \epsilon_2)\,. \tag{65}$$

By using (10), (45), (59), (62), (63), (64) and by repeatedly applying the Cauchy-Schwartz inequality and the orthogonality of the projections $\hat{a}_i \in \mathcal{M}$, $\hat{p}_j \in \mathcal{B}$, it follows:

$$\begin{aligned}
&\Big|\omega(\hat{a}_i\Theta^k(\hat{a}_j)\hat{a}_i\Theta^k(\hat{a}_j)) - \omega(\hat{a}_i)\omega(\hat{a}_j)\Big| \stackrel{(59)}{\leq} \\
&\Big|\omega(\gamma_n(\hat{p}_i)\Theta^k(\hat{a}_j)\hat{a}_i\Theta^k(\hat{a}_j)) - \omega(\hat{a}_i)\omega(\hat{a}_j)\Big| + \delta_4(\epsilon_1, \epsilon_2) \stackrel{(62)}{=} \\
&\Big|\sum_l \omega\Big(\hat{x}_{il}^{n,k}\Theta^k(\hat{a}_j)\hat{a}_i\Theta^k(\hat{a}_j)\Big) - \omega(\hat{a}_i)\omega(\hat{a}_j)\Big| + \delta_4(\epsilon_1, \epsilon_2) \stackrel{(64)}{\leq} \\
&\Big|\omega\Big(\hat{x}_{ij}^{n,k}\hat{a}_i\Theta^k(\hat{a}_j)\Big) - \omega(\hat{a}_i)\omega(\hat{a}_j)\Big| + \delta_4(\epsilon_1, \epsilon_2) + \delta_6(\epsilon_1, \epsilon_2) \stackrel{(63)}{\leq} \\
&\Big|\omega\Big(\hat{x}_{ij}^{n,k}\Theta^k(\hat{a}_j)\Big) - \omega(\hat{a}_i)\omega(\hat{a}_j)\Big| + \delta_4(\epsilon_1, \epsilon_2) + 2\delta_6(\epsilon_1, \epsilon_2) \stackrel{(64)}{\leq} \\
&\Big|\omega\Big(\hat{x}_{ij}^{n,k}\Big) - \omega(\hat{a}_i)\omega(\hat{a}_j)\Big| + \delta_4(\epsilon_1, \epsilon_2) + 3\delta_6(\epsilon_1, \epsilon_2) \stackrel{(45),(65)}{\leq} \\
&\Big|\mu(\hat{p}_i\sigma^k(\hat{p}_j)) - \mu(\hat{p}_i)\mu(\hat{p}_j)\Big| + 3\delta_4(\epsilon_1, \epsilon_2) + 3\delta_6(\epsilon_1, \epsilon_2)\,.
\end{aligned}$$

The last quantity is smaller than $\delta_2(\epsilon_1,\epsilon_2)+3\delta_4(\epsilon_1,\epsilon_2)+3\delta_6(\epsilon_1,\epsilon_2)$, because of Corollary 1. Thus, the result comes from letting $\epsilon_{1,2}\to 0$, together with $k\to+\infty$. Finally, we notice that the correlation functions factorize for $k\to-\infty$ as ω is Θ-invariant. In fact:

$$\lim_{k\to-\infty}\omega(\hat a_i\Theta^k(\hat a_j)\hat a_i\Theta^k(\hat a_j))=\lim_{k\to+\infty}\omega(\Theta^k(\hat a_i)\hat a_j\Theta^k(\hat a_i)\hat a_j)\,.$$

In order to arrive at the estimates (63) and (64), we notice that (61) above establishes an order relation among the positive linear functionals (not normalized states on $\mathcal{M}$)

$$\begin{aligned}\omega_{ij}(\hat m)&=\omega(\tau_\omega^{i/2}(\hat x_{ij}^{n,k})\hat m)\\ \omega_i(\hat m)&=\omega(\tau_\omega^{i/2}(\hat x_i^{n})\hat m)\,,\quad \hat x_i^n=\gamma_n(\hat p_i)\\ \omega_i^k(\hat m)&=\omega(\tau_\omega^{i/2}(\hat x_i^{n,k})\hat m)\,,\quad \hat x_i^{n,k}=\gamma_n(\sigma^k(\hat p_i))\,.\end{aligned}$$

Namely, $\omega_{ij}\le\omega_i$, respectively $\omega_{ij}\le\omega_i^k$. We can thus apply the noncommutative Radon-Nykodim theorem ([14, Par. 1.10]) which ensures us that there exist positive $\hat t_{ij}$ and $\hat t_{ij}^k$ in $\mathcal{M}$ such that:

$$\omega_{ij}(\hat m)=\omega_i(\hat t_{ij}\hat m\hat t_{ij})\,,\quad \omega_{ij}(\hat m)=\omega_i^k(\hat t_{ij}^k\hat m\hat t_{ij}^k)\,. \tag{66}$$

We prove that for all $\hat m\in\mathcal{M}$

$$\Big|\omega(\hat x_{ij}^{n,k}\tau_\omega^{-i/2}(\hat m))-\omega(\hat a_i\hat x_{ij}^{n,k}\hat a_i\tau_\omega^{-i/2}(\hat m))\Big|\le\delta(\epsilon_1,\epsilon_2)\,,$$

where $\delta(\epsilon_1,\epsilon_2)$ vanishes with $\epsilon_{1,2}\to 0$, the other case following along the same lines. Let J_ω, $J_\omega^2=\hat{1\!\!1}$, $J_\omega|\Omega>=|\Omega>$, be the modular conjugation on the GNS Hilbert space $\mathcal{H}_\omega$ for ω: $\mathcal{M}'=J_\omega\mathcal{M}J_\omega$. Then,

$$J_\omega\hat m J_\omega\in\mathcal{M}'\,,\quad \tau_\omega^{-i/2}(\hat n)|\Omega>=J_\omega\hat n J_\omega|\Omega>\,,\ \hat m,\hat n=\hat n^*\in\mathcal{M}\,.$$

By using (66) and the modular relations $\omega(\hat m_1\hat m_2)=\omega(\hat m_2\tau_\omega^{-i}(\hat m_1))$, we can write:

$$\begin{aligned}\omega\Big(\tau_\omega^{i/2}(\hat x_{ij}^{n,k})\tau_\omega^{i/2}(\hat a_i)\hat m\tau_\omega^{-i/2}(\hat a_i)\Big)&=\omega\Big(\tau_\omega^{i/2}(\hat x_i^n)\hat t_{ij}\tau_\omega^{i/2}(\hat a_i)\hat m\tau_\omega^{-i/2}(\hat a_i)\hat t_{ij}\Big)\\&=\omega\Big(\tau_\omega^{i}(\hat t_{ij})\hat t_{ij}J_\omega\hat x_i^nJ_\omega\tau_\omega^{i/2}(\hat a_i)\hat mJ_\omega\hat a_iJ_\omega\Big)\\&=\omega\Big(\tau_\omega^{i/2}(\hat a_i\hat x_i^n\hat a_i)\hat m\hat t_{ij}\tau_\omega^{-i}(\hat t_{ij})\Big)\,.\end{aligned}$$

Because of (59), we can find a suitable $\delta(\epsilon_1,\epsilon_2)\stackrel{\epsilon_{1,2}\to 0}{\longrightarrow}0$ and estimate

$$\Big|\omega\Big(\tau_\omega^{i/2}(\hat a_i\hat x_i^n\hat a_i)\hat m\hat t_{ij}\tau_\omega^{-i}(\hat t_{ij})\Big)-\omega\Big(\tau_\omega^{i/2}(\hat x_i^n)\hat m\hat t_{ij}\tau_\omega^{-i}(\hat t_{ij})\Big)\Big|\le\delta(\epsilon_1,\epsilon_2)\,.$$

The conclusion follows from (66) and the equality

$$\begin{aligned}\omega\Big(\tau_\omega^{i/2}(\hat x_i^n)\hat m\hat t_{ij}\tau_\omega^{-i}(\hat t_{ij})\Big)&=\omega\Big(\hat t_{ij}J_\omega\hat x_i^nJ_\omega\hat m\hat t_{ij}\Big)\\&=\omega\Big(\tau_\omega^{i/2}(\hat x_i^n)\hat t_{ij}\hat m\hat t_{ij}\Big)\,.\end{aligned}$$

□

As for Theorem 1, strong asymptotic Abelianness can be extended from the particular class of projections $\hat{p}^{\pm}_{\phi}, \hat{f}$ to all matrix units in $\mathcal{M}$ and thus to the entire set of its projections.

Theorem 2 *If the dynamical system $(\mathcal{M}, \Theta, \omega)$ in the class defined by (49) and (50) is an entropic K-system, then it is strongly clustering.*

Proof: see the appendix.

6 Generalization to Quasifree States

The von Neumann algebra $\mathcal{M}$ of the previous sections arises as a useful mathematical tool when dealing with a system of infinitely many fermions in a state ω. Let $\hat{a}(f)$, $\hat{a}^*(f)$ be the creation and annihilation operators of a fermion in the state $|f>$ of the one-particle (separable) Hilbert space $\mathcal{H}$. Choose an orthonormal basis $\{|h_j>\}_{j=1}^{\infty}$ of $\mathcal{H}$ and construct the operators:

$$\begin{aligned}
\hat{e}_{11}^{(k)} &= \hat{a}^*(h_k)\hat{a}(h_k) \\
\hat{e}_{22}^{(k)} &= \hat{a}(h_k)\hat{a}^*(h_k) \\
\hat{e}_{12}^{(k)} &= \prod_{j=1}^{k-1}\left[\hat{1\!\!1} - 2\hat{a}^*(h_j)\hat{a}(h_j)\right]\hat{a}^*(h_k) \\
\hat{e}_{21}^{(k)} &= \prod_{j=1}^{k-1}\left[\hat{1\!\!1} - 2\hat{a}^*(h_j)\hat{a}(h_j)\right]\hat{a}(h_k)\,.
\end{aligned}$$

Because $\hat{a}^*(h_i)\hat{a}(h_j) + \hat{a}(h_j)\hat{a}^*(h_i) = \delta_{ij}$, the $\hat{e}_{ij}^{(k)}$ constitute a system of matrix units (see for instance [8]) generating a two dimensional matrix algebra $\mathbf{M}^{(k)}$ isomorphic to $\mathbf{M}_2(\mathbb{C})$. As for different k and l the corresponding algebras $\mathbf{M}^{(l)}$ and $\mathbf{M}^{(k)}$ commute, the algebra $\mathcal{A}_{F,n}(\mathcal{H})$ generated by the $\hat{a}^\natural(h_k)$, $k = 1, \ldots, n$ is isomorphic to $\bigotimes_{k=1}^{n}(\mathbf{M}_2(\mathbb{C}))_k$. We call $\mathcal{A}_{F,\infty}$ the algebra of polynomials of whatever degree in $\hat{a}^\natural(h_k)$ and consider the von Neumann algebra $\mathcal{A}_F = \pi_\omega(\mathcal{A}_\infty)''$ that arises from the strong closure of $\mathcal{A}_{F,\infty}$ in the GNS representation based on a faithful state ω. Be it the quasifree state defined by the two-point functions

$$\omega(\hat{a}^*(f)\hat{a}(g)) = (g, \frac{\hat{1\!\!1}}{\hat{1\!\!1} + \mathrm{e}^{\hat{h}}} f)\,. \tag{67}$$

Then, it is invariant under the group of quasifree automorphisms τ_ω^t of $\mathcal{A}_\infty$ given by

$$\tau_\omega^t(\hat{a}^\natural(f)) = \hat{a}^\natural(\mathrm{e}^{it\hat{h}} f)\,.$$

Moreover, $\tau_\omega^i(\hat{a}^*(f)) = \hat{a}^*(\mathrm{e}^{-\hat{h}}f)$, hence τ_ω, when implemented on the GNS Hilbert space $\mathcal{H}_\omega$, coincides with the (unique) modular automorphism of ω:

$$\pi_\omega\Big(\tau_\omega^t(\hat{a}^\natural(f))\Big) = \Delta_\omega^{it}\pi_\omega(\hat{a}^\natural(f))\Delta_\omega^{-it}\,,$$

where Δ_ω is the modular operator of ω on $\mathcal{H}_\omega$ (see [15] for an analogous treatment).

If we now assume that the one-particle Hamiltonian $\hat{h}$ has a purely point spectrum with eigenvectors $|h_j>$, then we are exactly in the case discussed in section 4 and 5.1. Therefore, if $\mathcal{A}_F$ is equipped with an automorphism Θ that commutes with τ_ω and such that $(\mathcal{A}_F, \Theta, \omega)$ is an entropic K-system, then the quasifree Fermi system is strongly clustering.

In classical K-systems a memory-loss mechanism manifests itself in that the average information provided by repeated experiments gets lost if the interval between them lasts long enough. To the effectiveness of such a mechanism there correspond clustering properties, a sign that events tend to become independent, which are the strongest in a hierarchy of possibilities (see [6]). The same characterization carried over to quantum dynamical systems and the previous results imply that a Fermi system in a quasifree state ω whose modular operator Δ_ω has pure point spectrum is strongly clustering whenever its evolution makes it an entropic K-system. Hence, it is strongly asymptotically Abelian, which in quantum mechanics well expresses the increasing independence of time-separated physical occurrences.

The restriction on the spectrum of Δ_ω is rather severe, for in statistical mechanics one is usually confronted with absolutely continuous spectrum as is the case when the quasifree state reads:

$$\omega(\hat{a}^*(f)\hat{a}(g)) = \int_{\mathrm{I\!R}^3}\mathrm{d}\mathbf{p}\, g^*(\mathbf{p})f(\mathbf{p})\frac{1}{1+\mathrm{e}^{h(\mathbf{p})}}\,,$$

$(\hat{h}f)(\mathbf{p}) = h(\mathbf{p})f(\mathbf{p})$ in momentum space representation.

None the less, the quasifree case can be handled in full generality by means of the previous results.

Let $\hat{h} = \int h\,\mathrm{d}\hat{H}_h$ be the spectral resolution of the one-particle Hamiltonian, subdivide its continuous spectrum into a sequence of disjoint intervals $[h_j-\delta, h_j+\delta)$ in such a way that the sequence of orthogonal projections $\hat{H}_{j,\delta} = \int_{h_j-\delta}^{h_j+\delta}\mathrm{d}\hat{H}_h$ fulfills:

$$\sum_j \bigoplus \hat{H}_{j,\delta} = \hat{1\!\!1} \tag{68}$$

$$\|(\hat{h}-h_j)\hat{H}_{j,\delta}|f>\| \leq \delta \quad \forall f\in\mathcal{H}\,. \tag{69}$$

If we now choose orthonormal bases $\{|h_{j,k}>\}$ in each one of the orthogonal subspaces $\hat{H}_{j\delta}\mathcal{H}$, we can construct $\mathcal{A}_F$ as done before by considering the matrix units $\hat{e}_{rs}^{j,k}$ associated with the creation and annihilation operators $\hat{a}^\natural(h_{j,k})$.

The main difference is that, unlike the previous case, the two dimensional algebra $\mathbf{M}^{j,k}$ spanned by the $\hat{e}^{j,k}_{rs}$'s is not invariant under the modular automorphism τ_ω, so that a conditional expectation $E : \mathcal{A}_F \to \mathbf{M}^{j,k}$ that preserves the state ω as that used in Proposition 2 cannot exist because of Takesaki's theorem. What we can do is to consider the restriction $\omega^{j,k}$ of the state ω to the subalgebra $\mathbf{M}^{j,k}$ and construct the adjoint $i^\dagger : \mathcal{A}_F \to \mathbf{M}^{j,k}$ of the embedding $i : \mathbf{M}^{j,k} \to \mathcal{A}_F$ according to (41):

$$\omega(\tau_\omega^{i/2}(\hat{x})\hat{y}) = \omega^{j,k}(\tau^{i/2}_{\omega^{j,k}}(i^\dagger(\hat{x}))\hat{y}) \quad \forall \hat{y} \in \mathbf{M}^{j,k}\,,\ \hat{x} \in \mathcal{A}_F\,. \tag{70}$$

The adjoint map $i^\dagger$ respects the state, but does not fulfil

$$i^\dagger(\hat{y}_1\hat{x}\hat{y}_2) = \hat{y}_1 i^\dagger(\hat{x})\hat{y}_2 \quad \forall \hat{y}_1, \hat{y}_2 \in \mathbf{M}^{j,k}\,,\ \forall \hat{x} \in \mathcal{A}_F\,. \tag{71}$$

Such a property is satisfied by the conditional expectation of Proposition 2 and was used to arrive at an asymptotic behaviour like in (48).

Takesaki's result states that $i^\dagger$ would be a conditional expectation fulfilling (71) and $\omega \circ i^\dagger = \omega$ if and only if $\tau^t_\omega(\mathbf{M}^{j,k}) \subseteq \mathbf{M}^{j,k}$ for all $t \in \mathbb{R}$, equivalently, if and only if the restriction $\tau_\omega|_{\mathbf{M}^{j,k}}$ coincided with $\tau_{\omega^{j,k}}$. If we had $\mathbf{M}^{j,k}$ nearly invariant under τ_ω, we expect $i^\dagger$ would nearly behave as a conditional expectation (see [2], Lemma $VIII$.10). We observe, indeed, that the requests in (69) amount to the fact that the norms $\|\tau^t_\omega(\hat{e}^{j,k}_{11}) - \hat{e}^{j,k}_{11}\|$, $\|\tau^t_\omega(\hat{e}^{j,l}_{22}) - \hat{e}^{j,k}_{22}\|$, $\|\tau^t_\omega(\hat{e}^{j,k}_{12}) - \mathrm{e}^{i\,t\,h_j}\hat{e}^{j,k}_{12}\|$ and $\|\tau^t_\omega(\hat{e}^{j,k}_{21}) - \mathrm{e}^{-i\,t\,h_j}\hat{e}^{j,k}_{21}\|$ tend to zero when $\delta \to 0$. Consequently, the action of τ_ω on $\mathbf{M}^{j,k}$ does not differ too much from that of the modular automorphism $\tau_{\omega^{j,k}}$ of the state $\omega^{j,k}$. As we are in finite dimension we can find a suitable function $\eta_1(\delta)$ that tends to zero with $\delta \to 0$ and estimate:

$$\|\tau^{i/2}_\omega \circ \tau^{-i/2}_{\omega^{j,k}}(\hat{x}) - \hat{x}\| \le \eta(\delta)\,,\ \|\tau^{-i/2}_\omega \circ \tau^{i/2}_{\omega^{j,k}}(\hat{x}) - \hat{x}\| \le \eta(\delta)\,, \tag{72}$$

for all $\hat{x} \in \mathbf{M}^{j,k}$.

Lemma 9 *Let $\mathcal{M}$ be a von Neumann algebra with a faithful state ω. Let $\hat{n}_1$, $\hat{n}_2$ and $\hat{n}$ belong to a finite dimensional subalgebra $\mathbf{N} \subseteq \mathcal{M}$ and ν denote the restriction of ω to $\mathbf{N}$. Let the estimates (72) hold. Then, for all $\hat{m} \in \mathcal{M}$, there can be found a suitable function $\eta_1(\delta)$ that depends only on δ and on the dimension of $\mathbf{N}$ and tends to zero when $\delta \to 0$, such that: $\|i^\dagger(\hat{n}_1\hat{m}\hat{n}_2) - \hat{n}_1 i^\dagger(\hat{m})\hat{n}_2\| \le \eta_1(\delta)$.*

Proof: By means of the KMS condition and of the fact that $\nu(\hat{n}) = \omega(\hat{n})$ when $\hat{n} \in \mathbf{N}$, for all $\hat{m} \in \mathcal{M}$ we derive:

$$\begin{aligned}
\nu\Big(\hat{n}_1 i^\dagger(\hat{m})\hat{n}_2\tau_\nu^{-i/2}(\hat{n})\Big) &= \nu\Big(i^\dagger(\hat{m})\tau_\nu^{-i/2}\Big(\tau_\nu^{i/2}(\hat{n}_2)\hat{n}\tau_\nu^{-i/2}(\hat{n}_1)\Big)\Big) \\
&= \omega\Big(\tau_\omega^{i/2}(\hat{m})\tau_\nu^{i/2}(\hat{n}_2)\hat{n}\tau_\nu^{-i/2}(\hat{n}_1)\Big) \\
&= \omega\Big(\tau_\omega^{i/2}\Big(\tau_\omega^{i/2}\circ\tau_\nu^{-i/2}(\hat{n}_1)\hat{m}\tau_\omega^{-i/2}\circ\tau_\nu^{i/2}(\hat{n}_2)\Big)\hat{n}\Big) \\
&= \nu\Big(i^\dagger\Big(\tau_\omega^{i/2}\circ\tau_\nu^{-i/2}(\hat{n}_1)\hat{m}\tau_\omega^{-i/2}\circ\tau_\nu^{i/2}(\hat{n}_2)\Big)\tau_\nu^{-i/2}(\hat{n})\Big)\,.
\end{aligned}$$

Therefore $\left|\nu\left(i^\dagger\left(\hat{n}_1\hat{m}\hat{n}_2\right) - \hat{n}_1 i^\dagger(\hat{m})\hat{n}_2\right)\tau_\nu^{-i/2}(\hat{n})\right)\right|$ equals

$$\left|\nu\left(i^\dagger\left(\hat{n}_1\hat{m}\hat{n}_2 - \tau_\omega^{i/2}\circ\tau_\nu^{-i/2}(\hat{n}_1)\hat{m}\tau_\omega^{i/2}\circ\tau_\omega^{-i/2}(\hat{n}_2)\right)\tau_\nu^{-i/2}(\hat{n})\right)\right|.$$

The set $\tau_\nu^{-i/2}(\hat{n})$ is dense in $\mathbf{N}$ when we vary $\hat{n}$ in $\mathbf{N}$, thus, using the inequalities (72) and taking the supremum of the last expression over all $\hat{n}\in\mathbf{N}$ with $\|\hat{n}\| = 1$, we can construct the required function $\eta_1(\delta)$ and estimate

$$\|i^\dagger(\hat{n}_1\hat{m}\hat{n}_2) - \hat{n}_1 i^\dagger(\hat{m})\hat{n}_2\| \leq \eta_1(\delta)\,.$$

□

Next, we consider the Abelian algebra $\mathbf{A}_{rs}^{j,k}\subseteq\mathbf{M}^{j,k}$ generated by the minimal projections

$$\begin{aligned}
\hat{a}_1 &= \frac{\hat{1\!\!1} - \hat{f}^{j,k} + \mathrm{e}^{-i\phi}\hat{e}_{rs}^{j,k} + \mathrm{e}^{i\phi}\hat{e}_{rs}^{j,k}}{2}\\
\hat{a}_2 &= \frac{\hat{1\!\!1} - \hat{f}^{j,k} - \mathrm{e}^{-i\phi}\hat{e}_{rs}^{j,k} - \mathrm{e}^{i\phi}\hat{e}_{rs}^{j,k}}{2}\\
\hat{a}_3 &= \hat{1\!\!1} - \hat{e}_{rr}^{j,k} - \hat{e}_{ss}^{j,k}\,,
\end{aligned}$$

we assume $(\mathcal{A}_F,\Theta,\omega)$ to be an entropic K-system and study the expectation values

$$\omega\Big((\gamma_n(\hat{p}_i) - \hat{x}_i)^*(\gamma_n(\hat{p}_i) - \hat{x}_i)\Big)\,,$$

in the limit of large n, the meaning of the operators $\gamma_n(\hat{p}_i)$ and $\hat{x}_i$ being explained in Section 4. We estimate them by means of $i^\dagger:\mathcal{A}_F\to\mathbf{M}^{j,k}$. Thus:

$$\begin{aligned}
&\omega\Big((\gamma_n(\hat{p}_i) - \hat{x}_i)^*(\gamma_n(\hat{p}_i) - \hat{x}_i)\Big) =\\
&\omega(\gamma_n(\hat{p}_i)^2) + \omega(\hat{x}_i^2) - \omega\Big(i^\dagger\circ\tau_\omega^{i/2}(\gamma_n(\hat{p}_i)\hat{x}_i)\Big) - \omega\Big(i^\dagger\circ\tau_\omega^{i/2}(\hat{x}_i\gamma_n(\hat{p}_i))\Big) \leq\\
&\Big|\omega(\gamma_n(\hat{p}_i)^2) - \omega\Big(i^\dagger\circ\tau_\omega^{i/2}(\gamma_n(\hat{p}_i))\tau_\omega^{i/2}(\hat{x}_i)\Big)\Big| +\\
&\Big|\omega(\hat{x}_i^2) - \omega\Big(\tau_\omega^{i/2}(\hat{x}_i)i^\dagger\circ\tau_\omega^{i/2}(\gamma_n(\hat{p}_i))\Big)\Big| +\\
&\|i^\dagger\circ\tau_\omega^{i/2}(\gamma_n(\hat{p}_i)\hat{x}_i) - i^\dagger\circ\tau_\omega^{i/2}(\gamma_n(\hat{p}_i))\tau_\omega^{i/2}(\hat{x}_i)\| +\\
&\|i^\dagger\circ\tau_\omega^{i/2}(\hat{x}_i\gamma_n(\hat{p}_i)) - \tau_\omega^{i/2}(\hat{x}_i)i^\dagger\circ\tau_\omega^{i/2}(\gamma_n(\hat{p}_i))\|\,.
\end{aligned}$$

The expectation values tend to zero by letting $n\to+\infty$ (more precisely $\epsilon_{1,2}\to 0$) and $\delta\to 0$. This is so because of Lemma 9, (72) and of the asymptotic behaviour

$$\|i^\dagger(\tau_\omega^{i/2}(\gamma_n(\hat{p}_i))) - \tau_\omega^{i/2}(\hat{a}_i)\| \overset{\substack{n\to+\infty\\ \delta\to 0}}{\longrightarrow} 0$$

which follows from

$$\begin{aligned}
&\Big|\omega\Big(i^\dagger\Big(\tau_\omega^{i/2}(\gamma_n(\hat{p}_i))\Big)\hat{a}_j\Big) - \omega\Big(i^\dagger(\tau_\omega^{i/2}(\hat{x}_i)\hat{a}_j\Big)\Big| \leq\\
&\Big|\omega\Big(\tau_\omega^{i/2}(\gamma_n(\hat{p}_i))\hat{a}_j\Big) - \omega(\tau_\omega^{i/2}(\hat{x}_i)\hat{a}_j)\Big| +\\
&\|i^\dagger\Big(\tau_\omega^{i/2}(\gamma_n(\hat{p}_i))\hat{a}_i\Big) - i^\dagger\Big(\tau_\omega^{i/2}(\gamma_n(\hat{p}_i))\Big)\hat{a}_j\| + \|i^\dagger(\tau_\omega^{i/2}(\hat{x}_i)\hat{a}_j) - \tau_\omega^{i/2}(\hat{x}_i)\hat{a}_j\|
\end{aligned}$$

and from the finite dimensionality of $\mathbf{A}_{rs}^{j,k}$ as in (48) of Proposition 2.

7 Appendix

Proof of Theorem 2.

We show that, for all i,j and r,s, $\mathrm{st}-\lim_{k\to\pm\infty}\left[\hat{e}_{ij},\Theta^k(\hat{e}_{rs})\right]=0$, by using that, for all i,j and $\phi\in[0,2\pi)$,

$$\mathrm{st}-\lim_{k\to\pm\infty}\Big[\frac{\hat{1\!\!1}-\hat{f}\pm\mathrm{e}^{-i\phi}\hat{e}_{ij}\pm\mathrm{e}^{i\phi}\hat{e}_{ji}}{2}\,,\,\Theta^k\Big(\hat{1\!\!1}-\hat{f}\pm\mathrm{e}^{-i\phi}\hat{e}_{ij}\pm\mathrm{e}^{i\phi}\hat{e}_{ji}\Big)\Big]=0\,,$$

where $\hat{f}=\hat{e}_i+\hat{e}_j$, $\hat{e}_i=\hat{e}_{ij}\hat{e}_{ji}$ and $\hat{e}_j=\hat{e}_{ji}\hat{e}_{ij}$.

We must distinguish among five different possible cases:

1. $\left[\hat{e}_{ij}\,,\,\Theta^k(\hat{e}_{ij})\right]$
2. $\left[\hat{e}_{ij}\,,\,\Theta^k(\hat{e}_{ji})\right]$
3. $\left[\hat{e}_{ii}\,,\,\Theta^k(\hat{e}_{ij})\right]$
4. $\left[\hat{e}_{ij}\,,\,\Theta^k(\hat{e}_{rs})\right]\quad i\neq r,s\,;\,j\neq r,s$
5. $\left[\hat{e}_{ij}\,,\,\Theta^k(\hat{e}_{is})\right]\quad j\neq s\,.$

Case 1.

We can represent $\hat{e}_{ij}$ as $\begin{pmatrix}0&1\\0&0\end{pmatrix}$.

¿From the previous lemma we deduce that $\left[\sigma_x\,,\,\Theta^k(\sigma_x)\right]$ and $\left[\sigma_y\,,\,\Theta^k(\sigma_y)\right]$ tend strongly to zero for $k\to\pm\infty$. In fact,

The result follows if we show that $\left[\hat{\sigma}_x,\Theta^k(\hat{\sigma}_y)\right]$ does the same. This is, indeed, the case because $\hat{p}_0^+ + \hat{p}_{\pi/2}^+ = \dfrac{\sqrt{2}-1}{\sqrt{2}}\hat{1\!\!1}+\sqrt{2}\hat{p}_{\pi/4}^+$, and $\mathrm{st}-\lim_{k\to\pm\infty}\left[\hat{p}_{\pi/4}^+\,,\,\Theta^k(\hat{p}_{\pi/4}^+)\right]=0$ implies

$$\mathrm{st}-\lim_{k\to\pm\infty}\left[\hat{p}_0^+ + \hat{p}_{\pi/2}^+\,,\,\Theta^k(\hat{p}_0^+ + \hat{p}_{\pi/2}^+)\right]=0$$

$$\mathrm{st}-\lim_{k\to\pm\infty}\left[\hat{p}_0^+\,,\,\Theta^k(\hat{p}_{\pi/2}^+)\right]=0\,.$$

Case 2.

Same as in the previous case.

Case 3.

Because of the properties (52) of a system of matrix units we can write

$$\left[\hat{e}_{ii}\,,\,\Theta^k(\hat{e}_{ij})\right]=\hat{e}_{ij}\left[\hat{e}_{ji}\,,\,\Theta^k(\hat{e}_{ij})\right]+\left[\hat{e}_{ij}\,,\,\Theta^k(\hat{e}_{ij})\right]\hat{e}_{ji}\,.$$

Thus, the previous two cases show that

$$\mathrm{st}-\lim_{k\to\pm\infty}\left[\hat{e}_{ii}\,,\,\Theta^k(\hat{e}_{ij})\right]=0\,.$$

Case 4.

Construct the projections

$$\hat{f} = \hat{1\!\!1} - \hat{e}_i - \hat{e}_j\,, \qquad \hat{p}^{\pm} = \frac{\hat{1\!\!1} - \hat{f} \pm \mathrm{e}^{-i\theta}\hat{e}_{ij} \pm \mathrm{e}^{i\theta}\hat{e}_{ji}}{2}$$
$$\hat{g} = \hat{1\!\!1} - \hat{e}_r - \hat{e}_s\,, \qquad \hat{q}^{\pm} = \frac{\hat{1\!\!1} - \hat{g} \pm \mathrm{e}^{-i\phi}\hat{e}_{rs} \pm \mathrm{e}^{i\phi}\hat{e}_{sr}}{2}\,.$$

As $i \neq r, s$ and $j \neq r, s$, they commute and the algebra $\mathbf{A}^{+,-}_{\theta,\phi}$ generated by $\{\hat{p}^+, \hat{q}^-, \hat{1\!\!1}\}$ is Abelian and plays, with respect to the state ω on $\mathcal{M}$ the same role as $\mathbf{A}_\theta$ in Lemma 5. Thus, we can apply the same lemma to deduce that $\hat{p}^+_\theta$ and $\hat{q}^-_\phi$ contribute to the unique decomposition of ω such that $H_\omega(\mathbf{A}^{+,-}_{\theta,\phi})$ is attained. The argument of Lemma 8 applies. Thus:

$$\mathrm{st} - \lim_{k\to\pm\infty} \left[\hat{p}^+\,,\,\Theta^k(\hat{q}^-)\right] = 0\,.$$

This is true for all couples $\hat{p}^{\pm}$, $\hat{q}^{\pm}$ and all $\theta, \phi \in [0, \pi)$, so that

$$\mathrm{st} - \lim_{k\to\pm\infty} \left[\hat{e}_{ij}\,,\,\Theta^k(\hat{e}_{rs})\right] = 0$$

follows by exploiting the freedom in the parameters as done in the proof of Case 1.

Case 5.

We can work within the framework of 3×3 by representing the restricted state $\omega|_{\mathbf{M}_{ij}}$ as (compare Lemma 5) $\hat{\rho} = \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix}$ and the set of matrix units $\hat{e}_{ij}$, $i, j = 1, 2, 3$, accordingly.

We consider the commutator $\left[\hat{e}_{12}\,,\,\Theta^k(\hat{e}_{13})\right]$ and define

$$\hat{q}_{12}(\gamma) = \begin{pmatrix} 0 & \mathrm{e}^{-i\gamma} & 0 \\ \mathrm{e}^{i\gamma} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \qquad \hat{q}_{13}(\alpha) = \begin{pmatrix} 0 & 0 & \mathrm{e}^{-i\alpha} \\ 0 & 0 & 0 \\ \mathrm{e}^{i\alpha} & 0 & 0 \end{pmatrix}\,.$$

As in Theorem 1, the result follows by variation of the parameters, if

$$\mathrm{st} - \lim_{k\to\pm\infty} \left[\hat{q}_{12}(\gamma)\,,\,\hat{q}^k_{13}(\alpha)\right] = 0\,, \tag{73}$$

for all α, $\gamma \in [0, 2\pi)$, where $\hat{q}^k_{13}$ (and similar notations) is short for Θ^k acting on $\hat{q}_{13}(\alpha)$ In fact,

$$\hat{e}_{12} = \frac{i\hat{q}_{12}(\pi/2) + \hat{q}_{12}(0)}{2}\,, \qquad \hat{e}_{13} = \frac{i\hat{q}_{13}(\pi/2) + \hat{q}_{13}(0)}{2}\,.$$

The result in (73) comes from the following strong-operator limit

$$\mathrm{st} - \lim_{k\to\pm\infty} \left\{\left[\hat{q}_{12}(\delta)\,,\,\hat{q}^k(\beta,\delta)\right] + \left[\hat{q}(\beta,\delta)\,,\,\hat{q}^k_{12}(\delta)\right]\right\} = 0\,, \tag{74}$$

for all $\beta,\delta\in[0,2\pi)$ where $\hat q(\beta,\delta)=\begin{pmatrix}0&0&\mathrm{e}^{-i\beta}\\0&0&\mathrm{e}^{i(\delta-\beta)}\\\mathrm{e}^{i\beta}&\mathrm{e}^{-i(\delta-\beta)}&0\end{pmatrix}$.

We shall derive it after examining its consequences. Analogously to what already seen above, it turns out that

$$\Big[\hat q_{12}(\gamma)\,,\,\hat q^k_{13}(\alpha)\Big]=\frac12\Big[\hat q_{12}(\gamma)\,,\,\hat q^k(\alpha,\gamma)\Big]-\frac12\Big[\hat q_{12}(\gamma+\pi)\,,\,\hat q^k(\alpha,\gamma+\pi)\Big]\,.$$

These linear combinations tend to zero in the strong-operator limits, whence

$$\mathrm{st}-\lim_{k\to\pm\infty}\Big\{\Big[\hat q_{12}(\gamma)\,,\,\hat q^k_{13}(\alpha)\Big]+\Big[\hat q_{13}(\alpha)\,,\,\hat q^k_{12}(\gamma)\Big]\Big\}=0\,.$$

Upon using: 1.) the first two cases discussed in this theorem, 2.) that $\hat q_{12}(\gamma)$ and $\hat q_{13}(\alpha)$ strongly commute with $\hat q^k_{12}(\gamma)$, respectively $\hat q^k_{13}(\alpha)$ and 3.) that weak clustering holds because of Theorem 1, we finally arrive at

$$\lim_{k\to\pm\infty}\omega\Big(\Big[\hat q_{12}(\gamma)\,,\,\hat q^k_{13}(\alpha)\Big]^*\Big[\hat q_{12}(\gamma)\,,\,\hat q^k_{13}(\alpha)\Big]\Big)=0\,,$$

namely at strong clustering.

In order to have (74) hold, we consider the projections

$$\hat p_\pm(\delta)=\frac12\begin{pmatrix}1&\pm\mathrm{e}^{-i\delta}&0\\\pm\mathrm{e}^{i\delta}&1&0\\0&0&0\end{pmatrix}\,,\quad \hat e_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\,,$$

and the Abelian algebra $\mathbf{A}$ they generate. Then, we transform $\mathbf{A}$ into a new Abelian algebra $\mathbf{A}_\epsilon$ by means of an infinitesimal rotation $\hat V_\epsilon\simeq\hat{1\!\!1}+i\epsilon\hat a$.

¿From Lemma 1 and Lemma 5 we know that $H_\omega(\mathbf{A})$ is attained at the decomposition of ω that uses $\hat p^\pm(\delta)$ and $\hat e_3$. By continuity arguments (see for instance [2, 12]), in the limit of $\epsilon\to0$, the optimal decomposition for $H_\omega(\mathbf{A}_\epsilon)$ will be given by a triple of minimal projections $\hat p_\pm(\epsilon)\,,\hat e_3(\epsilon)$ with the following expansion up to order ϵ:

$$\hat p_\pm(\epsilon)\simeq\hat p_\pm(\delta)+\epsilon\hat q_\pm\,,\qquad \hat e_3(\epsilon)\simeq\hat e_3+\epsilon\hat q_3$$

$$\hat q_\pm=\begin{pmatrix}\pm q&\pm v&u_\pm\\\pm v^*&\mp q&u_\pm\mathrm{e}^{i\delta}\\u^*_\pm&u^*_\pm\mathrm{e}^{-i\delta}&0\end{pmatrix}\,,\qquad \hat q_3=\begin{pmatrix}0&0&q_3\\0&0&q_3\mathrm{e}^{i\delta}\\q^*_3&q^*_3\mathrm{e}^{-i\delta}&0\end{pmatrix}\,,$$

together with the constraints $q\,\mathrm{real}\,,\ \mathrm{Re}\,\{v\mathrm{e}^{i\delta}\}=0\,,\ u_++u_-=-q_3\,.$

¿From Lemma 8 it follows $\mathrm{st}-\lim_{k\to\pm\infty}\Big[\hat p_\pm(\epsilon)\,,\,\Theta^k(\hat p_\pm(\epsilon))\Big]=0$, and this property must hold at all orders ϵ^k, hence

$$\mathrm{st}-\lim_{k\to\pm\infty}\Big\{\Big[\hat p_\pm(\delta)\,,\,\Theta^k(\hat q_\pm)\Big]+\Big[\hat q_\pm\,,\,\Theta^k(\hat p_\pm(\delta))\Big]\Big\}=0\,.$$

Because of the first cases discussed above, after writing $u_\pm=|u\pm|\mathrm{e}^{-i\beta_\pm}$ and absorbing the modulus in the infinitesimal angle ϵ, we can concentrate on the operators

$$\hat{q}_{12}(\delta) = \begin{pmatrix} 0 & \mathrm{e}^{-i\delta} & 0 \\ \mathrm{e}^{i\delta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{q}(\beta_+, \delta) = \begin{pmatrix} 0 & 0 & \mathrm{e}^{-i\beta_+} \\ 0 & 0 & \mathrm{e}^{i(\delta-\beta_+)} \\ \mathrm{e}^{i\beta_+} & \mathrm{e}^{-i(\delta-\beta_+)} & 0 \end{pmatrix},$$

where $\beta_\pm$ depends on the state $\hat{\rho}$, on δ, on the rotation $\hat{V}_\epsilon$ and is determined by the request that the corresponding decomposition be optimal for $H_\omega(\mathbf{A}_\epsilon)$ in the limit $\epsilon \to 0$.

We need only to prove that, say, β_+ takes every value in $[0, 2\pi)$ since, then, a free variation of the parameters is possible and the proof is completed. In order to ensure that this is indeed the case we recall Lemma 5 and use the unitary operator

$$U = \begin{pmatrix} \mathrm{e}^{i\phi} & 0 & 0 \\ 0 & \mathrm{e}^{i\phi} & 0 \\ 0 & 0 & \mathrm{e}^{i\psi} \end{pmatrix},$$

to rotate the algebra $\mathbf{A}_\epsilon \to U\mathbf{A}_\epsilon U^{-1}$ while leaving the state ω unaltered. The rotated projections $U\hat{p}_\pm(\epsilon)U^{-1}$, $U\hat{p}_3(\epsilon)U^{-1}$ are optimal for $H_\omega(U\mathbf{A}_\epsilon U^{-1})$ in the limit $\epsilon \to 0$ and $U\hat{p}_\pm(\delta)U^{-1} = \hat{p}_\pm(\delta)$,whereas

$$U\hat{q}_\pm U^{-1} = \begin{pmatrix} \pm q & \pm v & u_\pm \mathrm{e}^{i(\phi-\psi)} \\ \pm v^* & \mp q & u_\pm \mathrm{e}^{i(\delta+\phi-\psi)} \\ u_\pm^* \mathrm{e}^{-i(\phi-\psi)} & u_\pm^* \mathrm{e}^{-i(\delta+\phi-\psi)} & 0 \end{pmatrix},$$

whence any value $\beta_+ \in [0, 2\pi)$ is available by varying $\phi - \psi$ in the same interval.

□

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