COHOMOLOGY EQUATIONS NEAR HYPERBOLIC POINTS AND GEOMETRIC VERSIONS OF STERNBERG LINEARIZATION THEOREM.

A. Banyaga

Math Dept.

Penn State Univ.

University Park, PA 16802

R. de la Llave

Math Dept.

Univ. of Texas

Austin, TX 78712

C.E. Wayne

Math Dept.

Penn State Univ.

University Park, PA 16802

Abstract.

We prove that if two germs of diffeomorphisms preserving a volume, symplectic or

contact structure are tangent to a high enough order and the linearization is hyperbolic, it is possible to find a smooth change of variables preserving the same structure that sends one into the other. This result is a geometric version of Sternberg's linearization theorem which we recover as a particular case.

An analogous result is also proved for flows.

1. Introduction and statement of results.

The celebrated Sternberg linearization theorem states that, given a local diffeomorphism with a fixed point, if the eigenvalues of its linearization satisfy certain non-resonance conditions, it is possible to find a differentiable change of variables making it linear. That is, given f, $f(0) = 0$, it is possible to find h in such a way that $h^{-1} \circ f \circ h(x) = Df(0)x$.

In many applications, f preserves a geometric structure — symplectic, volume or contact — and it is natural to require that h also does.

In the case that f preserves a geometric structure, it necessarily violates several of the non-resonance conditions and, there are easy examples where it is impossible to reduce to the linear part. Nevertheless, very frequently, one can hope to reduce to a much simpler form — usually called a "normal form."

Typically, it is not difficult to find polynomial germs h_k of degree k in such a way that

$$h_k^{-1} \circ f \circ h_k(x) = N_k(x) + o(\|x\|^k)$$

with N_k a much simpler diffeomorphism. (These eliminations usually entail only power matchings.)

The main goal of this paper is to prove a theorem stating that if the formal eliminations can be carried to a large enough order, and that $Df(0)$ is hyperbolic, there is a diffeomorphism h, which reduces f to exactly the normal form N_n. In the case that f preserves a symplectic or volume form or a contact structure so does h.

We also discuss an analogue for flows.

Such theorems were sketched in [St3]. A proof by other methods appeared in [Ch].

The method of proof that we use is based on the deformation method of singularity theory [Ma]. This method is ideally suited to discussing conjugacy problems in which a geometric structure is preserved. In principle, the preservation of a geometric structure is a non-linear non-local problem, but with the use of deformations, the preservation of the geometric structure is implemented by considering equations in an appropriate linear space.

In a first section, we describe the basic formalism of deformations for general diffeomorphisms as well as for diffeomorphisms preserving symplectic or volume forms or contact structures. The basic idea of the deformation method is to embed the problem into a family of problems, that includes also a trivially solvable one. Then, we study the derivatives of the quantities involved. The advantage is that the equations involved are always linear cohomology equations. (If we think of derivatives as infinitesimal quantities, it is clear that the only equations we can form among infinitesimal quantities are linear.) The method is also well suited for geometric problems since the non-linear and non-local constraints which are imposed by the preservation of the geometric structure also become linear constraints, which can be implemented by considering the cohomology equations among linear spaces of local objects.

In a second section, we provide estimates for the cohomology equations of the previous sections and, establish the main theorems of this paper.

We point out that from the point of view of group theory, the problem just discussed is the problem of classifying the conjugacy classes of the group of germs of diffeomorphisms preserving a geometric structure, or classifying the orbits under the natural action.

The reduction of the problem to the study of cohomology equations in the Lie algebra is quite standard in finite dimensional Lie groups, so in a certain sense, the method considered here is the extension to an infinite dimensional situation of methods that had been successful in the finite dimensional situation. This point of view is emphasized in [St2].

For problems of conjugacy, the deformation method was introduced in the context of singularity theory. Some early refences are [Ma] and the notes of the lectures of R. Thom by H. Levine [Le].

For the symplectic case, a discussion of the formal normal forms using this method can be found in [Mo]. For problems like ours, the main technical tools of the method are C^k estimates for solutions of cohomology equations. In a global context, these estimates were introduced in [LMM] and, in a context very similar to ours in [BLW]. The paper [Il] uses the deformation method to prove the Sternberg theorem for general diffeomorphisms without regard to geometric structures.

We point out that the deformation method can also be used to study the formal eliminations and the classification of normal forms. For example, the papers [Ta] and [Ro] used the deformation methods to classify germs of vector fields, forms, and diffeomorphisms to a finite order and the paper [Mo] discusses normal forms of symplectic diffeomorphisms using the deformation method. The advantages of the method appear not only in theoretical treatments, but it also allows effective numerical implementations [DL].

In relation with the problem of convergence of symplectic normal forms of hyperbolic maps we also point out that for analytic, two dimensional, symplectic mappings a proof of the existence of analytic changes of variables reducing to a normal form has also been presented in [Mo3]. The method of proof is, nevertheless quite different and many of the techniques we use such as cut-off functions etc. clearly do not apply for analytic regularities.

Remark. We will consider the volume preserving problem only on manifolds of dimension 3 or bigger. The case when the dimension is 1 is completely trivial and, for the case in which the dimension is 2 volume preserving is the same as preserving a symplectic structure. The results for symplectic structures are sharper than those we obtain for general volume preserving case particularized for dimension 2.

More precisely, we will prove:

Theorem 1.1. *Let f, N be C^r diffeomorphisms of $\mathbb{R}^n$, $f(0) = N(0) = 0$, and let A and B be as defined below.*

Assume

i) $D^i f(0) = D^i N(0) \quad i = 0, \ldots, k < r - 1,$

ii) $\operatorname{Spec} Df(0) \subset \{z \in \mathbb{C} \mid \lambda_-^{-1} \le |z| \le \lambda_+\} \cup \{z \in \mathbb{C} \mid \mu_-^{-1} \le |z| \le \mu_+\}$

for some $0 < \lambda_-^{-1} < \lambda_+ < 1 < \mu_-^{-1} < \mu_+$.

Then, provided that $1 \le \ell < kA - B$, for some integer ℓ, we can find a C^ℓ diffeomorphism h such that $h^{-1} \circ f \circ h = N$ on a neighborhood of the origin, $h(0) = 0$, $Dh(0) = Id$.

If both f, N preserve symplectic, contact or volume structures h can be chosen to preserve the same structure.

Remark. In this paper we will show in detail that, in all the cases we can take

$$A = \frac{|\ln \lambda_+|}{\ln \mu_+} \, \frac{|\ln \mu_-|}{(\ln \lambda_- + |\ln \mu_-|)}.$$

Morever, we can take :

$$B = B_{\text{no}-\text{structure}} \equiv \frac{(\ln \mu_+)^2 + |\ln \mu_-|(\ln \lambda_- - |\ln \lambda_+|)}{(\ln \mu_+)(\ln \lambda_- + |\ln \mu_-|)}$$

in the case that there is no geometric structure to be preserved.

$$B = B_{\text{symplectic}} \equiv 1 - 2\frac{|\ln \lambda_+|}{(\ln \mu_+)} \frac{|\ln \mu_-|}{(\ln \lambda_- + |\ln \mu_-|)}$$

when we require preservation of a symplectic structure ;

$$B = B_{\text{volume}} \equiv 1 + (n-2)\frac{\ln \lambda_- - |\ln \mu_-|}{(\ln \mu_+)(\ln \lambda_- + |\ln \mu_-|)} + (n-2)\frac{\ln \mu_+}{(\ln \lambda_- + |\ln \mu_-|)}$$

for volume preserving case and

$$B = B_{\text{contact}} \equiv 1 + B_{\text{no}-\text{structure}}$$

in the contact case.

Analogously, we have for vector fields:

Theorem 1.2. *Let X, Y be C^r vector fields of $\mathbb{R}^n$, $X(0) = Y(0) = 0$.*

Assume

i) $D^i X(0) = D^i Y(0) \quad i = 0, \dots, k < r - 1$

ii) $\operatorname{Spec} DX(0) \subset \{z \in \mathbb{C} \mid -\lambda_- \le Re(z) \le \lambda_+\} \cup \{z \in \mathbb{C} \mid -\mu_- \le Re(z) \le \mu_+\}$

for some $-\lambda_- < \lambda_+ < 0 < -\mu_- < \mu_+$

Then, provided that $\ell = kA - B \geq 1$ we can find a C^ℓ diffeomorphism h such that $h_* X = Y$ on a neighborhood of the origin, $h(0) = 0$, $Dh(0) = Id$.

In case that both X, Y preserve a volume, symplectic or contact structure h can be chosen to preserve the same structure.

Remark. *In all the above cases, we can take the values of A and B to be equal to their values in the diffeomorphism case, except that we omit all logarithms. For example,*

$$A = \frac{\lambda_+}{\mu_+} \frac{|\mu_-|}{(\lambda_- + |\mu_-|)} \quad ,$$

$$B = B_{\text{no}-\text{structure}} \equiv \frac{(\mu_+)^2 + |\mu_-|(\lambda_- - |\lambda_+|)}{(\mu_+)(\lambda_- + |\mu_-|)} \quad ,$$

and so on.

Remark. The values of A and, especially, of B that we have obtained are not optimal even using the method of proof in this paper. At the end of the proof we indicate in remarks how the results could have been improved by using more complicated spaces to measure the regularity. For example, one can obtain better results if one keeps track separately of the derivatives with respect to parameters and with respect to the variables in the manifold. A further improvement can be obtained if one keeps track of the derivatives along stable and unstable manifolds in the way that it was done in [LMM]. We also point out that in the contact and volume cases, it is possible to obtain more precise formulas that involve the eigenvalues of the derivative at 0 and not just bounds on their size. Since these improvements require a lot of technical complication and are not central to the main goal of this paper, we have just relegated them to remarks along the side of the main proof.

Remark. Notice that the expressions for B in Theorem 1.1 and Theorem 1.2 change if we substitute for f and N, f^{-1}, N^{-1} respectively. The h obtained in one case would work in the other. Given a particular pair f, N we could choose whichever of the expressions gives the more favorable values of A and B.

Remark. The method of proof we present carries over to infinite dimensions. The only properties of $\mathbb{R}^n$ which are used is that it is a Banach space and that it has the "approximation property" — that is, that there exist C^∞ functions with bounded support and taking the value 1 in an open ball.

Even if the generalization of Theorem 1.2 and Theorem 1.1 to Banach spaces satisfying the approximation property is quite straightforward, and natural when there are no geometric structures present, we point out that generalization of geometric structures to infinite dimensions is somewhat delicate. There is no natural definition of volume forms and, even if it is possible to define symplectic and contact forms by extending straightforwardly the finite dimensional definitions, these extensions are not natural in many applications (see e.g., [CM], [M] for a discussion of natural extensions of symplectic forms to infinite dimensions).

2. The deformation method

The basic idea of the deformation method is to consider a family of problems that interpolates between a trivial one and the problem we want to solve.

Assuming that the problem is solved for all values of the deformation parameter, we can derive equations between the infinitesimal deformations of the problem and the solution.

From a heuristic point of view, the deformation method can be considered as an "infinitesimal" version of induction. Assuming that the problem is solved for all values of the parameter between $[0, \epsilon]$ we study what it would take to solve it for $[0, \epsilon +$ infinitesimal$]$. (This amounts to first order perturbation theory.)

The advantage of this procedure is that the equations between the infinitesimal deformations are linear. More importantly from our point of view, they are geometrically natural and it is possible to perform geometrically natural constructions on them that force the preservation of geometric structures. The preservation of geometric structures is, in principle, a non-linear and non-local problem. Nevertheless, by considering the infinitesimal equation, we can formulate it as a problem about quantities belonging to appropriate linear spaces.

We will be concerned with the so-called *hyperbolic* diffeomorphisms and flows. The following definition is quite standard.

Definition 2.1. *We say that a diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$ is hyperbolic at 0 if the spectrum of $Df(0)$ does not intersect the unit circle. We say that a vector field $L(x)$ in $\mathbb{R}^n$ with $L(0) = 0$, is hyperbolic if the spectrum of $DL(0)$ does not intersect the imaginary axis.*

As is well known, (see *e.g.* [Ni]) f is a hyperbolic diffeomorphism if and only if one can write $\mathbb{R}^n = E^s \oplus E^u$, and, denoting $A = Df(0)$, we have $AE^s = E^s$, $AE^u = E^u$ and, for some norm $\|A|_{E^s}\| < 1$; $\|(A|_{E^u})^{-1}\| < 1$.

Similarly, a flow is hyperbolic if, denoting $A = DL(0)$ we have $\mathbb{R}^n = E^s \oplus E^u$, $AE^s = E^s$, $AE^u = E^u$ and $\|\exp(A|_{E^s})\| < 1$, $\|\exp(-A|_{E^u})\| < 1$, for some norm.

This characterization makes it clear that a vector field is hyperbolic at zero if and only if its time t map defines a hyperbolic map.

In this paper, we will consider mainly two problems:

Problem 2.2. *Given f, N, germs of C^r diffeomorphisms in $(\mathbb{R}^n, 0)$, f – and therefore N- hyperbolic diffeomorphisms at 0, $\|f(x) - N(x)\| = o(\|x\|^s)$, $s \geq 1$, determine g, the germ of a diffeomorphism in $(\mathbb{R}^n, 0)$ with $Dg(0) = Id$, in such a way that $g^{-1} \circ f \circ g = N$.*

Problem 2.3. *Given $\mathcal{X}, \mathcal{Y}$, germs of C^r vector fields at $0 \in \mathbb{R}^n$, $\mathcal{X}(0) = \mathcal{Y}(0) = 0$, and $\|\mathcal{X}(x) - \mathcal{Y}(x)\| \leq o(\|x\|^s)$, with $\mathcal{X}$ – and hence $\mathcal{Y}$ – hyperbolic flows at 0, determine g a germ of a diffeomorphism $Dg(0) = Id$ such that*

$$g_* \mathcal{X} = \mathcal{Y} \ .$$

If we have a differentiable family f_ϵ with $f_0 = N$, $f_1 = f$, it is natural to try to find a family g_ϵ with $g_0 = Id$ satisfying

$$g_\epsilon^{-1} \circ f_\epsilon \circ g_\epsilon = f_0 \ . \tag{2.1}$$

Analogously, for a family of vector fields, $\mathcal{X}_0 = \mathcal{X}$, $\mathcal{X}_1 = \mathcal{Y}$ (for example, $\mathcal{X}_\epsilon = \epsilon\mathcal{Y} + (1-\epsilon)\mathcal{X}$) it is natural to try to find g_ϵ with $g_0 = Id$ in such a way that

$$g_{\epsilon *} \mathcal{X}_\epsilon = \mathcal{X}_0 \ . \tag{2.2}$$

Since f_ϵ is a smooth family of diffeomorphisms, we can find a family of vector fields $\mathcal{F}_\epsilon$ defined by

$$\frac{d}{d\epsilon} f_\epsilon = \mathcal{F}_\epsilon \circ f_\epsilon \ . \tag{2.3}$$

By the uniqueness theorem of differential equations, if $f_0, \mathcal{F}_\epsilon$ are C^1, there is one and only one family f_ϵ with the given f_0 and $\mathcal{F}_\epsilon$. We also note that, if $D^i f_\epsilon(0) = D^i f_0(0),\ i = 1, \cdots, k$, then, $D^i \mathcal{F}_\epsilon(0) = 0\ ,\ i = 1, \cdots, k$.

If we assume that g_ϵ can also be written in the form (2.3), taking derivatives with respect to ϵ in (2.1), we obtain

$$\frac{d}{d\epsilon}(g_\epsilon^{-1} \circ f_\epsilon \circ g_\epsilon) = (g_\epsilon^{-1})_*(\mathcal{F}_\epsilon - \mathcal{G}_\epsilon + f_{\epsilon *}\mathcal{G}_\epsilon) \circ (g_\epsilon^{-1} \circ f_\epsilon^{-1} \circ g_\epsilon)\ .$$

So that, if, given $\mathcal{F}_\epsilon$, we can find a C^1 family of vector fields $\mathcal{G}_\epsilon$ satisfying

$$\mathcal{F}_\epsilon - \mathcal{G}_\epsilon + (f_\epsilon)_* \mathcal{G}_\epsilon = 0\ , \tag{2.4}$$

the family of diffeomorphisms g_ϵ determined by $g_0 = Id;\ \ \frac{d}{d\epsilon} g_\epsilon = \mathcal{G}_\epsilon \circ g_\epsilon$ — whose existence is guaranteed by the fundamental existence theorem of O.D.E.'s — will satisfy:

$$\frac{d}{d\epsilon} g_\epsilon^{-1} \circ f_\epsilon \circ g_\epsilon = 0 \quad ; \quad g_0^{-1} \circ f_0 \circ g_0 = f_0 \quad ,$$

hence, $g_\epsilon^{-1} \circ f_\epsilon \circ g_\epsilon = f_0$.

Analogously, denoting $\dot{\mathcal{X}}_\epsilon = \frac{d}{d\epsilon}\mathcal{X}_\epsilon$, we have:

$$\frac{d}{d\epsilon}(g_\epsilon)_* \mathcal{X}_\epsilon = (g_\epsilon)_*(L_{\mathcal{G}_\epsilon}\mathcal{X}_\epsilon + \dot{\mathcal{X}}_\epsilon).$$

Where, we recall, we have $L_{\mathcal{G}_\epsilon}\mathcal{X}_\epsilon = [\mathcal{G}_\epsilon, \mathcal{X}_\epsilon] = -L_{\mathcal{X}_\epsilon}\mathcal{G}_\epsilon$. Therefore, finding C^1 solutions of

$$-L_{\mathcal{X}_\epsilon}\mathcal{G}_\epsilon + \dot{\mathcal{X}}_\epsilon = 0, \tag{2.5}$$

solves the original problem.

Notice that both (2.4) and (2.5) are linear in the unknowns — the original problems (2.1) and (2.2) are not — and are geometrically natural. The fact that the equations are linear, is to be expected since the vector fields can be considered as infinitesimal diffeomorphisms and the equations between infinitesimal quantities are typically linear.

We emphasize that both (2.4) and (2.5) were shown to be equivalent to the original problems independently of the fact that there were geometric structures present. Our next task is to introduce geometric structures and to show that the deformation method provides a natural calculus to study them.

Notice that all the calculations we have done so far only require that one can take derivatives and that the solutions of O.D.E's are unique. If we consider higher regularities we find it convenient to define precisely.

Definition 2.4. *We say that f_ϵ is a C^r family if $f_\epsilon(x)$ is C^r jointly in ϵ and x and, moreover*

$$\sup_{x,\epsilon} \sup_{\ell+|\alpha|\leq r} \left| \left(\frac{\partial}{\partial\epsilon}\right)^\ell \frac{\partial^\alpha}{\partial x^\alpha} f_\epsilon(x) \right| \equiv \|f_\epsilon\|_{C^r} < \infty \ .$$

Remark. Notice that, with this definition when the domain is $\mathbb{R}^n$, we can have infinitely differentiable functions such as x^2 which are not C^1 in the sense above since we require uniform control of the derivatives over all of the space. Also, it could have been natural to require a different number of derivatives with respect to the parameters and with respect to the variables.

Remark. We recall that it is a result in harmonic analysis (see [Kr]) that if $r+1$ derivatives with respect to the parameter exist and are continuous and $r+1$ derivatives with respect to the variables exist and are continuous, then the function is jointly C^r. Unfortunately, (see again [Kr]) it is not true that the function is jointly C^{r+1}. These complications in the regularity are due to the fact that the C^r spaces $r \in \mathbb{N}$ are not well behaved under many of the operations that appear naturally in harmonic analysis. Nevertheless, they are quite well behaved under composition and, for the purposes of this paper, this is quite important.

Remark. We also recall that the theorem of existence and regularity on initial conditions for O.D.E's shows that if $\mathcal{F}_\epsilon(x)$ is C^r in ϵ and C^ℓ in x, then the solutions f_ϵ of $\frac{d}{d\epsilon} f_\epsilon(x) = \mathcal{F}_\epsilon(f_\epsilon(x))$; $\quad f_0(x) = x$ exist and are unique when $\ell \geq 1$, $r \geq 0$. Moreover they are C^{r+1} in ϵ, C^ℓ in x. (See [Ha], Chapter 5.)

3. Volume and symplectic geometry

A k-form γ on an n-dimensional manifold M is said to be *nondegenerate* if the mapping $X \to i(X)\gamma$ sending a vector field X to the $(k-1)$-form $i(X)\gamma$ is an isomorphism: this means that for all $x \in M$, $X_{(x)} \to i(X_{(x)})[\gamma_{(x)}]$ is an isomorphism of the tangent space T_xM with the vector space $\Lambda^{k-1}(T_x^*M)$ of alternating $(k-1)$-linear forms on T_xM. This can happen only if $n = \dim(T_xM) = \dim(\Lambda^{k-1}(T_x^*M)) = \binom{n}{k-1}$, i.e., if $k = 2$ or n.

Let γ be a nondegenerate k-form; then any $(k-1)$-form ω defines a unique vector field X such that $i(X)\gamma = \omega$.

A $(k-2)$-form F is called a *hamiltonian* of the unique vector field X_F such that

$$i(X_F)\gamma = dF \ .$$

Clearly, $X_F = X_{F'}$ if and only if $F - F'$ is a closed form. The vector field X_F is called a *globally hamiltonian vector field with hamiltonian* F.

The vector field X^ω defined by a closed $(k-1)$-form ω according to $i(X^\omega)\gamma = \omega$ is said to be a *locally hamiltonian* vector field. Indeed if U is any contractible open subset of M, the Poincaré lemma says that $\omega|_U = dF$, for some $(k-2)$-form, F, on U. Hence on U, $X^\omega = X_F$.

Recall that on an n-dimensional manifold only 2-forms or n-forms may be nondegenerate. A nondegenerate n-form is called a volume form. This form is automatically closed. For 2-forms, if some critical calculations are to go through, we must require this as an independent hypothesis and get the notion of symplectic form: i.e., a nondegenerate closed 2-form. From now on, γ stands for a nondegenerate closed k-form (which is a symplectic form, or a volume form); to emphasize the distinction, we reserve the letter ω for symplectic forms and μ for volume forms.

Note that even if a hamiltonian determines a unique vector field, the hamiltonian of a vector field is only determined up to forms with zero exterior derivative. When γ is a 2-form, the hamiltonians are 0-forms, that is functions and the the only functions with zero differential are constants. Nevertheless, if γ is a k form with $k \geq 3$, there are much more complicated $(k-2)$-forms with zero exterior derivative.

Let γ be a C^∞ nondegenerate k-form and consider the equation:

$$i(X)\gamma = dF$$

If X is C^r, so is $i(X)\gamma = dF$, hence, using the same construction of integrating along lines, used in the standard proof of Poincaré lemma (see [Sp] among others), we obtain that the hamiltonian F is C^r and the C^r-norm of F can be estimated by a constant times the C^r-norm of X. In the symplectic case, it is possible to do better. If F is a function, the fact that dF is C^r allows us to conclude that F is C^{r+1} and that the C^{r+1} norm of F is bounded by the C^r norm of dF. Conversely if the hamiltonian F is C^r, the vector field is C^{r-1} and the C^{r-1} norm of X is estimated by the C^r-norm of the hamiltonian times a constant (depending on γ).

Using the standard formulas relating the interior product $i(\cdot)$, the exterior derivative d, and the Lie derivative,

$$i\big([X,Y]\big)\alpha = L_X\big(i(Y)\alpha\big) - i(Y)(L_X\alpha)$$
$$L_X\alpha = i(X)\,d\alpha + d\,i(X)\alpha$$

for all k-forms α, we get the well known:

Lemma 3.1. *Let γ be a nondegenerate closed k-form.*

(i) If X, Y are locally hamiltonian vector fields, then $[X,Y]$ is a globally hamiltonian vector field with hamiltonian $i(X)(i(Y)\gamma)$.

(ii) If X has G as hamiltonian and Y is locally hamiltonian, then $L_Y G$ is a hamiltonian for $[Y,X]$.

(iii) If Y has F as a hamiltonian, then $-L_X F$ is a hamiltonian of $[Y,X]$.

Remark. In the Hamiltonian case, $i(X)(i(Y)\omega) = \omega(Y,X)$

Proof. Observe first that a locally hamiltonian vector field X satisfies:

$$L_X\gamma = 0\ .$$

Indeed

$$L_X\gamma = d\,i(X)\gamma + i(X)\,d\gamma = 0$$

since $d\gamma = 0$ and $i(X)\gamma$ is closed.

To prove (i), compute:

$$\begin{aligned} i\big([X,Y]\big)\gamma &= L_X\big(i(Y)\gamma\big) - i(Y)(L_X\gamma) \\ &= L_X i(Y)\gamma \\ &= di\big(i_X i_Y \gamma\big) + i_X d\big(i(Y)\gamma\big) \\ &= d\big[i(X)i(Y)\gamma\big] \end{aligned} .$$

This proves (i).

For (ii) we know that $i(X)\gamma = dG$, so

$$\begin{aligned} d(L_Y G) &= L_Y(dG) + L_Y\big[i(X)\gamma\big] \\ &= i\big([Y,X]\big)\gamma + i\big(X\big)(L_Y\gamma) \\ &= i\big([Y,X]\big)\gamma \end{aligned} .$$

For (iii), $i(Y)\gamma = dF$, so:

$$d(L_X F) = L_X\, dF = L_X i\big(Y\big)\gamma = i\big([X,Y]\big)\gamma + i\big(Y\big)\big(L_X\gamma\big) = i\big([X,Y]\big)\gamma \ .$$

A differentiable family f_ϵ of diffeomorphisms preserves γ if and only if the associated family $\mathcal{F}_\epsilon$ of vector fields is locally hamiltonian: indeed, if $f_\epsilon^*\gamma = \gamma$, taking derivatives with respect to ϵ, we get:

$$0 = \frac{d}{d\epsilon}(f_\epsilon^*\gamma) = f_\epsilon^*(L_{\mathcal{F}_\epsilon}\gamma)$$

and since $d\gamma = 0$, we conclude:

$$L_{\mathcal{F}_\epsilon}\gamma = d\,i(\mathcal{F}_\epsilon)\gamma = 0 \ .$$

In this paper we are interested in what happens near a critical point. Hence, in a contractible neighborhood of the point, $\mathcal{F}_\epsilon$ is globally hamiltonian and we denote by F_ϵ its hamiltonian:

$$i(\mathcal{F}_\epsilon)\gamma = dF_\epsilon$$

The hamiltonian F_ϵ determines $\mathcal{F}_\epsilon$ which in turn determines f_ϵ given f_0.

If f_ϵ is a family of γ-preserving local diffeomorphisms with $f_0 = N$, $f_1 = f$ it is natural to try to find γ-preserving g_ϵ such that $g_\epsilon^{-1} \circ f_\epsilon \circ g_\epsilon = f_0$, by first determining its hamiltonian G_ϵ.

We recall the following property of the interior product whose proof is a straightforward computation:

Proposition 3.2. *If f is a diffeomorphism, γ a p-form, and X a vector field then,*

$$i(f_* X)\gamma = (f^{-1})^* \left[i(X)(f^*\gamma) \right] \ .$$

If we apply Proposition 3.2 to simplify the result of taking the interior product of all the terms in (2.4) we obtain:

$$F_\epsilon - G_\epsilon + (f_\epsilon^{-1})_* G_\epsilon = 0 \ . \tag{3.1}$$

This equation is equivalent to (2.4) and hence to the solution of the main problem. In the symplectic case, G_ϵ is a function and (3.1) can be written as:

$$F_\epsilon - G_\epsilon + G_\epsilon \circ f_\epsilon^{-1} = 0 \ . \tag{3.2}$$

An analogous calculation shows that if one is given a family $\mathcal{X}_\epsilon$ of γ-preserving vector fields with hamiltonian X_ϵ, it is equivalent to find a family g_ϵ of γ-preserving maps solving (2.2) and to solve:

$$d\left(-L_{\mathcal{X}_\epsilon} G_\epsilon + \dot{X}_\epsilon\right) = 0 \tag{3.3}$$

(Where by $\dot{X}_\epsilon$ we mean $\frac{\partial}{\partial \epsilon} X_\epsilon$.) and then determine g_ϵ by

$$\frac{dg_\epsilon}{d\epsilon} = G_\epsilon \circ g_\epsilon \ ; \qquad g_0 = \text{ identity.}$$

Clearly, a G_ϵ solving

$$-L_{\mathcal{X}_\epsilon} G_\epsilon + \dot{X}_\epsilon = 0 \tag{3.4}$$

is a solution of (3.3) and, therefore, we will concentrate our efforts in solving (3.4). This does not incurr any loss of generality for the original problem because hamiltonians are really defined up to exact forms. If $-L_{\mathcal{X}_\epsilon} G_\epsilon + \dot{X}_\epsilon = R_\epsilon$ with $dR_\epsilon = 0$, then $X_\epsilon + \int_0^\epsilon R_s ds$ generates the same vector field that X_ϵ.

The previous discusion shows that we can reduce the problem of conjugacy of families, all of whose elements preserve a geometric structure to the study of cohomology equations. To study Problem 2.2 and Problem 2.3, where we are not given families but just two diffeomorphisms or vector fields, we need only to show that given such problems, we can find families whose initial and final points are the given diffeomorphisms or vector fields and such that all the intermediate vector diffeomorphisms and vector fields in the family preserve the same geometric structure.

For vector fields we observe that the space of locally or globally hamiltonian vector fields is a subspace of the vector space of all vector fields. Given $\mathcal{X}$ and $\mathcal{Y}$ locally or globally hamiltonian vector fields, then:

$$\mathcal{X}_\epsilon = \epsilon\mathcal{Y} + (1-\epsilon)\mathcal{X}$$

is a smooth path of locally or globally hamiltonian vector fields such that $\mathcal{X}_0 = \mathcal{X}$ and $\mathcal{X}_1 = \mathcal{Y}$.

For diffeomorphisms, we need to interpolate $f, N : \mathbb{R}^n \to \mathbb{R}^n$ with the properties:

$$f(0) = N(0) = 0 \qquad (D^i f)(0) = (D^i N)(0) \ , \qquad i = 0, \dots, k < r \ .$$

In addition we may require that the interpolating maps preserve the form γ when γ is either the standard volume form $dx_1 \wedge \cdots \wedge dx_n$ or the standard symplectic form $dx_1 \wedge dx_2 + \cdots + dx_{2p-1} \wedge dx_{2p}$ $(n = 2p)$.

Simply consider $\hat{f} = f \circ N^{-1}$ and $\hat{f}_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$:

$$\hat{f}_\epsilon(x) = \begin{cases} \epsilon^{-1}\hat{f}(\epsilon x) & \text{if } \epsilon \neq 0 \ . \\ x & \text{if } \epsilon = 0 \ . \end{cases} \tag{3.5}$$

It is easy to see that $\hat{f}_\epsilon$ is a smooth family of diffeomorphisms interpolating between the identity and $\hat{f}$ [Mi] and preserving the volume or symplectic form γ. The required family f_ϵ is defined by:

$$f_\epsilon = \hat{f}_\epsilon \circ N$$

Clearly

$$f_1 = f \quad ; \quad f_0 = N; \qquad f_\epsilon(0) = N(0) = 0$$

and

$$(D^i f_\epsilon)(0) = (D^i N)(0) \ , \qquad i = 0, 1, \ldots, k < r \ .$$

Moreover if f and N were γ-preserving, so are f_ϵ for all ϵ. If f and N are C^r, $r \geq 1$, f_ϵ is a C^{r-1} family.

4. Contact geometry

A contact form on a $(2n+1)$-dimensional manifold is a 1-form θ such that $\theta \wedge (d\theta)^n$ is a volume form. The codimension 1 distribution D_θ whose sections are vector fields X with $i(X)\theta = 0$, is the contact structure defined by θ. A diffeomorphism f preserves the contact structure D_θ if and only if $f^*\theta = \lambda\theta$, for some nowhere zero function λ.

If we have a family of diffeomorphisms f_ϵ preserving the contact structure D_θ, then we have: $f_\epsilon^*\theta = \lambda_\epsilon\theta$ and taking derivatives with respect to ϵ:

$$L_{\mathcal{F}_\epsilon}\theta = (f_\epsilon^{-1})^* \left[\frac{d\lambda_\epsilon}{d\epsilon} \cdot \theta\right] \ .$$

Since $(f_\epsilon^{-1})^*\theta = (\lambda_\epsilon \circ f_\epsilon^{-1})^{-1}\theta$, we have

$$L_{\mathcal{F}_\epsilon}\theta = \mu_\epsilon\theta \tag{4.1}$$

with

$$\mu_\epsilon = \left(\frac{1}{\lambda_\epsilon} \cdot \frac{d\lambda_\epsilon}{d\epsilon}\right) \circ f_\epsilon^{-1} \ .$$

Since the factor μ_ϵ and λ_ϵ obviously depend on the family f_ϵ, when there is risk of confusion, we will denote them by $\mu_\epsilon^{f_\epsilon}$, $\lambda_\epsilon^{f_\epsilon}$ if there is any danger of confusion.

A vector field $\mathcal{F}$ satisfying (4.1) is called a contact vector field. Again, we will denote the dependence of the factor μ_ϵ on the vector field by $\mu_\epsilon^{\mathcal{F}_\epsilon}$

We recall the following fundamental result in contact geometry.

Lemma 4.1. *Let (M, θ) be a contact manifold.*

Given a function F there is one and only one contact vector field $\mathcal{F}$ satisfying

$$F = \theta(\mathcal{F}) \ .$$

Such an F is called the generating function of the vector field $\mathcal{F}$.

Proof. On the distribution D_θ, $d\theta$ is a symplectic structure. Hence it defines an isomorphism $\widetilde{d\theta}$ between sections of D_θ (called horizontal vector fields) and sections of its dual D_θ^*, which can be identified to 1-forms α vanishing on the orthogonal complement of D_θ. These forms are called semi-basic forms. The orthogonal complement of D_θ is spanned by the Reeb field Z: i.e., the unique vector field Z such that $i(Z)\theta = 1$ and $i(Z)\, d\theta = 0$.

Given F a smooth function then $\alpha_F = (i(Z)\, dF)\theta - dF$ is a semi-basic form. Hence $(\widetilde{d\theta})^{-1}[\alpha_F]$ is a well defined horizontal vector field. One verifies that

$$\mathcal{F} = FZ + (d\tilde{\theta})^{-1}\Big[\big(i(Z)\, dF\big)\theta - dF\Big]$$

is a contact vector field with $\theta(\mathcal{F}) = F$.

■

For a 1-form α and a vector field X we write $i(X)\alpha$ or $\alpha(X)$ or $\langle \alpha, X\rangle$ interchangeably. If f_ϵ and $\mathcal{G}_\epsilon$ are families of contact diffeomorphisms and vector fields respectively, we have:

$$\begin{aligned}\theta(f_{\epsilon *}\mathcal{G}_\epsilon)(x) &= \langle (f_\epsilon^*\theta)(f_\epsilon^{-1}(x), \mathcal{G}_\epsilon(f_\epsilon^{-1}(x))\rangle \\ &= \lambda_\epsilon(f_\epsilon^{-1}(x)) \,.\, (\theta(\mathcal{G}_\epsilon))(f_\epsilon^{-1}(x)) \\ &= \lambda_\epsilon(f_\epsilon^{-1}(x)) \;\; G_\epsilon(f_\epsilon^{-1}(x)) \\ &= [(\lambda_\epsilon G_\epsilon)\circ f_\epsilon^{-1}](x)\end{aligned}$$

Therefore taking generating functions (2.4) becomes:

$$F_\epsilon - G_\epsilon + (\lambda^{f_\epsilon} G_\epsilon)\circ f_\epsilon^{-1} = 0\ . \tag{4.2}$$

Analogously, for vector fields, we observe that the generating function of $L_\mathcal{F}\mathcal{G} = [\mathcal{F}, \mathcal{G}]$ is

$$\begin{aligned} i([\mathcal{F},\mathcal{G}])\theta &= L_\mathcal{F} i(\mathcal{G})\theta - i(\mathcal{G})L_\mathcal{F}\theta \\ &= L_\mathcal{F} G - i(\mathcal{G})[\mu_\mathcal{F}\theta] \qquad . \\ &= L_\mathcal{F} G - \mu^\mathcal{F} G\end{aligned}$$

Hence (2.5) becomes, for contact fields:

$$-L_{\mathcal{X}_\epsilon} G + \mu^{\mathcal{X}_\epsilon} + \dot{\mathcal{X}}_\epsilon = 0\ . \tag{4.3}$$

To connect a contact diffeomorphism $f : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ with $f(0) = 0$, and $Df(0) = Id$ to the identity, the simple formula (3.5) that worked for symplectic and volume preserving diffeomorphisms, does not work since f_ϵ is not a contact diffeomorphism.

Instead we use a chart on a neighborhood $\mathcal{U}$ of the set of contact diffeomorphisms with the set $\mathcal{J}^1_M$ of 1-jets of functions on M. This chart [Ly] is the equivalent of the Weinstein chart for symplectic diffeomorphisms and it is built exactly the same way, that we now describe.

Let (M, θ) be a contact manifold. Consider $\hat{M} = M \times M \times \mathbb{R}$ and denote by π_1, π_2, t, the projection of $\hat{M}$ onto the first, second, and third factors. On $\hat{M}$ we have the contact form $\hat{\theta} = t \cdot (\pi_1^* \theta) - \pi_2^* \theta$. Let f be a contact diffeomorphism of (M, θ), i.e., $f^* \theta = \lambda \theta$. We define the graph Γ_f of f as the mapping:

$$\Gamma_f : M \to \hat{M} : x \mapsto (x, f(x), \lambda(x)) \ .$$

Clearly, Γ_f is a Legendre embedding, i.e.,

$$\Gamma_f^* \hat{\theta} = 0$$

Let $L = \Gamma_f(M) \subseteq \hat{M}$ and $L_0 = \Gamma_{id}(M) \subseteq \hat{M}$. Both L, L_0 are Legendre submanifolds. We identify L_0 with M as the zero section of the 1-jet bundle J^1M over M. There exists a diffeomorphism σ of a neighborhood M_1 of L_0 in $\hat{M}$ with a neighborhood M_2 of $M = L_0$ in J^1M such that $\sigma|_{M_0} =$ identity and $\sigma^* \theta_M = \hat{\theta}$, where θ_M is the canonical contact form of J^1M. Then if f is C^2 close enough to the identity, $L \subseteq M_1$ and $\sigma(L)$ is a Legendre submanifold of J^1M, which is C^1-close to L_0, hence $\sigma(L) = j^1(\mu_f)$ for some function on M. The desired contact isotopy f_ϵ will have as graph:

$$L_\epsilon = \sigma^{-1}[j^1(\epsilon \mu_{id} + (1 - \epsilon)\mu_f)]$$

where

$$\sigma(L) = j^1(\mu_f) \quad \text{and} \quad \sigma(L_0) = j(\mu_{id}) \ .$$

Observe that for points $x \in M$ such that

$$f(x_0) = x_0 \quad \text{and} \quad (Df)(x_0) = \text{ Identity}, \quad (f^* \theta)(x_0) = \theta(x_0)$$

(i.e., $\lambda(x_0) = 1$), hence $\Gamma_{id}(x_0) = \Gamma_f(x_0)$. Since σ is a diffeomorphism the corresponding functions have the same jet at the point x_0. Therefore the deformation L_ϵ is fixed at x_0, i.e., $L_\epsilon(x_0) = \Gamma_{id}(x_0) = \Gamma_f(x_0)$ for all ϵ. Therefore $D^\ell f_\epsilon(x_0) = D^\ell f(x_0)$, for all $\ell = 0, \cdots, k$. If f and N are C^r, $r \geq 2$, then f_ϵ will be a C^{r-1} family.

5. Regularity results for cohomology equations

In this section we study the solution of the cohomology equations. We emphasize that the cohomology equations are linear and relate functions taking values in linear spaces. The geometric properties derived for the objects we are interested in are consequences of the choices of spaces. In particular, it is not necessary to produce solutions using only geometrically natural methods. From the point of view of analysis, it is useful to take coordinates, add elements or apply smoothing operators. Although those operations are not geometrically natural, they can be used in the present context.

We observe that the cohomology equations we have encountered ((2.4), (3.1), (3.2), (4.2)) can be written in the form

$$\varphi_\epsilon(x) - M_\epsilon(x)\varphi_\epsilon \circ f_\epsilon(x) = \eta_\epsilon(x) \tag{5.1}$$

where $\eta_\epsilon : \mathbb{R}^n \to \mathbb{R}^m$, $M : \mathbb{R}^n \to \mathcal{M}^{m\times m}$ (We denote by $\mathcal{M}^{m\times m}$ the space of $m \times m$ real valued matrices.) and $f_\epsilon : \mathbb{R}^n \to \mathbb{R}^n$ are given and we have to determine φ_ϵ.

Moreover, for the applications we are considering, we have that $f_\epsilon(0) = 0$ and that η_ϵ vanishes up to some given order.

More generally, in the following theorems, we will give conditions under which we can solve (5.1) assuming only that for some Banach space E, with the approximation property mentioned in the introduction, $\eta_\epsilon : \mathbb{R}^n \mapsto E$, $M : \mathbb{R}^n \mapsto \mathcal{L}(E, E)$, $f_\epsilon : \mathbb{R}^n \mapsto \mathbb{R}^n$. (Here $\mathcal{L}(E, E)$ is the set of bounded linear operators from E to itself.) This additional generality will be of use later in the construction.

The cohomology equations for vector fields can, likewise be written

$$L_\epsilon\varphi_\epsilon(x) + M_\epsilon(x)\varphi_\epsilon(x) = \eta_\epsilon(x) \tag{5.2}$$

where $L = \sum_i A_i(x)\frac{\partial}{\partial x_i}$ where $A(x) \in \mathcal{M}^{m\times m}$.

We will start discussing both equations in the contractive case ($\|Df\| < 1$). This will have direct applications for the general deformation method and for the contact cases. It will also be an intermediate step in the symplectic and volume preserving cases.

Theorem 5.1. *Assume that in (5.1) we have:*

i) $f_\epsilon, \eta_\epsilon, M_\epsilon$ *are* C^r *and* $f_\epsilon(0) = 0$ *for all* ϵ, $D^i f_\epsilon(0) = D^i f_0(0)$, $i \le k$,

ii) $D^i \eta_\epsilon(0) = 0$, $i \le k < r$,

iii) $\|Df_\epsilon(0)\| \le \tilde{\alpha} < 1$,

iv) $\|M_\epsilon(0)\| \le \tilde{\beta}$,

v) $\tilde{\alpha}^{k+1}\tilde{\beta} < 1$.

Then there exists one and only one continuous solution φ_ϵ *of (5.1) defined in a small enough neighborhood of zero which satisfies*

a) $\|\varphi_\epsilon(x)\| \le K||x||^{k+1}$. *Moreover*

b) φ_ϵ *is a* C^r *family.*

Remark. Since it is always possible to choose norms in $\mathbb{R}^n$, $\mathbb{R}^m$ in such a way that the norms of matrices are as close as desired to the spectral radius, conditions iii), iv) are really conditions about the spectrum of $Df_\epsilon(0)$ and $M_\epsilon(0)$.

Proof. Valuable intuition about the proof can be obtained by considering first the particular case in which $M_\epsilon(x) \equiv M$ does not depend on x or on ϵ and, likewise, $f_\epsilon(x) \equiv Ax$ is just a linear map.

In that case, we claim that

$$\varphi_\epsilon = \sum_{n=0}^{\infty} M^n \eta_\epsilon(A^n x) \tag{5.3}$$

is a solution of (5.1). Since $\|A^n x\| \le \|A\|^n \|x\|$, we have $\|\eta_\epsilon(A^n x)\| \le K \|A\|^{n(k+1)} \|x\|$ so that the series converges uniformly whenever $\|M\| \|A\|^{k+1} < 1$. The uniform convergence justifies the rearrangements needed to show it is indeed a true solution.

The norm of ℓ^{th} derivative with respect to the variables of the n term in (5.3) $\|M^n D^\ell \eta_\epsilon(A^n x) \left(A^{\otimes \ell}\right)^n \|$ can be bounded by $\|M\|^n \|A\|^{n(k+1-\ell)} K \|x\|^{k+1-\ell} \|A\|^{\ell n}$ if $\ell \le k+1$ and by $\|M\|^n K \|A\|^{\ell n}$ if $\ell \ge k+1$. Hence, the series obtained by differentiating term by term the series (5.3) converges uniformly and, hence the series defines a C^r

function. The proof in the general case will follow the same pattern, even if some of the estimates are considerably more involved.

We can choose a ball B_ρ of of radius $\rho \leq 1$ centered at $0 \in \mathbb{R}^n$ in such a way that

$$\begin{aligned} &\sup_{|x|\leq\rho} \|Df_\epsilon(x)\| \leq \alpha < 1 \ , \\ &\sup_{|x|\leq\rho} \|M_\epsilon(x)\| \leq \beta, \qquad \beta\alpha^{k+1} < 1. \end{aligned} \tag{5.4}$$

Since, by applying (5.1) N times we have

$$\varphi_\epsilon(x) = \sum_{i=0}^{N} \left[\prod_{j=0}^{i-1} M_\epsilon\big(f_\epsilon^j(x)\big) \right] \eta_\epsilon\big(f_\epsilon^i(x)\big) + \left[\prod_{j=0}^{N} M_\epsilon\big(f_\epsilon^j(x)\big) \right] \varphi_\epsilon\big(f_\epsilon^{N+1}(x)\big).$$

By (5.4) we have

$$\left\| \prod_{j=0}^{N} M_\epsilon\big(f_\epsilon^j(x)\big) \right\| \leq \beta^{N+1},$$

and if φ_ϵ is to satisfy a) $|\varphi_\epsilon(f_\epsilon^{N+1}(x))| \leq K|\alpha|^{(N+1)(k+1)}$. The only possible solution of of (5.1) satisfying a) is

$$\varphi_\epsilon(x) = \sum_{i=0}^{\infty} \left[\prod_{j=0}^{i-1} M_\epsilon\big(f_\epsilon^j(x)\big) \right] \eta_\epsilon\big(f_\epsilon^i(x)\big) \ . \tag{5.5}$$

The proof of Theorem 5.1 will be complete once we show that the sum in (5.5) is well defined and that it satisfies a) and b). In the course of this proof we will denote by K constants that are independent of i and x. The actual value of the constants may change from line to line.

We start by observing that the series (5.5) converges uniformly on B_ρ. Since $\|f^i(x)\| \leq \rho\alpha^i$, hypothesis ii) implies that we have $\|\eta_\epsilon(f_\epsilon^i(x))\| \leq K(\rho\alpha^i)^{k+1}$. On the other hand, $\| \prod_{j=0}^{i-1} M_\epsilon(f_\epsilon^j(x))\| \leq \beta^i$, so that the supremum of the general term can be estimated by $K\rho^{k+1}(\alpha^{k+1}\beta)^i$, which, by (5.4) is a geometric series of ratio smaller than 1. By the Weierstrass M-test, the series converges uniformly.

Once we have established uniform convergence, it is easy to show that it satisfies (5.1) by substituting it in the equation and rearranging the order of the terms.

Moreover,

$$\|\varphi_\epsilon(x)\| \le \sum_{i\ge 0} \beta^i K\big(\|x\|\alpha^i\big)^{k+1} \le \|x\|^{k+1}K\ .$$

To establish existence and continuity of higher derivatives, we take derivatives term by term in (5.5) and show that the resulting series converge uniformly.

We observe that by the product rule for derivatives,

(5.6)

$$\begin{aligned}
D(&\left[\prod_{j=0}^{i-1} M_\epsilon\big(f_\epsilon^j(x)\big)\right]\eta_\epsilon\big(f_\epsilon^i(x)\big)) = \\
&\sum_{j'=0}^{i-1}\left[\prod_{j=0}^{j'-1} M_\epsilon\big(f_\epsilon^j(x)\big)\right] DM_\epsilon(f_\epsilon^{j'}(x))Df_\epsilon(f_\epsilon^{j'-1}(x))\cdots Df_\epsilon(x) \\
&\qquad\times\left[\prod_{j=j'+1}^{i-1} M\big(f_\epsilon^j(x)\big)\right]\eta_\epsilon\big(f_\epsilon^i(x)\big) \\
&+\prod_{j=0}^{i-1} M\big(f_\epsilon^j(x)\big)D\eta_\epsilon(f_\epsilon^i(x))Df_\epsilon(f_\epsilon^{i-1}(x))\cdots Df_\epsilon(x),
\end{aligned}$$

and that the supremum over B_ρ of each of the terms in the sum can be bounded by $\beta^{i-1}K(\rho\alpha^i)^{k+1}$. The last term can be bounded by $\beta^i K(\rho\alpha^i)^k\alpha^i$. Hence, the derivative can be bounded by

$$(\beta\alpha^{k+1})^i[i\rho^{k+1}K\beta^{-1} + \tilde{K}\rho^k]$$

which converges absolutely if summed over i.

We also observe that $\|D\varphi(x)\| \le K\|x\|^k$.

For fixed ϵ we have established that the solution is C^1.

To prove that the solution is C^r as claimed in the theorem, we can use the "tangent functor trick." We observe that by taking derivatives of (5.1) we obtain:

$$D\varphi_\epsilon(x) - M_\epsilon(x)D\varphi_\epsilon\big(f_\epsilon(x)\big)Df_\epsilon(x) = D\eta_\epsilon(x) + DM_\epsilon(x)\varphi_\epsilon\big(f_\epsilon(x)\big)\ . \tag{5.7}$$

Note that the equation for $D\varphi_\epsilon$ is of the form of cohomology equations considered in the theorem. It satisfies conditions analogous to those satisfied by φ_ϵ, except that β is replaced by $\beta\cdot\alpha$ (since the linear operator is replaced by $D\varphi_\epsilon \to M_\epsilon(x)D\varphi_\epsilon(f_\epsilon(x))Df_\epsilon(x)$)

and condition (ii) is replaced by $D^i R_\epsilon(0) = 0$, $i \leq (k-1)_+ < r$, where $R_\epsilon(x)$ refers to the right hand side of (5.7) and $(k-1)_+ = \max(k-1, -1)$. This in turn means that condition (v) is replaced by $\alpha^{(k-1)_+ + 1}(\beta\alpha) < 1$. Clearly if $\alpha^{k+1}\beta < 1$ then $\alpha^{(k-1)_+ + 1}(\beta\alpha) < 1$, so we can repeat the same construction as outlined above to construct $D\varphi_\epsilon$. Uniqueness implies that the solution of (5.7) is indeed the derivative of $\varphi_\epsilon(x)$. Then a calculation like (5.6) implies φ_ϵ is C^2.

If $r \geq 2$, we can repeat this procedure, differentiating (5.7) a second time with respect to x. In this case β is replaced by $\beta\alpha^2$, while in condition (ii), k is replaced by $(k-2)_+$. Once again if $\beta\alpha^{k+1} < 1$, then $(\alpha^2\beta)\alpha^{(k-2)_+ + 1} < 1$, so condition (v) is satisfied and we can construct $D^2\varphi_\epsilon$ and using a calculation like (5.6) conclude that φ_ϵ is C^3.

Similarly, we will show that we can bootstrap the number of derivatives with respect to ϵ.

We compute the derivative with respect to ϵ of one of the terms in the sum (5.5)

$$
\begin{aligned}
&\frac{\partial}{\partial \epsilon}\left[\prod_{j=0}^{i-1} M_\epsilon\big(f_\epsilon^i(x)\big)\right] \eta_\epsilon\big(f_\epsilon^i(x)\big) = \\
&= \sum_{j_1=0}^{i-1} \prod_{j=0}^{j_1-1} M_\epsilon\big(f_\epsilon^j(x)\big)\left[\left(\frac{\partial}{\partial\epsilon} M_\epsilon\right)\left(f_\epsilon^{j_1}(x)\right) + \right. \\
&\qquad \left. DM_\epsilon\big(f_\epsilon^{j_1}(x)\big) \sum_{0\leq j_2 \leq j_1-1} Df_\epsilon^{j_1-j_2-1}(f_\epsilon^{j_2+1}(x))\left(\frac{\partial}{\partial\epsilon} f_\epsilon\right)\left(f_\epsilon^{j_2}(x)\right)\right] \\
&\qquad \prod_{j=j_1+1}^{i-1} M_\epsilon\big(f_\epsilon^j(x)\big)\eta_\epsilon\big(f_\epsilon^i(x)\big) + \\
&+ \left[\prod_{j=0}^{i-1} M_\epsilon\big(f_\epsilon^j(x)\big)\right]\left[\frac{\partial \eta_\epsilon}{\partial\epsilon}\big(f_\epsilon^i(x)\big) + \right. \\
&\qquad \left. D\eta_\epsilon\big(f_\epsilon^i(x)\big) \sum_{j_1=0}^{i-1} Df_\epsilon^{i-j_1-1}(f_\epsilon^{j_1+1}(x))\left(\frac{\partial}{\partial\epsilon} f_\epsilon\right)\left(f_\epsilon^{j_1}(x)\right)\right]
\end{aligned}
\qquad (5.8)
$$

Observe that since $D^i\eta_\epsilon(0) = 0$ $i = 0, \cdots, k$ for all values of ϵ, we have $\frac{\partial^\ell}{\partial\epsilon^\ell} D^i\eta_\epsilon(0) = 0$, $D^i \frac{\partial^\ell}{\partial\epsilon^\ell}\eta_\epsilon(0) = 0$. Hence,

$$
\left\|\left(\frac{\partial^\ell}{\partial\epsilon^\ell}\eta_\epsilon\right)\left(f^j(x)\right)\right\| \leq K\left(\alpha^j \|x\|\right)^{k+1} .
$$

Similarly,

$$\left\| \left(\frac{\partial^\ell}{\partial \epsilon^\ell} f_\epsilon \right) \left(f^j(x) \right) \right\| \le K \left(\alpha^j \|x\| \right)^{k+1} .$$

Hence, it is not difficult to establish again that the norm of the derivative in (5.8) is less than

$$K \|x\|^{k+1} (\alpha^{k+1} \beta)^i i^2 .$$

Now we observe that the derivative of φ with respect to parameters satisfies

$$\dot{\varphi}_\epsilon(x) - M_\epsilon(x) \dot{\varphi}_\epsilon\big(f_\epsilon(x)\big) = \dot{M}_\epsilon(x) \varphi_\epsilon\big(f_\epsilon(x)\big) + M_\epsilon(x) D\varphi_\epsilon\big(f_\epsilon(x)\big) \dot{f}_\epsilon(x) + \dot{\eta}_\epsilon(x). \tag{5.9}$$

The first term in the right hand side vanishes at zero to order k and, using that $\dot{f}_\epsilon$ vanishes to order k, we see that the second one vanishes up to order $2k - 1 \ge k$ and that it is is C^{r-1}. Since we established that $\dot{\varphi}_\epsilon$ vanishes up to order k, we conclude that $\dot{\varphi}_\epsilon$ is C^{r-1} in the x variables.

Applying the result just established to (5.9) we will conclude that $\dot{\varphi}_\epsilon$ admits one derivative with respect to ϵ.

Hence, we can proceed by induction. If $D^a_\epsilon D^b_x \varphi_\epsilon$ exist and are continuous and vanish at the origin, we conclude that $D^{a+1}_\epsilon D^b_x \varphi_\epsilon$ and $D^a_\epsilon D^{b+1}_x \varphi_\epsilon$ exist. The induction stops only when the corresponding derivatives of η_ϵ cease to exist.

Remark. It is also possible to prove that the function is C^r by taking formal derivatives term by term and showing that the resulting series converges uniformly. The estimates needed are developed in [BLW].

■

Remark. Notice that the size of the ball on which the solution is defined is independent of η. It depends only on f and M. This is, of course, to be expected since η_ϵ enters only linearly.

To study the solution of (5.1) in the general hyperbolic case, we will find it convenient to perform several initial reductions that will make it possible to simplify the notation in the proof, but which do not incurr a loss of generality.

We observe that, if (5.1) holds and h_ϵ is a diffeomorphism, we have

$$\varphi_\epsilon \circ h(x) - M_\epsilon \circ h_\epsilon(x)\varphi_\epsilon \circ h_\epsilon\big(h_\epsilon^{-1} \circ f_\epsilon \circ h_\epsilon(x)\big) = \eta_\epsilon \circ h_\epsilon(x) \tag{5.10}$$

so that $\tilde{\varphi}_\epsilon = \varphi_\epsilon \circ h_\epsilon$ satisfies (5.1) with data $\tilde{M}_\epsilon = M_\epsilon \circ h_\epsilon(x)$, $\tilde{f}_\epsilon = h_\epsilon^{-1} \circ f_\epsilon \circ h_\epsilon$, $\tilde{\eta}_\epsilon = \eta_\epsilon \circ h_\epsilon$.

In particular, if we take $h_\epsilon(x) = ax$ for some small positive number, a, we have $D\tilde{f}_\epsilon(0) = Df_\epsilon(0)$ and $D^i\tilde{f}_\epsilon(x) = a^{i-1}D^i f_\epsilon(ax)$,

$$\left(\frac{\partial}{\partial\epsilon}\right)^\ell D^i\tilde{f}_\epsilon(x) = a^{i-1}\left(\frac{\partial}{\partial\epsilon}\right)^\ell D^i f_\epsilon(ax) \ ,$$

so that, for $i > 1$, $i + \ell \le r$

$$\sup_{\substack{\epsilon\in[0,1] \\ \|x\|<1}} \left\| \left(\frac{\partial}{\partial\epsilon}\right)^\ell D^i\tilde{f}_\epsilon(x) \right\| \ ,$$

can be made as small as we please by taking a sufficiently small.

Furthermore,

$$\sup_{\substack{\epsilon\in[0,1] \\ \|x\|<1}} \left\| \left(\frac{\partial}{\partial\epsilon}\right)^\ell \left[D\tilde{f}_\epsilon(x) - D\tilde{f}_\epsilon(0)\right] \right\| \le \sup_{\substack{\epsilon\in[0,1] \\ \|x\|\le a}} \left\| \left(\frac{\partial}{\partial\epsilon}\right)^\ell \left[Df_\epsilon(x) - Df_\epsilon(0)\right] \right\| \ ,$$

which also can be made arbitrarily small.

If $\alpha(x)$ is a C^∞ function with $\alpha(x) = 1$ when $\|x\| \le 1$, $\alpha(x) = 0$, when $\|x\| \ge 3/2$, we see that defining,

$$\begin{aligned} \tilde{\tilde{f}}_\epsilon(x) &= \alpha(x)\tilde{f}_\epsilon(x) + \big(1 - \alpha(x)\big)Df_\epsilon(0)x \ , \\ \tilde{\tilde{M}}_\epsilon(x) &= \alpha(x)\tilde{M}_\epsilon(x) + \big(1 - \alpha(x)\big)\tilde{M}_\epsilon(0) \ , \\ \tilde{\tilde{\eta}}_\epsilon(x) &= \alpha(x)\tilde{\eta}_\epsilon(x) + \big(1 - \alpha(x)\big)\tilde{\eta}_\epsilon(0) \ , \end{aligned} \tag{5.11}$$

finding a solution of (5.1) defined in a sufficiently small neighborhood of the origin can be accomplished by finding global solutions of (5.1) under the assumption that $\|f_\epsilon(x) - Df_\epsilon(0)x\|_{C^r(\mathbb{R}^n\times[0,1])}$ is sufficiently small and that $M_\epsilon, \eta_\epsilon$ are in $C^r(\mathbb{R}^n \times [0,1])$.

We emphasize that the solutions we obtain in this case may depend on the cut-off function used. There are easy examples that show that indeed different cut-off functions lead to different solutions. Hence, the solutions produced by Theorem 5.1 are highly non-unique.

There are some further reductions that we will use.

We recall the stable/unstable manifold theorem for diffeomorphisms.

Theorem 5.2. *Let $A : \mathbb{R}^s \oplus \mathbb{R}^u$ be a linear map and f_ϵ a C^r family of diffeomorphisms $f_\epsilon(0) = 0$ in such a way that*

i) The splitting $\mathbb{R}^s \oplus \mathbb{R}^u$ is invariant under A,

ii) $\|A|_{\mathbb{R}^s}\| < 1$, $\|A^{-1}|_{\mathbb{R}^u}\| < 1$.

iii) $\|f_\epsilon - A\|_{C^r(\mathbb{R}^s\oplus\mathbb{R}^u\times[0,1])} \leq \delta$ where $\delta > 0$ is a number that can be computed explicitley depending only on A.

Then, there exist unique C^r families $W^s_\epsilon : \mathbb{R}^s \to \mathbb{R}^u$, $W^u_\epsilon : \mathbb{R}^u \to \mathbb{R}^s$ such that

a) The graphs of W^s_ϵ, W^u_ϵ are invariant under f_ϵ,

b) $\|W^s_\epsilon\|_{C^r(\mathbb{R}^s\times[0,1])} \leq K\delta$,

$\|W^u_\epsilon\|_{C^r(\mathbb{R}^u\times[0,1])} \leq K\delta$.

where K, can be chosen independently of δ.

Proofs of invariant manifold theorems with dependence on parameters can be found in [La] using the graph transform method and in [LW] using Irwin's method.

We note that a corollary of the estimates b) in Theorem 5.2 is that if $D^i f_\epsilon(0) = 0$ for $i = 1, \cdots, k$ then, $D^i W^s_\epsilon(0) = 0$, $D^i W^u_\epsilon(0) = 0$, $i = 1, \cdots, k$. In effect, if we consider

$\tilde{f}_\epsilon(x) = a^{-1} f_\epsilon(ax)$ we see that $||\tilde{f}_\epsilon - A||_{C^r} \le K a^{k-1}$. On the other hand, the function $\tilde{W}^s_\epsilon(y) = a^{-1} W^s_\epsilon(ay)$ has a graph invariant under $\tilde{f}_\epsilon$ and, by the uniqueness in the conclusions of Theorem 5.2, we obtain that it should satisfy estimates b). We conclude that $||a^{-1} W^s_\epsilon(ay)||_{C^r} \le K a^{k-1}$ which, can only happen if the derivatives of W^s_ϵ of order less or equal than k vanish at zero. Of course, an analogous argument works for W^u_ϵ. The same result can be obtained by examining the graph transform equations in [La]. By repeated differentiation we can obtain equations satisfied by the derivatives of W^s_ϵ and we can check by inspection that if the derivatives at the origin of f_ϵ vanish so do the derivatives of W^s_ϵ.

This theorem allows us to choose the change of variables (5.10) in such a way that the invariant manifolds are the coordinate axes.

That is, $\mathbb{R}^n = \mathbb{R}^s \oplus \mathbb{R}^u$ and $f_\epsilon(x, 0) = (f^s_\epsilon(x), 0)$, $f_\epsilon(0, y) = (0, f^u_\epsilon(y))$ $\forall x \in \mathbb{R}^s$, $y \in \mathbb{R}^u$.

Therefore, we have established:

Lemma 5.3. *Let f_ϵ be a C^r family of diffeomorphisms defined in a neighborhood of the origin in $\mathbb{R}^n$ such that*

i) $f_\epsilon(0) = 0, \quad Df_\epsilon(0) = A,\ D^i f_\epsilon(0) = D^i f_0(0), \quad 2 \le i \le k,$

ii) *A is hyperbolic.*

Then, if s and u are the dimensions of the stable and unstable subspaces for A, $\mathbb{R}^n = \mathbb{R}^s \oplus \mathbb{R}^u$, for every $\delta > 0$ we can find a C^r family h_ϵ and a C^r family $\tilde{f}_\epsilon$ defined on the whole of $\mathbb{R}^n$ such that:

a) $h_\epsilon(0) = 0, \quad Dh_\epsilon(0) = Id, \quad D^i h_\epsilon(0) = 0, \quad 2 \le i \le k,$

b) $h_\epsilon^{-1} \circ f_\epsilon \circ h_\epsilon = \tilde{f}_\epsilon$ *on a neighborhood of the origin,*

c) $||\tilde{f}_\epsilon - A||_{C^r} \le \delta$,

d) $\tilde{f}_\epsilon(\mathbb{R}^s \oplus \{0\}) = \mathbb{R}^s \oplus \{0\},$

$\tilde{f}_\epsilon^{-1}(\{0\} \oplus \mathbb{R}^u) = \{0\} \oplus \mathbb{R}^u$.

Since for our purposes it is enough to have solutions of (5.1) on a neighborhood of the origin, we can use cut-off functions and study (5.1) defined in the whole $\mathbb{R}^n$, but nevertheless assume that f_ϵ satisfies the conclusions of Lemma 5.3 and that η_ϵ has compact support around the origin.

Since the stable and unstable directions will play an important role, we will introduce the notation $x = (x_s, x_u)$, with $x \in \mathbb{R}^n$, $x_s \in \mathbb{R}^s$, $x_u \in \mathbb{R}^u$. We will also write

$$A(x_s, 0) = (A_s x_s, 0), \quad A(0, x_u) = (0, A_u x_u) \ .$$

Inspired by the method used in [St] we start by solving the equation on the neighborhood of $\mathbb{R}^s$ up to high orders in x_u.

Lemma 5.4. *Let f_ϵ, η_ϵ be C^r families as above. Assume that:*

i) $\|A_s\| \le \lambda_+ < 1$,

$\|A_u\| \le \mu_+ > 1$,

$D^i \eta_\epsilon(0) = 0$; $i = 0, \dots, k$,

$\|M_\epsilon(0)\| \le \beta_+$.

Let ℓ be such that:

ii) $\lambda_+^{k+1} \beta_+ \mu_+^\ell < 1$, *i.e.* $\ell < -\frac{\ln \beta_+}{\ln \mu_+} + (k+1) \frac{|\ln \lambda_+|}{\ln \mu_+}$.

Then, we can find a C^ℓ family $\tilde{\varphi}_\epsilon$ of compactly supported functions such that

a) $\frac{\partial^j}{\partial x_u^j} (\tilde{\varphi}_\epsilon(x) - M_\epsilon(x) \tilde{\varphi}_\epsilon \circ f_\epsilon(x) - \eta_\epsilon(x))|_{x_u = 0} = 0, \quad j = 0, \cdots, \ell$.

Proof. Since we have good control of the problem on the stable manifold, our first goal is to rewrite the problem in an expansion in powers of x_u. We will expand the original equation in powers of x_u and try to match the coefficients of corresponding powers.

Using a partial Taylor expansion

$$\eta_\epsilon(x) = \sum_{|i|=0}^{\ell} \eta_\epsilon^{[i]}(x_s) x_u^{\otimes i} + \eta_\epsilon^{[>]}(x).$$

Here $\eta_\epsilon^{[i]}(x_s) \in \mathcal{L}((\mathbb{R}^u)^{\otimes i}, R^m)$ is given by Taylor's formula, $\eta_\epsilon^{[i]}(x_s) = \frac{1}{i!}\frac{\partial^i}{\partial {x_u}^i}\eta_\epsilon(x_s, 0)$.

Note further that the Taylor remainder satisfies:

$$\frac{\partial^i}{\partial x_u^i}\eta_\epsilon^{[>]}(x_s, 0) = 0 \qquad i = 0, \ldots, \ell \ ,$$

and that $\eta_\epsilon^{[i]}$ is an $r-i$ family with $\|\eta_\epsilon^{[i]}\|_{C^{r-i}} \le \|\eta_\epsilon\|_{C^r}$. We introduce similar notations for $M_\epsilon(x)$ and its Taylor expansion. We will prove that one can solve recursively for the coefficients in an analogous expansion of $\tilde{\varphi}_\epsilon$. Once we succeed in doing this, we can cut off the Taylor polynomial by a function tangent to the identity to infinite order. Since the conclusion a) depends only on the Taylor expansions up to finite order, it will also be satisfied.

If $\tilde{\varphi}_\epsilon^i(x) = \tilde{\varphi}_\epsilon^{[i]}(x_s)x_u^{\otimes i}$ is the monomial of degree i in the Taylor expansion, then

$$\begin{aligned}\tilde{\varphi}_\epsilon^i\big(f_\epsilon(x)\big) &= \tilde{\varphi}_\epsilon^{[i]}\big(f_\epsilon^s(x_s, x_u)\big) f_\epsilon^u(x_s, x_u)^{\otimes i} \\ &= \tilde{\varphi}_\epsilon^{[i]}\big(f_\epsilon^s(x_s, 0)\big)\big(\frac{\partial}{\partial x_u} f_\epsilon^u(x_s, 0)\big)^{\otimes i} x_u^{\otimes i} + R_\epsilon^{[i]}(x) \quad ,\end{aligned}$$

where $\left(\frac{\partial}{\partial x_u}\right)^j R_\epsilon^{[i]}(x_s, 0) = 0 \qquad j < i+1$.

The operator $\tilde{\varphi}_\epsilon^{[i]}\big(f_\epsilon^s(x_s, 0)\big)\big(\frac{\partial}{\partial x_u} f_\epsilon^u(x_s, 0)\big)^{\otimes i}$ belongs to $\mathcal{L}((\mathbb{R}^u)^{\otimes i}, \mathbb{R}^m)$ and maps $x_u^{\otimes i}$ to $\tilde{\varphi}_\epsilon^{[i]}\big(f_\epsilon^s(x_s, 0)\big)\big(\frac{\partial}{\partial x_u} f_\epsilon^u(x_s, 0)x_u\big)^{\otimes i}$ If $\tilde{\varphi}_\epsilon^i(x)$ is a C^ℓ family, then $R_\epsilon(x)$ is a $C^{\ell-1}$ family, and if

$$\left(\frac{\partial}{\partial x_s}\right)^j \tilde{\varphi}_\epsilon^{[i]}(0) = 0 \qquad j \le k \ ,$$

then

$$\left(\frac{\partial}{\partial x_s}\right)^j R_\epsilon^{[i]}(0, 0) = 0 \qquad j \le k \ .$$

The derivatives of order ℓ of $R_\epsilon^{[i]}$ with respect to $\frac{\partial}{\partial x_u}$ can also be evaluated in terms of tensor products of derivatives of f up to order ℓ and $\varphi_\epsilon^{[i]}(x_s)$.

Therefore, if

$$\varphi_\epsilon^{[\le]}(x) = \sum^{\ell} \varphi_\epsilon^{[i]}(x_s) x_u^{\otimes i}, \tag{5.12}$$

then,

$$\varphi_\epsilon^{[\le]}\big(f_\epsilon(x)\big) = \sum_{i=0}^{\ell} T_\epsilon^{[i]}(x_s) x_u^{\otimes i} + R_\epsilon^{[\ge]}(x)$$

where, $R_\epsilon^{[\ge]}$ is of high order and

$$T_\epsilon^{[i]}(x_s) = \varphi_\epsilon^{[i]}\big(f_\epsilon^s(x_s,0)\big)\big(\frac{\partial}{\partial x_u} f_\epsilon^u(x_s,0)\big)^{\otimes i} + C_\epsilon^{[i]}(x_s) \quad ,$$

where $C^{[i]}$ is an expression that involves derivatives of f and $\varphi_\epsilon^{[j]}$ for $j < i$.

If we substitute the expression (5.12) for $\varphi_\epsilon^{[\le]}$ into (5.1) and equate similar powers we obtain for $i \le \ell$ fixed,

$$\begin{aligned} \eta_\epsilon^{[i]}(x_s) &= \varphi_\epsilon^{[i]}(x_s) - M_\epsilon^{[0]}(x_s)\varphi_\epsilon^{[i]}\big(f_\epsilon^s(x_s,0)\big)\big(\frac{\partial}{\partial x_u} f_\epsilon^u(x_s,0)\big)^{\otimes i} \\ &\quad - \sum_{j=0}^{i-1} M_\epsilon^{[i-j]}(x_s) C_\epsilon^{[j]}(x_s) \quad . \end{aligned} \tag{5.13}$$

Remark. The $\tilde{C}_\epsilon^{[i]}$'s in (5.13) are redefined with respect to the $C_\epsilon^{[i]}$'s in the previous equation by absorbing additional terms of lower order in the derivatives of f and $\varphi^{[j]}$ with $j < i$.

We observe that (5.13) is of the form of the cohomology equations we considered in Theorem 5.1. The data, $C^{[i]}$, $\eta_\epsilon^{[i]}$ and the unknowns, $\varphi^{[i]}$ are defined on the stable manifold and $f_\epsilon^s(x_s,0)$ maps the stable manifold into itself and is a contraction. Therefore, we want to verify the quantitative hypothesis of Theorem 5.1.

Note that the linear operator M_ϵ that appears in Theorem 5.1 is replaced by the operator $\varphi^{[i]}(\cdot) \mapsto M_\epsilon^{[0]}(x_s)\varphi^{[i]}(\cdot)\big(\frac{\partial}{\partial x_u} f_\epsilon^u(x_s,0)\big)^{\otimes i}$ The norm of this operator is bounded in a neighborhood of the origin by supremum of the norm of $M_\epsilon^{[0]}$ and by the supremum of the norm of $\frac{\partial}{\partial x_u} f_\epsilon^u(x_s,0)$ raised to the power i. Hence, by choosing the neighborhood sufficiently small, we can bound the operator norm by $\beta\mu^i$ where β and μ are as close as one likes to β_+ and μ_+. Similarly, we can assume that $||\frac{\partial}{\partial x_s} f^s_\epsilon(x_s, 0)|| \le \lambda$, where λ can be made arbitrily close to λ_+ by considering a sufficiently small neighborhood of the origin. Hypothesis (v) of Theorem 5.1 becomes $\lambda^{k+1}\beta\mu^i < 1$, which is satisfied because of $ii)$ of the hypothesis of the Lemma, provided that we consider only a sufficiently small neighborhood of the origin.

Once we have candidates for what the derivatives should be, the rest of the proof of Lemma 5.4 is just verifying that we can extend them using a cutoff function.

If $\gamma : \mathbb{R}^u \to \mathbb{R}$ is a C^∞ function taking the value 1 in a neighborhood of zero and with compact support, then

$$\tilde{\varphi}_\epsilon(x) = \varphi_\epsilon^{[\le]}(x)\gamma(x_s) \ ,$$

where $\varphi_\epsilon^{[\le]}$ is as in (5.12) and the coefficients are obtained solving (5.13) is a solution of the problem.

We observe that, if

$$\varphi_\epsilon(x) - M_\epsilon(x)\varphi_\epsilon\big(f_\epsilon(x)\big) = \eta_\epsilon(x)$$

and

$$\Phi_\epsilon(x) - M_\epsilon(x)\Phi_\epsilon\big(f_\epsilon(x)\big) = \alpha_\epsilon(x) \ ,$$

then

$$(\varphi_\epsilon - \Phi_\epsilon)(x) - M_\epsilon(x)(\varphi_\epsilon - \Phi_\epsilon)\big(f_\epsilon(x)\big) = \eta_\epsilon(x) - \alpha_\epsilon(x) \ . \tag{5.14}$$

Hence, to find solutions of (5.1), by considering $\varphi_\epsilon - \tilde{\varphi}_\epsilon$ where φ_ϵ is produced using Lemma 5.4 we can consider only the case where $(\frac{\partial}{\partial x_u})^i \eta_\epsilon(x)$ vanishes on the stable direction for $|i| \le \ell$.

∎

Now, we try to solve the equation (2.4) for functions which vanish to a high order in the unstable direction.

Lemma 5.5. *Let f_ϵ, M_ϵ be as before. Assume that η_ϵ is a C^r family with support in a neighborhood of the origin and that, in addition to the hypothesis of Lemma 5.4 we have:*

i) $\|A_s^{-1}\| \le \lambda_- > 1$

$\|A_u^{-1}\| \le \mu_- < 1$

ii) $D^i\eta_\epsilon(x_s, x_u) \le K|x_u|^{k-i+1}$

iii) $\|M_\epsilon(0)^{-1}\| \le \beta_-$

Let $\ell \in \mathbb{N}$ be such that

iv) $\lambda_-^\ell \beta_- \mu_-^{k-\ell+1} < 1$; that is, $\ell < \frac{(k+1)|\ln \mu_-| - \ln \beta_-}{(\ln \lambda_- + \ln \mu_-)}$.

Then, there exists a C^ℓ family φ_ϵ solving (5.1) on a neighborhood of the origin.

Remark. The conditions imposed in *iv)* are too conservative. The natural conditions for the method of proof presented here seem to be:

iv′) $\lambda_-^\ell \beta_- \mu_-^{k+1} < 1$; $\ell < \frac{-\ln \beta_-}{\ln \lambda_-} + (k+1)\frac{|\ln \mu_-|}{\ln \lambda_-}$.

At the end of the proof we will sketch how to improve the argument to get this result.

Proof. The proof is very similar to Theorem 5.1.

Again, considerable intuition can be obtained considering the case where M is constant and, likewise, $f_\epsilon \equiv A$.

We just check that

$$\varphi_\epsilon(x) = \sum_{i=1}^{\infty} M^{-n}\eta_\epsilon(A^{-n}x) \tag{5.15}$$

is a solution.

In effect, since $\|(A^{-n}x)_u\| \le \mu_-^n \|x_u\|$, we have that $\eta_\epsilon(A^{-n}x) \le K\mu_-^{n(k+1)}\|x_u\|$. So that the series (5.15) converges provided that $\mu_-^{k+1}\beta_- < 1$.

As before, if we take a derivatives with respect to x and b derivatives with respect to ϵ in the general term, we obtain a general term:

$$M^{-n} D_x^a D_\epsilon^b \eta_\epsilon(A^{-n}x)\left(A^{\otimes a}\right)^{-n}$$

whose norm can be bounded by $\left(\beta_- \mu_-^{k+1-a} \lambda_-^a\right)^n K\|x_u\|$ if $a \le k+1$.

To prove the general case, observe that, by cutting off the function M and f_ϵ as we did in (5.11) we can assume that the bounds assumed for the derivative at zero are valid globally, with slightly worse constants. We can arrange that these new constants also satisfy assumption iv).

We also observe that (5.1) can also be written

$$\varphi_\epsilon(x) - M_\epsilon^{-1}\big(f_\epsilon^{-1}(x)\big)\varphi_\epsilon(f_\epsilon^{-1}(x)) = -M_\epsilon^{-1}\big(f_\epsilon^{-1}(x)\big)\eta_\epsilon\big(f_\epsilon^{-1}(x)\big) \quad . \tag{5.16}$$

We claim that a solution of (5.16) can be obtained by setting

$$\varphi_\epsilon(x) = \sum_{i=1}^{\infty} \left[\prod_{j=1}^{i} M_\epsilon^{-1} \circ f_\epsilon^{-j}(x)\right] \eta_\epsilon \circ f_\epsilon^{-i}(x) \quad . \tag{5.17}$$

The estimates to show that the sum in (5.17) converges are very similar to those used in the study of (5.5). Hence, it will be important to find out how fast $f_\epsilon^{-i}(x)$ approaches the set where η_ϵ vanishes.

Proposition 5.6. *Under the conditions of Lemma 5.5 denote $f_\epsilon^{-n}(x_s, x_u) = (x_n^s, x_n^u)$. Provided that f_ϵ is C^1-close enough to A, we can find α as close as desired to $|A^{-1}|_{E^u}|$ and $K > 1$ so that $|x_n^u| \le K\alpha^n \|x_0^u\|$.*

Proof. We denote:

$$f_\epsilon^{-1}(x_s, x_u) = (A_s^{-1}x_s + N^s(x_s, x_u)\ ,\ A_u^{-1}x_u + N^u(x_s, x_u)) \quad . \tag{5.18}$$

We note that $N^u(x_s, 0) = 0$ due to the invariance of the unstable manifold and that the C^1 norm of N^s and N^u can be assumed to be as small as desired.

Hence, we can estimate

$$\begin{aligned} |A_u^{-1}x_u + N^u(x_s, x_u)| &\le |A_u^{-1}x_u + N^u(x_s, x_u) - N^u(x_s, 0)| \\ &\le (||A_u^{-1}|| + ||N^u||_{C^1})||x_u|| \quad . \end{aligned}$$

■

To show that the R.H.S. of (5.17) converges, we just observe that

$$\left\| \prod_{j=1}^{i} M_\epsilon^{-1} \circ f_\epsilon^{-j}(x) \right\| \leq (\beta_- + \delta)^i$$

where δ can be made arbitrarily small by making the neighborhood under consideration sufficiently small. Moreover, by Proposition 5.6 and the assumptions on η we have

$$\|\eta_\epsilon \circ f_\epsilon^{-i}(x)\| \leq K(\mu_- + \delta)^{i(k+1)} \ .$$

By assumption iv) the general term of (5.17) is bounded by a geometric series of ratio less than 1 if δ is sufficiently small.

The rest of the argument is very similar to that in the proof of Theorem 5.1. We observe that (5.17) has the same form as (5.5) with M_ϵ replaced by M_ϵ^{-1}, f_ϵ replaced by f_ϵ^{-1} so that, if we take derivatives term by term we obtain analogues of (5.6), (5.8).

Consider, for example, the analogue of (5.6). We have:

$$\begin{aligned}
& D\left(\prod_{j=1}^{i} M_\epsilon^{-1} \circ f_\epsilon(x)\right) \eta \circ f_\epsilon^{-i}(x) = \\
& = \sum_{j'=0}^{i-1} \left[\prod_{j=0}^{j'-1} M_\epsilon^{-1}(f_\epsilon^{-j}(x))\right] DM_\epsilon^{-1}(f_\epsilon^{-j'}(x)) \times \\
& \times Df_\epsilon^{-1}(f_\epsilon^{-(j'-1)}(x) \cdots Df_\epsilon^{-1}(x)) \quad \left[\prod_{j=j'+1}^{i-1} M_\epsilon^{-1}(f_\epsilon^{-j}(x))\right] \eta_\epsilon(f_\epsilon^{-1}(x)) + \\
& + \prod_{j=0}^{i-1} M_\epsilon^{-1}(f_\epsilon^{-j}(x)) D\eta_\epsilon(f_\epsilon^{-i}(x)) Df_\epsilon^{-1}(f_\epsilon^{-i+1}(x)) \cdots Df_\epsilon^{-1}(f_\epsilon^{-1}(x))
\end{aligned} \tag{5.19}$$

Most factors in this expression have been estimated before. The only additional comment we need to make is that, in any neighborhood of the origin, we can bound $\|Df_\epsilon^{-1}\| \leq (\lambda_- + \delta)$ and δ can be chosen arbitrarily small by taking the neighborhood of the origin small. Thus, the norm of (5.19) can be bounded by

$$\sum_{j'=0}^{i-1} K_1(\beta_- + \delta)^{i-1}(\lambda_- + \delta)^{j'}(\mu_- + \delta)^{i(k+1)} + K_2(\beta_- + \delta)^i(\lambda_- + \delta)^i(\mu_- + \delta)^{ik} \tag{5.20}$$

Note that, to estimate $D\eta_\epsilon(f_\epsilon^{-i})$ we have used hypothesis ii) and Proposition 5.6.

We can bound (5.20) by $K(\beta_- + \delta)^i(\lambda_- + \delta)^i(\mu_- + \delta)^{ik}$ and, if δ is sufficiently small, hypothesis iv) will ensure that the sum over i converges. Higher derivatives and derivatives with respect to ϵ are handled in a similar fashion.

■

As remarked before, the conditions imposed in Lemma 5.5 are too conservative.

In the following paragraphs, we will give a proof only in the case where f is linear and just sketch the modifications needed in the general situation. Even if these imporvements lead to better values of A, we did not think it was worth to write a whole proof. Of course, these remarks are only meant for the very motivated reader.

For the case that $M_\epsilon(x) \equiv M$ and $f_\epsilon(x) = Ax$, we observe that, if we denote by D_s and D_u derivatives along the stable and unstable directions respectively, the results in Lemma 5.4, allow us to assume:

ii′) $\|D_u^\ell \eta_\epsilon(x)\| \le K\|x_u\|^{k+1-\ell}$;

$\|D_s^\ell \eta_\epsilon(x)\| \le K\|x_u\|^{k+1}$; rather than the more conservative *ii*) in Lemma 5.5.

As before, the solution of (2.4) is given by $\sum_{i=1}^{\infty} M^{-n}\eta_\epsilon(A^{-n}x)$. If we estimate the terms that we obtain when we apply D_s^ℓ and D_u^ℓ to the general term, we obtain:

$$\|M^{-n}D_s^\ell\eta_\epsilon(A^{-n}x)(A_s^{\otimes \ell})^{-n}\| \le (\beta_-\mu_-^{k+1}\lambda_-^\ell)^n K\|x_u\|^{k+1}$$
$$\|M^{-n}D_u^\ell\eta_\epsilon(A^{-n}x)(A_u^{\otimes \ell})^{-n}\| \le (\beta_-\mu_-^{k+1-\ell}\mu_-^\ell)^n K\|x_u\|^{k+1-\ell}$$

Hence, under the hypothesis *iv′*), we know that ℓ derivatives along complementary directions exist and are uniformly bounded. It is well known (see e.g. [Kr]) that, if a function has ℓ derivatives along complementary directions and those are uniformly bounded, the function is $C^{\ell-\epsilon}$. Of course, if we consider $\ell \notin \mathbb{N}$ with the usual meaning of Hölder regularity for the derivative of order $[\ell]$, then this lemma allows us to remove the ϵ in the conclusions.

In the case that f_ϵ is not a constant linear map, the argument can be generalized. We remark that, since f_ϵ is uniformly close to A in the whole space, then, one can prove stable and unstable foliation theorems completely similar to the ones usually stated for compact manifolds. These foliations, have smooth leaves but are not very smooth in transverse directions. Nevertheless, it is shown in [LMM] that one can consider differential operators along the leaves. Moreover, one has regularity results analogous to those for coordinate foliations, namely that regularity along both the stable and unstable foliations implies global regularity – with a loss of ϵ in the integer case and no loss in the non-integer case – (See [LMM], [Jo].) The existence of derivatives along the stable and unstable directions can be used by estimating the derivatives term by term. The details of these estimates are in [LMM]. Moreover, one can use regularity lemmas that show that when a function is differentiable along the stable and unstable leaves, then it is differentiable. This completes the sketch of the proof that (iv) in Lemma 5.5 can be replaced by (iv').

As a consequence of Lemma 5.4 and Lemma 5.5 we have the main result of the section.

Theorem 5.7. *Let f_ϵ, M_ϵ, η_ϵ be C^r families as before. Assume that $Df_\epsilon(0) = A$ is hyperbolic, and that*

i) $\|A_s\| \leq \lambda_+ \quad , \quad \|A_s^{-1}\| \leq \lambda_-$.

$\|A_u\| \leq \mu_+ \quad , \quad \|A_u^{-1}\| \leq \mu_-$,

$\|M_\epsilon(0)\| \leq \beta_+ \quad , \quad \|M_\epsilon(0)^{-1}\| \leq \beta_-$,

where $\lambda_+, \mu_- < 1$, $\lambda_-, \mu_+ > 1$.

ii) $D^i\eta_\epsilon(0) = 0 \quad , \quad i \leq k < r$.

Let $\ell \in \mathbb{N}$ be such that:

$$\ell < (k+1)\frac{|\ln\lambda_+||\ln\mu_-|}{\ln\mu_+(\ln\lambda_- + |\ln\mu_-|)} - \frac{\ln\beta_+|\ln\mu_-|}{\ln\mu_+(\ln\lambda_- + |\ln\mu_-|)} - \frac{\ln\beta_-}{(\ln\lambda_- + |\ln\mu_-|)} .$$

Then, it is possible to find a C^ℓ family φ_ϵ satisfying (5.1) on a neighborhood of the origin.

Proof. Using Lemma 5.4 we can construct a $C^{\tilde\ell}$ family $\tilde\phi_\epsilon$ that solves (5.1) when restricted to the stable manifold and that satisfies $a)$ of Lemma 5.4 vanishing up to order $\tilde{\ell}$, where $\tilde{\ell}$ is any integer satisfying:

$$\tilde{\ell} < \frac{-\ln \beta_+}{\ln \mu_+} + (k+1)\frac{|\ln \lambda_+|}{\ln \mu_+}. \tag{5.21}$$

Hence, we can choose $\tilde{\ell}$ to be:

$$\tilde{\ell} \geq \frac{-\ln \beta_+}{\ln \mu_+} + (k+1)\frac{|\ln \lambda_+|}{\ln \mu_+} - 1. \tag{5.22}$$

If we now consider the family $\phi_\epsilon - \tilde{\phi}_\epsilon$, we see that it will satisfy the same equation (5.1) but with a right hand side that satisfies the hypothesis of Lemma 5.5.

We can now obtain a solution which is C^ℓ for any ℓ such that:

$$\ell \leq \frac{-\ln \beta_-}{(\ln \lambda_- + \ln \mu_-|)} + (\tilde{\ell}+1)\frac{\ln \mu_-}{(\ln \lambda_- + |\ln \mu_-|)} \quad . \tag{5.23}$$

If we now take a $\tilde{\ell}$ satisfying (5.22), we see that we can take ℓ as claimed in the statement of Theorem 5.7.

■

Remark. The assumption that $\ell \in \mathbb{N}$ does not enter in the study of the cohomology equation. It is quite possible to develop a theory of cohomology equations when the coefficients are in the usual Hölder spaces. To use them in the deformation method, we would need a theory of solutions of ordinary differential equations when the coefficients are Hölder.

¿From Theorem 5.7 we can deduce the results claimed in the main theorems if we observe that in all cases, if we have a C^r family of mappings, the generators of the flow are a C^{r-1} family and, hence, the hamiltonians are also a C^{r-1} family. (This is the reason why the condition $k < r-1$ appears in Theorem 1.1 rather than $k < r$ in Theorem 5.7.) Furthermore, if our family of diffeomorphisms f_ϵ is tangent to order k at the origin, the vector field $\mathcal{F}$ will vanish also to order k. In the symplectic case, this implies that the hamiltonian F_ϵ vanishes up to order $k+1$. In the volume preserving case, since we need to integrate, we can only conclude that the hamiltonian vanishes up to order $(k-1)$ at the origin. In the contact case, since the hamiltonian is obtained just taking interior products, it vanishes up to order k.

Now we turn to compute the values for A, B for the different geometric structures that we claimed in the remarks after Theorem 1.1 To do that, we just need to relate the bounds on M to the bounds of the derivatives in the original problem – the operators M are constructed out of the derivatives of the original problem – and to study the relation of the regularity of the hamiltonian to the regularity of the original problem.

If we look at the way that the cohomology equations were derived, we see that in the case that no structure is preserved, the equation (2.4) can be written in components as $\mathcal{F}_\epsilon(x)-\mathcal{G}_\epsilon(x)+(Df_\epsilon\circ f_\epsilon^{-1}(x))\mathcal{G}_\epsilon(f_\epsilon^{-1}(x))=0$. In order to compare it more easily with the equation (5.1) discussed in Theorem 5.7, we write it as $\mathcal{F}_\epsilon(f_\epsilon(x))-Df_\epsilon^{-1}(x)\mathcal{G}_\epsilon(f_\epsilon(x))+\mathcal{G}_\epsilon(x)=0$ so that $\lambda_\pm,\mu_\pm$ have the same meaning in Theorem 5.7 and in Theorem 1.1. In that case, $M=Df^{-1}$ and, hence we can take $\beta_+=\lambda_-$ and $\beta_-=\mu_+$. Substituting this into the claim of Theorem 5.7 leads to the value in the remarks after Theorem 1.1.

Similarly, in the case that the mapping is symplectic, we take $M_\epsilon=1$ and hence $\beta_+=\beta_-=1$. The order of tangency in the cohomology equation is one more than that appearing in the hypothesis of Theorem 1.1.

In the case that the flow is volume preserving, we take $M_\epsilon=\left(Df_\epsilon^{-1}\right)^{\wedge(n-2)}$ – notice that we are acting on antisymmetric forms – hence, we can take $\beta_+=\lambda_-^{(n-2)}$, $\beta_-=\mu_+^{(n-2)}$. Notice that the previous bounds are very conservative and that we could take as β_+ the product of the $(n-2)$ – maybe after changing the norm to an equivalent one – largest absolute values of points in the spectrum. An analogous statement holds for β_-. This is the improvement alluded to in the remarks at the end of the statement of Theorem 1.1. The order of tangency in the cohomology equation is one less than the order of tangency of the diffeomorphisms. So that the k appearing in Theorem 5.7 in this case is one less than the k appearing in the hypothesis of Theorem 1.1.

In the contact case, we take M to be the factor by which the push forward multiplies a form. hence, we can take $\beta_+=\lambda_-$, $\beta_-=\mu_+$

Note that in the symplectic, volume-preserving, and contact cases, to construct the flow from the solution of the cohomology equation we have to take one derivative. Thus, if we know that solution of the cohomology equation is C^ℓ, we can conclude that the conjugating diffeomorphism is $C^{\ell-1}$.

For the case of flows, it is quite possible to give a very similar treatment which we only sketch.

We observe that if L_ϵ, φ_ϵ, M_ϵ, η_ϵ are as in (5.2), if we call f_ϵ^t the time t map of the flow and introduce the matrix valued function Γ_ϵ^t by:

$$\left.\begin{aligned} \frac{d}{dt}\Gamma_\epsilon^t(x) &= \Gamma_\epsilon^t(x) M_\epsilon(f_\epsilon^t(x)) \\ \Gamma_\epsilon^0(x) &= Id \end{aligned}\right\} .$$

then (5.2) can be written as

$$\begin{aligned} \frac{d}{dt}\Gamma_\epsilon^t(x)\varphi_\epsilon(f_\epsilon^t(x)) = \Gamma_\epsilon^t(x) M_\epsilon(f_\epsilon^t(x))\varphi_\epsilon(f_\epsilon^t(x)) \\ + \Gamma_\epsilon^t(x)[L_\epsilon\varphi_\epsilon]\circ f_\epsilon^t(x) = \\ = \Gamma_\epsilon^t(x)\, \eta_\epsilon(f_\epsilon^t x) \end{aligned} \cdot$$

Hence

$$\varphi_\epsilon(x) = -\int_0^s \Gamma_\epsilon^s(x)\eta_\epsilon(f_\epsilon^s(x))\, ds + \Gamma_\epsilon^t(x)\varphi_\epsilon(f_\epsilon^t(x)) \quad .$$

As in the case of the discrete equation this suggests that, if $f_\epsilon^t(x)$ is a contradiction we can write a solution as

$$\varphi_\epsilon(x) = -\int_0^\infty \Gamma_\epsilon^s(x)\eta_\epsilon(f_\epsilon^s(x))\, ds \; . \tag{5.24}$$

Notice that this equation is quite similar to (5.5) and an analysis analogous to the one we performed there can establish that the improper integral above converges and is differentiable with respect to x and to parameters.

For the case of flows, it is also possible to prove an invariant manifold theorem and the same argument we gave can be adapted without difficulty. We leave the details to the reader.

Theorem 5.8. *Let L_ϵ, φ_ϵ, M_ϵ, η_ϵ be C^r families of vector fields, vector valued as before. Assume that $DL_\epsilon(0) = A$ is hyperbolic in the sense of flows and that if $\sigma(A)$ denotes the spectrum of the operator A, we have:*

i) $-\lambda_- \le Re(\sigma(A_s)) \le \lambda_+$,

$-\mu_- \le Re(\sigma(A_u)) \le \mu_+$,

$\|M_\epsilon(0)\| \le \beta_+$,

$\|M_\epsilon(0)^{-1}\| \le \beta$,

ii) $D^i\eta_\epsilon(0) = 0 \quad i \le k < r$.

Let

$$\ell < (k+1)\frac{|\lambda_+||\mu_-|}{(\mu_+)(\lambda_- + |\mu_-|)} - \frac{\beta_+|\mu_-|}{(\mu_+)(\lambda_- + |\mu_-|)} - \frac{\beta_-}{(\lambda_- + |\mu_-|)} \ .$$

Then, it is possible to find a C^ℓ family φ_ϵ satisfying (2.5) on a neighborhood of the origin.

¿From this, we can derive the values of A and B in the remarks after Theorem 1.2 in a way analogous to the way we derived the values of in Theorem 1.1 from those in Theorem 5.7.

6. References

[B] A. Belitskii: Equivalence and Normal Forms of Germs of Smooth Mappings. Russ. Math. Surv. **33**, 107-177 (1978).

[BLW] A Banyaga, R. de la Llave, C.E. Wayne: Cohomology equations and commutators of germs of contact diffeomorphisms. Trans. A.M.S. **312**, 755–778

[Ch] M. Chaperon: Géométrie differéntielle et singularités des systèmes dynamiques. Astérisque 138–139 (1986).

[Che] Chen, K. T.: Equivalence and decomposition of vector fields about an elementary critical point. Amer. J. Math. **85**, 693–722 (1963).

[CM] P. Chernoff, J. Marsden: Properties of infinite dimensional hamiltonian systems. Springer, Lecture Notes in Math. **425**, (1974).

[DL] A. Delshams, R. de la Llave: Some algorithms for computations of normal forms and their object oriented implementation. Manuscript

[Gr] J. Grey: Some global properties of contact structures. Ann. of Math. **69**, 421–450 (1959).

[Har] P. Hartman: "*Ordinary Differential Equations*", Birkhauser, Boston (1982).

[I] Y. Ilyashenko and S. Yakovenko: Finitely smooth normal forms of local families of diffeomorphisms and vector fields. Russ. Mat. Surv. **46**, 1–43 (1991).

[Jo] J–L. Journé: A regularity lemma for functions of several variables. Rev. Mat. Iber. **4**, 187-193 (1988).

[Kr] S. Krantz: Lipschitz spaces, smoothnes of functions and approximation theory. Expo. Mat. **3**, 193–260 (1983).

[La] O. Lanford: Bifurcation of periodic solutions into invariant tori: The work of Ruelle and Takens. Springer, New York, Lecture Notes in Math. **322**, (1972).

[Le] H.I. Levine: Singularities of differentiable mappings, notes of a course by R. Thom. In "*Singularities at Liverpool I, C. T. Wall ed.*". Springer Verlag, Berlin, Lec. Notes in Math. **192**, (1971).

[Li] P. Liberman: Sur les automorphismes infinitesimaux des structures symplectiques et des structure de contact. Louveim, Colloque de Geometrie différentielle globale Bruxelles 1958 (1959).

[Li2] P. Liberman and Ch-M. Marle: Symplectic geometry and analytical mechanics. D. Reidel, Dordrecht (1987).

[LW] R. de la Llave, C.E. Wayne: On Irwin's proof of the pseudostable manifold theorem. Math. Zeit., In Press

[LMM] R. de la Llave, J.M. Marco, R. Moriyón: Canonical perturbation theories of Anosov diffeomorphisms and regularity results for the Livsic cohomology equation. Ann. of Math. **123**, 537–611 (1986).

[Ly] V.V. Lychiagin: On sufficient orbits of a group of contact diffeomorphisms. Math.

USSR Sbornik **33**, 233–242 (1977).

[M] J. Marsden: "*Applications of Global Analysis in Mathematical Physics*", Publish or Perish, Waltham (1974).

[Ma] J. Mather: Stability of C^∞ mappings II: Infinitesimal stability implies stability. Ann. of Math. **89**, 254-291 (1969).

[Mi] J. Milnor: "*Topology from a differenatiable viewpoint*", Univ. of Va. Press, Charlotsvile (1965).

[Mo] J. Moser: Proof of a generalized form of a fixed point theorem. in "Geometry and Topology Rio 1976", J. Palis, M. do Carmo, eds, Lecture Notes in Math. (597), Springer, N.J. (1977).

[Mo2] J. Moser: On the volume elements of a manifold. Trans. Amer. Math. Soc. **120**, 286–294 (1965).

[Mo3] J. Moser: The analytic invariants of an area preserving mapping near a hyperbolic fixed point. Comm. Pure Appl. Math. **9**, 673-693 (1956).

[Ni] Z. Nitecki: "*Differentiable Dynamics*", M.I.T. Press, Cambridge, MA (1971).

[Pa] J. Palis: On the local structure of hyperbolic fixed points in Banach spaces. Anais Acad. Brazil **40**, (1968).

[Ro] Modèles locaux des champs et des formes. Asterisque **30**, (1975).

[Ru] D. Ruelle: "*Elements of Differential dynamics and bifurcation theory*", Academic Press, New York (1990).

[Sp] M. Spivak: "*Calculus on manifolds*", Benjamin (1965).

[St] S. Sternberg: On the structure of local homeomorphisms II. Amer. J. of Math. **80**, 623–632 (1958).

[St2] S. Sternberg: Infinite Lie groups and the formal aspects of dynamical systems. Jour. Math. Mech. **10**, 451–474 (1961).

[St3] S. Sternberg: The structure of local homeomorphism III. Amer. J. of Math. **81**, 578–604 (1959).

[Ta] F. Takens: Singularities of vector fields. Pub. Mat. IHES **43**, 47-100 (1974).