GROUP COHOMOLOGY, MODULAR THEORY AND SPACE-TIME SYMMETRIES

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Abstract. The Bisognano-Wichmann property on the geometric behavior of the modular group of the von Neumann algebras of local observables associated to wedge regions in Quantum Field Theory is shown to provide an intrinsic sufficient criterion for the existence of a covariant action of the (universal covering of) the Poincaré group. In particular this gives, together with our previous results, an intrinsic characterization of positive-energy conformal pre-cosheaves of von Neumann algebras. To this end we adapt to our use Moore theory of central extensions of locally compact groups by polish groups, selecting and making an analysis of a wider class of extensions with natural measurable properties and showing henceforth that the universal covering of the Poincaré group has only trivial central extensions (vanishing of the first and second order cohomology) within our class.

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Introduction

In this paper we discuss the problem of characterizing the existence of a Poincaré covariant action for a net of local observable algebras in terms of net-intrinsic algebraic properties. In other words, given a state with a Reeh-Schlieder cyclicity property, we look for a condition that ensures it to be a relativistic vacuum.

The work done by Bisognano and Wichmann [1] during the seventies about the essential duality has shown the geometrical character of the Tomita-Takesaki modular operators associated with von Neumann algebras generated by Wightman fields with support based on wedge regions and the vacuum vector.

While an algebraic version of the Bisognano-Wichmann theorem in the Poincaré covariant case is not yet available, there are general results in this direction given by the recent work of Borchers [2], see also [5,19,21] for related results. Based on the result of Borchers, there is a complete algebraic analysis in [4] and [8] showing that, for a conformally invariant theory, the modular groups associated with the von Neumann algebras of double cones are associated with special one-parameter subgroups of the conformal group leaving invariant the given double cone (cf. [11]). Applications of the Bisognano-Wichmann theorem to the analysis of the relations between the space-time symmetries and particle-antiparticle symmetry in quantum field theory are contained in [9].

Here we show how to reconstruct the Poincaré group representation from the net-intrinsic property of *modular covariance* (assumption 2.1), i.e. the modular operators associated with the algebras of the wedge regions act geometrically as boosts.

In our previous paper [4] we showed in particular that, for a Poincaré covariant net, the modular unitaries associated with wedge regions should necessarily coincide with the boost unitaries in the given representation if the modular covariance and the split property are assumed. Here, we do not have any given unitary representation of the Poincaré group from the start. Instead we recover a canonical representation of the Poincaré group, which is unique if the split property holds, just by using the modular unitaries corresponding to all wedge algebras.

This step forward makes two significant changes. The first concerns physics, since our result shows that the net itself contains all the space-time symmetry informations, and that the modular covariance property gives rise to a canonical representation even when (non-split case) it is not unique.

The second concerns mathematics, providing a new application of the group cohomology describing central extensions of groups.

As is well known, central extensions of groups appear naturally in physics, as shown for example by the Wigner theorems [17] on the (anti-)unitary realization of the symmetries in quantum physics.

The class of results in this context which is relevant for our purposes concerns the triviality of (suitable classes of) central extensions of the universal covering of the Poincaré group. One of the first results of this type is due to Michel [13], but his hypotheses seems unapplicable to our problem.

Another result in this direction is a particular case of the theory of Moore [15,16] on central extensions of locally compact groups via polish groups. The extensions described by Moore are continuous and open, and his theorem on the universal extension concerns extensions by closed subgroups of $\mathcal{U}(\mathcal{H})$, the group of unitary operators on a separable Hilbert space $\mathcal{H}$.

As a main step in our analysis, we study a class of extensions with natural measurable properties, which we call weak Lie central extensions. These extensions are given by a (not necessarily closed) subgroup G of $\mathcal{U}(\mathcal{H})$, and we assume that there are continuous one-parameter subgroups in G which correspond to a set of Lie-algebra generators (see section 2 for the precise definition). We show that weak Lie central extensions of a simply connected perfect Lie group are trivial, and this applies to the universal covering of the Poincaré group.

Our proof strongly relies on Moore theory. In fact, even though the continuity properties of the central extensions described by Moore are too restrictive for our purposes, we prove that his theorem on the structure of the first and second order (measurable) cohomology of a connected perfect Lie group (see [16] or theorem 1.3) can be generalized to the weak Lie central extensions (see theorem 1.7). The cases of the Poincaré group and of the conformal group find application in section 2.

Finally we mention that our results, together with theorem 2.3 in our previous paper [4], give a complete characterization of positive-energy conformal pre-cosheaves as those where the modular groups associated with double cones act geometrically as expected (see corollary 2.9).

The plan of the paper is the following: in the first section we review some known facts about cohomology and extensions of groups, recall elements of Moore theory and provide a generalization of results needed in the following.

In the second section we present our results about the generation of the unitary representation of the group of space-time symmetries by modular groups, in case of local algebras on a separable Hilbert space.

The last section contains an outlook.

1. Group extensions and cohomology

We begin by shortly reviewing basic elements of central extensions of groups (see [12,3]). We first deal with *algebraic central extensions*, i.e. with short exact sequences of groups

$$1 \to A \xrightarrow{i} G \xrightarrow{\pi} P \to 1 \tag{1.1}$$

where P is the group to extend and A is a central subgroup of G. The pair (G,π) determines the extension, and we often refer to it as a central extension of P (by $A = ker(\pi)$).

Two extensions (G_1,π_1), (G_2,π_2) are called *equivalent* if there is an isomorphism between G_1 and G_2 such that the following diagram commutes:

$$\begin{array}{ccccccccc} & & & & G_1 & & & & \\ & & & \nearrow & & \searrow & & & \\ 1 & \to & A & & \updownarrow & & P & \to & 1 \\ & & & \searrow & & \nearrow & & & \\ & & & & G_2 & & & & \end{array} \tag{1.2}$$

An extension of P via A is *trivial* if it is equivalent to the direct product $(A\times P,\pi)$ where π is the projection onto P, i.e. sequence (1.1) splits.

A *section* of an extension (G,π), i.e. a map $s : P \to G$ such that $\pi\cdot s = id_P$, determines an identification of G with $A \times P$ as sets given by $g \to (s(\pi(g))^{-1}g, \pi(g))$. Then, the multiplication rule on $A\times P$ is given by

$$(a,p)\cdot(b,q) = (ab\omega(pq)^{-1}, pq), \tag{1.3}$$

where $\omega(p,q) = s(q)s(pq)^{-1}s(p)$ satisfies the 2-*cocycle* condition $\delta_2\omega = 0$, where δ_2 is defined in (1.4). Conversely a 2-cocycle gives an associative multiplication on $A\times P$, thus defining an extension. Two cocycles ω_1, ω_2 give rise to equivalent extensions *iff* there is a group automorphism of $A \times P$ which is compatible with diagram (1.2) and intertwines the products given by ω_1 and ω_2. This means that there is a map $\varphi : P \to A$ such that the morphism between $(A \times P, \omega_1)$ and $(A \times P, \omega_2)$ is given by $(a,p) \mapsto (a\varphi(p), p)$, and $\omega_1 = \omega_2 \cdot \delta_1\varphi$, with δ_1 defined in (1.4), i.e. ω_2 differs by ω_1 of a coboundary, showing that equivalence classes of extensions are indeed cohomology classes.

The *n-cochains* $C^n(P,A)$ of P with values in A are maps from P^n to A, and the n-th *coboundary map* $\delta_n : C^n(P,A) \to C^{n+1}(P,A)$ is given by

$$\begin{aligned} \delta_n f(p_1,\ldots,p_{n+1}) = f(p_2,\ldots,p_{n+1}) &- f(p_1p_2,p_3,\ldots,p_{n+1}) \\ &+ f(p_1,p_2p_3,\ldots,p_{n+1}) + \ldots \\ &+ (-1)^n f(p_1,\ldots,p_np_{n+1}) - (-1)^n f(p_1,\ldots,p_n) \end{aligned} \tag{1.4}$$

and verifies $\delta_{n+1}\delta_n = 1$.

The range of δ_{n-1} is denoted by $B^n(P, A)$, the kernel of δ_n by $Z^n(P, A)$, and their quotient by $H^n(P, A)$. Since $C^n(P, A)$ is an abelian group with respect to the pointwise multiplication, $H^n(P, A)$ is a group too.

The above discussion shows that equivalence classes of central extensions of P via A are in $1-1$ correspondence with elements of $H^2(P, A)$, and therefore form a group. Note that the cocycle equation for 1-cochains means exactly that $Z^1(P, A) \equiv H^1(P, A)$ is the group of homomorphisms from P to A, therefore the vanishing of the first cohomology group for all abelian A is equivalent to the fact that P coincides with its commutator $[P, P]$, i.e. P is *perfect.*

The groups G for which all central extensions split in a unique way, namely $H^n(P, A) = 0, n = 1, 2$ for all A as above, are called *algebraically simply connected* (see [14,3]).

A *universal central extension* (E, σ) of P is a central extension such that for each central extension (G, π) of P there is a homomorphism $U : E \to G$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "U"] \arrow[dr, "\sigma"'] & G \arrow[d, "\pi"] \\ & P \end{tikzcd} \quad . \tag{1.5}$$

If a universal central extension exists, it is unique (up to equivalence) and the extensions of P via A may be easily described. Indeed consider the following commutative diagram:

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, equal] & E \times A \arrow[r, "id_E \cdot 1_A"] \arrow[d, "U \cdot id_A"] & E \arrow[r] \arrow[d, "\sigma"] & 1 \\ 1 \arrow[r] & A \arrow[r] & G \arrow[r, "\pi"] & P \arrow[r] & 1 \end{tikzcd} \quad . \tag{1.6}$$

By a classical diagram chasing argument, it is easy to see that $U \cdot id_A$ is surjective, and that the restriction of the projection $id_E \cdot 1_A$ to $\ker(U \cdot id_A)$ is an isomorphism onto $S := \ker(\sigma)$. Moreover the embedding of S in $E \times A$ via the preceding isomorphism has the form $id_S \times \psi$, where ψ is a morphism from S to A. As a consequence, there is an isomorphism

$$E \times A\big/_{S} \to G \tag{1.7}$$

where S is embedded into $E \times A$ as described, therefore the extension (G, π) is described in terms of this embedding, i.e. in terms of the morphism ψ, which implies that $H^1(S, A) \simeq H^2(P, A)$.

A central extension (E, σ) of P where E is algebraically simply connected is the universal central extension of P. Indeed fix an extension (G, π) of P, choose a section s of π and consider the 2-cocycle ω defined in equation (1.3). Then the map

$$\begin{array}{rcl} \tilde{\omega} \quad : E \times E & \to & A \\ (g, h) & \mapsto & \omega(\sigma(g)\sigma(h)) \end{array} \tag{1.8}$$

is indeed a cocycle in $Z^2(E, A)$. Since $H^2(E, A)$ is trivial, there is a 1-cochain φ s.t. $\delta_1 \varphi = \tilde{\omega}$. Then, setting

$$U(g) = s(\sigma(g))\varphi(g)^{-1}, \qquad g \in E \tag{1.9}$$

we easily get that U is a homomorphism from E to G and makes diagram (1.5) commutative. We note that since σ is surjective and E is perfect, necessarily P is perfect too.

If P is a topological group, extensions with given topological properties are of interest. A theory in this sense has been developed by Moore in a series of papers [15,16]. We recall that a topological group A is called *polish* if its topology may be obtained by a separable complete metric (that can be chosen compatible with the uniformities of A). Moore considers *topological central extensions* of a locally compact group P via a polish group A, i.e. exact sequences (1.1) such that i is a homeomorphism into its image and π is continuous and open. He denotes by $Ext(P, A)$ the equivalence classes of topological extensions of P via A, where the isomorphism in diagram (1.2) is asked to be a homeomorphism. In analogy with the algebraic case, topological cohomology groups are to be defined in such a way that the identifications of $Ext(P, A)$ with $H^2_{top}(P, A)$, and of $H^1_{top}(P, A)$ with the continuous homomorphisms from P to A hold. Moore proves this to be the case if cochains are Borel measurable. With this hypothesis, the previously mentioned identifications hold, cohomology groups are polish groups and all usual functorial properties are satisfied. From now on we consider topological cohomology groups only, therefore we drop the subscript $_{top}$.

Let us call *unitary group* a subgroup of $\mathcal{U}(\mathcal{H})$, the group of the unitary operators on a separable Hilbert space $\mathcal{H}$ equipped with the weak topology. It turns out that closed (i.e. complete) unitary groups are polish. Concerning universal extensions, Moore deals with *unitary topological extensions*, namely A in (1.1) is a closed unitary group. We shall say that E is *$\mathcal{U}$-simply connected* if all central unitary extensions split in a unique way, that is to say $H^1(E, A)$ and $H^2(E, A)$ vanish for all closed abelian unitary groups A.

1.1 Theorem [16]. *Let the locally compact group P have a universal topological central extension (E, σ) with E a polish $\mathcal{U}$-simply connected group. Then (E, σ) is a universal central extension for the class of all unitary central extensions, namely for* *every topological central extension (G,π) via a closed unitary group A there exists a continuous map $U : E \to G$ s.t. the diagram (1.5) commutes. As a consequence, the isomorphism (1.7) is a homeomorphism and $H^1(S,A)$ is isomorphic to $H^2(P,A)$, where $S := \ker(\sigma)$.*

According to theorem 1.1, we call the $\mathcal{U}$-covering of P the (unique) central extension of P via a simply connected group (if it exists). We need however a generalization of theorem 2 to a larger class of extensions.

1.2 Theorem. *Let P be a locally compact group which admits a $\mathcal{U}$-covering (E,σ). Let (G,π) be an algebraic central extension of P with G a unitary group and $A := \ker(\pi)$ a topological subroup of G, closed as unitary group. If π has a Borel section s, then π splits on E, i.e there exists a continuous map $U : E \to G$ s.t. diagram (1.5) commutes.*
As a consequence, the group $E \times A \big/ S$ is polish and isomorphism (1.7) is continuous.
If, in addition, the map π is continuous, then G is a closed unitary group and the extension (G,π) is a topological extension in the sense of definition 1.1.

The last statement of the theorem shows that the difference between the extensions in theorems 1.1 and 1.2 lies in the continuity of the map π.

Proof. Since s is a measurable section, the 2-cocycle defined in (1.3) is measurable too, and it is an element of $Z^2(P,A)$. Then we define $\tilde{\omega}$ as in (1.8), and since σ is continuous, $\tilde{\omega}$ turns out to be an element of $Z^2(E,A)$. Since A is a closed unitary group and E is $\mathcal{U}$-simply connected, $\tilde{\omega}$ is indeed a coboundary, i.e. there exists $\varphi \in C^1(E,A)$ s.t. $\tilde{\omega} = \delta_1\varphi$. Then U defined by equation (1.9) is a homomorphism from E to G, it is measurable by construction, and makes diagram (1.5) commute. By proposition $5(a)$ in [15], U is indeed continuous.
Now we come back to diagram (1.6). Surjectivity of $U \cdot id_A$ and the existence of the isomorphism $j : \ker(U \cdot id_A) \to S$ follow by algebraic reasons. Then we observe that j is simply the restriction to $\ker(U \cdot id_A)$ of the projection $id_E \cdot 1_A$, which is continuous and open, therefore j is a homeomorphism. Then $E \times A \big/ S$ is a polish group and the isomorphism (1.7) is continuous.
If the map π is continuous, then

$$g \in G \to \left(\sigma^{-1} \circ \pi(g)\ ,\ \left[U \circ \sigma^{-1} \circ \pi(g)\right]^{-1}\ g\right) \in E \times A$$

gives a map from the subsets of G to the subsets of $E \times A$ for which the preimage of an open set is open. On the other hand, composing such a map with the projection on $E \times A \big/ S$, we get a map of points which is the inverse of the isomorphism (1.7). Such isomorphism is therefore a homeomorphism, and all other properties follow.

□

Note that, in the topological setting, the only property that $[P, P]$ is dense in P is needed for the existence of a universal central extension. A remarkable result of Moore shows that a perfect connected group P admits the $\mathcal{U}$-covering. Moreover, if P is a Lie group, central extensions may be completely described in terms of the Lie algebra cohomology and the fundamental group $\pi_1(P)$.

We recall that when $\mathbf{p}$ is a Lie algebra, a central extension of $\mathbf{p}$ via a vector space $\mathbf{a}$ is an exact sequence $0 \to \mathbf{a} \to \mathbf{g} \xrightarrow{\pi} \mathbf{p} \to 0$ where $\mathbf{a}$ is a central Lie subalgebra of $\mathbf{g}$ and π is a Lie algebra homomorphism. Extensions are still described in terms of 2-cocycles, and equivalence classes of extensions in terms of 2-cohomology classes. We denote by $H^n(\mathbf{p}, \mathbf{a})$ the cohomology groups of $\mathbf{p}$ with respect to $\mathbf{a}$.

1.3 Theorem [16]. *The $\mathcal{U}$-covering (E, σ) of a perfect connected Lie group P is a perfect Lie group, and*

$$\ker(\sigma) \simeq \pi_1(P) \times H^2(\mathbf{p}, \mathbf{R})$$

where $\mathbf{p}$ is the Lie algebra of P. If $H^2(\mathbf{p}, \mathbf{R}) = 0$, E coincides with the universal covering of P.

In the case of Lie groups, the existence of a measurable section in theorem 1.2 may be replaced by a natural condition.

1.4 Definition. *Let (G, π) be an algebraic extension of the Lie group P where G is a unitary group. We say that π is a weak Lie extension if there exists a set $\{L_1 \dots L_n\}$ of generators of $\mathbf{p}$ as a Lie algebra ad a corresponding set $\{V_1(t), \dots, V_n(t)\}$ of strongly continuous 1-parameter groups in G such that*

$$\pi(V_i(t)) = \exp(tL_i), \qquad i = 1, \dots, n.$$

where we have denoted by $\mathbf{p}$ the Lie algebra of P.

1.5 Lemma. *Let (G, π) be a central weak Lie extension of the Lie group P. Then there is a finite set of strongly continuous 1-parameter groups in G such that the generators of their images in P are a basis for $\mathbf{p}$ as a vector space.*

Proof. Let us consider the map v which associates to a given set of one parameter groups in P the (finite dimensional) vector space spanned by their generators.
We observe that given $A, B \in \mathbf{p}$, the commutator $[A, B]$ belongs to

$$X := v(\{t \to \exp(sA)\ \exp(tB)\ \exp(-sA) : s \in \mathbf{R}\}).$$

Indeed, X is spanned, by definition, by the range of the Lie algebra valued function $s \to \exp(sA)\ B\ \exp(-sA)$, whose derivative in 0 is $[A, B]$.

As a consequence, by finite dimensionality, we may find a finite set $\{s_i^j\}$ such that the vector space

$$v \circ \pi \left\{ t \to V_{h_1}(s_1^j) \cdot \ldots \cdot V_{h_p}(s_p^j) \cdot V_{h_0}(t) \cdot V_{h_p}(s_p^j) \cdot \ldots \cdot V_{h_1}(s_1^j) : j = 1, 2, \ldots \right\}$$

contains the element $[L_{h_1}, [\ldots [L_{h_p}, L_{h_0}]\ ]$, where the V_i are described in definition 1.4. Then the thesis easily follows.

□

1.6 Proposition. *Let (G, π) be a central weak Lie extension of the connected Lie group P, with G a unitary group. Then there exists a Borel measurable section $S : P \to G$ of the extension π.*

Proof. According to the previous lemma, we may suppose that $\{L_1, \ldots L_n\}$ is a basis for $\mathbf{p}$ as a vector space.
The map $\alpha : \mathbf{R}^n \to P$ given by $\alpha(t_1, \ldots t_n) = \prod_{i=1}^n \exp(t_i L_i)$ has non trivial Jacobian at the origin because $\{L_1 \ldots L_n\}$ is a basis. Therefore there there exist two open sets, $U \subset \mathbf{R}^n$ and $V \equiv \alpha(U) \subset P$, where U is a neighborhood of zero in $\mathbf{R}^n$ and, consequently, V is a neighborhood of the identity element in P, such that the map $\alpha : U \to V$ is a diffeomorphism.
Now we define the (strongly) continuous map $\beta : \mathbf{R}^n \to G$ given by $\beta(t_1, \ldots t_n) = \prod_{i=1}^n V_i(t_i)$, and observe that the map $S_0 := \beta\ \alpha^{-1} : V \to G$ is a (strongly) continuous section of π on V, i.e. $\pi \cdot S_0 \equiv id_V$.
Since V is an open neighborhood of the origin and P is a connected group, then P is algebraically generated by V, i.e., each element of P can be written as a finite product of elements in V.
Now we consider the open covering $\bigcup_{g \in P} gV = P$. Since P is σ-compact, we may extract a countable sub-covering,

$$P = \bigcup_{k \in \mathbf{N}} g_k V.$$

Finally, define the measurable partition of P given by,

$$A_1 = g_1 V$$

$$A_{n+1} = g_{n+1} V \cap \left(\bigcup_{k=1}^{n} g_k V \right)^c \qquad n \in \mathbf{N}.$$

For each $n \in \mathbf{N}$, we may write $g_n = v_{n,1} \cdots v_{n,m_n}$, $v_{n,k} \in V$, hence, since $A_n \subset g_n V$, any element of A_n may be written as $v_{n,1} \cdots v_{n,m_n} v$, when v varies in V. Then, we define

$$\begin{array}{rccc} S_n : & A_n & \longrightarrow & G \\ & v_{n,1} \cdots v_{n,m_n} v & \longmapsto & S_0(v_{n,1}) \cdots S_0(v_{n,m_n}) S_0(v) \end{array} .$$

As before, S_n is a strongly continuous section of π on A_n. Therefore it is easy to understand that the map $S = \sum_{n \in \mathbf{N}} \chi_{A_n} S_n$, where χ_B is the characteristic function of B, is the desired measurable section.

□

Making use of theorems 1.2 and 1.3 and proposition 1.6, we can prove the following theorem on weak Lie extensions. We remark that no condition on the closure of A is needed.

1.7 Theorem. *Let P be a perfect connected Lie group. Then all central weak Lie extensions (G, π) of P, with G a unitary group, split on the $\mathcal{U}$-covering (E, σ), i.e. there exist a continuous homomorphism $U : E \to G$ such that diagram (1.5) commutes.*

Proof. Let $\tilde{G}$ be the group algebraically generated by G and $\overline{A}$, where $\overline{A}$ is the weak closure of A in $\mathcal{U}(H)$, and extend π to a morphism $\tilde{\pi}$ in such a way that $\tilde{\pi}|_{\overline{A}} = 0$. Then, applying proposition 1.6 and theorem 1.2 to the extension $(\tilde{G}, \tilde{\pi})$ we get a continuous map $U : E \to \tilde{G}$ such that $\tilde{\pi} \cdot U = \sigma$.
By theorem 1.3 E is perfect, therefore $U(E) = U([E, E]) \subseteq [\tilde{G}, \tilde{G}]$. Since $\overline{A}$ commutes with G and $\tilde{G}$, equality $[\tilde{G}, \tilde{G}] \equiv [G, G]$ holds. As a consequence, $Rg(U) \subseteq G$, and the proof is concluded.

□

We recall that the (4-dimensional) Poincaré group $\mathcal{P}_+^\uparrow$ is perfect and its Lie algebra $\wp$ satisfies $H^2(\wp, \mathbf{R}) = 0$. Therefore, next corollary immediately follows.

1.8 Corollary. *Let (G, π) be a central weak Lie extension of the Poincaré group $\mathcal{P}_+^\uparrow$ where G is a unitary group. Then there exists a strongly continuous unitary representation U of the universal covering $\widetilde{\mathcal{P}}$ of $\mathcal{P}_+^\uparrow$ such that $\pi \cdot U = \sigma$, where σ is the covering map, and there is a continuous isomorphism*

$$\widetilde{\mathcal{P}} \times A \big/ _{\mathbf{Z}_2} \to G$$

where $\mathbf{Z}_2$ is a suitable order two central subgroup of $\widetilde{\mathcal{P}} \times A$.

Proof. The corollary easily follows by theorems 1.2, 1.3 and 1.7, the previous observations on the Lie algebra $\wp$ and the fact that the kernel of the covering map from $\widetilde{\mathcal{P}}$ to $\mathcal{P}_+^\uparrow$ is $\mathbf{Z}_2$.

□

2. Modular covariance and the reconstruction of space-time symmetries.

In this section we study how the unitary representation of the group of the space-time symmetries of a Quantum Field Theory [10] may be generated by the modular unitaries associated with the algebras of a suitable class of regions. Local Quantum Theories are described by a local pre-cosheaf of von Neumann algebras (see [9]), i.e by a map

$$\mathcal{A} : \mathcal{O} \to \mathcal{A}(\mathcal{O}), \qquad \mathcal{O} \in \mathcal{K}$$

where $\mathcal{K}$ is the family of the double cones in the Minkowski space M, such that

$$\begin{aligned} \mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathcal{A}(\mathcal{O}_1) \subset \mathcal{A}(\mathcal{O}_2) \qquad & \text{(isotony)} \\ \mathcal{A}(\mathcal{O}) \subset \mathcal{A}(\mathcal{O}')' \qquad & \text{(locality).} \end{aligned}$$

Local algebras are supposed to act on a *separable* Hilbert space $\mathcal{H}$, and $\mathcal{A}_0$ denotes the C^*-algebra of quasi-local observables generated by the local algebras. There is a vector Ω (vacuum) which is cyclic for all of them.

The pre-cosheaf is extended by additivity to general open regions of M.

As a consequence of locality, Ω turns out to be cyclic and separating for the algebras associated with all non empty open regions whose complement has non-empty interior (Reeh-Schlieder property).

We recall that a *wedge* region is any Poincaré transformed of the region $W_1 := \{x \in \mathbf{R}^n : |x_0| < x_1\}$. The *boosts* preserving W_1 are the elements of the one-parameter subgroup $\Lambda_{W_1}(t)$ of $\mathcal{P}_+^\uparrow$ which acts on the coordinates x_0, x_1 via the matrix

$$\begin{pmatrix} \cosh 2\pi t & -\sinh 2\pi t \\ -\sinh 2\pi t & \cosh 2\pi t \end{pmatrix}$$

and leaves the other coordinates unchanged. One-parameter boosts $\Lambda_W(t)$ for any wedge W are defined by Poincaré conjugation. We denote by $\mathcal{W}$ the family of all wedges in M.

We make here our main assumption:

2.1 Assumption: Modular Covariance. *Given any wedge region W, the modular unitaries Δ_W^{it} associated with $(\mathcal{A}(W), \Omega)$ act geometrically on the pre-cosheaf $\mathcal{A}$, i.e.*

$$\Delta_W^{it} \mathcal{A}(\mathcal{O}) \Delta_W^{-it} = \mathcal{A}(\Lambda_W(t)\mathcal{O}), \qquad \mathcal{O} \in \mathcal{K}, \quad W \in \mathcal{W} .$$

2.2 Proposition. *Let $\mathcal{A}$ be a local pre-cosheaf on M satisfying modular covariance. Then essential duality holds, i.e.*

$$\mathcal{A}(W)' = \mathcal{A}(W')$$

for each wedge region W.

Proof. The proof is identical to that of Theorem 2.3[1], part (i) in [4].

$\square$

Now we consider the universal covering $\widetilde{\mathcal{P}}$ of the Poincaré group $\mathcal{P}_+^\uparrow$. Given a wedge W, we shall indicate with $\tilde{\Lambda}_W(t)$ the lifting of $\Lambda_W(t)$ on $\widetilde{\mathcal{P}}$. If $g \to U(g)$ is a unitary representation of $\widetilde{\mathcal{P}}$, we shall say that it is $\mathcal{A}$-covariant if

$$U(g)\mathcal{A}(\mathcal{O})U(g)^* = \mathcal{A}(\sigma(g)\mathcal{O}), \qquad g \in \widetilde{\mathcal{P}}$$

where $\sigma : \widetilde{\mathcal{P}} \to \mathcal{P}_+^\uparrow$ is the covering map, and that it has positive energy if the restriction of U to the translation subgroup satisfies the spectrum condition.

2.3 Theorem. *Let $\mathcal{A}$ be a modular covariant, local pre-cosheaf on M, dim$(M) > 2$. Then there exists a vacuum preserving, positive energy, $\mathcal{A}$-covariant, unitary representation U of $\widetilde{\mathcal{P}}$ which is canonically determined by the property*

$$U(\tilde{\Lambda}_W(t)) = \Delta_W^{it}, \qquad W \in \mathcal{W}$$

In order to prove Theorem 2.3 we need some results about the structure of $\mathcal{P}_+^\uparrow$ (Lemma 2.5), and about the group generated by modular operators associated with wedge regions.

2.4 Proposition. *The generators of the boost transformations generate the Lie algebra of the Poincaré group. As a consequence, the boosts algebraically generate $\mathcal{P}_+^\uparrow$.*

Proof. The first statement is well known, and the second follows by the connectedness of $\mathcal{P}_+^\uparrow$.

$\square$

2.5 Lemma. *Let $\mathcal{A}$ be a local pre-cosheaf on M satisfying assumption 2.1, and consider the set*

$$H = \{g \in \mathcal{P}_+^\uparrow : \mathcal{A}(g\mathcal{O}) = \mathcal{A}(\mathcal{O}) \quad \forall \mathcal{O} \in \mathcal{K}\}$$

Then, H is a normal subgroup of $\mathcal{P}_+^\uparrow$ and either
(a) $H = \{1\}$

[1] We take this opportunity to point out that the positivity of the energy is implicitly assumed in corollary 2.2 and theorem 2.3 of [4] in order to ensure the uniqueness of the conformal group representation. We thank H. Borchers for a comment in this sense. However, positivity of the energy is not used in the derivation of the essential duality in [4, theorem 2.3(i)]

or
(b) *for each* $\mathcal{O} \in \mathcal{K}$, $\mathcal{A}(\mathcal{O})$ *is equal to* $\mathcal{A}_0$ *and is a maximal abelian subalgebra of* $B(\mathcal{H})$.

Proof. H is clearly a group, and we prove that it is indeed normal in $\mathcal{P}_+^\uparrow$. In fact, by modular covariance, $\forall g \in H$ we get

$$\mathcal{A}(\Lambda_W(t) g \Lambda_W(-t)\mathcal{O}) = Ad\Delta_W^{it} \cdot \mathcal{A}(g\Lambda_W(-t)\mathcal{O}) = Ad\Delta_W^{it} \cdot \mathcal{A}(\Lambda_W(-t)\mathcal{O}) = \mathcal{A}(\mathcal{O})$$

therefore, since the boosts generate $\mathcal{P}_+^\uparrow$ (proposition 2.4), H is normal.
Now we prove that each non trivial normal subgroup K of $\mathcal{P}_+^\uparrow$ contains the translation subgroup $T = \{T(a) : a \in \mathbf{R}^n\}$. Let $1 \neq g \in K$. If g is not a translation then, by normality,

$$K \ni gT(a)g^{-1}T(-a) = T(g(a))T(-a) \in T$$

i.e. $K \cap T$ contains at least a non trivial element g. Again by normality, $K \cap T$ is globally invariant under the action of the Lorentz group. Since the (minimal) Lorentz-invariant subsets of $\mathbf{R}^n$ are the future light cone, its surface, the past light cone, its surface, and the complement of the preceding ones, and since any of these generates $\mathbf{R}^n$ as a group, then $K \supset T$. In particular, we proved that if H is non trivial it contains T. In this case, for each $\mathcal{O} \in \mathcal{K}$ we can find a translation $T(a)$ such that $T(a)\mathcal{O}$ is spatially separated by $\mathcal{O}$, and therefore, by locality, $\mathcal{A}(\mathcal{O})$ is abelian. Then it is easy to see that all local algebras coincide, and they are maximal abelian sub-algebras [18] because the vacuum is cyclic.

□

Remark. If alternative (b) of Lemma 2.5 holds, then Theorem 2.3 is true by taking the trivial representation of the universal covering of the Poincaré group. In the rest of the paper we shall discuss alternative (a), i.e. we shall suppose $H = \{1\}$.

2.6 Lemma. *Let* G *be the subgroup of* $\mathcal{U}(\mathcal{H})$ *which is algebraically generated by the modular groups of the algebras* $\mathcal{A}(W)$, $W \in \mathcal{W}$. *Then* G *is an algebraic central extension of the Poincaré group* $\mathcal{P}_+^\uparrow$.

Proof. First we prove that the map π which satisfies

$$\pi(\Delta_W^{it}) = \Lambda_W(t) \qquad W \in \mathcal{W}, \quad t \in \mathbf{R} \tag{2.1}$$

extends to a well defined surjective group homomorphism $\pi : G \to \mathcal{P}_+^\uparrow$.
Indeed let $\Delta_1^{it_1} \cdot \ldots \cdot \Delta_n^{it_n} = 1$ be a non trivial identity in G, where Δ_i is the modular operator of $\mathcal{A}(W_i)$ and $\{W_i, i = 1, \ldots, n\}$ is any collection of n wedges in $\mathcal{W}$. Then

$$\mathcal{A}(\mathcal{O}) = \Delta_1^{it_1} \cdot \ldots \cdot \Delta_n^{it_n} \mathcal{A}(\mathcal{O}) \Delta_n^{-it_n} \cdot \ldots \cdot \Delta_1^{-it_1} = \mathcal{A}(\pi(\Delta_1^{it_1} \cdot \ldots \cdot \Delta_n^{it_n})\mathcal{O}), \quad \forall \mathcal{O} \in \mathcal{K}.$$

From the Lemma 2.5 we get $\pi(\Delta_1^{it_1} \cdot \ldots \cdot \Delta_n^{it_n}) = 1$. As a consequence π is a well-defined homomorphism. Since the boosts algebraically generate $\mathcal{P}_+^\uparrow$ (proposition 2.4), π is surjective.

Now we observe that if $U \in \ker(\pi)$, then $\mathrm{ad}U(\mathcal{A}(W)) = \mathcal{A}(\pi(U)W) = \mathcal{A}(W)$ for each wedge W, therefore U commutes with the modular group of any wedge algebra [18], and hence with any element in G. As a consequence, the exact sequence $1 \longrightarrow \ker(\pi) \longrightarrow G \longrightarrow \mathcal{P}_+^\uparrow \longrightarrow 1$ gives the announced central extension.

□

Proof of Theorem 2.3. Equation (2.1) and proposition 2.4 implies that the extension π described in lemma 2.6 is a weak Lie extension. Since $\dim(M) > 2$, the Poincaré group is perfect and its Lie algebra has trivial second cohomology. Therefore, by lemma 2.6 and corollary 1.8, we get a strongly continuous unitary representation U of $\widetilde{\mathcal{P}}$. Setting $U_W(t) := U(\tilde{\Lambda}_W(t))$, $W \in \mathcal{W}$, $t \in \mathbf{R}$, the unitary operator

$$z(t) = \Delta_W^{it} U_W(-t)$$

implements internal symmetries by modular covariance, therefore is in the center of G. Let us denote by θ_g, $g \in \mathcal{P}_+^\uparrow$, the action of $\mathcal{P}_+^\uparrow$ on the central extension G. It follows that $\theta_g(\Delta_W^{it}) = \Delta_{gW}^{it}$, $\theta_g(U_W(t)) = U_{gW}(t)$. Since θ_g acts trivially on the center of G and $\mathcal{P}_+^\uparrow$ acts transitively on the family $\mathcal{W}$, then $z(t)$ does not depend on $W \in \mathcal{W}$. Moreover

$$z(s)z(t) = \Delta_W^{it} z(s) U_W(-t) = z(s+t) \qquad s, t \in \mathbf{R}$$

i.e. $z(t)$ is a one-parameter group. Finally, since $\Delta_{W'}^{it} = \Delta_W^{-it}$ by essential duality (theorem 2.2), and $U_{W'}(t) = U_W(-t)$ because $\tilde{\Lambda}_W(t)$ and $\tilde{\Lambda}_{W'}(-t)$ are the unique lifting of $\Lambda_W(t) = \Lambda_{W'}(-t)$, we have $z(t) = z(-t)$, i.e. $z(t) \equiv I$.

In order to check the positivity of the energy-momentum, we observe that the generator of any time-like translation is a convex combination of generators of light-like translations. Moreover each generator of a light-like translation gives rise to a one-parameter semigroup of endomorphisms of a suitable wedge. Therefore the result follows by proposition 2.7.

□

The following proposition is a partial converse to Borchers theorem, in a slightly different form it appeared also in Wiesbrock paper [19], with the difference that we do not assume the commutation relations of the modular conjugation with the one-parameter group.

2.7 Proposition. *(Converse of Borchers theorem) Let $\mathcal{R}$ be a von Neumann algebra standard with respect to the vector Ω. If $U(a)$ is a one-parameter semigroup ($a \in \mathbf{R}_+$) of endomorphysms of $\mathcal{R}$ such that*

(i) $U(a)\Omega = \Omega$
(ii) $ad\Delta^{it}_{\mathcal{R}} U(a) = U(e^{-2\pi t}a)$
then $U(a)$ *has positive generator.*

We give here an easy proof of the preceding Theorem.

Proof. From Lemma II.3 in [2] we know that the operator valued function $t \to U(e^{-2\pi t}a)$ can be analitically extended on the strip $\{z \in \mathbf{C}; -1/2 < Imz < 0\}$, where it satisfies the bound $\|U(e^{-2\pi z}a)\| \leq 1$. Then, taking $z = -i/4$, we get

$$\|U(ia)\| = \|e^{-aP}\| \leq 1$$

hence P, being the generator of U, has non-negative spectrum.

$\square$

2.8 Corollary. *If the distal split property (cf. [6]) holds, the representation described in Theorem* 2.3 *is the only* $\mathcal{A}$*-covariant unitary representation of the universal covering group* $\widetilde{\mathcal{P}}$ *of* $\mathcal{P}_+^\uparrow$ *on* $\mathcal{H}$*.*

Proof. The proof is identical to that of Theorem 3.1 in [4].

$\square$

We conclude this section observing that our techniques work also for conformally covariant precosheaves (see e.g. [4] for definitions and properties). Moreover, in this case, the (conformal) modular covariance property for a local precosheaf on (some covering of) the Dirac-Weyl compactification of the Minkowski space is a necessary and sufficient condition for the existence of a covariant unitary representation of the conformal group.

For simplicity, we illustrate this for conformal theories on S^1. Such theories are described by a local precosheaf of von Neumann algebras on the family $\mathcal{K}$ of all proper open intervals in S^1. Here the causal complement I' denotes the interior of the complement of I. As before, the vacuum vector is supposed to be cyclic for the local algebras associated with subsets in $\mathcal{K}$.

With each $I \in \mathcal{K}$ one associates the one-parameter subgroup of the Möbius group $\Lambda_I(t)$ which preserves I [4]. The following theorem holds:

2.9 Theorem. *Let* $I \to \mathcal{A}(I)$ *be a local precosheaf of von Neumann algebras on* S^1 *and* Ω *a cyclic vector for each* $\mathcal{A}(I)$*. Then* $\mathcal{A}$ *is conformally covariant if and only if the modular groups associated to* $(\mathcal{A}(I), \Omega)$, $I \in \mathcal{K}$, *verify (conformal modular covariance)*

$$\Delta_I^{it}\mathcal{A}(L)\Delta_I^{-it} = \mathcal{A}(\Lambda_I(t)L), I, L \in \mathcal{K}.$$

In this case the unitary representation $g \to U(g)$ of the Möbius group $SL(2,\mathbf{R})/\mathbf{Z}_2$ is generated by the modular groups of the algebras associated with open intervals and $U(\Lambda_I(t)) = \Delta_I^{it}$, $I \in \mathcal{K}$.

Proof. With the same arguments used in the Poincaré covariant case, the unitary group generated by the modular automorphisms associated with double cones generates a weak Lie central extension of the Möbius group which, because of theorem 1.7, splits on the universal covering. Again, we get a unitary representation of the covering group of the Möbius group where the action of one-parameter group which preserves an open interval is implemented by the corresponding modular group. Then, applying the same argument as in [4,8], the modular conjugations associated with open intervals implement the reflection with respect to the end points of the interval. An argument of Wiesbrock [20] finally implies that the constructed representation is indeed a representation of the Möbius group.

□

3. Final comments.

In a forthcoming paper [7] our analysis will be completed in two directions. First, a discussion on the relationship between the modular covariance (assumption 2.1) and the geometrical behaviour of the modular conjugations associated with wedge regions will be discussed, with applications to the construction of a PCT symmetry.

Secondly, notice that we have obtained a representation of the universal covering group $\widetilde{\mathcal{P}}$ of the Poincaré group $\mathcal{P}_+^\uparrow$, therefore the center $\mathbf{Z}_2$ of $\widetilde{\mathcal{P}}$ might act as a non–trivial gauge symmetry. Our net being local, this possibility is incompatible with the spin-statistics correspondence [10] and will be analyzed in terms of an algebraic spin-statistics theorem. Moreover, we expect the latter to follow from the split property [6], that selects physically relevant nets and implies the uniqueness of the Poincaré action.

Note however that the product of the modular conjugations associated with an inclusion of wedge regions is necessarily a translation [2], and our analysis should be compared with the characterization of translation invariant theories given in [5].

Finally we mention that essential duality may fail in general, as shown recently in [21].

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