SUPEREXPONENTIAL STABILITY OF KAM TORI

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Abstract. We study the dynamics in the neighbourhood of an invariant torus of a nearly integrable system. We provide an upper bound to the diffusion speed, which turns out to be of superexponentially small size $\exp(\exp(-1/\varrho))$, ϱ being the distance from the invariant torus. We also discuss the connection of this result with the existence of many invariant tori close to the considered one.

1. Introduction and results

We consider the problem of Arnold diffusion in nearly integrable Hamiltonian systems, with the aim of producing bounds on the diffusion speed in the spirit of Nekhoroshev's theory. We concentrate our attention in the neighbourhood of the invariant tori, the existence of which is guaranteed by KAM theory. We show that a clever use of the known results of both KAM theory and Nekhoroshev's theory leads to strong consequences, although of local character.

The key remark is that in the neighbourhood of an invariant KAM torus it is natural to introduce the distance from the torus as a new perturbation parameter. Thanks to this change of perspective we can apply a Birkhoff procedure, thus reducing the perturbation to an exponentially small size in $1/\varrho$, ϱ being the distance from the invariant torus. The new action variables introduced by the Birkhoff normalization are good adiabatic invariants, since they remove most of the quasiperiodic oscillation of the old action variables due to perturbation, namely, the so called deformation. At this point we change again our perspective, and investigate the dynamics with the global approach of Arnold on the one hand, and of Nekhoroshev on the other hand. All the results follow from a direct application of known theorems. In particular, a relevant consequence is that the speed of the Arnold diffusion (if any) turns out to be of superexponentially small size $\exp(\exp(-1/\varrho))$. This is our main contribution.

In view of this result, we can consider an invariant KAM torus as the head of a structure which dominates the dynamics in its neighbourhood. It appears that such a structure has a typical radius which depends on the size of the perturbation, and decreases to zero as the perturbation increases towards the critical size corresponding to the destruction of the torus. The neighbourhood of the torus turns out to contain many other invariant tori, constituting a set the relative volume of which tends to 1 when approaching the head torus. More precisely, the relative volume of the complement of the set of invariant tori turns out to be as small as $\exp(-1/\varrho)$. This represents a local improvement with respect to the estimates more or less explicitly contained in many previous statements: see for instance ref. [13]. The new aspect that we point out is that the dynamics is strongly affected by the existence of such a structure, inasmuch as the chaotic diffusion of orbits starting in the gaps between tori is thus forced to require a very long time. According to the known Nekhoroshev's estimates, such a time is exponentially large with the inverse of the perturbation. In our case, as we already remarked, the perturbation is exponentially small in $1/\varrho$. This determines the superexponential estimate for the diffusion time.

The picture resulting from the discussion above contrasts with the quite widespread opinion, especially among physicists, that the existence of invariant tori in systems with more than two degrees of freedom is not so relevant (see for instance [10]). Such an opinion is supported by the known fact that the tori do not isolate separated regions in phase space, thus allowing for the so called Arnold's diffusion. More recently, some authors have pointed out that KAM tori are very sticky, by applying locally the Nekhoroshev theory to the neighbourhood of an invariant torus: see for instance [16]. Their approach is actually equivalent to the application of the Birkhoff normalization procedure using the distance from the invariant torus as a perturbation parameter. However, we go beyond this level. We show in fact that although the KAM tori are not isolating, they form nevertheless a kind of impenetrable structure that the orbits cannot escape (nor enter, of course) for an exceedingly long time, very large even with respect to the known Nekhoroshev's estimates. Thus, the behaviour of the system in the region containing invariant KAM tori can be said to be effectively integrable.

We come now to a formal statement of our result. We consider a canonical system of differential equations with an Hamiltonian of the form

$$H(p,q,\varepsilon) = H_0(p) + \varepsilon H_1(p,q,\varepsilon) \ , \tag{1}$$

where $p \in \mathcal{G} \subset \mathbf{R}^n$, $\mathcal{G}$ being an open set, and $q \in \mathbf{T}^n$ are the action–angle variables, and ε a small parameter. The Hamiltonian is assumed to be a real analytic function of all its variables. The frequencies of the unperturbed system will be denoted by

$\omega(p) = \frac{\partial H_0}{\partial p}$, and the Hessian matrix of H_0 with respect to p will be denoted by $\mathcal{C}(p)$. Also, we shall denote by $|\cdot|$ a norm on functions over their domain of analyticity, for instance the usual supremum norm, and by $\|\cdot\|$ a norm for vectors in $\mathbf{R}^n$ or $\mathbf{C}^n$, for instance the Euclidean norm.

We pick up a point p^* such that the corresponding frequency $\omega^* := \omega(p^*)$ satisfies the diophantine condition

$$(2) \qquad |k \cdot \omega^*| \geq \gamma |k|^{-\tau} \quad \text{for all } k \in \mathbf{Z}^n \ , \quad k \neq 0 \ ,$$

for some $\gamma > 0$ and $\tau > n-1$; here, we define $|k| = |k_1| + \ldots + |k_n|$. We also assume that the Hamiltonian admits an analytic bounded extension to a domain

$$(3) \qquad D_\delta(p^*) = B_\delta(p^*) \times \mathbf{T}^n_\delta \ ,$$

for some positive δ. Here, $B_\delta(p^*)$ is the open ball of radius δ and center p^* in $\mathbf{C}^n$, and $\mathbf{T}^n_\delta = \{q \in \mathbf{C}^n \ : \ |\mathrm{Im}\,(q)| < \delta\}$. By the analyticity of the Hamiltonian, such a δ exists. With this setting we can state our

Theorem: *Consider the Hamiltonian (1) in the domain $D_\delta(p^*)$ defined by (3), with $\omega^* \equiv \omega(p^*)$ satisfying (2), and assume that the following conditions hold with positive constants ε, $d < 1$ and m:*
(a) $|H_1| < \varepsilon$ in $D_\delta(p^*)$;
(b) $\mathcal{C}(p)$ satisfies the nondegeneracy condition $d\|v\| < \|\mathcal{C}\, v\| < d^{-1}\|v\|$;
(c) the matrix $C = \mathcal{C}(p^*)$ satisfies $|Cv \cdot v| > m|v \cdot v|$ for all $v \in \mathbf{R}^n$ with $v \perp \omega^*$.
Then there exists a positive ε^* such that for every $\varepsilon < \varepsilon^*$ the following statement holds true: there is a positive ϱ^* depending on ε such that for every $\varrho < \varrho^*$ there is an analytic canonical transformation mapping $(J, \psi) \in B_\varrho(0) \times \mathbf{T}^n$ to $(p, q) \in B_\delta(p^*) \times \mathbf{T}^n$ with the properties:
(i) $J = 0$ is an invariant torus carrying a quasiperiodic flow with frequencies ω^*.
(ii) The domain $D_\varrho(0)$ contains an infinity of invariant tori the relative volume of which tends to 1 as $\varrho \to 0$, being estimated by $1 - \exp(-1/\varrho)$; the structure of these invariant tori is close to that of the tori $J = \mathrm{const}$, in the sense that every invariant torus lies in a neighbourhood of some torus $J = \mathrm{const}$, the radius of the neighbourhood being estimated by $\exp[-(1/\varrho)]$.
(iii) For every initial datum $J_0 \in B_\varrho(0)$ one has that $|J(t) - J_0|$ is estimated by $\exp\bigl(-(1/\varrho)^{1/(\tau+1)}\bigr)$ for $|t| \leq T$, where

$$T \simeq \exp\bigl[\exp(1/\varrho)^{1/(\tau+1)}\bigr] \ .$$

Let us add a comment concerning the local quasi–convexity hypothesis (c). We use this hypothesis in connection with Nekhoroshev's theorem. It should be remarked that the original formulation of Nekhoroshev requires the less stringent condition of steepness, which however is more difficult to handle. Later formulations of the theorem make use of the easier condition of convexity, i.e., $|\mathcal{C}(p)v \cdot v| > mv \cdot v$ for all $p \in \mathcal{G}$ and for all $v \in \mathbf{R}^n$. However, as stressed by Nekhoroshev himself, it is enough to require that the latter inequality holds for all $v \perp \omega(p)$. This is the so called condition of quasi–convexity. The fact that we restrict our attention to the neighbourhood of the invariant torus p^* allows us to further relax the quasi–convexity condition to the form (c).

2. Scheme of the proof

The proof of our theorem relies on a composition of known results, namely of KAM theorem[9] and Nekhoroshev's theorem[14][15], both in a local and a global formulation. More precisely, we need two preliminary steps in order to prove the statement (i), and two independent steps in order to prove the statements (ii) and (iii) respectively. The first step reduces the Hamiltonian to Kolmogorov's normal form, thus ensuring the existence of an invariant torus (the head torus). The second step is the construction of a Birkhoff normal form up to a finite order in the neighbourhood of the head torus, with exponential estimates of Nekhoroshev's type. The mapping $(J,\psi) \to (p,q)$ referred to in the statement of the theorem is actually the composition of the mapping leading to Kolmogorov's normal form and of that leading to Birkhoff's normal form. The third step is the application of KAM theorem in the global version due to Arnold[1]. The fourth and last step is the application of Nekhoroshev's theorem. We perform the four steps above in separate subsections. We omit all the unnecessary technical details, making a direct use of the known theorems in some available form, without attempting to find optimal estimates.

2.1 Use of Kolmogorov's normal form

We follow the formulation of Kolmogorov's theorem provided in [3]. The hypotheses (a) and (b) allow to apply the theorem. Accordingly, there exists a positive ε^* such that for every $\varepsilon < \varepsilon^*$ there exists a positive $\delta' < \delta$ and an analytic canonical transformation $(p',q') \to (p,q)$, mapping $D_{\delta'}(0)$ into $D_\delta(p^*)$, which gives the Hamiltonian the Kolmogorov's normal form

$$H'(p',q') = \sum_i \omega_i p'_i + \frac{1}{2}\sum_{i,j} C'_{i,j} p'_i p'_j + f(p',q') \tag{4}$$

with f at least quadratic in p'. In particular, the real matrix C'_{ij} satisfies the conditions (b) and (c) with new constants $m' < m$ and $d' < d$, while the quadratic part of f has zero average over the angles q'.

2.2 Use of the local formulation of Nekhoroshev's theorem

We expand now the perturbation f in (4) in power series about the origin (i.e., the invariant torus). We write the Hamiltonian as

$$H(p',q') = \sum_i \omega_i p'_i + H_2(p',q') + H_3(p',q') + \ldots \tag{5}$$

where $H_s(p',q')$ is for $s \geq 2$ a homogeneous polynomial of degree s in p'. As f is analytic in $D_{\delta'}(0)$, the expansion is convergent, and the norm of H_s decreases at least geometrically with s, being of order δ'^s. Remark that the Hamiltonian (5) resembles that of a system of perturbed harmonic oscillators, as considered for instance in [8]. Thus, we can apply the local formulation of Nekhoroshev's theorem. To this end, as usual, we perform a Birkhoff normalization up to some finite order r, thus giving the Hamiltonian the form

$$H(J,\psi) = \sum_i \omega_i J_i + Z^{(r)}(J) + \mathcal{R}^{(r)}(J,\psi) \ , \tag{6}$$

which is analytic in $D_\varrho(0)$ for some positive $\varrho < \delta'$, depending on r. Here, $Z^{(r)}(J)$ is the normalized part of the new Hamiltonian, while $\mathcal{R}^{(r)}(J,\psi)$ is the still non normalized remainder. In particular the quadratic part of $Z^{(r)}$ is nothing but $(1/2)\sum_{i,j} C'_{ij} J_i J_j$, i.e., it coincides with the average quadratic part of the Kolmogorov's normal form (4). Thus, the validity of the conditions (b) and (c) is preserved by the Birkhoff's normalization procedure.

Standard estimates (see for instance [5], [7], [8], [17]) allow to prove that in $D_\varrho(0)$ the size of the remainder is of order $(r!)^{\tau+1}\varrho^r$. An optimal choice of r as a function of ϱ, i.e., $r \simeq (1/\varrho)^{1/(\tau+1)}$, allows one to prove that there exists a positive $\overline{\varrho}$ such that for $\varrho < \overline{\varrho}$ one has

$$\left|\mathcal{R}^{(r+1)}\right| < A\,|f| \exp\left[-\left(\frac{\overline{\varrho}}{\varrho}\right)^{1/(\tau+1)}\right] \ , \tag{7}$$

with a constant A depending on the number n of degrees of freedom.

2.3 Use of Arnold's formulation of KAM theorem

We rewrite the Hamiltonian (6) in the form $H(J,\psi) = H_0(J) + H_1(J,\psi)$, where

$$H_0(J) = \sum_i \omega_i J_i + Z^{(r)}(J) \ , \quad H_1(J,\psi) = \mathcal{R}^{(r)}(J,\psi) \ . \tag{8}$$

To this Hamiltonian we apply the statement of the main theorem in [1]. The analyticity hypothesis is clearly satisfied, as well as the condition $|H_1| < M$ for some M. Indeed, in view of the result of the previous section, the remainder $\mathcal{R}^{(r)}(p,q)$ is an analytic function in a domain D_ϱ for some positive ϱ, and is bounded there by $A\,|f|\exp\left[-\left(\overline{\varrho}/\varrho\right)^{1/(\tau+1)}\right]$. We stress that, choosing ϱ small enough, the size of H_1 can be made arbitrarily small, exponentially with $1/\varrho$. Concerning the nondegeneracy condition

$$\det|\mathcal{C}'(J)| \neq 0\ , \quad \text{with } \mathcal{C}'_{i,j} = \frac{\partial^2 H_0}{\partial J_i \partial J_j}\ ,$$

which must hold for all J in the domain where the theorem is applied, we remark that it holds in some neighbourhood of the torus $J = 0$. Indeed, it holds for $J = 0$ since one clearly has $\mathcal{C}'(0) = C'_{ij}$, where C'_{ij} is just the matrix appearing in (4), and so, by continuity, it holds in a neighbourhood of $J = 0$. Thus, a straightforward application of Arnold's theorem shows that there exists a positive $\varrho_1 < \overline{\varrho}$ such that for $\varrho < \varrho_1$ the relative volume of the KAM tori in the neighbourhood of $J = 0$ is large, and tends to 1 as $\varrho \to 0$. According to Neishtadt[13], the volume of the complement of the set of invariant tori is estimated by $\sqrt{\varepsilon}$, where ε is the size of the perturbation. In our case we replace ε by $\exp(-1/\varrho)$. This proves the statement (ii).

2.4 Use of Nekhoroshev's theorem

As in the last section, we rewrite the Hamiltonian as $H(J,\psi) = H_0(J) + H_1(J,\psi)$, with H_0 and H_1 given by (8). We use the formulation of Nekhoroshev's theorem given in [4]. Again, the analyticity and nondegeneracy conditions are satisfied, as was already remarked. The condition on the smallness of the perturbation is satisfied too, due to the exponential decrease of the remainder. Instead of the convexity condition required in [4],

$$\|\mathcal{C}'(p)v\cdot v\| \geq mv\cdot v \text{ for all } v \in \mathbf{R}^n \tag{9}$$

we use the quasi–convexity, i.e., we add the condition $v \perp \omega(p)$ as remarked in the introduction. In view of the hypothesis (c), such a condition is clearly satisfied in some neighbourhood of $p = 0$. Thus, we apply Nekhoroshev's theorem, according to which, denoting by ε the size of the perturbation, for ε smaller than a positive $\overline{\varepsilon}$ one has

$$|J(t) - J(0)| < \mathcal{P}\varepsilon^{1/c} \quad \text{for} \quad |t| \leq \mathcal{T}\left(\frac{1}{\varepsilon}\right)^{1/2} \exp\left[\left(\frac{1}{\varepsilon}\right)^{1/c}\right]\ , \tag{10}$$

with constants $\mathcal{P}$, $\mathcal{T}$ and $c \simeq n^2$. (We stress that the estimates are not optimal: for better estimates see for instance [11], [17].) In our case the size of the perturbation is $\varepsilon = A\,|f|\exp\left[-\left(\overline{\varrho}/\varrho\right)^{1/(\tau+1)}\right]$. We conclude that there exists a positive ϱ_2 such that for every $\varrho < \varrho_2$ the upper bound to the diffusion time is of order $\exp\left[\exp(1/\varrho)\right]$, as claimed. This proves the statement (ii). The constant ϱ^* in the statement of the theorem must be identified with the minimum between ϱ_1 and ϱ_2.

3. Discussion

We discuss here some points that we consider relevant for interpretation and use of our result. We discuss in particular three points. (i) The possible optimality of our estimate, and the relation with some recent results of Chierchia and Gallavotti[6] on the existence of diffusion. (ii) The possible interpretation of the known phenomenon of the existence of a quite sharp threshold separating order from chaos in nearly integrable systems. (iii) The applicability of the same approach to different situations, for instance, the case of an elliptic equilibrium.

Concerning the first point, namely the optimality of the results, a remark is in order. There is a widespread opinion that Nekhoroshev's theorem is in some sense optimal, although evidence of this fact has never been produced. Thus, the following question naturally comes to mind: the Nekhoroshev's procedure has been already used in sect. 2.2; how can we go further in optimization using again the Nekhoroshev's approach (as is done in sect. 2.4)?

To clarify this point, let us consider again the Hamiltonian (1). Attempting to transform that system into an integrable one up to terms of order ε^2 is clearly impossible, as remarked long ago by Poincaré, unless some restriction is imposed either on the domain, or on the Fourier terms to be removed, or both. Indeed, the existence of resonances actually causes a change of the actions (in resonant zones) with speed of order ε. Nekhoroshev's theory, in its most general formulation, states essentially that the main effect of the perturbations reduces to an oscillation along a direction of fast drift, just due to the resonance. Such an oscillation is necessarily bounded as far as the resonant zones do not overlap: actually, the strongest condition in the so called geometric part of the Nekhoroshev's theorem is precisely the non overlapping of resonances. Superimposed to the oscillation there could be a very slow diffusion, but the diffusion speed is exponentially slow with the inverse of the perturbation. In brief, a perturbation of order ε causes an oscillation of order, e.g., $\varepsilon^{1/2}$, but a diffusion of order $\exp(-1/\varepsilon)$ only.

Let us now come back to our problem. The procedure illustrated in sect. 2.2 is an attempt to prove that in the neighbourhood of a torus the system is still integrable, in Birkhoff's spirit. Such a procedure is successful up to a certain finite order, because the diophantine condition on the frequencies ensures that in the neighbourhood of the torus there are no resonances of low order. This can be understood on the basis of the following heuristic argument. An elementary estimate shows that inside a ball of radius ϱ there are no resonances of order lower than $\varrho^{-1/(\tau+1)}$. On the other hand, due to the analyticity of the Hamiltonian, the coefficient of a resonance of order s is of order $\exp(-s)$. Thus, in a ball of radius ϱ, Birkhoff's procedure stops when one encounters a resonance of order $\varrho^{-1/(\tau+1)}$, causing a perturbation of order $\exp(-\varrho^{-1/(\tau+1)})$. This is indeed the exponential remainder given by the local Nekhoroshev's estimate in the neighbourhood of radius ϱ of the torus.

At this point comes the new idea that we introduce in the present work: we add the geometric part of the Nekhoroshev's theorem. This is justified by the following standard argument. It is known that the number of resonances of order s increases as s^n, where n is the number of degrees of freedom; on the other hand, the size of a resonant region is of order $\exp(-s)$, being essentially controlled by the coefficient of the resonant term. Thus, approaching the torus, resonances do accumulate geometrically with their order, but each of them controls a region of an exponentially small volume. One thus concludes that the total relative volume of the resonant regions is small, which precludes overlapping. This fact, on the one hand, leaves enough space free from resonances to allow the existence of a set of invariant tori of large relative measure. On the other hand, the general conclusion of Nekhoroshev's theorem that a perturbation of order ε causes a diffusion at most of order $\exp(-1/\varepsilon)$ applies here too. The superexponential estimate in this case follows from $\varepsilon \simeq \exp(-1/\varrho)$.

We stress that the superexponential estimate cannot be directly obtained from Kolmogorov's normal form, without the Birkhoff normalization of the second step. Heuristically, this can be justified as follows. Nekhoroshev's theory, including the geometric part, is developed making always reference to the unperturbed action variables. The main contribution to the change in time of such quantities is due to the so called deformation, which in the case of Kolmogorov's normal form is estimated by some power of ϱ. On the other hand, when the distance ϱ from the invariant torus decreases to zero a deformation bounded only by a power of ϱ would forbid a consistent construction of the geography of resonances, as required by the geometric part of Nekhoroshev's theorem. The good action variables introduced by Birkhoff's normalization procedure allow us to overcome this difficulty.

The natural question now is: is our new result optimal? We believe that the answer is yes. Indeed, the recent paper by Chierchia and Gallavotti[6], although not applicable to our case in a straightforward manner, suggests that a resonance of multiplicity one and size ε gives macroscopic diffusion along the resonance line in a time proportional to $\exp(1/\varepsilon)$. Since in the vicinity of a KAM torus the size of the strongest resonances is $\exp(-1/\varrho)$, one has to expect that diffusion, if any, actually requires a time of order $\exp(\exp(1/\varrho))$.

We come now to the point (ii), namely the problem of the existence of thresholds for transition from order to chaos. Here, our argument is completely heuristic. The classical exponential bound, as given by local Nekhoroshev–like results, suggests that diffusion takes place in a rather smooth way, although quite rapidly, when the perturbation parameter is increased. Conversely, the numerical simulations shows that diffusion takes place in a very sharp fashion, hardly compatible with the Nekhoroshev's exponential. Here we understand that such a sharp change from order to chaos corresponds to a transition from resonance non–overlapping to resonance overlapping, which completely changes the structure of orbits in phase space and activates different mechanisms of chaotic diffusion, the diffusion speed passing from exponentially slow with respect to resonance strength to directly proportional to resonance strength.

Concerning the point (iii), we discuss on the one hand the application to the case of an elliptic equilibrium, and, on the other hand, a possible extension to our previous work [12].

We first consider the case of an elliptic equilibrium point with the frequencies of the harmonic part of the Hamiltonian satisfying a diophantine condition (2). The main remark is that such an equilibrium is nothing but a degenerate invariant torus. Thus, one can follow the same steps as above, just skipping the construction of Kolmogorov's normal form. The application of Arnold's theorem then leads to the existence of a set of invariant tori, the relative volume of which tends to 1 as $1-\exp(-1/\varrho)$ in the limit $\varrho \to 0$, (ϱ being in this case the distance from the equilibrium point); this improves a little, from a quantitative viewpoint, the statement in [2], appendix 8. The application of Nekhoroshev's theorem is instead new: if the convexity condition (c) of the theorem is satisfied, then one obtains a superexponential estimate of the diffusion time, instead of the usual exponential one.

We come now to our previous result in [12]. We considered there an Hamiltonian of the form (1), and proved that the diffusion speed can be made arbitrarily small provided one starts close enough to an invariant torus. However, we could not recover a superexponential estimate because we did not actually use in a complete fashion the powerfulness of the geometric part of the Nekhoroshev's theorem. A straightforward application of this idea leads to a quantitative reformulation of that result, stating that for ε sufficiently small there exists a domain, D_ε say, which contains open balls of radius ε, characterized by a diffusion speed of order $\exp\bigl(-\exp(1/\varepsilon)\bigr)$. The domain D_ε can be identified with the nonresonant region of the geometric construction by Nekhoroshev.

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