ON THE DIMENSION OF DETERMINISTIC AND RANDOM CANTOR-LIKE SETS, SYMBOLIC DYNAMICS, AND THE ECKMANN-RUELLE CONJECTURE

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ABSTRACT. In this paper we unify and extend many of the known results on the dimensions of deterministic and random Cantor-like sets in $\mathbb{R}^n$ using their symbolic representation. We also construct several new examples of such constructions that illustrate some new phenomena. These sets are defined by geometric constructions with arbitrary placement of subsets. We consider Markov constructions, general symbolic constructions, nonstationary constructions, random constructions (determined by a very general distribution), and combinations of the above. One application to dynamical systems is a counterexample to the C^0 version of the Ruelle-Eckmann conjecture.

Introduction

In this paper we unify and extend many of the known results on the dimension of deterministic and random Cantor-like sets in $\mathbb{R}^n$. These sets are defined by geometric constructions of different types.

Our most basic construction, which we call a *simple geometric construction*, defines a Cantor-like set of the form

$$F = \bigcap_{n=1}^{\infty} \bigcup_{(i_1 \cdots i_n)} \Delta_{i_1 \cdots i_n},$$

where the basic sets on the nth step of the construction, $\Delta_{i_1 \cdots i_n}$, $i_k = 1, \cdots, p$, are closed, disjoint, and $\Delta_{i_1 \cdots i_n j} \subset \Delta_{i_1 \cdots i_n}$, for $j = 1, \cdots, p$. We assume that the $\operatorname{diam}(\Delta_{i_1 \cdots i_n}) \to 0$ as $n \to \infty$ and we emphasize that the placement of the sets $\Delta_{i_1 \cdots i_n}$ can be arbitrary as long as they satisfy the above conditions. See Figure 1.

Key words and phrases. Hausdorff dimension, box dimension, Cantor-like set, geometric construction, random geometric construction, gauge function, Eckmann-Ruelle Conjecture.

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FIGURE 1. SIMPLE GEOMETRIC CONSTRUCTION

We also consider a much broader class of geometric constructions where the admissible basic sets $\Delta_{i_1 \cdots i_n}$ are determined by a compact shift-invariant subset of the full shift on p symbols. Our main results provide lower and upper estimates for the Hausdorff dimension and the box dimension of Cantor-like limit sets for large classes of geometric constructions. We also prove strict positivity and boundedness of the Hausdorff measure of the limit set for a broad class of constructions.

We will use two methods for obtaining a lower bound for the Hausdorff dimension of a set: the uniform and non-uniform mass distribution principles. See Appendix 2. The uniform mass distribution principle requires the existence of a measure m for which

$$m(B(x,r)) \leq Cr^s, \tag{1}$$

where $B(x,r)$ is the ball of radius r centered at the point $x \in F$ and C is a constant. Then s produces a lower bound for $\dim_H F$, the Hausdorff dimension of F. We stress that (1) must hold for all $x \in F$ and the constant C must be independent of x and r. The non-uniform mass distribution principle requires that (1) hold for $m-$almost every $x \in F$ with $C = C(x)$ a measurable function. Again, s provides a lower bound for the Hausdorff dimension of the limit set. If the uniform mass distribution principle holds, one can obtain refined information about the $s-$Hausdorff measure of F. Although uniform mass distribution is stronger than the non-uniform version, we will establish it for a broard class of geometric constructions.

One particular case is when a geometric construction has sets at the nth step that are strictly geometrically similar to the corresponding sets at the $(n-1)$th step. This means that there is a collection of similarity maps (affine contractions) or more generally contraction maps $h_1, \cdots, h_p$ such that

$$\Delta_{i_1 \cdots i_n} = h_{i_1} \circ \cdots \circ h_{i_n}(\Delta),$$

where Δ denotes a ball in $\mathbb{R}^n$ (see Section 4). This construction is called a *similarity construction* since it exposes a self-similar character of the geometric process. These special constructions have been one of the main objects of study in dimension theory for many years.

About 50 years ago, Moran [Mo] computed the Hausdorff dimension of geometric constructions in $\mathbb{R}^n$ given by p non-overlapping balls $\Delta_{i_1 \ldots i_n}$ satisfying $\operatorname{diam}(\Delta_{i_1 \ldots i_n j}) = \lambda_j \operatorname{diam}(\Delta_{i_1 \ldots i_n})$ where $0 < \lambda_j < 1$ for $j = 1, \ldots, p$ are the *ratio coefficients.* We call this *a Moran construction.* It *need not* be a similarity construction. Moran discovered the formula $s = \dim_H F$, where s is the unique root of the equation

$$\sum_{i=1}^{p} \lambda_i^t = 1. \tag{2}$$

He also showed that the $s-$dimensional Hausdorff measure of the limit set is finite and strictly positive.

Moran's great insight was to realize that the similarity maps, or even the spacing of the balls $\{\Delta_{i_1\cdots i_n}\}$ are not important in the calculation of the Hausdorff dimension of the limit set: the dimension depends only on the ratio coefficients. Moran proved this using the (uniform) mass distribution principle applied to the $s-$dimensional Hausdorff measure. We believe this is a seminal paper that should be much better known. There are very few papers in the literature dealing with geometric constructions that do not involve similarity maps.

In this paper we extend the Moran approach to much broader classes of geometric constructions. We study constructions with general *geometry* of the basic sets and general *symbolic representations*.

One can not expect to obtain any refined estimates for the Hausdorff and box dimensions of the limit set F of a construction with arbitrary shape and spacing of the basic sets. We control the geometry of the construction by either restricting the shapes or sizes of the basic sets, the spacing of the basics sets, or both. If one has strong control over the sizes of the basic sets, then the spacing can be fairly arbitrary, and vice-versa. In this paper we introduce a new approach to studying geometric constructions having complicated geometry. Our approach is based on the notions of regularity and boundedness of the construction. *Regular and bounded constructions* are those where the control over the geometry is effected in the spirit of Moran processes by *generalized ratio coefficients* that encode the information about *both* the shape and spacing of the basic sets. Example 3 in Section 5 illustrates the role of spacing.

For some constructions these coefficients are determined by the largest inscribed balls and smallest circumscribed balls for the basic sets. However, we construct an example where these numbers are completely independent of these balls (see Section 5, Example 8). We will compute the generalized ratio coefficients for all previously studied classes of geometric constructions as well as several new ones.

One new class of geometric constructions that we introduce is *asymptotic geometric constructions* where the ratio coefficients depend on steps of the construction and admit a *good* asymptotic behavior. We apply our techniques to study asymptotic constructions and show that the non-uniform mass distribution principle can be used to estimate the Hausdorff dimension of the limit set. For asymtotic constructions, one can not expect the uniform mass distribution principle to hold. Hence the Hausdorff measure is usually zero or infinite, even in simple examples (see Section 5, Example 4). In order to obtain more information about the Hausdorff measure one can use a *gauge function* (see Appendix 4). In Section 5, we present a family of simple asymptotic geometric constructions which admit infinitely many gauge functions depending on the rate of convergence of ratio coefficients. This illustrates that the asymptotic category of constructions is quite rich in that there exists a large variety of different limit sets exhibiting many different structural properties.

We apply our study of asymptotic constructions to analyze a new class of geometric constructions which are *random Moran geometric constructions*. These constructions are essentially Moran construction with ratio coefficients chosen randomly from an arbitrary ergodic stationary process. We also consider random one-dimensional constructions. In the literature, random constructions are usually considered as a distinct class of geometric constructions. We stress that our approach completely unifies the analysis of many types of geometric constructions.

The Moran process has a natural symbolic description by the full shift on p symbols, where p is the number of basic sets on the first step of the construction. We consider constructions modeled by a general symbolic dynamical system.

For the Moran process, the crucial observation is the existence of a measure m on the set F such that the Hausdorff dimension of F is equal to the Hausdorff dimension of m. This measure is the pullback of the equilibrium state for the function $\phi(x) = \log \lambda_{i_1}$ where x is associated with a sequence $(i_1 i_2 \cdots)$ and s is the unique root of the equation

$$P(s\phi) = 0. \tag{3}$$

For the Moran constructions, s is the Hausdorff dimension of the limit set as well as its box dimension. We show that this is true for general symbolic Moran constructions (see Proposition 3 and Theorem 5.)

Equation (3) was discovered by Bowen [Bo2] and seems to be universal. We will show that all known equations previously used to compute the Hausdorff dimension (for example equation (2)) coincide with or are particular cases of (3).

For the general symbolic geometric constructions the measure m is an equilibrium measure and admits the non-uniform mass distribution principle. There is a crucial difference between Gibbs measures and equilibrium measures in Statistical Physics (see [R] and Appendix 3; see also Section 1, Remark 1). These notions coincide for subshifts of finite type, but need not coincide for general symbolic systems. We show that if m is a Gibbs measure, then the Hausdorff dimension of the limit set, s, can be studied using the uniform mass distribution principle, and the $s-$dimensional Hausdorff measure is positive and finite (see Theorems 2 and 4). If m is an equilibrium state, then a non-uniform mass distribution principle can be applied (see Theorem 1).

Theorem 5 establishes the coincidence of the Hausdorff dimension and the box dimension of a measure and is similar in spirit to some results of Ledrappier and Young [LY]. It supports a general belief that the coincidence of the Hausdorff dimension and box dimension of a *set* is a rare phenomenon and requires *rigid* geometric constructions like some of those considered in this paper. In Section 5, we provide an example of a simple geometric construction in the plane with basic sets being rectangles for which the Hausdorff dimension $\dim_H F$, the lower box dimension $\underline{\dim}_B F$, and the upper box dimension $\overline{\dim}_B F$, are distinct. The construction uses different ratio coefficients in different directions and combines both vertical and horizontal stacking of rectangles to effect the noncoincidence of the dimensions. In [PW], the authors found that number theoretic properties of the ratio coefficients, related to Pisot numbers, is another mechanism to cause noncoincidence of dimensions. Another example which illustrates the non-coincidence of the Hausdorff and box dimensions was constructed in [Mc].

The coincidence of the Hausdorff dimension and box dimension of a *measure* is more common. There is a general criterion proved by Young [Y] that guarantees the coincidence. Namely, let m be a Borel measure on F such that for $m-$almost every $x \in F$ the limit

$$\lim_{r\to 0} \frac{\log m(B(x,r))}{\log r} = d_m(x)$$

exists. The limit $d_m(x)$ is called the *pointwise dimension* at x. In this case m is called *exact dimensional*. If the limit does not exist, one can consider the lower and upper limits

$\underline{d}_m(x), \overline{d}_m(x)$ to obtain the *lower and upper pointwise dimensions* at x. If for m-almost every x

$$d_m(x) = \text{const} \stackrel{\text{def}}{\equiv} s$$

then the Hausdorff dimension of m, $\dim_H m$, and the lower and upper box dimensions of m, $\underline{\dim}_B m$ and $\overline{\dim}_B m$, coincide and have the common value s.

This statement poses the problem of whether a given measure m is exact dimensional and moreover whether $d_m(x) = const$ almost everywhere. In [C], Cutler constructed an example of a continuous map that preserves an ergodic exact dimensional measure m with $d_m(x)$ essentially non-constant. In Section 6.4, we present a much simplier version of the construction of such a map that uses a special simple geometric construction. If the map is smooth and m is exact dimensional and ergodic, then $d_m(x) = const$ almost everywhere (since $d_m(x)$ is invariant under the map and measurable). Ledrappier and Misiurewicz [LM] constructed a one-dimensional smooth map preserving an ergodic measure that is not exact dimensional. Eckmann and Ruelle conjectured that an ergodic measure invariant under a C^2-diffeomorphism with non-zero Lyapunov exponents is exact dimensional (and, hence, $d_m(x) = const$, see [ER]). This was proved in [Y] for two-dimensional maps and in [PY] for some measures including Gibbs measures for Axiom A diffeomorphisms (see also [L]). In Section 6, we construct a homeomorphism having an ergodic invariant Gibbs measure with positive entropy that has different upper and lower pointwise dimensions almost everywhere. In other words, the *continuous* version of the Eckmann-Ruelle Conjecture fails.

Mauldin and Williams [MW1] computed the Hausdorff dimension for a Markov process given by similarity maps. They used symbolic dynamics, large deviation theory, and graph theory. We give a considerable generalization of their result as well as a considerable simplification of their proof. The Moran construction in the Markov case was studied by Stella [St] with some additional strong assumptions. Afraimovich and Shereshevsky [AS] found a lower estimate for the Hausdorff dimension of some simple geometric constructions. Similar types of simple geometric constructions, given by two-dimensional self-affine maps related to graphs of functions, were considered by Bedford and Urbanski in [BU]. Shereshevsky [S] also considered some one-dimensional Markov geometric constructions.

Geometric constructions of random type have been considered by Falconer [F2], Graf [G], Kahane [K], Graff, Mauldin and Williams [GMW], and Mauldin and Williams [MW2]. These authors studied special types of branching processes that correspond to the full shift on p symbols with p^n ratio coefficients at step n chosen randomly, essentially independently and with the same distribution on $(0, 1)$. They also assume some independence conditions over n. In this paper, we consider branching processes that are associated with arbitrary compact shift-invariant subsets. We generate the ratio coefficients by chosing p random numbers on the interval (a, b) where $0 < a \leq b < 1$. We do not require that these numbers be independent nor be indentically distributed. A good general reference for dimension theory is [F1].

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Section 1: Geometric Constructions

We define geometric constructions of Cantor-like subsets of $\mathbb{R}^d$ by using a symbolic description in the space of all one-sided infinite sequences $(i_1 i_2 \cdots)$ on p symbols. We denote this space by Σ_p^+ and endow it with its usual topology (see Appendix 3).

A **symbolic construction** is defined by

a) a compact set $Q \subset \Sigma_p^+$ invariant under the shift σ (i.e., $\sigma(Q) = Q$)

b) a family of closed sets called **basic sets** $\{\Delta_{i_1 \cdots i_n}\} \subset \mathbb{R}^d$ for $i_j = 1, 2, \ldots, p$ and $n \in \mathbb{N}$ where the n-tuples $(i_1 \cdots i_n)$ are admissible with respect to Q (i.e., there exists $\omega = (i'_1, i'_2, \cdots) \in Q$ such that $i'_1 = i_1, i'_2 = i_2, \cdots, i'_n = i_n$) and these sets satisfy

$$\lim_{n \to \infty} \max_{(i_1 \cdots i_n)} \operatorname{diam}(\Delta_{i_1 \cdots i_n}) = 0. \tag{4}$$

For any admissible sequence $(i_1 \cdots i_{n+1}) \in \{1, \cdots, p\}^n$, we require that

$$\Delta_{i_1 \cdots i_n i_{n+1}} \subset \Delta_{i_1 \cdots i_n}$$

and

$$\Delta_{i_1 \cdots i_n} \cap \Delta_{i'_1 \cdots i'_n} = \emptyset \quad \text{if } (i_1 \cdots i_n) \neq (i'_1, \cdots, i'_n). \tag{5}$$

The **limit set** F for this construction is defined by

$$F = \bigcap_{n=1}^{\infty} \bigcup_{\substack{(i_1 \cdots i_n) \\ \text{admissible}}} \Delta_{i_1 \cdots i_n}.$$

The set F is a generalized Cantor set, i.e., it is a perfect, nowhere dense, and totally disconnected set. We stress that the placement of the sets $\{\Delta_{i_1 \cdots i_n}\}$ is completely arbitrary, and we make no assumptions on the regularity of the boundaries of the sets $\{\Delta_{i_1 \cdots i_n}\}$ which can be fractal. The basic sets need not even be connected.

Given $x \in F$ and $n > 0$, there exists a unique set $\Delta_{i_1 \cdots i_k}(x)$ that contains x and hence $x = \bigcap_{k=1}^{\infty} \Delta_{i_1 \cdots i_k}(x)$. This gives a unique one-sided infinite sequence $(i_1 i_2 \cdots)$ such that the mapping $\chi : F \to Q$ defined by $x \mapsto (i_1 i_2 \cdots)$ is a homeomorphism from F onto Q. The associated symbolic dynamics is one of our main tools to compute the Hausdorff dimension of the limit set F.

Consider the symbolic dynamical system (Q, σ), where $Q \subset \Sigma_p^+$. Given a $p-$tuple $\alpha = (\alpha_1, \cdots, \alpha_p)$ such that $0 < \alpha_i < 1$, there exists a uniquely defined number s_α such that $P(s_\alpha \log \alpha_{i_1}) = 0$, where P denotes the topological pressure (see Appendix 3). Let μ_α denote an equilibrium measure for the function $(i_1, i_2, \cdots) \mapsto s_\alpha \log \alpha_{i_1}$ on Q, and let m_α be the pull back measure on F under the coding map χ.

1. Lower estimates for the Hausdorff dimension.

Since we allow arbitrary spacing and shape of the basic sets, one can not expect, in general, to obtain any refined estimates for the Hausdorff and box dimensions of the limit set F (see Section 5, Example 8.) To obtain refined estimates, one needs to control the *geometry* of the construction by either controlling the spacing of the basic sets, sizes or shapes of the basic sets, or both. If one has strong control over the sizes of the basic sets, then the spacing can be fairly arbitrary, and vice-versa. One well known method to control the geometry is to effect the construction using similarity maps (see Section 4). This class of construction is very restrictive since one has strong control over both the spacing of the basic sets and their sizes in all directions. In [Mo], Moran presented a more general construction where one has complete control over the sizes of the basic sets but where the spacing is arbitrary. Moran computed the Hausdorff dimension of the limit set for such a construction using the uniform mass distribution principle. We will present a more general method to control the geometry of the construction and extend the Moran approach from the full shift to a general symbolic system. This requires the non-uniform mass distribution principle (instead of the uniform one). We will estimate the dimensions for a large class of constructions called regular constructions, which include many constructions where one has only moderate control over the spacing and/or the sizes of the basic sets.

In a regular construction, control over the geometry is given by numbers $\gamma_1, \cdots, \gamma_p$ such that one can approximate the basic sets $\Delta_{i_1 \cdots i_n}$ by balls of radius $\prod_{j=1}^{n} \gamma_{i_j}$. We believe that one can compute the numbers $\gamma_1, \cdots, \gamma_p$ and construct the balls using detailed information about shape and spacing of the basic sets. In some cases these balls coincide with the largest balls that can be inscribed in the basic sets. However, we construct an example where the optimal numbers $\gamma_1, \cdots, \gamma_p$ are completely independent of the radius of the largest inscribed balls (see Section 5, Example 8). We will compute the numbers $\gamma_1, \cdots, \gamma_p$ for all previously studied classes of geometric constructions as well as several new ones, and we will produce refined estimates for the Hausdorff dimension from below.

Given $0 < r < 1$ and a vector of numbers $\gamma = (\gamma_1, \cdots, \gamma_p)$, $0 < \gamma_i < 1, i = 1, \ldots, p$, we first define a special cover $\mathfrak{U}_r = \mathfrak{U}_r(\gamma)$ of the limit set F. For any $x \in F$, let $n(x)$ denote the unique positive integer such that $\gamma_{i_1}\gamma_{i_2} \cdots \gamma_{i_{n(x)}} > r$ and $\gamma_{i_1}\gamma_{i_2} \cdots \gamma_{i_{n(x)+1}} \leq r$ where $\chi(x) = (i_1 i_2 \cdots)$. It is easy to see that $n(x) \to \infty$ as $r \to 0$ uniformly in x. Fix $x \in F$. Consider the set $\Delta_{i_1 \cdots i_{n(x)}}$. We have $x \in \Delta_{i_1 \cdots i_{n(x)}}$, and if $y \in \Delta_{i_1 \cdots i_{n(x)}}$ and $n(y) \geq n(x)$, then

$$\Delta_{i_1 \cdots i_{n(y)}} \subset \Delta_{i_1 \cdots i_{n(x)}}.$$

Let $\Delta(x)$ be the largest set containing x with the property that $\Delta(x) = \Delta_{i_1 \cdots i_{n(z)}}$ for some $z \in \Delta(x)$ and $\Delta_{i_1 \cdots i_{n(y)}} \subset \Delta(x)$ for any $y \in \Delta(x)$. The sets $\Delta(x)$ corresponding to different $x \in F$ either coincide or are disjoint. We denote these sets by $\Delta_r^{(j)}$, $j = 1, \cdots, N_r$. There exist points x_j such that $\Delta_r^{(j)} = \Delta_{i_1 \cdots i_{n(x_j)}}$. These sets form a disjoint cover of F.

Consider the open Euclidean ball $B(x, r)$ of radius r centered at a point x. Let $N(x, r)$ denote the number of sets $\Delta_r^{(j)}$ that have non-empty intersection with $B(x, r)$. We call a vector γ **l-estimating** if $N(x, r) \leq$ constant, uniformly in x and r. We call a symbolic construction **regular** if it admits an l-estimating vector. If $\gamma = (\gamma_1, \ldots, \gamma_p)$ is an l-estimating vector for a regular symbolic construction, then any vector $\tilde{\gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_p)$ for which $\gamma_i \geq \tilde{\gamma}_i, i = 1, \ldots, p$ is also l-estimating. It is easy to see that $s_\gamma \leq s_{\tilde{\gamma}}$. On the other hand a vector $\tilde{\gamma} = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_p)$ for which $\tilde{\gamma}_i$ is sufficiently close to 1 for some i is not l-estimating.

Below we will give several examples of regular constructions although there are many constructions which are not regular. For example, if the Hasudorff dimension of the limit set is zero, then the corresponding construction is not regular. Let F_1, F_2 be two limit sets for two symbolic constructions on the line: the first construction is defined on the interval $[0,1]$ and the second one on the interval $[2,3]$. It is easy to see that if $\dim_H F_1 = 0$ and $\dim_H F_2 > 0$ then the set $F_1 \cup F_2$ is the limit set for a construction which is not regular but whose limit set has positive Hausdorff dimension.

In Section 2 we present conditions which guarantee regularity of a large class of one-dimensional geometric constructions. We show that, in particular, if the geometric construction satisfies

$$\inf_n \inf_{(i_1 i_2 \ldots i_n)} \frac{\log \operatorname{diam}(\Delta_{i_1 i_2 \ldots i_n})}{n} \geq constant > 0,$$

then the construction is regular. This condition provides strong control over the ratio coefficients and requires them to be uniformly bounded away from zero. It is easy to show that if the ratio coefficients go to zero uniformly in n, then the Hausdorff dimension of the limit set is zero.

Barreira [Ba] has shown that the above assumption is almost optimal. He exhibited a one dimensional simple geometric construction with ratio coefficients not uniformly bounded away from zero such that: (1) the construction is not regular, (2) the limit set F satisfies the following strong homogeneity property

$$\dim_H F \cap B(x,r) = constant > 0$$

for all $x \in F$ and $r > 0$.

If one considers a geometric construction in the plane with limit set $F = F_1 \times F_2$ where F_1 and F_2 are limit sets for one-dimensional geometric constructions and $\dim_H F_1 = 0$, then the construction is not regular. We believe that a geometric construction is regular if the projection of the basic sets at step n onto every line satisfies the above uniformity condition uniformly over the direction of the projection.

Theorem 1. *Let F be the limit set for a regular symbolic construction. Then $s_\gamma \leq dim_H F$ for any l-estimating vector γ. Hence $\sup s_\gamma \leq dim_H F$ where the supremum is taken over all l-estimating vectors γ.*

The lower bound for the Hausdorff dimension of the limit set follows from the lower bound for the lower pointwise dimension of the measure m_γ restricted to a sequence of sets that exhaust F. A more delicate question in dimension theory is whether the Hausdorff measure of the limit set is positive. The answer is presumably negative for a general symbolic geometric construction. For a regular constrution, the positivity of the Hausdorff measure may depend on whether the equilibrium measure m_γ is a Gibbs measure. If this measure is Gibbs, then it satisfies the uniform mass distribution principle and we have our most refined estimates. If the construction is Markov, then the measure m_γ is Gibbs. Little is known about the existence of Gibbs measures for general symbolic dynamical systems (see Remark 1). We denote by $m_H(t,F)$ the $t-$dimensional Hausdorff measure of F (see Appendix 1).

Theorem 2. *Let F be the limit set a regular symbolic construction. Assume that there exists an l-estimating vector γ such that the measure m_γ is a Gibbs measure. Then*

1) *the measure m_γ satisfies the uniform mass distribution principle*
2) $0 < m_{\mathrm{H}}(s_\gamma, F)$, *moreover,* $m_\gamma(Z) \le C\, m_{\mathrm{H}}(s_\gamma, Z)$ *for any* $Z \subset F$ *where* $C > 0$ *is a constant*
3) $s_\gamma \le \underline{d}_{m_\gamma}(x)$ *for every* $x \in F$.

The second statement in Theorem 2 is nontrivial only when $s_\gamma = \dim_H F$. Otherwise, $m_H(s_\gamma, F) = \infty$. If $s_\gamma < s = \dim_H F$, then the $s-$Hausdorff measure may be zero or infinite.

2. Upper estimates for the upper box dimension.

In order to obtain upper estimates for the upper box dimension, we require that the diameters of the basic sets decrease exponentially.

More precisely, we say that a vector $\lambda = (\lambda_i, \ldots, \lambda_p)$, $0 < \lambda_i < 1$ is a **u-estimating** vector for a construction if

$$\mathrm{diam}(\Delta_{i_1\cdots i_n}) \le C \prod_{j=1}^{n} \lambda_{i_j}$$

where $C > 0$ is a constant. The symbolic construction is called **bounded** if there exists a u-estimating vector λ. It is easy to see that if $\lambda = (\lambda_1, \ldots, \lambda_p)$ is a u-estimating vector for a bounded construction, then any vector $\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_p)$ for which $\tilde{\lambda}_i \ge \lambda_i$, $i = 1, \ldots, p$ is a u-estimating vector and $s_\lambda \le s_{\tilde{\lambda}}$.

We now wish to estimate the upper box dimension of the limit set F for a symbolic geometric construction. We need not assume that the construction is regular, but only bounded.

Theorem 3. *Let F be the limit set for a bounded symbolic construction. Then*

(1) $\overline{dim}_B F \le s_\lambda$ *for any u-estimating vector* λ. *Hence* $\overline{dim}_B F \le \inf s_\lambda$, *where the infinum is taken over all u-estimating vectors* λ
(2) $\overline{d}_{m_\lambda}(x) \le s_\lambda$ *for any u-estimating vector* λ *and* $m_\lambda-$*almost every* $x \in F$.

The arguments in the proof of the upper bound for the upper box dimension are essentially due to Bowen and were pointed out to us by Ledrappier. In the case when the measure m_λ is a Gibbs measure, we can establish finiteness of the $s_\lambda-$Hausdorff measure and obtain a stronger statement about the upper pointwise dimensions.

Theorem 4. *Let F be the limit set for a bounded symbolic construction. Assume that there exists a u-estimating vector λ such that the measure m_λ is a Gibbs measure. Then*

(1) $m_{\mathrm{H}}(s_\lambda, F) < \infty$, *moreover* $m_{\mathrm{H}}(s_\lambda, Z) \le C m_\lambda(Z)$ *for any* $Z \subset F$ *where* $C > 0$ *is a constant*
(2) $\overline{d}_{m_\lambda}(x) \le s_\lambda$ *for every* $x \in F$.

The next statement provides an upper estimate for the number s_λ.

Proposition 1. *Let F be the limit set for a bounded symbolic construction and λ a u-estimating vector.*

(1) *We have*

$$s_\lambda \le \frac{h(\sigma \,|\, Q)}{-\log \lambda_{\max}}$$

where $\lambda_{\max} = \max_{k=1}^{p}\{\lambda_k\}$, and $h(\sigma \,|\, Q)$ denotes the topological entropy. Equality occurs if $\lambda_i = \lambda$ for $i = 1, \cdots, p$.

(2) *See* [Fu]. *If $\lambda_i = \lambda$ for $i = 1, \cdots, p$, then*

$$s_\lambda = dim_H F = \underline{dim}_B F = \overline{dim}_B F = \frac{h(\sigma \,|\, Q)}{-\log \lambda}.$$

(3) *If $h(\sigma \,|\, Q) = 0$, then $s_\lambda = dim_H F = \underline{dim}_B F = \overline{dim}_B F = 0$.*

The following Theorem is an immediate consequence of Theorems 1–4.

Theorem 5. *Let F be the limit set for a regular and bounded symbolic construction. Then for any l-estimating vector γ and any u-estimating vector λ we have*

(1) $s_\gamma \le dim_H F \le \underline{dim}_B F \le \overline{dim}_B F \le s_\lambda$ *where s_γ and s_λ are the unique roots of the equations $P(s_\gamma \log \gamma_{i_1}) = 0$ and $P(s_\lambda \log \lambda_{i_1}) = 0$, respectively*

(2) *if $\gamma_i = \lambda_i$ for $i = 1, \cdots, p$ then*

$$s = s_\gamma = s_\lambda = dim_H F = \underline{dim}_B F = \overline{dim}_B F.$$

Moreover, $m = m_\gamma = m_\lambda$, and if the measure m is Gibbs, then $m_H(s, *)$ is equivalent to m and $\underline{d}_m(x) = \overline{d}_m(x) = s$ for every $x \in F$.

The coincidence of the Hausdorff dimension and the box dimension was established by Falconer [F3], however he did not compute the actual value of the dimension.

In general, the lower and upper pointwise dimensions $\underline{d}_{m_\gamma}(x)$ and $\overline{d}_{m_\gamma}(x)$ (as well as $\underline{d}_{m_\lambda}(x)$ and $\overline{d}_{m_\lambda}(x)$) do not coincide almost everywhere (see Sections 6.2 and 6.3).

3. Markov construction.

We will consider two important special cases of symbolic constructions specified by a subshift of finite type or the full shift. Let A denote a $p \times p$ transfer matrix with entries $A(i,j) = 0$ or 1 and let Σ_A^+ consists of admissible sequences $(i_1 i_2 \cdots)$ with respect to A (i.e., $A(i_j, i_{j+1}) = 1$ for $j = 1, 2 \cdots$). The construction is called **Markov** if $Q = \Sigma_A^+$. In the case when the set $Q = \Sigma_p^+$ the construction is called **simple**.

Consider a subshift of finite type (Σ_A^+, σ). Given p numbers $0 < \alpha_1, \cdots, \alpha_p < 1$, we define a $(p \times p)$ diagonal matrix $M_t(\alpha) = \mathrm{diag}(\alpha_1^t, \cdots, \alpha_p^t)$. Let $\rho(B)$ denote the spectral radius of the matrix B.

Proposition 2. *The equation $P(s\log\alpha_{i_1}) = 0$ is equivalent to the equation $\rho(A^*M_s(\alpha)) = 1$, where A^* denotes the transpose of the matrix A.*

We will assume that the transfer matrix A is transitive, i.e., there exists $N \in \mathbb{N}$ such that $(A^N)_{i,j} > 0$ for all i and j (see Remark (4) below).

The following corollaries are immediate consequences of Theorem 5 and Proposition 2.

Corollary 1. *Let F be the limit set for a* **Markov** *regular and bounded construction. Then for any l-estimating vector γ and any u-estimating vector λ we have*

$$s_\gamma \leq dim_H F \leq \underline{dim}_B F \leq \overline{dim}_B F \leq s_\lambda$$

where s_γ and s_λ are the unique roots of the equations

$$\rho(A^*M_s(\gamma)) = 1, \quad \rho(A^*M_s(\lambda)) = 1$$

respectively.

Corollary 2. *Let F be the limit set for a* **simple** *regular and bounded construction. Then for any l-estimating vector γ and any u-estimating vector λ we have*

(1) *$s_\gamma \leq dim_H F \leq \underline{dim}_B F \leq \overline{dim}_B F \leq s_\lambda$ where s_γ and s_λ are the unique roots of the equations*

$$\sum_{i=1}^{p} \gamma_i^t = 1 \quad \text{and} \quad \sum_{i=1}^{p} \lambda_i^t = 1$$

respectively

(2) *the Gibbs measures μ_γ and μ_λ on Σ_p^+ satisfy*

$$\mu_\gamma(C_{i_1\cdots i_n}) = \prod_{j=1}^{n} \gamma_{i_j}^{s_\gamma} \quad \text{and} \quad \mu_\lambda(C_{i_1\cdots i_n}) = \prod_{j=1}^{n} \lambda_{i_j}^{s_\lambda}$$

where $C_{i_1\cdots i_n}$ is a cylinder set.

Remarks.

(1) It follows from the expansiveness of the shift map that for any general symbolic system and any continuous function, there exists an equilibrium measure corresponding to this function. However, in general, this measure need not be a Gibbs measure. It is well known that if a symbolic system has the specification property, then the equilibrium measure is Gibbs and is unique provided the function is Hölder continuous [R]. Thus, Theorem 5 provides refined estimates for the Hausdorff and box dimensions as well as the Hausdorff measure for a symbolic geometric construction modeled by a symbolic system which satisfies specification.

Subshifts of finite type and their finite factors (sophic systems) are known to satisfy specification [R, We]. The only other example we know is the beta-shift [BM] for special values of β.

If the topological entropy of $\sigma \mid Q$ is zero, then by Proposition 1, the Hausdorff dimension and box dimension coincide and are zero. In this case the measure μ_λ is a measure of maximal entropy and is not a Gibbs measure.

We believe that one can find a symbolic system with positive topological entropy and can build a $\gamma-$regular and $\lambda-$bounded symbolic construction such that the two measures m_γ and m_λ are not Gibbs. Moreover, m_λ will not be equivalent to the $s_\lambda-$Hausdorff measure. There is an interesting question of whether one can build a regular (or bounded) symbolic construction admitting two l-estimating vectors γ_1, γ_2 (two u-estimating vectors λ_1, λ_2) such that m_{γ_1} is Gibbs and m_{γ_2} is not (m_{λ_1} is a Gibbs measure and m_{λ_2} is not).

(2) There exists a weaker version of regularity of a construction which enables one to prove lower estimates for the Haudorff dimension of the limit set. This weaker condition is sufficient to prove Theorem 1.

Given $0 < r < 1$ and a vector of numbers $\gamma = (\gamma_1, \cdots, \gamma_p)$, $0 < \gamma_i < 1$, $i = 1, \dots, p$, consider the special cover $\mathfrak{U}_r = \mathfrak{U}_r(\gamma)$ of the limit set F by basic sets $\Delta_r^{(j)} = \Delta_{i_1 \cdots i_{n(x)}}$. Let $N(x, r)$ denote the number of sets $\Delta_r^{(j)}$ that have non-empty intersection with $B(x, r)$. We call a vector γ **weak l-estimating** if for every $\epsilon > 0$ there exists a $m-$measurable function $N(x, \epsilon)$ such that for all $x \in F$ and $r > 0$

$$N(x, r) \leq N(x, \epsilon)\, r^{-\epsilon}. \tag{6}$$

We call a symbolic construction **weakly regular** if it admits a weakly l-estimating vector. Obviously, a regular symbolic construction is weakly regular.

Theorem 1'. *Let F be the limit set for a weakly regular symbolic construction. Then $s_\gamma \leq dim_H F$ for any weakly l-estimating vector γ. Hence* $\sup s_\gamma \leq dim_H F$ *where the supremum is taken over all weakly l-estimating vectors γ.*

One can also obtain an approriate version of Theorem 5 under this weaker hypothesis.

(3) Two closed sets are called *non-overlapping* if their intersection consists of only boundary points. By examining the proofs of Theorems 1–5, one can show that they are valid for one-dimensional constructions with non-overlapping basic sets. See [Mo, MW1].

(4) In the Markov constructions above, we assumed that the transfer matrix A was transitive. For an arbitrary transfer matrix A, one can decompose the set Σ_A^+ into two shift-invariant subsets: the wandering set Q_1 and the non-wandering set Q_2. The latter can be further partitioned into finitely many shift-invariant subsets of the form $\Sigma_{A_i}^+$ where each matrix A_i is transitive and corresponds to a class of equivalent recurrent states [AJ]. The limit set F contains disjoint sets $F_i = \chi^{-1}(\Sigma_{A_i}^+)$. Each set F_i is the limit set for a Markov construction defined by the transitive matrix A_i and hence admits lower and upper estimates for the Hausdorff and box dimensions stated in Corollary 1. In [MW1], the authors discuss the effect of the wandering set Q_1 on the dimension of the limit set F.

(5) Let F be the limit set specified by a symbolic construction. Define the map $G : F \to F$ by $G(x) = \chi^{-1} \circ \sigma \circ \chi(x)$. It is easy to see that G is a continuous endomorphism such that the set $G^{-1}(x), x \in F$ consists of finitely many points. If μ is a $\sigma-$invariant measure then its pull back measure $m = \chi^*\mu$ is $G-$invariant and has the same ergodic properties as μ.

(6) Consider the full shift σ on p symbols with the standard metric d_β, $\beta > 1$. Let μ be a $\sigma-$invariant ergodic measure, and let $r = \beta^{-n}$. Since the ball centered at ω with radius r, $B(\omega, r)$, is a cylinder set, we have by a theorem of Brin and Katok [BK] that for μ almost every ω,

$$\frac{\log \mu(B(\omega, r))}{\log r} = -\frac{\log \mu(B(\omega, \beta^{-n}))}{n \log \beta} \overset{n\to\infty}{\longrightarrow} \frac{h_\mu(\sigma)}{\log \beta},$$

hence

$$\frac{h_\mu(\sigma)}{\log \beta} = \underline{d}_\mu(\omega) = \overline{d}_\mu(\omega) = \dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu,$$

where $h_\mu(\sigma)$ denotes the Kolmogorov-Sinai entropy of the shift map. It is evident that the common value depends on β, which is not surprising since the two metrics d_{β_1} and d_{β_2} are not equivalent for $\beta_1 \neq \beta_2$. Now consider the Moran simple construction with parameters $\lambda = (\lambda_1, \cdots, \lambda_p)$. Since μ_λ is Gibbs we have that $h_{\mu_\lambda}(\sigma) + s \int \log \lambda_{\omega_1} d\mu_\lambda = 0$, which gives

$$s = -\frac{h_{\mu_\lambda}(\sigma)}{\int \log \lambda_{i_1} d\mu_\lambda} = \frac{h_{\mu_\lambda}(\sigma)}{\log \beta}$$

if

$$\beta = \exp\left(-\int \log \lambda_{i_1} d\mu_\lambda\right). \tag{7}$$

Thus the Hausdorff dimension of μ_λ calculated with respect to the two metrics d_β and ρ (the Euclidean metric on $\mathbb{R}^d$) coincide. The two metrics $\chi^*\rho$ and d_β with β satisfying (7) are not equivalent if not all the λ_i coincide. To see this, notice that $\chi : F \to \Sigma_p^+$ is onto. We have with respect to the metric ρ

$$\dim_H F = s = \frac{h_{\mu_\lambda}(\sigma)}{\log \beta}$$

and with respect to the metric d_β

$$\dim_H \Sigma_p^+ = \frac{\log p}{\log \beta} = \frac{h(\sigma)}{\log \beta}$$

where $h(\sigma)$ denotes the topological entropy of σ. It is easy to see that $h_{\mu_\lambda}(\sigma) = h(\sigma)$ if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_p = \lambda$.

Section 2: Examples of Symbolic Constructions

Here we consider several classes of symbolic constructions with restrictions on the shapes of basic sets or restrictions on the gaps between basic sets.

1: Moran constructions.

The simpliest case is when the basic sets are *essentially* balls. Such constructions were considered by Moran [Mo]. A **Moran symbolic construction** is a symbolic construction such that each basic set $\Delta_{i_1\cdots i_n}$ satisfies

$$D(C_1\prod_{j=1}^{n}\lambda_{i_j})\subset\Delta_{i_1\cdots i_n}\subset D(C_2\prod_{j=1}^{n}\lambda_{i_j}) \tag{8}$$

where $0<\lambda_i<1$, $i=1,\cdots,p$ and C_1,C_2 are positive constants.

Proposition 3. *Let F be the limit set for a Moran symbolic construction. Then*

(1) *the construction is regular and bounded with l-estimating vector and u-estimating vector equal to $\lambda=(\lambda_1,\ldots,\lambda_p)$*

(2) *$s=s_\lambda=dim_H F=\underline{dim}_B F=\overline{dim}_B F$. Moreover, if the measure m_λ is Gibbs, then $m_H(s,*)$ is equivalent to m_λ, $0<m_H(s,F)<\infty$, and $s=\underline{d}_m(x)=\overline{d}_m(x)$ for every $x\in F$.*

The first statement of Proposition 3 is obvious, and the second statement follows from Theorem 5.

2: Constructions with rectangles.

We now consider geometric constructions where the basic sets are (multi-dimensional) rectangles. More precisely, we call a symbolic construction a **construction with rectangles** if there exist $2p$ numbers $\underline{\lambda}_i,\overline{\lambda}_i$, $i=1,\cdots,p$, $0<\underline{\lambda}_i\le\overline{\lambda}_i<1$ such that the basic set $\Delta_{i_1\cdots i_n}\subset\mathbb{R}^d$ is a rectangle (the direct product of intervals, called sides, lying on n orthogonal lines) with the largest side equal to $C_1\prod_{j=1}^n\overline{\lambda}_{i_j}$ and the smallest side equal to $C_2\prod_{j=1}^n\underline{\lambda}_{i_j}$ where C_1,C_2 are positive constants.

Proposition 4. *Let F be the limit set for a symbolic construction with rectangles. Then*

(1) *the construction is regular and bounded with l-estimating vector $\underline{\lambda}=(\underline{\lambda}_1,\ldots,\underline{\lambda}_p)$ and u-estimating vector $\overline{\lambda}=(\overline{\lambda}_1,\ldots,\overline{\lambda}_p)$*

(2) *$s_{\underline{\lambda}}\le dim_H F\le\underline{dim}_B F\le\overline{dim}_B F\le s_{\overline{\lambda}}$. Moreover, if the measures $m_{\underline{\lambda}}$ and $m_{\overline{\lambda}}$ are Gibbs, then $s_{\underline{\lambda}}\le\underline{d}_{m_{\underline{\lambda}}}(x)$ and $\overline{d}_{m_{\overline{\lambda}}}(x)\le s_{\overline{\lambda}}$ for every $x\in F$.*

The first statment is obvious and the second statement follows from Theorems 1-4. In Section 5, Example 6, we will exhibit a simple construction with rectangles for which $\dim_H F<\underline{\dim}_B F<\overline{\dim}_B F$.

In [Mc], McMullen studied a special example of a geometric construction in the plane with rectangles. Given positive integers m and $n, m\le n$, he considers the partition of the unit square into rectangles of size $\frac{1}{m}\times\frac{1}{n}$. One chooses any r, $1\le r\le m\cdot n$ of these rectangles and colors them. One then linearly contracts the unit square by a factor of $\frac{1}{m}\times\frac{1}{n}$ and inserts a copy of this set into each of the original colored rectangles. One keeps going and obtains a limit set F. This is not a similarity construction since one does not insert a copy of this set into all of the original rectangles, just the shaded ones. McMullen proves that $\underline{\dim}_B F = \overline{\dim}_B F \stackrel{\text{def}}{\equiv} \dim_B F$ and finds explicit formulars for the box dimension and the Hausdorff dimension which use information about the initial configuration. It follows from his results that for *most* initial configurations, $\dim_H F < \dim_B F$.

The vectors $\gamma = (\frac{1}{n}, \frac{1}{n})$ and $\lambda = (\frac{1}{m}, \frac{1}{m})$ are l-estimating and u-estimating vectors respectively. Proposition 4 is applicable to this example and gives the following estimate

$$\frac{\log r}{-\log n} \le \dim_H F \le \dim_B F \le \frac{\log r}{-\log m}.$$

It follows from McMullen's formulas that for a most initial configuration, the three inequalities above are strict inequalities.

In [PW], the authors studied another type of simple geometric constructions with rectangles in the plane. Their construction is the similarity construction with two rectangles, each of size $\lambda_1 \times \lambda_2$, $\lambda_1 \le \lambda_2$, whose boundaries are alligned with the coordinate axes. The vectors $\gamma = (\lambda_1, \lambda_1)$ and $\lambda = (\lambda_2, \lambda_2)$ are l-estimating and u-estimating vectors respectively. Proposition 4 is applicable to this example and gives the following estimate

$$\frac{\log 2}{-\log \lambda_1} \le \dim_H F \le \underline{\dim}_B F \le \overline{\dim}_B F \le \frac{\log 2}{-\log \lambda_2}.$$

The authors proved that: 1) $\underline{\dim}_B F = \overline{\dim}_B F \stackrel{\text{def}}{\equiv} \dim_B F$, 2) for "almost all" initial configurations with $\lambda_2 \le \frac{1}{2}$, $\dim_H F = \dim_B F = \frac{\log 2}{-\log \lambda_2}$, 3) for "almost all" initial configurations with $\lambda_2 > \frac{1}{2}$, $\dim_B F = \frac{\log(\frac{2\lambda_2}{\lambda_1})}{-\log \lambda_1}$ and for a set of λ_2 of positive measure, $\dim_H F = \dim_B F = \frac{\log(\frac{2\lambda_2}{\lambda_1})}{-\log \lambda_1}$, 4) if λ_2 is the reciprical of a Pisot number, then the Hausdorff dimension is strictly less than $\frac{\log(\frac{2\lambda_2}{\lambda_1})}{-\log \lambda_1}$. Hence the Hausdorff dimension of the limit set may depend on delicate number theoretic properties of the ratio coefficients.

3: Constructions with exponentially large gaps.

We now consider constructions where we have a strong control over the gaps, but no control over the shape and the size of the basic sets. We call a symbolic construction a **construction with exponentially large gaps** if there exists a number $0 < \beta < 1$ such that the (Euclidean) distance between any two basic sets $\Delta_{i_1 \cdots i_n}$ and $\Delta_{j_1 \cdots j_n}$ is exponentially bounded away from 0, i.e.,

$$dist(\Delta_{i_1 \cdots i_n}, \Delta_{j_1 \cdots j_n}) \ge C\beta^n$$

where $C > 0$ is a constant.

Proposition 5. *Let F be the limit set for a symbolic construction with exponentially large gaps. Then*

(1) *the construction is regular with l-estimating vector $\gamma = (\beta, \ldots, \beta)$.*
(2) $s_\gamma \le dim_H F$.

Statement 1 is obvious and the proof of Statement 2 follows immediately from Theorem 1.

4: One-dimensional constructions.

We now consider one-dimensional symbolic constructions. We will assume that each basic set $\Delta_{i_1\cdots i_n}$ is an interval that satisfies

$$D(C_1\prod_{j=1}^{n}\underline{\lambda}_{i_j}) \subset \Delta_{i_1\cdots i_n} \subset D(C_2\prod_{j=1}^{n}\overline{\lambda}_{i_j})$$

where $0<\underline{\lambda}_i<\overline{\lambda}_i<1$, $i=1,\cdots,p$ and C_1, C_2 are positive constants.

Proposition 6. *Let F be the limit set for a one-dimensional symbolic construction defined above. Then*

(1) *the construction is regular with l-estimating vector $\underline{\lambda}=(\underline{\lambda}_1,\ldots,\underline{\lambda}_p)$ and bounded with u-estimating vector $\overline{\lambda}=(\overline{\lambda}_1,\ldots,\overline{\lambda}_p)$.*

(2) $s_{\underline{\lambda}} \le dim_H F \le \underline{dim}_B F \le \overline{dim}_B F \le s_{\overline{\lambda}}$.

(3) *if $\underline{\lambda}_i=\overline{\lambda}_i=\lambda_i$, then $s=s_\lambda=dim_H F=\underline{dim}_B F=\overline{dim}_B F$. Moreover, if the measure m_λ is Gibbs, then $m_H(s,*)$ is equivalent to m_λ, $0<m_H(s,F)<\infty$, and $s=\underline{d}_m(x)=\overline{d}_m(x)$ for every $x\in F$.*

The proof of Statement 1 is obvious and Statements 2 and 3 follow immediately from Theorems 1-5.

Section 3: Asymptotic and Random Symbolic Constructions

3.1. We now consider more general types of geometric constructions where the basic sets are *asymptotically* balls. These constructions are not weakly regular, but are weakly regular on each subset of an increasing sequence of subsets which exhaust the limit set up to a set of measure zero. More precisely we say that a vector γ is a **conditionally l-estimating** vector for a symbolic construction if there exists a sequence of subsets $\{Q_l\}, l=1,2,\ldots$ such that

(1) $Q_l\subset Q_{l+1}$ and $\cup_l Q_l=Q$ up to a set of $\mu_\gamma-$measure zero

(2) for every $l=1,2,\ldots$ and $\epsilon>0$ there exists $N(l,\epsilon)$ such that for all $x\in F$ with $\chi(x)\in Q_l$ and $r>0$

$$N(x,r)\le N(l,\epsilon)\,r^{-\epsilon},$$

where $N(x,r)$ denotes the number of sets in the special cover $\mathfrak{U}_r(\gamma)$ of the limit set F that have non-empty intersection with $B(x,r)\cap Q_l$.

We call a symbolic construction **conditionally regular** if it admits a conditionally l-estimating vector.

Note that a conditionally regular construction is weakly regular with respect to each set Q_l. Thus by Theorem 1' one can obtain lower estimates for the Hausdorff dimension of the set $\chi^{-1}(Q_l)$ which is uniform over l. This immediately implies the following result:

Theorem 6. *Let F be the limit set for a conditionally regular symbolic construction. Then $s_\gamma\le dim_H F$, γ is a conditionally l-estimating vector.*

We begin with an asymptotic version of the Moran symbolic constructions. We will show that these constructions are conditionally regular.

1: Asymptotic Moran constructions.

An **asymptotic Moran symbolic construction** is a symbolic construction for which there exist two sequences of numbers

$$\underline{\lambda}_{i,n} = \lambda_i \exp(\underline{a}_{i,n}), \quad \overline{\lambda}_{i,n} = \lambda_i \exp(\overline{a}_{i,n})$$

where $0 < \lambda_i < 1, i = 1, \cdots, p$ such that

a) for $m_\lambda-$almost every $x \in F$ with $\chi(x) = (i_1 i_2 \cdots)$ and $\lambda = (\lambda_1, \ldots, \lambda_p)$,

$$\frac{1}{n}\sum_{j=1}^{n} \underline{a}_{i_j,j} \to 0 \quad \text{and} \quad \frac{1}{n}\sum_{j=1}^{n} \overline{a}_{i_j,j} \to 0 \quad \text{as } n \to \infty.$$

b) each basic set $\Delta_{i_1,\cdots,i_n}$ satisfies

$$D(C_1 \prod_{j=1}^{n} \underline{\lambda}_{i_j,j}) \subset \Delta_{i_1 \cdots i_n} \subset D(C_2 \prod_{j=1}^{n} \overline{\lambda}_{i_j,j})$$

where C_1, C_2 are positive constants (compare with (8)).

Proposition 7. *Let F be the limit set for an asymptotic Moran symbolic construction. Then construction is conditionally regular with the conditionally l-estimating vector equal to $\lambda = (\lambda_1, \ldots, \lambda_p)$. Hence $s_\lambda \le dim_H F$.*

Condition (a) in the definition of asymptotic Moran symbolic construction is quite weak; one can obtain more information about the Hausdorff and box dimension of the limit set if the construction satisfies the following uniform version of (a)

a1)

$$\sup_{(i_1,\ldots,i_n)} \frac{1}{n}\sum_{j=1}^{n} \underline{a}_{i_j,j} \to 0 \quad \text{as } n \to \infty$$

a2)

$$\sup_{(i_1,\ldots,i_n)} \frac{1}{n}\sum_{j=1}^{n} \overline{a}_{i_j,j} \to 0 \quad \text{as } n \to \infty.$$

Proposition 8. *Let F be the limit set for an asymptotic Moran symbolic construction. Assume that the construction satisfies condition (a2). Then $s = s_\lambda = dim_H F = \underline{dim}_B F = \overline{dim}_B F$.*

Unlike the Moran symbolic construction, the limit set for an asymptotic Moran symbolic construction may have zero Hausdorff measure (even in the case when the construction satisfies the strong asymptotic conditions (a1) and (a2) and the measure μ_λ is Gibbs, see Section 5, Example 4 and compare with Statement 2 of Proposition 3).

2: Asymptotic one-dimensional symbolic construction.

We now consider an **asymptotic one-dimensional symbolic construction**. This is a one-dimensional construction for which there exist two sequences of numbers

$$\underline{\lambda}_{i,n} = \underline{\lambda}_i \exp(\underline{a}_{i,n}), \quad \overline{\lambda}_{i,n} = \overline{\lambda}_i \exp(\overline{a}_{i,n})$$

where $0 < \underline{\lambda}_i \le \overline{\lambda}_i < 1$, $i = 1, \cdots, p$ such that

a) for $m_{\underline{\lambda}}-$almost every $x \in F$ with $\chi(x) = (i_1 i_2 \cdots)$ and $\underline{\lambda} = (\underline{\lambda}_1, \ldots, \underline{\lambda}_p)$,

$$\frac{1}{n}\sum_{j=1}^{n} \underline{a}_{i_j,j} \to 0 \text{ as } n \to \infty$$

and for $m_{\overline{\lambda}}-$almost every $x \in F$ with $\chi(x) = (i_1 i_2 \cdots)$ and $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_p)$,

$$\frac{1}{n}\sum_{j=1}^{n} \overline{a}_{i_j,j} \to 0 \text{ as } n \to \infty$$

b) each basic set $\Delta_{i_1 \cdots i_n} \subset \mathbb{R}$ satisfies

$$D(C_1 \prod_{j=1}^{n} \underline{\lambda}_{i_j,j}) \subset \Delta_{i_1 \cdots i_n} \subset D(C_2 \prod_{j=1}^{n} \overline{\lambda}_{i_j,j})$$

where $C_1, C_2 > 0$ are positive constants.

Proposition 9. *Let F be the limit set for an asymptotic one-dimensional symbolic construction. Then the construction is conditionally regular with the conditionally l-estimating vector $\underline{\lambda} = (\underline{\lambda}_1, \ldots, \underline{\lambda}_p)$. Hence $s_{\underline{\lambda}} \le dim_H F$.*

As in Proposition 8, one can obtain more information about the Hausdorff and box dimension of the limit set if the construction satisfies the uniform conditions (a1) and (a2).

Proposition 10. *Let F be the limit set for an asymptotic one-dimensional symbolic construction. Assume that the construction satisfies condition (a2). Then $s_{\underline{\lambda}} \le dim_H F \le \underline{dim}_B F \le \overline{dim}_B F \le s_{\overline{\lambda}}$.*

3: Random symbolic construction.

We now consider a random version of the Moran symbolic construction. In this case the sizes of the basic sets are chosen randomly with respect to some stationary ergodic distribution. We will use an ergodic theorem in [BFKO] to reduce the study of these constructions to the asymptotic Moran constructions.

A **random symbolic construction** is defined by

a) a stochastic vector process $(\Lambda, \mathfrak{F}, \nu)$ with $\Lambda = \{\vec{\lambda} = (\underline{\lambda}_{i,n}, \overline{\lambda}_{i,n}),\ i = 1, \cdots, p$ and $n \in \mathbb{N}\}$ where $0 < \alpha \le \underline{\lambda}_{j,n} \le \overline{\lambda}_{j,n} \le \beta < 1$, $\mathfrak{F}$ is the $\sigma-$algebra of Borel sets in Λ, and ν is an arbitrary stationary, shift-invariant ergodic Borel probability measure on Λ

b) a compact set $Q \subset \Sigma_p^+$ invariant under the shift (i.e., $\sigma(Q) = Q$)

c) for $\nu-$almost every $\vec{\lambda} \in \Lambda$, a family of sets $\{\Delta_{i_1 \cdots i_n}\}(\vec{\lambda}) \subset \mathbb{R}^n$ for $i_j = 1, 2, \ldots, p$, where the n-tuple $(i_1 \cdots i_n)$ is admissible with respect to Q and satisfies

$$D(C_1 \prod_{j=1}^{n} \underline{\lambda}_{i_j,j}(\vec{\lambda})) \subset \Delta_{i_1 \cdots i_n}(\vec{\lambda}) \subset D(C_2 \prod_{j=1}^{n} \overline{\lambda}_{i_j,j}(\vec{\lambda}))$$

where $C_1 > 0$ and $C_2 > 0$.

d) for any sequence $(i_1 \cdots i_n) \in \{1, \cdots p\}^n$, we require that $\Delta_{i_1 \cdots i_{n+1}}(\vec{\lambda}) \subset \Delta_{i_1 \cdots i_n}(\vec{\lambda})$ and

$$\Delta_{i_1 \cdots i_n}(\vec{\lambda}) \cap \Delta_{i'_1 \cdots i'_n}(\vec{\lambda}) = \emptyset, \ \text{ if } (i_1 \cdots i_n) \neq (i'_1, \cdots i'_n).$$

For every $\vec{\lambda} \in \Lambda$, the limit set

$$F(\vec{\lambda}) = \bigcap_{n=1}^{\infty} \bigcup_{\substack{(i_1 \cdots i_n) \\ \text{admissible}}} \Delta_{i_1 \cdots i_n}(\vec{\lambda})$$

is a perfect, nowhere dense, totally disconnected set.

The following Lemma describes the limiting behavior of the numbers $\underline{\lambda}_{i,n}, \overline{\lambda}_{i,n}$ in the random symbolic construction:

Lemma 1. *Let F be the limit set specified by a random symbolic geometric construction. Then there are numbers $\underline{\lambda}_i, \overline{\lambda}_i, 0 < \underline{\lambda}_i \leq \overline{\lambda}_i < 1, i = 1, \cdots, p$ such that for $\nu-$almost every $\vec{\lambda} \in \Lambda$ the following limits exist:*

(1) *for $\mu_{\underline{\lambda}}-$almost every sequence $(i_1 i_2 \cdots) \in Q(\vec{\lambda})$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \underline{a}_{i_k,k} = 0$$

(2) *for $\mu_{\overline{\lambda}}-$almost every sequence $(i_1 i_2 \cdots) \in Q(\vec{\lambda})$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \overline{a}_{i_k,k} = 0$$

where $\underline{a}_{i,j} = \log\left(\frac{\underline{\lambda}_{i,j}}{\underline{\lambda}_i}\right), \overline{a}_{i,j} = \log\left(\frac{\overline{\lambda}_{i,j}}{\overline{\lambda}_i}\right), \underline{\lambda} = (\underline{\lambda}_1, \ldots, \underline{\lambda}_p), \overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_p).$

The next statement immediately follows from Lemma 1 and Propositions 7 and 9.

Proposition 11. *Let F be the limit set specified by a random symbolic construction. Assume that either the construction is one-dimensional or $\underline{\lambda}_i = \overline{\lambda}_i$ for $i = 1, \ldots, p$. Then for $\nu-$almost every $\vec{\lambda} \in \Lambda$,*

$$s_{\underline{\lambda}} \leq dim_H F(\vec{\lambda}).$$

Section 4: Generating Maps and Codings

One particular but important case of a geometric construction is when the sets $\Delta_{i_1\cdots i_n}$ are given by

$$\Delta_{i_1\cdots i_n} = h_{i_n} \circ h_{i_{n-1}} \circ \cdots \circ h_{i_1}(\Delta) \tag{9}$$

where $h_1, \cdots, h_p : \Delta \to \Delta$ are contraction maps, i.e., $dist(h_i(x), h_i(y)) \le L_i dist(x, y)$ with $L_i < 1$ and $x, y \in \Delta$ (a ball in $\mathbb{R}^n$). Most of the results in the literature in dimension theory require that the process be described in this way. We stress that this is a very special case and that the following situations can (and do) occur:

(1) *The construction can not be described by any continuous maps*, i.e., there are no continuous maps satisfying (9). This can occur if the boundary of a set $\Delta_{i_1\cdots i_n}$ is fractal. In the one-dimensional case the maps h_j always exist and are continuous, so the above mentioned pathology does not occur. In this case the maps are well defined by

$$h_j(\partial\Delta_{i_1\cdots i_n}) = \partial\Delta_{i_1\cdots i_n j}$$

for any admissible sequence $(i_1 \cdots i_n j)$, where $\partial\Delta_{i_1\cdots i_n j}$ denotes the boundary of the set $\Delta_{i_1\cdots i_n j}$.

(2) *There are continuous maps satisfying (9) that are not Lipschitz.* One obstruction is that the boundary of a set $\Delta_{i_1\cdots i_n}$ can be a continuous but not Lipschitz image of $\partial\Delta$. In the one-dimensional case, this obstruction cannot occur. However, one can construct a one-dimensional example where the process is defined by continuous but not Lipschitz maps. Namely, there exists a simple geometric construction on $[0,1]$ with $p = 2$, $\underline{\lambda}_1 = \overline{\lambda}_1 = \lambda_1$, $\underline{\lambda}_2 = \overline{\lambda}_2 = \lambda_2$, $0 < \lambda_1 < \lambda_2 < 1$ and the locatios of the intervals $\Delta_{i_1\cdots i_n}$ are such that the map h_1 is not Lipschitz. Choose

$$\frac{d(\Delta_{\underbrace{1\cdots 1}_{n_k}}, \Delta_{\underbrace{1\cdots 1}_{n_k-1},2})}{d(\Delta_{\underbrace{1,\cdots,1}_{n_k+1}}, \Delta_{\underbrace{1\cdots 1}_{n_k-1},2,1})} \overset{n_k\to\infty}{\longrightarrow} \infty.$$

(3) *There are Lipschitz maps satisfying (9) but these maps are not contractions.* To see this, choose intervals Δ_1, Δ_2 and maps h_1, h_2 such that $\Delta_{11} = h_1(\Delta_1)$, $\Delta_{21} = h_1(\Delta_2)$ and $d(\Delta_1, \Delta_2) << d(\Delta_{11}, \Delta_{21})$. Then the map h_1 is not a contraction.

(4) *There are contraction maps satisfying (9) whose inverse maps are not Lipschitz.* The Lipschitz constants for the inverse maps depend on the gaps between the sets $\Delta_{i_1\cdots i_n}$. To see this, choose intervals $\Delta_{i_1\cdots i_n}$ such that

$$\frac{d(\Delta_{\underbrace{1\cdots 1}_{n}}, \Delta_{\underbrace{1\cdots 1}_{n-1},2})}{d(\Delta_{\underbrace{1\cdots 1}_{n+1}}, \Delta_{\underbrace{1\cdots 1}_{n-1},2,1})} \overset{n\to\infty}{\longrightarrow} 0.$$

Then the inverse of the map h_1 is not Lipschitz.

(5) *There are contraction maps satisfying (9) whose inverse maps are Lipschitz but the maps are not similarities* (i.e., $d(h_i(x), h_i(y)) = L_i d(x, y)$ for all $x \in \Delta$).

As we saw in case (4) the Lipschitz constants for the inverse maps may depend on the gaps between the sets $\Delta_{i_1 \cdots i_n}$. For this reason, even if the process can be described using contraction maps with Lipschitz inverses, they may be of no use in estimating the Hausdorff dimension of the limit set.

We point out that the coding we employ in our definitions of geometric constructions is not the obvious generalization of the *natural* coding for a similarity process. The natural coding is $\Delta_{i_1 \cdots i_n} = g_{i_n} \circ g_{i_{n-1}} \circ \cdots \ g_{i_1}(\Delta)$ where the maps g_k are affine contractions. It follows that $\Delta_{i_1 \cdots i_n} \subset \Delta_{i_2 \cdots i_n}$. This coding has several undesirable properties which the coding that we adopted does not share. One undesirable property is that for a given $x \in F$ with coding $(i_1 i_2 \cdots)$, one can not determine which Δ_k the point x lies in.

The maps h_k have an interesting manifestation on the level of symbolic dynamics. The mapping $G : F \to F$ defined in Section 1, Remark (5) can be described as

$$G(x) = h_k^{-1}(x) \quad \text{if } x \in \Delta_k.$$

If the maps h_k are all similarities, then the map G is a contraction.

Section 5: More Examples

FIGURE 2. SIERPIŃSKI GASKETS A), B), C)

1) Sierpiński Gaskets

a) It is well known that the Hausdorff dimension of the Sierpiński gasket (Figure 2a) coincides with the box dimension and is $\frac{\log 3}{\log 2}$. This immediately follows from Corollary 2 and Proposition 3 since $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}$ and $p = 3$.

b) Suppose that in the construction of the Sierpiński gasket we forbid all configurations whose codings contain a 1 followed by a 2 (Figure 2b). The spectral radius of $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is $\frac{3+\sqrt{5}}{2}$. Hence by Proposition 1 and Corollary 1, $\dim_H F = \frac{\log(\frac{3+\sqrt{5}}{2})}{\log 2}$ ≈ 1.38848.

c) A simple construction of the Sierpiński gasket with $\lambda_i = \frac{1}{3}$ for $i = 1, 2, 3$ is illustrated in Figure 2c). The sets $\Delta_{i_1 \cdots i_n}$ are *asymptotically congruent* to the corresponding triangles in the usual constructions and possess *wiggly boundaries* that become *asymptotically straight.* As long as the approximation is sufficiently fast and uniform, such that the construction satisfies the hypotheses of Proposition 8, then the Hausdorff dimension of the limit set is $\frac{\log 3}{\log 2}$. One can even choose the sets $\Delta_{i_1 \cdots i_n}$ in the construction to have fractal boundaries.

2) General Smale-Williams solenoid

Let P be a solid torus embedded in $\mathbb{R}^3$. We represent points on P by means of coordinates (θ, r, s) where $\theta \in S^1$the unit circle, $-1 \leq r, s \leq 1$ such that $r^2 + s^2 \leq 1$. The point x with coordinates (θ, r, s) belongs to the plane orthogonal to the core of the torus through the point $\theta \in S^1$ having position (r, s) relative to the standard frame (e_1, e_2). We define a mapping $f : P \to P$ by

$$f(\theta, r, s) = (2\,\theta, \lambda_1 r + \epsilon \cos\theta, \lambda_2 s + \epsilon \sin\theta)$$

where $p > 0$ is an integer, ϵ is a small positive constant and $0 < \lambda_1, \lambda_2 < 1$. The image $f(P)$ is contained in P and wraps twice around P. See Figure 3a). The set $\Delta = \cap_{n=1}^{\infty} f^n(P)$ is called a *solenoid*. See [Sh] for more details and nice pictures.

FIGURE 3. SMALE-WILLIAMS SOLENOID a), b)

Let D_θ be the section of P determined by the plane perpendicular to the core at θ. The set $\Delta_\theta = \Delta \cap D_\theta$ is the Cantor-like set obtained by the simple geometric construction with $\underline{\lambda}_i = \lambda_1$ and $\overline{\lambda}_i = \lambda_2$ for i = 1, 2. See Figure 3b. Since the basic sets in the construction of Δ_θ are rectangles, it follows from Proposition 4 that

$$\frac{\log 2}{\log(\frac{1}{\lambda_1})} \leq \dim_H \Delta_\theta \leq \frac{\log 2}{\log(\frac{1}{\lambda_2})}.$$

Appling Marstrand's theorem [F1], we obtain

$$1 + \frac{\log 2}{\log(\frac{1}{\lambda_1})} \leq \dim_H \Delta.$$

A very simple argument [F1] shows that $\dim_H \Delta \leq 1 + \frac{\log 2}{\log(\frac{1}{\lambda_2})}$.

If $\lambda_1 = \lambda_2 = \lambda$, we have that $\dim_H \Delta_\theta = \frac{\log 2}{\log(\frac{1}{\lambda})}$ and $\dim_H \Delta = \frac{\log 2}{\log(\frac{1}{\lambda})} + 1$, which is the result obtained by Falconer [F1].

3) Geometric Construction With Rectangles in the Plane.

The following simple example illustrates the fact that the regularity of the construction depends not only on the sizes and shapes of the basic sets but also may depend on their spacing. Consider the two similarity constructions where all the basic sets at step n are congruent rectangles with width $\underline{\lambda}^n$ and length $\overline{\lambda}^n$. The rectangles are stacked horizontally in the first construction and vertically in the second construction. The limit sets of both constructions are one-dimensional Cantor sets. The first has Hausdorff dimension $\frac{\log 2}{-\log \overline{\lambda}}$ and the second has Hausdorff dimension $\frac{\log 2}{-\log \underline{\lambda}}$. The first construction is regular with l-estimating vector $\gamma = (\overline{\lambda}, \overline{\lambda})$. The vector $(\overline{\lambda}, \overline{\lambda})$ is not l-estimating for the first construction but is l-estimating for the second construction.

4) A simple asymptotic Moran construction.

This example shows that the second statement of Proposition 3 may fail for an asymptotic Moran construction, i.e., the measure of the limit set may be zero.

Let $p = 2$ and suppose $\lambda_{i,n} = \lambda_n = \lambda \exp(a_n)$ for $i = 1, 2$. Let F denote the limit set. Clearly

$$\operatorname{diam}(\Delta_{i_1 \cdots i_n}) = \prod_{j=1}^{n} \lambda_{i_j, j} = \lambda^n \exp(A_n),$$

where $A_n = \sum_{k=1}^n a_k$. For fixed n, the sets $\{\Delta_{i_1 \cdots i_n}\}$ give a cover of F with

$$\sum_{i_1 \cdots i_n} \operatorname{diam}(\Delta_{i_1 \cdots i_n})^s = 2^n \lambda^{ns} \exp(sA_n) = \exp(sA_n),$$

where $s = \frac{\log 2}{\log(\frac{1}{\lambda})}$. There is now a trichotomy:

(1) if $\sum_{k=1}^\infty a_k < \infty$, then one can easily show that the geometric process satisfies conditions a1) and a2), and by Proposition 8, $s = \dim_H F = \dim_B F = \frac{\log 2}{\log(\frac{1}{\lambda})}$.
(2) if $\sum_{k=1}^\infty a_k = \infty$, then one can show that $m_H(s, F) = \infty$.
(3) if $\sum_{k=1}^\infty a_k = -\infty$, then $m_H(s, F) = 0$.

The sequence $\{a_k\}$ satisfies the condition in the definition of asymptotic Moran construction if and only if $\frac{1}{n} A_n \to 0$. For example, choose $a_n = \frac{1}{n}$ in the case (2) and $a_n = -\frac{1}{n}$ in case (3). One can show that s is still the Hausdorff dimension of the limit set F. The mass distribution principle does not hold in case 3) since otherwise we would have $m_H(s, F) > 0$.

We now construct gauge functions for certain sequences $\{a_j\}$ where $\sum_{k=1}^\infty a_j = -\infty$. Let $A_n = \sum_{k=1}^n a_k$. We seek a function $h(t)$ such that $0 < 2^n h(\lambda^n \exp(A_n)) < \infty$. We will find $h(t)$ in the form $h(t) = t^s \exp(\phi(t))$. Then we would have $-\infty < sA_n + \phi(\lambda^n \exp(A_n)) < \infty$. Define $t = \lambda^n \exp(A_n)$. Since $\lim_{n \to \infty} \frac{A_n}{n} = 0$, then $t \asymp \lambda^n$, and hence we can set $\phi(t) = -sA_{\frac{\log t}{\log \lambda}}$.

If $a_n = -\frac{1}{n}$ then $A_n = O(-\log n)$ and $\phi(t) = \log(\frac{\log t}{\log \lambda})$. If $a_n = -\frac{1}{n \log n}$ then $A_n = O(-\log(\log n))$ and $\phi(t) = \log(\log(\frac{\log t}{\log \lambda}))$. More generally, if $a_n = -\frac{1}{n \log^{(i)} n}$, where $\log^{(i)} n$ denotes the $i-$fold composition of $\log n$, then $A_n = O(-\log^{(i+1)} n)$ and hence $\phi(t) = \log^{(i+1)}(\frac{\log t}{\log \lambda})$. We can thus obtain gauge functions with *arbitrarily many logs* from this basic construction. For these different sequences, the coefficients $\lambda_n = \lambda \exp(a_n)$ converge to λ, but of course with different speeds, and hence, require different gauge functions.

5) Random version of Example 4

The following example is a special case of random constructions that was pointed out to the authors by Peres. We consider the construction in Example 4 where the numbers $\{a_n\}$ are independent and identically distributed random variables on the interval $\infty < a \le a_n \le b < \log(\frac{1}{\lambda})$ having mean 0. If we define the random variable $A_n = \sum_{k=1}^{n} a_k$, then the law of the iterated logarithm implies that $\liminf A_n = -\infty$. The law of large numbers shows that $\frac{A_n}{n} \to 0$. It immediately follows from a simple calculation as in Example 4 that $s = \dim_H F = \frac{\log 2}{-\log \lambda}$ and $m_H(s, F) = 0$.

6) Simple geometric construction with rectangles having limit set F for which $\mathbf{dim_H F < \underline{dim}_B F < \overline{dim}_B F}$

We describe a simple construction with rectangles in $\mathbb{R}^2$ with $p = 2$, $\underline{\lambda}_1 = \underline{\lambda}_2 = \underline{\lambda}$, $\overline{\lambda}_1 = \overline{\lambda}_2 = \overline{\lambda}$, $0 < \underline{\lambda} < \overline{\lambda} < \frac{1}{3}$ such that the limit set F satisifes

$$\dim_H F = \frac{\log 2}{-\log \underline{\lambda}}, \qquad \underline{\dim}_B F = \gamma \frac{\log 2}{-\log \underline{\lambda}}, \qquad \overline{\dim}_B F = \frac{\log 2}{-\log \overline{\lambda}},$$

where $\gamma \in (1, \alpha)$ is an arbitrary number and $\alpha = \left[\frac{\log \underline{\lambda}}{\log \overline{\lambda}}\right] > 1$.

Let $n_0 = 0$ and for $k = 0, 1, 2, \cdots$, let $n_{k+1} = [\alpha n_k] + 2$ and $\beta_k = 2^{(\gamma-\alpha)n_{3k+1}}$. In order to describe the nth step of the construction we use the basic types of spacings: vertical stacking (A) and horizontal staking (B). See Figure 4.

FIGURE 4 VERTICAL STACKING (A) HORIZONTAL STACKING (B)

(1) We start with two horizontally stacked rectangles. During steps $n_{3k} < n \le n_{3k+1}$ we use (B).
(2) We begin with $2^{n_{3k+1}}$ rectangles. Choose β_k percent of these rectangles arbitrarily and paint them blue; paint the others green. During steps $n_{3k+1} < n \le n_{3k+2}$, we use (B) in all blue rectangles and use (A) in all green rectangles.
(3) During steps $n_{3k+2} < n \le n_{3k+3}$, we use (A) in all blue rectangles and use (B) in all green rectangles.
(4) Repaint all $2^{n_{3k+3}}$ rectangles white; repeat steps 1 to 4.

The collection of rectangles at the nth step contains 2^n rectangles each with vertical and horizontal sides of size $\underline{\lambda}^n \times \overline{\lambda}^n$ (the size in the vertical direction is $\underline{\lambda}^n$ and the size in the horizontal direction is $\overline{\lambda}^n$). Any two subrectangles on step $n+1$ that are contained in the same rectangle at step n are stacked either horizontally or vertically and the distance between them is at least $\frac{1}{3}\underline{\lambda}^n$. The projections of any two distinct rectangles on step n onto the two coordinate axes either coincide or are disjoint.

We need the following three lemmas:

Lemma 2. *Let $\mathfrak{U}$ be a covering of the limit set F by disjoint closed balls $B(x_i, r), i = 1, \cdots, p$, where r is fixed. Assume that every ball in the covering has a non-trivial intersection with the limit set F. Then $N(r) \geq \frac{1}{6}\#(\mathfrak{U})$, where $N(r)$ denotes the smallest number of balls of radius r needed to cover F.*

Proof. Let $\mathfrak{V} = \{B(x_i', r)\}, i = 1, \cdots, q$ be a covering of F. Then by elementary geometry, for all x_j', there are at most 6 points $x_1, \cdots, x_6$ such that $F \cap B(x_j', r) \subset \cup_{k=1}^{6} B(x_k, r)$. $\square$

Lemma 3. *For any $n > 0$, $dist(\Delta_{i_1' \cdots i_n'}, \Delta_{i_1'' \cdots i_n''}) \geq \frac{1}{3}\underline{\lambda}^n$.*

Proof. The proof is obvious. $\square$

Lemma 4. *If $0 < r < \epsilon$ and an optimal covering of the limit set F by balls of radius r consists of disjoint balls $B(x_i, r)$, then $N(\epsilon) \geq \frac{1}{4}N(r)\left(\frac{r}{\epsilon}\right)^2$.*

Proof. Let $\{B(y_j, \epsilon)\}$ be a covering of F. For any point x_i one can find a point y_i such that $x_i \in B(y_i, \epsilon)$. Hence, $B(x_i, r) \subset B(y_i, 2\epsilon)$. Therefore $\mathrm{vol}(B(x_i, r)) \leq \mathrm{vol}(B(y_i, 2\epsilon))$. Since the balls $B(x_i, r)$ are disjoint, this implies that $N(r)r^2 \leq N(\epsilon)(2\epsilon)^2$. The lemma immediately follows. $\square$

We now calculate the Hausdorff dimension and the lower and upper box dimensions of the limit set F.

a) Calculation of Upper Box Dimension.

Let $r_k = \overline{\lambda}^{n_{3k+1}}$. Consider the covering of F by rectangles $\Delta_{i_1 \cdots i_{n_{3k+1}}}$. Each rectangle can be covered by a square of side length less than or equal to $\overline{\lambda}^{n_{3k+1}}$ that is centered at some point $x \in \Delta_{i_1 \cdots i_{n_{3k+1}}}$. These squares are disjoint since $\overline{\lambda}^{n_{3k+1}} = \overline{\lambda}^{[\alpha n_{3k}]+2} \leq \frac{1}{3}\underline{\lambda}^{n_{3k}}$, and by Lemma 3, $\mathrm{dist}(\Delta_{i_1' \cdots i_{n_{3k}}'}, \Delta_{i_1'' \cdots i_{n_{3k}}''}) \geq \frac{1}{3}\underline{\lambda}^{n_{3k}}$. By Lemma 2, $N(r_k) \geq \frac{1}{6}2^{n_{3k+1}}$ and hence

$$\overline{\dim}_B F \geq \overline{\lim}_{k\to\infty} \frac{\log N(r_k)}{-\log r_k} = \frac{\log 2}{-\log \overline{\lambda}}.$$

On the other hand, by Proposition 4, $\overline{\dim}_B F \leq \frac{\log 2}{-\log \overline{\lambda}}$. It follows that $\overline{\dim}_B F = \frac{\log 2}{-\log \overline{\lambda}}$.

b) Calculation of Hausdorff Dimension.

Given $\epsilon > 0$ choose $k > 0$ such that $\underline{\lambda}^{n_{3k+1}} \leq \epsilon$. Consider the covering of F consisting of green rectangles for $n = n_{3k+1}$ and blue rectangles for $n = n_{3k+2}$.

Consider a green rectangle $\Delta_{i_1\cdots i_{n_{3k+1}}}$. By construction, the intersection $A = \Delta_{i_1\cdots i_{n_{3k+1}}} \cap F$ is contained in $2^{n_{3k+2}-n_{3k+1}}$ small green rectangles corresponding to $n = n_{3k+2}$. These rectangles are vertically aligned and have size $\underline{\lambda}^{n_{3k+2}} \times \overline{\lambda}^{n_{3k+2}}$. Since $\overline{\lambda}^{n_{3k+2}} = \overline{\lambda}^{[\alpha n_{3k+1}]+2} \leq \frac{1}{3}\underline{\lambda}^{n_{3k+1}}$, the set A is contained in a square of size $\underline{\lambda}^{n_{3k+1}}$. Thus we have $(1-\beta_k)2^{n_{3k+1}}$ green squares of length $\underline{\lambda}^{n_{3k+1}}$.

Now consider a blue rectangle $\Delta_{i_1\cdots i_{n_{3k+1}}}$. By our construction, the intersection $B = \Delta_{i_1\cdots i_{n_{3k+2}}} \cap F$ is contained in $2^{n_{3k+3}-n_{3k+2}}$ small blue rectangles corresponding to $n = n_{3k+3}$. They are vertically aligned and have size $\underline{\lambda}^{n_{3k+3}} \times \overline{\lambda}^{n_{3k+3}}$. Since $\overline{\lambda}^{n_{3k+3}} \leq \frac{1}{3}\underline{\lambda}^{n_{3k+2}}$, the set B is contained in a square of size $\underline{\lambda}^{n_{3k+2}}$. Thus we have $\beta_k 2^{n_{3k+1}} 2^{n_{3k+2}-n_{3k+1}} = \beta_k 2^{n_{3k+2}}$ blue squares of length $\underline{\lambda}^{n_{3k+1}}$.

The collection of green and blue rectangles comprise a covering $\mathfrak{G} = \{U_i\}$ of F such that

$$\sum_{U_i\in\mathfrak{G}} (\mathrm{diam}\ U_i)^s = (1-\beta_k)2^{n_{3k+1}}(2\underline{\lambda}^{n_{3k+1}})^s + \beta_k 2^{n_{3k+2}}(2\underline{\lambda}^{n_{3k+2}})^s = 1 > 0$$

if $s = \frac{\log 2}{-\log\underline{\lambda}}$. This implies that $\dim_H F \leq \frac{\log 2}{-\log\overline{\lambda}}$. On the other hand, by Proposition 4, we know that $\dim_H F \geq \frac{\log 2}{-\log\underline{\lambda}}$, hence $\dim_H F = \frac{\log 2}{-\log\underline{\lambda}}$.

c) Calculation of Lower Box Dimension.

Consider $r_k = \underline{\lambda}^{n_{3k+1}}$ and the covering $\mathfrak{G}_k$ of F by green rectangles for $n = n_{3k+1}$ and blue rectangles for $n = n_{3k+2}$.

There are $(1-\beta_k)2^{n_{3k+1}}$ green rectangles and the intersection of each of them with F is contained in a square of size $2\underline{\lambda}^{n_{3k+1}}$. There are $\beta_k\, 2^{n_{3k+1}}\, 2^{n_{3k+2}-n_{3k+1}} = \beta_k 2^{n_{3k+2}}$ blue rectangles each of which is contained in a square of size $\overline{\lambda}^{n_{3k+2}}$. It is easy to see that

$$L_1\underline{\lambda}^{n_{3k+1}} \leq \overline{\lambda}^{n_{3k+2}} \leq L_2\underline{\lambda}^{n_{3k+1}},$$

where L_1, L_2 are positive constants. Therefore $\mathfrak{G}_k$ induces a covering $\mathfrak{H}_k = \{U_i\}$ of F by squares of size $L\underline{\lambda}^{3k+3}$, where $L > 0$ is a constant. The cardinality N_k of this covering is

$$N_k = ((1-\beta_k)2^{n_{3k+1}} + \beta_k 2^{n_{3k+2}}) \leq ((1-\beta_k)2^{n_{3k+1}} + 4\beta_k 2^{\alpha n_{3k+1}}).$$

Since $\alpha > \gamma$, we have that

$$(1-\beta_k)2^{(\alpha-\gamma)n_{3k+1}} \to 0$$

as $k \to \infty$ and by the definition of β_k,

$$\beta_k 2^{(\alpha-\gamma)n_{3k+1}} = 1.$$

Therefore

$$\underline{\dim}_B F \leq \lim_{k\to\infty} \frac{\log N(r_k)}{-\log r_k} = \gamma\frac{\log 2}{-\log\underline{\lambda}}.$$

Choose a sequence $\epsilon_m \to 0$. Given $m > 0$, one can choose an integer $k = k(m)$ satisfying $\underline{\lambda}^{n_{3k+1}} \leq \epsilon_m \leq L_3\underline{\lambda}^{n_{3k+1}}$, where L_3 is a positive constant that is independent of m. We wish to compare $N(\epsilon_m)$ and $N(r_k)$. Since the squares comprising the covering $\mathfrak{H}_k$ are disjoint, Lemma 3 implies

$$N(\epsilon_m) \geq \frac{1}{4} N(r_k) \left(\frac{r_k}{\epsilon_m}\right)^2 .$$

It follows that

$$\underline{\dim}_B F = \lim_{m\to\infty} \frac{\log N(\epsilon_m)}{-\log \epsilon_m} \geq \limsup_{m\to\infty} \frac{\log N(r_k)}{-\log r_k} = \gamma \frac{\log 2}{-\log \overline{\lambda}}.$$

8) Example where diameters of inscribed balls in basic sets is not an l-estimating vector

Let $\gamma_1, \gamma_2, \gamma_3, \lambda$ be any numbers in $(0,1)$ and let $A(\gamma_i)$ denote a simple geometric construction on the interval $[0,1] \times \{i\}$, $i = -1, 0, 1$ with 2^n basic sets of size γ_i^n at step n. We wish to use these three one-dimensional constructions to define a simple geometric construction in the square $[0,1] \times [-1,1]$. Since the 2^n intervals at step n in each of the one-dimensional constructions are clearly ordered, we may refer to the ith subinterval at step n, $1 \leq i \leq 2^n$ of these constructions. Consider the 2^n polygons in $[0,1] \times [-1,1]$ having six vertices which consist of the two endpoints of the ith subinterval at step n for all three constructions. We define the 2^n basic sets of our construction at step n by intersecting these 2^n polygons with the square $[0,1] \times [-\lambda^n, \lambda^n]$. See Figure 5.

Figure 5.

It is easy to see that the limit set F of this construction coincides with the limit set of the construction $A(\gamma_2)$. Hence $\dim_H F = \frac{\log 2}{-\log \gamma_2}$ and does not depend on γ_1, γ_3 or λ. If we choose $\gamma_2 < \gamma_1 = \gamma_3 < \lambda$, then the inscribed and circumscribed balls of the basic sets at step n have radii $C_1 \gamma_1^n$ and $C_2 \lambda^n$, where C_1 and C_2 are positive constants that are independent of n. Thus these balls cannot be used to determine the Hausdorff dimension of the limit set.

Section 6: Pointwise Dimension of Measures Concentrated on General Cantor-like Sets

6.1. Let F be the limit set for an asymptotic Moran or one-dimensional symbolic construction. We formulate a powerful criterion that allows one to estimate the lower and upper pointwise dimensions with respect to a Borel probability measure ν on F. Given $x \in F, n > 0$ consider the unique set $\Delta_{i_1 \cdots i_n} = \Delta_n(x)$ that contains the point x. Denote

$$\underline{d}(x) = \liminf_{n\to\infty} \frac{\log \nu(\Delta_n(x))}{\log \mathrm{diam} \Delta_n(x)};$$

$$\overline{d}(x) = \limsup_{n\to\infty} \frac{\log \nu(\Delta_n(x))}{\log \mathrm{diam} \Delta_n(x)}.$$

The following theorem is related to a result in [C]:

Theorem 7.

(1) $\overline{d}_\nu(x) \le \overline{d}(x)$ *and* $\underline{d}(x) \le \underline{d}_\nu(x)$ *for all* $x \in F$.

(2) *If* $\underline{d}(x) = \overline{d}(x) \stackrel{\text{def}}{\equiv} d(x)$ *for* $\nu-$*almost every* $x \in F$, *then*

$$\underline{d}_\nu(x) = \overline{d}_\nu(x) = d(x)$$

for $\nu-$*almost every* $x \in F$.

The next statement is an immediate corollary of Propositions 7 and 9.

Corollary 3. *Let F be the limit set for an asymptotic Moran or one-dimensional construction. Assume that there is a Borel measure ν on F such that* $\underline{d}(x) = \overline{d}(x) \stackrel{\text{def}}{\equiv} s$ *for* $\nu-$*almost every* $x \in F$. *Then* $s \le dim_H F$.

6.2. We present several computations of pointwise dimension. We start with Example 6 and show that:

a) $\underline{d}_{\underline{m}}(x) = \underline{s}$ *for* $\underline{m}-$*almost every* $x \in F$ *and* $\underline{d}_{\overline{m}}(x) = \underline{s}$ *for* $\overline{m}-$*almost every* $x \in F$, *where* $\underline{s} = \frac{\log 2}{\log \underline{\lambda}}$

b) $\overline{d}_{\underline{m}}(x) = \overline{d}_{\overline{m}}(x) = \overline{s}$ *for every* $x \in F$, *where* $\overline{s} = \frac{\log 2}{\log \overline{\lambda}}$.

The fact that $\dim_H F = \underline{s}$ immediately implies that

$$\underline{d}_{\underline{m}}(x) \le \underline{s} \ \text{ for } \underline{m} - \text{almost every } x \in F$$

and

$$\underline{d}_{\overline{m}}(x) \le \underline{s} \ \text{ for } \overline{m} - \text{almost every } x \in F.$$

Otherwise there would exist a set A of positive $\underline{m}-$measure (respectively $\overline{m}-$measure) with $\underline{d}_{\underline{m}}(x) \ge \underline{s} + \epsilon$ (respectively $\underline{d}_{\overline{m}}(x) \ge \underline{s} + \epsilon$) for any $x \in A$. The non-uniform mass distribution principle would then imply that $\dim_H F \ge \dim_H A \ge \underline{s} + \epsilon$. Statement (a) follows from Statement 2 in Proposition 4.

In order to prove (b), consider $r_k = \overline{\lambda}^{n_{3k+1}}$ and denote $\Delta_k(x)$ the unique cylinder set $\Delta_{i_1 \cdots n_{3k+1}}$ that contains $x \in F$. It is easy to see that $B(x, r_k) \bigcap F \subset \Delta_k(x) \bigcap F$. This implies that for all $x \in F$,

$$\overline{m}(B(x, r_k) \cap F) \le \overline{m}(\Delta_k(x) \cap F) \le D_1 \overline{\lambda}^{\overline{s} n_{3k+1}} = D_1 r_k^{\overline{s}}$$

and hence

$$\overline{d}_{\overline{m}}(x) \ge \limsup_{k \to \infty} \frac{\log \overline{m}(B(x, r_k) \cap F)}{\log r_k} \ge \overline{s}.$$

Similarly, we have that for all $x \in F$

$$\underline{m}(B(x, r_k) \cap F) \le \underline{m}(\Delta_k(x) \cap F) \le D_1 \underline{\lambda}^{\underline{s} n_{3k+1}}$$

and hence,

$$\overline{d}_{\underline{m}}(x) \ge \limsup_{k \to \infty} \frac{\log \underline{m}(B(x, r_k) \cap F)}{\log r_k} = \underline{s} \frac{\log \underline{\lambda}}{\log \overline{\lambda}} = \overline{s}.$$

Statement (b) now follows from the second statement of Proposition 4. $\square$

6.3. Let F be the limit set generated by the simple geometric construction in Example 6, $G : F \to F$ the map $G = \chi^{-1} \circ \sigma \circ \chi$, where χ denotes the coding map and $\sigma : \Sigma_p^+ \to: \Sigma_p^+$ is the full shift (see Remark 5). Consider the set $\tilde{F} = F \times F$ endowed with the metric

$$\tilde{\rho}((x_1, y_1), (x_2, y_2)) = \rho(x_1, x_2) + \rho(y_1, y_2), \qquad x_1, x_2, y_1, y_2 \in F$$

and the coding map $\tilde{\chi} : \tilde{F} \to \Sigma_p$ defined by

$$\tilde{\chi}(x, y) = (\cdots i_{-1} i_0 i_1 \cdots)$$

where Σ_p denotes the space of two-sided sequences $(\cdots i_{-1} i_0 i_1 \cdots)$, $i_j = 1 \cdots p$ and $\chi(x) = (\cdots i_{-1})$, $\chi(y) = (i_0 i_1 \cdots)$. Set $\tilde{G} = \tilde{\chi}^{-1} \circ \sigma \circ \tilde{\chi}$. It is easy to see that $\tilde{G}$ is a homeomorphism and that for any $(x, y) \in \tilde{F}$,

$$\pi_1 \tilde{G}(x, y) = G(x), \quad \pi_2 \tilde{G}^{-1}(x, y) = G(y)$$

where π_1, π_2 are the projections, $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Consider the measure $\underline{\tilde{m}} = \tilde{\chi}^* \underline{\tilde{\mu}}$ where $\underline{\tilde{\mu}}$ is the measure on Σ_p^+ defined by

$$\underline{\tilde{\mu}}(\Delta_{i_k \cdots i_n}) = \underline{\lambda}^{(n-k)\underline{s}}.$$

By virtue of Corollary 2, $\underline{\tilde{\mu}}$ is invariant under σ and hence $\underline{\tilde{m}}$ is invariant under $\tilde{G}$. It is easy to see that $\underline{\tilde{m}} = \underline{m} \times \underline{m}$. It follows that for $\underline{\tilde{m}}-$almost every (x, y)

$$\underline{d}_{\underline{\tilde{m}}}(x, y) = \underline{d}_{\underline{m}}(x) + \underline{d}_{\underline{m}}(y) = \underline{s}$$

$$\overline{d}_{\underline{\tilde{m}}}(x, y) = \overline{d}_{\underline{m}}(x) + \overline{d}_{\underline{m}}(y) = \overline{s}.$$

It is not difficult to check that $\underline{\tilde{m}}$ is an equilibrium measure (and is a Gibbs measure) corresponding to the function $\underline{s\phi}(x, y) = 2\underline{s} \log \underline{\lambda}$. Thus the map $\tilde{G}$ provides an example of a homeomorphism that is not a smooth map but possesses a Gibbs measure with different lower and upper pointwise dimension almost everywhere. The same is also true with respect to the measure $\overline{\tilde{m}} = \tilde{\chi}^* \overline{\tilde{\mu}}$ where $\overline{\tilde{\mu}}$ is defined by $\overline{\tilde{\mu}}(\Delta_{i_k \cdots i_n}) = \overline{\lambda}^{(n-k)\overline{s}}$. As we mentioned, this does not hold for smooth maps [L, PY].

6.4. We will construct a simple geometric construction for which the map G (see Remark (5)) possesses an ergodic invariant measure with positive entropy whose pointwise dimension exists almost everywhere, and is not constant. Our approach follows [C].

Let F be the limit set for simple geometric construction on $[0, 1]$ with $p = 3$ and ratio coefficients $\lambda_{i,n}$, $i = 1, 2, 3$ and $n = 1, 2, 3, \cdots$ given by

$$\lambda_{1,n} = \lambda_{2,n} = \begin{cases} \alpha & \text{if } n \text{ is even} \\ \beta & \text{if } n \text{ is odd,} \end{cases}$$

$$\lambda_{3,n} = \begin{cases} \gamma, & \text{if } n \text{ is even} \\ \delta, & \text{if } n \text{ is odd,} \end{cases}$$

where $0 < \alpha < \beta < 1$, $0 < \gamma < \delta < 1$ and $\alpha\beta \neq \gamma\delta$. It is easy to show that the following limit exists:

$$\log \lambda_i \stackrel{\text{def}}{\equiv} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log \lambda_{i,k} = \begin{cases} \frac{1}{2}\log(\alpha\beta), & \text{if } i = 1, 2 \\ \frac{1}{2}\log(\delta\gamma), & \text{if } i = 3, \end{cases}$$

and thus the construction is a one-dimensional asymptotic construction.

Let s be the unique root of the equation

$$2(\sqrt{\alpha\beta})^s + (\sqrt{\gamma\delta})^s = 1.$$

According to Proposition 9 we have that $s \leq \dim_H F$.

Consider the transitive stochastic matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}.$$

Let μ be the Markov measure determined by P and let $\nu = \chi^*\mu$. Consider the sets

$$A = \left\{x \in F \,:\, \chi(x) = (i_1 i_2 \cdots) \in \Sigma_P^+ \text{ with } i_j = 1 \text{ or } 2\right\}$$

and

$$B = \left\{x \in F \,:\, \chi(x) = (i_1 i_2 \cdots) \in \Sigma_P^+ \text{ with } i_1 = 3\right\}.$$

The sets A and B are not invariant under G and $\nu(A) > 0$, $\nu(B) > 0$. Noting that $P^3 = P$, and using arguments in [C], one can show that for $\nu-$almost every $x \in A$,

$$\lim_{n\to\infty} \frac{\log \operatorname{diam}\Delta_n(x)}{n} = \frac{1}{2}\log(\alpha\beta)$$

and for $\nu-$almost every $x \in B$,

$$\lim_{n\to\infty} \frac{\log \operatorname{diam}\Delta_n(x)}{n} = \frac{1}{2}\log(\gamma\delta).$$

Consider the map $G = \chi^{-1} \circ \sigma \circ \chi$. Since $\chi(\Delta_n(x))$ is a cylinder set, the Shanon-McMillan-Breiman Theorem implies that for $\nu-$almost every $x \in F$,

$$\lim_{n\to\infty} \frac{\log \nu(\Delta_n(x))}{n} = h_\nu(G) = h_\mu(\sigma) > 0$$

where $h_\nu(G)$ is the Kolmogorov-Sinai entropy of the map G. Thus

$$\underline{d}(x) = \overline{d}(x) = d(x) = \frac{h_\mu(\sigma)}{\lim_{n\to\infty}\left(\frac{\log \operatorname{diam}\Delta_n(x)}{n}\right)} = \frac{2h_\nu(G)}{\log(\alpha\beta)} \quad \text{if } x \in A$$

and

$$\underline{d}(x) = \overline{d}(x) = d(x) = \frac{h_\mu(\sigma)}{\lim_{n\to\infty}\left(\frac{\log \operatorname{diam}\Delta_n(x)}{n}\right)} = \frac{2h_\nu(G)}{\log(\gamma\delta)} \quad \text{if } x \in B.$$

Repeating the construction in Section 6.3, one can prove that the homeomorphism $\tilde{G}$ possesses an invariant ergodic measure $\tilde{\nu}$ for which $d_{\tilde{\nu}}(x,y) = \underline{d}_{\tilde{\nu}}(x,y) = \overline{d}_{\tilde{\nu}}(x,y)$ for $\nu-$almost every (x,y) and where $d_{\tilde{\nu}}(x,y)$ is not essentially constant.

Section 7: Proofs

Proof of Theorem 2.

Let γ be a strongly l-estimating vector. Given $r, 0 < r < 1$, consider a cover $\mathfrak{U}_r = \mathfrak{U}_r(\gamma)$ of the limit set F which consists of basic sets $\Delta^{(j)}, j = 1, \ldots, N$.

By the definition of a Gibbs measure (see Appendix 3) there exist positive constants D_1 and D_2 such that for $j = 1, \cdots, N$

$$D_1 \leq \frac{m_\gamma(\Delta_{i_1 \cdots i_{n(x_j)}})}{\prod_{k=1}^{n(x_j)} \gamma_{i_k}^{s_\gamma}} \leq D_2 \tag{10}$$

where $m_\gamma = \chi^* \mu_\gamma$ is the pull back of the Gibbs measure μ_γ corresponding to the function $\phi(i_1 i_2 \cdots) = s_\gamma \log \gamma_{i_1}$ on Q. Since the vector γ is strongly l-estimating, by (10), we have

$$m_\gamma(B(x,r)) \leq \sum_{j=1}^{N(x,r)} m_\gamma(\Delta^{(j)}) \leq \sum_{j=1}^{N(x,r)} D_2 \prod_{k=1}^{n(x_j)} \gamma_{i_k}^{s_\gamma}$$

$$\leq L_4 \sum_{j=1}^{N(x,r)} \prod_{k=1}^{n(x_j)+1} \gamma_{i_k}^{s_\gamma} \leq L_4 N(x,r) r^{s_\gamma} \leq L_5 r^{s_\gamma}, \tag{11}$$

where $L_4, L_5 > 0$ are constants. Hence the measure m_γ satisfies the uniform mass distribution principle. This proves statement (1). Moreover, (11) implies that $s_\gamma \leq \underline{d}_{m_\gamma}(x)$ for every $x \in F$. It follows that $s_\gamma \leq \dim_H F$ and $\mathrm{m_H}(s_\gamma, F) > 0$.

We now prove that $m_\gamma(Z) \leq m_H(s, Z)$. Given $\delta > 0$, there exists $\epsilon > 0$ and a covering $\mathfrak{U} = \{U^{(k)}\}$ of Z by open sets $U^{(k)}$ with $\operatorname{diam} U^{(k)} \leq \epsilon$ satisfying

$$\sum_{U^{(k)} \in \mathfrak{U}} (\operatorname{diam} U^{(k)})^s \leq m_H(s, Z) + \delta.$$

By (11) it follows that

$$m_\gamma(Z) \leq \sum_{U^{(k)} \in \mathfrak{U}} m_\gamma(U^{(k)}) \leq L_5 \sum_{U^{(k)} \in \mathfrak{U}} (\operatorname{diam} U^{(k)})^s \leq L_5 m_H(s, Z) + L_5 \delta.$$

Since δ is chosen arbitrarily this implies that $m_\gamma(Z) \leq m_H(s, Z)$. $\square$

Proof of Proposition 2.

We begin with the following general lemma:

Lemma 5. *Let $A = (A(i,j))$ be a transitive $(p \times p)$ matrix of 0s and 1s, and consider the subshift of finite type defined by A. Let $f : \Sigma_A^+ \to \mathbb{R}$ be a continuous function that depends only on the first coordinate. Let F denote the $(p \times p)$ diagonal matrix $diag(e^{f(1)}, e^{f(2)}, \cdots, e^{f(p)})$. Then $P_A(\log f(\omega)) = P_A(\log f(\omega_1)) = \log r$, where r denotes the spectral radius of the $(p \times p)$ matrix A^*F.*

Proof.

In the proof, we exploit the fact that the exponential of the pressure is the maximal eigenvalue of the transfer operator. Let $\phi : \Sigma_A^+ \to \mathbb{R}$ be a continuous function. Then the transfer operator

$$(L_f\phi)(x) \stackrel{\text{def}}{=} \sum_k \exp(f(kx))\phi(kx)A(k,x_1) = \sum_k \exp(f(k))\phi(kx)A(k,x_1),$$

where $x = (x_1x_2\cdots)$. The eigenvalue equation for L_f is

$$\sum_k \exp(f(k))h(k)A(k,j) = \eta h(j).$$

According to [PP, page 24 (note normalization of the transfer operator)], the largest eigenvalue of L_f is $\eta = \exp(P(f))$. Hence $\exp(P(f))$ is the spectral radius of the matrix A^*F.
□

Proposition 2 follows by applying Lemma 5 to the function $f(\omega) = f(\omega_1) = t\log\alpha_{\omega_1}$.
□

Proof of Theorem 4.

By the definition of a Gibbs measure, there exist constants $D_1, D_2 > 0$ such that for any $n \in \mathbb{N}$,

$$D_1 \leq \frac{m_\lambda(\Delta_{i_1\cdots i_n})}{\prod_{k=1}^n \lambda_{i_k}^{s_\lambda}} \leq D_2. \tag{12}$$

Fix $0 < r < 1$. For each $x \in F$ with $\chi(x) = (i_1i_2\cdots)$, choose $1 \leq i_{n(x)} \leq p$ such that $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n(x)}} > r$ and $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n(x)}}\lambda_{i_{n(x)+1}} \leq r$. It follows that $\Delta_{i_1\cdots i_{n(x)+1}} \subset B(x, 2C_2r)$, where $B(x,r)$ denotes the ball of radius r around the point x. Hence for all $x \in F$,

$$m_\lambda(B(x,2C_2r)) \geq m_\lambda(\Delta_{i_1\cdots i_{n(x)+1}}) \geq D_1 \prod_{k=1}^{n(x)+1} \lambda_{i_k}^{s_\lambda} \geq L_1 r^{s_\lambda}$$

where $L_1 > 0$ is a constant. It follows by (12) that for all $x \in F$,

$$\overline{d}_{m_\lambda}(x) = \limsup_{r\to 0} \frac{\log m_\lambda(B(x,r))}{\log r} \leq s_\lambda.$$

We now prove that $m_H(s,\cdot) \le const\, m_\lambda(\cdot)$. Let $Z \subset F$ be a closed subset. Given $\delta > 0$, there exists $\epsilon > 0$ such that for any covering $\mathfrak{U} = \{U^{(k)}\}$ of Z by open sets $U^{(k)}$ with $\operatorname{diam} U^{(k)} \le \epsilon$, we have

$$m_H(s,Z) \le \sum_{U^{(k)} \in \mathfrak{U}} (\operatorname{diam} U^{(k)})^s + \delta.$$

There exists a covering $\mathfrak{U}$ of Z by basic sets $\Delta^{(k)} = \Delta_{i_1 \cdots i_{n(k)}}$ satisfying $\operatorname{diam} \Delta^{(k)} \le \epsilon$ and

$$\sum_{\Delta^{(k)} \in \mathfrak{U}} m_\lambda(\Delta^{(k)}) \le m_\lambda(Z) + \delta.$$

Since λ is a u-estimating vector, it follows from (2) that

$$m_H(s,Z) \le \sum_{\Delta^{(k)} \in \mathfrak{U}} (\operatorname{diam} \Delta^{(k)})^s + \delta \le C_2 \sum_{\Delta^{(k)} \in \mathfrak{U}} \prod_{j=1}^{n(k)} \lambda_{i_j}^s + \delta$$

$$\le C_2 D_2 \sum_{\Delta^{(k)} \in \mathfrak{U}} m_\lambda(\Delta^{(k)}) + \delta \le C_2 D_2 m_\lambda(Z) + (C_2 D_2 + 1)\delta.$$

Since δ is chosen arbitrarily this implies the desired result.

Proof of Theorem 1.

Given numbers $0 < \lambda_1, \lambda_2, \cdots, \lambda_p < 1$, let μ_λ be an equilibrium measure on Q corresponding to the function $s \log \lambda_{i_1}$. By definition,

$$h_{\mu_\lambda}(\sigma|Q) + s \int_Q \log \lambda_{i_1} d\mu_\lambda = 0, \tag{13}$$

where $h_\mu(\sigma|Q) \stackrel{\text{def}}{=} h$ is the Kolmogorov-Sinai entropy. Let us first assume that μ_λ is ergodic. For fixed $\epsilon > 0$, it follows from the Shannon-McMillan-Breiman theorem that for $\mu_\lambda-$almost every $\omega \in Q$ one can find $N_1(\omega) > 0$ such that for any $n \ge N_1(\omega)$,

$$\exp(-(h+\epsilon)n) \le \mu_\lambda(C_{i_1 \cdots i_n(\omega)}) \le \exp(-(h-\epsilon)n) \tag{14}$$

where $C_{i_1 \cdots i_n(\omega)}$ is the cylinder set containing ω. It follows from the Birkhoff ergodic theorem that for $\mu_\lambda-$almost every $\omega \in Q$ there exists $N_2(\omega)$ such that for any $n \ge N_2(\omega)$,

$$\frac{1}{n} \log \prod_{j=1}^{n} \lambda_{i_j}^s - \epsilon \le s \int_Q \log \lambda_{i_1} d\mu_\lambda \le \frac{1}{n} \log \prod_{j=1}^{n} \lambda_{i_j}^s + \epsilon. \tag{15}$$

Combining (13), (14), and (15) we have that for $\mu_\lambda-$almost every $\omega \in Q$ and n sufficiently large,

$$\prod_{j=1}^{n} \lambda_{i_j}^{s+\alpha} \le \prod_{j=1}^{n} \lambda_{i_j}^s \exp(-2\epsilon n) \le \mu_\lambda(C_{i_1 \cdots i_n(\omega)}) \le \prod_{j=1}^{n} \lambda_{i_j}^s \exp(2\epsilon n) \le \prod_{j=1}^{n} \lambda_{i_j}^{s-\alpha},$$

where $\alpha = \frac{2\epsilon}{\min(\log\frac{1}{\lambda_j}, 1\le j\le p)} > 0$. This implies that for $\mu_\lambda-$almost every $\omega \in Q$ and any $n \ge \max\{N_1(\omega), N_2(\omega)\}$,

$$\prod_{j=1}^{n} \lambda_{i_j}^{s+\alpha} \le \mu_\lambda(C_{i_1\cdots i_n(\omega)}) \le \prod_{j=1}^{n} \lambda_{i_j}^{s-\alpha}. \tag{16}$$

If μ_λ is not ergodic, then (16) is still valid and can be shown by decomposing μ into its ergodic components.

Given $l > 0$ denote $Q_l = \{\omega \in Q : N_1(\omega) \le l \text{ and } N_2(\omega) \le l\}$. It is easy to see that $Q_l \subset Q_{l+1}$ and $Q = \cup_{l=1}^{\infty} Q_l$ (mod 0). Thus there exists $l_0 > 0$ such that $\mu(Q_l) > 0$ if $l \ge l_0$. Let $x \in Q_l, l \ge l_0, 0 < r < 1$.

Repeating the arguments in the proof of Theorem 2 with Q replaced by Q_l and applying (16) to the numbers $\gamma_1, \cdots, \gamma_p$ and measure μ_γ, one obtains that $m_\gamma(B(x,r) \cap \chi^{-1}(Q_l)) \le K r^{s_\gamma - \alpha}$, where $K = K(l) > 0$ is a constant and $x \in \chi^{-1}(Q_l))$ is any point. This implies that for any $l > 0$ and $x \in \chi^{-1}(Q_l)$

$$\underline{d}^{l}_{m_\gamma}(x) \stackrel{\text{def}}{\equiv} \liminf_{r\to 0} \frac{\log m_\gamma(B(x,r) \cap \chi^{-1}(Q_l))}{\log r} \ge s_\gamma - \alpha.$$

It follows that for all $l \in \mathbb{N}$, $\dim_H(F) \ge s_\gamma - \alpha$. This implies that $\dim_H(F) \ge s_\gamma - \alpha$. Since α can be arbitrarily small, this gives the desired result.

Proof of Theorem 3.

Fix $0 < r < 1$. For each x with $\chi(x) = (i_1 i_2 \cdots)$, choose the unique $n(x)$ such that $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n(x)}} > r$ and $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n(x)+1}} \le r$. It is easy to see that $n(x) \to \infty$ as $r \to 0$ uniformly in x. Fix $l > 0$ for which $\mu_\lambda(Q_l) > 0$, where Q_l are the sets constructed in the proof of Theorem 1. One can now choose $r = r(l) > 0$ sufficiently small such that $n(\omega)$ becomes large enough to satisfy (16) for any $\omega \in Q_l$. Repeating the arguments in the proof of Theorem 4 with Q replaced by Q_l, and applying (16) to the numbers $(\lambda_1, \ldots, \lambda_p)$ and the measure μ_λ, one can show that for any $x \in \chi^{-1}(Q_l)$ and any $r > 0$ sufficiently small,

$$m_\lambda(B(x,r)) \ge K r^{\lambda+\alpha}$$

where $K = K(l) > 0$ is a constant. This implies that $\overline{d}_{m_\lambda}(x) \le s_\lambda + \alpha$ and hence, $\overline{d}_{m_\lambda}(x) \le s_\lambda + \alpha$ for any $x \in F$. This completes the proof of statement (2).

We now show that $\overline{\dim}_B F \le s_\lambda$. It is sufficient to prove that $P(\overline{\dim}_B F \log \lambda_{i_1}) \ge 0$, since the map $t \to P(t \log \lambda_{i_1})$ is a decreasing function [Bo1].

Given $\delta > 0$, it follows from the definition of $\overline{\dim}_B F$ that there exists $\epsilon > 0$ such that $N_\epsilon(F) \ge \epsilon^{\delta - \overline{\dim}_B F}$. Consider the cover $\{\Delta^{(j)}\} = \{\Delta_{i_1,\cdots,i_{n(x_j)}}\}$, $j = 1, \cdots, N^\epsilon(F)$. Note that this covering need not be the optimal covering, i.e., $N^\epsilon(F) \ge N_\epsilon(F)$. There clearly exists $A = A(\epsilon) > 0$ such that for $j = 1, \cdots, N^\epsilon(F)$,

$$\frac{\epsilon}{A} \le \prod_{k=1}^{n(x_j)} \overline{\lambda}_{i_k} \le \epsilon$$

and hence

$$C_1 \log(\frac{1}{\epsilon}) \leq n(x_j) \leq C_2 \log(\frac{A}{\epsilon})$$

where $C_1 = \frac{1}{\log(\frac{1}{\lambda_{\max}})}$ and $C_2 = \frac{1}{\log(\frac{1}{\lambda_{\min}})}$. This implies that $n(x_j)$ can take on at most $C_2 \log(\frac{A}{\epsilon}) - C_1 \log(\frac{1}{\epsilon})$ possible values.

We now think of having $N_\epsilon(F)$ balls and $C_2 \log(\frac{A}{\epsilon}) - C_1 \log(\frac{1}{\epsilon})$ baskets. Then for $N^\epsilon(F) \geq C_2 \log(\frac{A}{\epsilon}) - C_1 \log(\frac{1}{\epsilon})$, there exists a basket containing at least $\frac{N^\epsilon(F)}{C_2 \log(\frac{A}{\epsilon}) - C_1 \log(\frac{1}{\epsilon})}$ balls. This implies that there exists a positive integer α, $C_1 \log(\frac{1}{\epsilon}) \leq \alpha \leq C_2 \log(\frac{A}{\epsilon})$ such that for ϵ sufficiently small,

$$\#\{x_j \text{ such that } n(x_j) = \alpha\} \geq \frac{N^\epsilon(F)}{C_2 \log(\frac{A}{\epsilon}) - C_1 \log(\frac{1}{\epsilon})}$$

$$\geq \frac{N_\epsilon(F)}{C_2 \log(\frac{A}{\epsilon}) - C_1 \log(\frac{1}{\epsilon})} \geq \frac{\epsilon^{\delta - \overline{\dim}_B F}}{C_3 \log(\frac{1}{\epsilon})} \geq \epsilon^{2\delta - \overline{\dim}_B F}.$$

Let $\phi(x) = (\overline{\dim}_B F - 2\delta) \log \lambda_{i_1}$, where $\chi(x) = (i_1 i_2 \ldots)$. Then

$$(S_n \phi)(x) = \sum_{k=1}^{n} \phi(\sigma^k x) = (\overline{\dim}_B F - 2\delta) \log \prod_{k=1}^{n} \lambda_{i_k}$$

and hence $\exp(S_n\phi)(x) = (\prod_{k=1}^{n} \lambda_{i_k})^{\overline{\dim}_B F - 2\delta}$. It follows that

$$P_\alpha((\overline{\dim}_B F - 2\delta) \log \lambda_{1_1}) \stackrel{\text{def}}{\equiv} \frac{1}{\alpha} \log \sum_{\substack{(i_1 \cdots i_\alpha) \\ \text{admissible}}} \inf_{x \in \Delta_{(i_1 \cdots i_\alpha)}} (\prod_{k=1}^{n} \lambda_{i_k})^{\overline{\dim}_B F - 2\delta}$$

$$\geq \frac{1}{\alpha} \log \sum_{\substack{(i_1 \cdots i_\alpha) \\ \text{in covering } \{\Delta^{(j)}\}}} \left(\frac{A}{\epsilon}\right)^{\overline{\dim}_B F - 2\delta}$$

$$\geq A^{\overline{\dim}_B F - 2\delta} \frac{1}{\alpha} \log(\epsilon^{\overline{\dim}_B F - 2\delta} \cdot \epsilon^{2\delta - \overline{\dim}_B F}) \geq 0.$$

Hence $P((\overline{\dim}_B F - 2\delta) \log \lambda_{i_1}) = \lim_{\alpha \to \infty} P_\alpha((\overline{\dim}_B F - 2\delta) \log \lambda_{i_1}) = 0.$ $\square$

Proof of Proposition 1.

Let μ_λ denote an equilibrium measure for $s_\lambda \log \lambda_{i_1}$. It immediately follows from the variational principle that

$$s_\lambda = \frac{h_{\mu_\lambda}(\sigma \mid Q)}{-\int \log \lambda_{i_1} d\mu_\lambda} \leq \frac{h(\sigma \mid Q)}{-\log \max \lambda_k}.$$

The case of equality is obvious.

If $\lambda_i = \lambda$ for $i = 1, \cdots, p$, then this immediately implies that $s_\lambda = \frac{h(\sigma\,|\,Q)}{-\log\lambda}$. This proves the desired results. $\square$

Proof of Proposition 7.

1) Condition a) implies that for any $\epsilon > 0$ and $\mu_\lambda-$almost every $\omega \in Q$, $\lambda = (\lambda_1, \ldots, \lambda_p)$, there exists $N_3(\omega) = N_3(\omega, \epsilon)$ such that for any $n \geq N_3(\omega)$,

$$\left| \frac{1}{n} \sum_{j=1}^{n} \underline{a}_{i_j, j} \right| \leq \epsilon. \tag{17}$$

The inequality (17) is equivalent to

$$\prod_{j=1}^{n} \lambda_{i_j} \exp(-\epsilon n) \leq \prod_{j=1}^{n} \underline{\lambda}_{i_j, j} \leq \prod_{j=1}^{n} \lambda_{i_j} \exp(\epsilon n). \tag{18}$$

It is sufficient to consider only the case when μ_λ is ergodic with respect to σ. Then for $\mu_\lambda-$almost every $\omega = (i_1 i_2 \cdots) \in Q$, the following limit exists:

$$\lim_{n\to\infty} \sum_{j=1}^{n} \log \lambda_{i_j} = \int_Q \log \lambda_{\omega_1} d\mu_\lambda(\omega) \equiv a < 0.$$

This implies that for any $\epsilon > 0$ and $\mu_\lambda-$almost every $\omega \in Q$ there exists $N_4(\omega) = N_4(\omega, \epsilon)$ such that for any $n \geq N_4(\omega)$,

$$\left| \frac{1}{n} \sum_{j=1}^{n} \log \lambda_{i_j} - a \right| \leq \epsilon. \tag{19}$$

Given $l > 0$ denote

$$Q_l = \{\omega \in Q : N_i(\omega) \leq l, i = 1, 2, 3, 4, 5\}$$

where $N_1(\omega), N_2(\omega)$ are the two functions constructed in the proof of Theorems 1 (see (16)). It is easy to see that $Q_l \subset Q_{l+1}$ and $Q = \cup_{l=1}^{\infty} Q_l \mod (0)$. Thus there exists $l_0 > 0$ such that $\mu_\lambda(Q_l) > 0$ if $l \geq l_0$.

Consider $\omega = (i_1 i_2 \cdots) \in Q_l, l \geq l_0$ and $0 < r < 1$. Let $n(\omega)$ denote the unique positive integer such that $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n(\omega)}} > r$ and $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_{n(\omega)+1}} \leq r$. Obviously, $n(\omega) \to \infty$ as $r \to 0$ uniformly in ω. Hence we can assume $r = r(l)$ is so small that $n(\omega) \geq \max\{N_3(\omega), N_4(\omega)\}$. Applying (19) to $n = n(\omega)$ we have that

$$n(\omega) \leq \frac{\log r}{(a + \epsilon)}. \tag{20}$$

Consider the cylinder set $C_{i_1 \cdots i_{n(\omega)}}$. We have that $\omega \in C_{i_1 \cdots i_{n(\omega)}}$ and if $\omega' \in C_{i_1 \cdots i_{n(\omega)}} \cap Q_l$ and $n(\omega') \leq n(\omega)$ then

$$C_{i_1 i_2 \cdots i_{n(\omega')}} \supset C_{i_1 i_2 \cdots i_{n(\omega)}}.$$

Let $C(\omega)$ be the largest cylinder set containing ω with the property that for every $\omega' \in C(\omega) \cap Q_l$ we have $C_{i_1 \cdots i_{n(\omega')}} \cap Q_l \subset C(\omega) \cap Q_l$ and there exists $\tilde{\omega} \in C(\omega)$ such that $C(\omega) = C_{i_1 \cdots i_{n(\tilde{\omega})}}$. It is easy to see that the sets $C(\omega) \cap Q_l$ corresponding to different $\omega \in Q_l$ either coincide or are disjoint. We denote these sets by $C^{(j)}, j = 1, \cdots N$. There exists points ω_j such that $C^{(j)} = C_{i_1 \cdots i_{n(\omega_j)}}$. For any $j = 1, \cdots, N$ we have

$$l^{-1} \prod_{k=1}^{n(\omega_j)} \lambda_{i_k}^{s_\lambda} \le \mu_\lambda(C^{(j)}) \le l \prod_{k=1}^{n(\omega_j)} \lambda_{i_k}^{s_\lambda}.$$

Consider a point $x \in F$ with $\chi(x) = \omega \in Q_l$. We estimate the number $N(x, r)$ of sets $C^{(j)}$ that intersect the set $\chi(B(x, r)) \cap Q_l$ for sufficiently small r. It follows from (19) and (20) that

$$\begin{aligned} N(x,r) &\le \frac{\operatorname{vol}(B(x, r + \max_{1\le j\le N}(\operatorname{diam}(\chi^{-1}(C^{(j)}))))}{\min_{1\le j\le N} \operatorname{vol}(\chi^{-1}(C^{(j)} \cap Q_l))} \\ &\le \frac{\max_{1\le j\le N} \operatorname{vol}(B(x, r + \frac{r}{\lambda_{\min}} \exp(\epsilon n(w_j))))}{\min_{1\le j\le N} \operatorname{vol} B(x, C_1 r^{\frac{a}{a+\epsilon}})} \\ &\le C_2 \frac{\left(r(1 + \frac{1}{\lambda_{\min}} r^{\frac{\epsilon}{a+\epsilon}})\right)^d}{(r^{\frac{\alpha}{a+\epsilon}})^d} \le C_3 r^{-b\epsilon}, \end{aligned}$$

where C_1, C_2, C_3, b are positive constants. This implies that the construction is conditionally regular. The proof of Proposition 8 is a slight modification of the proof of Proposition 7.

3) The proof follows from **2)**. $\square$

Proof of Lemma 1.

We first prove Statement (1). The proof of Statement (2) is similar. Applying the Birkhoff ergodic theorem for the stationary ergodic process to the functions $\underline{f}(\vec{\lambda}) = (\log \underline{\lambda}_{1,1}, \cdots, \log \underline{\lambda}_{p,1})$ and $\overline{f}(\vec{\lambda}) = (\log \overline{\lambda}_{1,1}, \cdots, \log \overline{\lambda}_{p,1})$ we obtain that

$$\frac{1}{n} \sum_{k=1}^{n} \log \underline{\lambda}_{i,k} \to \log \underline{\lambda}_i$$

for ν almost every $\vec{\lambda}$, where we define $\log \underline{\lambda}_i$ as the limiting value. We need to show that for $\underline{\mu}$ almost every $\omega = (i_1 i_2 \cdots) \in Q$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} \log\left(\frac{\underline{\lambda}_{i_k,k}}{\underline{\lambda}_{i_k}}\right) = 0.$$

We break up the sum

$$\frac{1}{N} \sum_{k=1}^{N} \log\left(\frac{\underline{\lambda}_{i_k,k}}{\underline{\lambda}_{i_k}}\right) = \frac{1}{N} \sum_{j=1}^{p} \sum_{\substack{i_k=j \\ 1\le k\le N}} \log\left(\frac{\underline{\lambda}_{j,k}}{\underline{\lambda}_j}\right)$$

$$= \sum_{j=1}^{p} \frac{\#\{i_k = j\}}{N} \left(\frac{1}{\#\{i_k = j\}} \sum_{\substack{i_k = j \\ 1 \le k \le N}} \log \left(\frac{\underline{\lambda}_{j,k}}{\underline{\lambda}_j} \right) \right).$$

We wish to show that for $\underline{\mu}-$almost every ω, each of the above terms in parentheses tends to zero as N tends to infinity. This immediately follows from the following generalization of the Birkhofff ergodic theorem for return times to a set.

Theorem [BFKO]**.** Let $(X, \mathfrak{B}, \nu, T)$ and $(Y, \mathfrak{C}, \mu, S)$ be two ergodic measure theoretic dynamical systems. Let $A \subset Y$ be of positive measure $\mu(A) > 0$. For every $y \in Y$ let A_y denote the return time sequence $\{n \in \mathbb{N} \,|\, S^n y \in A\}$. Then for $\mu-$almost every $y \in Y$ and for $f \in L^1(X)$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{\substack{1 \le k \le N \\ k \in A_y}} T^k f(x) = \int_X f d\nu$$

for $\nu-$almost every $x \in X$.

We apply this theorem for T the ergodic stationary process in the definition of random symbolic geometric construction, $S = \sigma : Q \to Q$ the shift map with invariant Gibbs measure $\underline{\mu}$, and $A = C_j = \{\omega \in Q \,|\, \omega_1 = j\}$. Clearly, the set $\{1 \le k \le N, \, i_k = j\} = \{1 \le k \le N, \, \sigma^k \omega \in C_j\}$. $\square$

Proof of Theorem 7.

Given $x \in F, \chi(x) = (i_1 i_2 \cdots)$ and $r > 0$, choose $n_r(x)$ such that $\prod_{k=1}^{n_r(x)} \lambda_{i_k,k} < r$ and $\prod_{k=1}^{n_r(x)-1} \lambda_{i_k,k} \ge r$. Since $x \in \Delta_{n_r(x)}(x)$ we have that $\Delta_{n_r(x)}(x) \subset B(x,r)$. This implies

$$\frac{\log \nu(B(x,r))}{\log r} \le \frac{\log \nu(\Delta_{n_r(x)}(x))}{\log r}.$$

We also have that

$$r \le \prod_{k=1}^{n_r(x)-1} \lambda_{i_k,k} = \prod_{k=1}^{n_r(x)} \lambda_{i_k,k} \frac{1}{\lambda_{i_{n_r(x)}}} \le \frac{1}{\beta} \mathrm{diam} \Delta_{n_r(x)}(x).$$

It follows that

$$\overline{d}_\nu(x) = \limsup_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} \le \limsup_{r \to 0} \frac{\log \nu(\Delta_{n_r(x)}(x))}{\log(\frac{1}{\beta} \mathrm{diam} \Delta_{n_r(x)}(x))}$$

$$\le \limsup_{r \to 0} \frac{\log \nu(\Delta_n(x))}{\log \mathrm{diam} \Delta_n(x)} = \overline{d}(x).$$

We now prove the second estimate in (1). Given $\alpha \ge 0, C > 0$, define $F_{\alpha,C} = \{x \in F :$ $\nu(\Delta_n(x)) \le C(\mathrm{diam}\Delta_n(x))^\alpha$ for all $n \ge 0\}$. Fix $x \in F_{\alpha,C}$ and $r > 0$. By repeating the arguments in the proof of Theorem 1, one can find points $x_j \in F_{\alpha,C}, j = 1, \cdots, N$ (where

N is independent of x and r) such that $x_j \in \Delta^{(j)} = \Delta_{i_1 \cdots i_{n(x_j)}}$ and such that the collection $\Delta^{(j)}$ are disjoint, $\prod_{k=1}^{n(x_j)} \lambda_{i_k,k} \leq r$, and $B(x,r) \cap F_{\alpha,C} \subset \cup_{j=1}^{N} (\Delta^{(j)} \cap F_{\alpha,C})$. If follows that $\nu(B(x,r)) \leq \sum_{j=1}^{N} \nu(\Delta^{(j)})$ and hence

$$\underline{d}_\nu(x) = \liminf_{r \to 0} \frac{\log \nu(B(x,r))}{\log r} \geq \liminf_{r \to 0} \frac{\log \sum_{j=1}^{N} \nu(\Delta^{(j)})}{\log r} \geq \alpha.$$

The result follows since $F = \cup_\alpha \cup_{C>0} F_{\alpha,C}$.

The last statement is a direct consequence of the preceeding statements. $\square$

Appendices.

Appendix 1: Hausdorff Dimension and Box Dimension

Let U $\subset \mathbb{R}^n$. The *diameter* of U is defined as $\mathrm{diam}(U) = \sup\{|x-y| : x, y \in U\}$. If $\{U_i\}$ is a countable collection of sets of diameter at most δ that cover Z, i.e., $Z \subset \cup_i U_i$ with $0 < |U_i| \le \delta$ for each i, we say that $\{U_i\}$ is a δ-cover of Z.

Suppose that $Z \subset \mathbb{R}^n$ and $s \ge 0$. For any $s > 0$ we define

$$\mathrm{m_H}(s, Z) = \lim_{\delta \to 0} \inf_{\{U_i\}} \left\{ \sum_i \mathrm{diam}(U_i)^s : \{U_i\} \text{ is a } \delta\text{-cover of } Z \right\}.$$

We call $\mathrm{m_H}(s, Z)$ the *s-dimensional Hausdorff measure of Z.* There exists a unique critical value of s at which $\mathrm{m_H}(s, Z)$ jumps from ∞ to 0. This critical value is called the *Hausdorff dimension* of Z and is written $\dim_H Z$. If $s = \dim_H Z$, then $\mathrm{m_H}(s, Z)$ may be $0, \infty$, or finite. Hence $\dim_H Z = \sup\{s : \mathrm{m_H}(s, Z) = \infty\} = \inf\{s : \mathrm{m_H}(s, Z) = 0\}$.

Let $N_\delta(Z)$ denote the minimum number of sets of diameter precisely δ needed to cover the set Z. We define the lower and upper box dimensions of Z by

$$\underline{\dim}_B Z = \liminf_{\delta \to 0} \frac{\log N_\delta(Z)}{\log(\frac{1}{\delta})}, \quad \text{and} \quad \overline{\dim}_B Z = \limsup_{\delta \to 0} \frac{\log N_\delta(Z)}{\log(\frac{1}{\delta})}.$$

It is easy to see that $\dim_H Z \le \underline{\dim}_B Z \le \overline{\dim}_B Z$.

Appendix 2: Two Methods of Obtaining Lower Bounds for $\mathbf{dim_H F}$

Uniform Mass Distribution Principle [Fr]. *Let μ be a probability measure of Z and suppose that for some s there are numbers $c > 0$, and $\delta > 0$ such that $\mu(U) \le c\,diam(U)^s$ for all sets U with $diam(U)^s \le \delta$. Then $\mathrm{m_H}(s, Z) \ge \mu(F)/c$ and $s \le dim_H Z$.*

Non-Uniform Mass Distribution Principle. *Let μ be a probability measure supported on $Z \subset R^n$. We define the* **Hausdorff dimension of the measure** μ, $dim_H \mu = \inf\{dim_H U,\ \mu(U) = 1\}$, *and the* **lower and upper pointwise dimensions of** μ, $\underline{d}_\mu(x) = \liminf_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon}$ *and* $\overline{d}_\mu(x) = \limsup_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon}$.

Proposition. *If $\underline{d}_\mu(x) \ge d \ge 0$ for a.e. $x \in Z$, then $dim_H \mu \ge d$, and hence $dim_H Z \ge d$.*

The uniform mass distribution principle implies the non-uniform mass distribution principle. In fact, $d_\mu^L(x) \ge d$ implies that for any $\alpha > 0$, $\mu(B(x, r)) \le C(x, \alpha) r^{d-\alpha}$.

Appendix 3: Thermodynamic Formalism

Good references for this material are [Bo1] and [PP]. Given a transitive $p \times p$ transfer matrix A of $0's$ and $1's$, consider the one-sided subshift of finite type (Σ_A^+, σ) with $\sigma : \Sigma_A^+ \to \Sigma_A^+$ the shift map. The set Σ_A^+ consists of all admissible words, i.e., a word $x = (x_1, x_2, \cdots) \in \{1, \cdots, p\}^{\mathbb{N}}$ is admissible if $a_{x_i, x_{i+1}} = 1$ for all $i \in \mathbb{N}$. The space Σ_A^+ has a natural family of metrics defined by $d(x, y) = \sum_{k=1}^{\infty} \frac{|x_k - y_k|}{\beta^k}$, where β is any number satisfying $\beta > 1$. The set Σ_A^+ is compact with respect to the topology induced by d and the shift map $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is a homeomorphism. We let $\Sigma_p^+ = \{1, \cdots, p\}^{\mathbb{N}}$.

More generally, we consider a general symbolic system, i.e., a compact subset $Q \subset \Sigma_p^+$ that is $\sigma-$invariant, i.e., $\sigma(Q) = Q$.

Let $C^0(Q)$, $C^\alpha(Q)$ denote the spaces of continuous and $\alpha-$Hölder continuous functions on Q. We define a mapping $P : C^0(Q) \to \mathbb{R}$ by

$$P(\phi) = \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{\substack{(i_1 \cdots i_n) \\ \text{admissible}}} \inf_{x \in \Delta_{(i_1 \cdots i_n)}} \exp(S_n \phi(x)) \right),$$

where $S_n\phi(x) = \sum_{i=0}^{n-1} \phi(\sigma^i x)$. We call $P(\phi)$ the *topological pressure* of ϕ.

The following result is a variational characterization of pressure that is valid for *all* topological dynamical systems. Let $\mathfrak{M}(Q)$ denote the space of all shift-invariant Borel probability measures on Q.

Variational Principle. Let $\phi \in C^0(Q)$. Then

$$P(\phi) = \sup_{\mu \in \mathfrak{M}(\Sigma_A)} \left(h_\mu(\sigma) + \int_{\Sigma_A} \phi d\mu \right).$$

Measures that realize the variational principle for topological pressure play crucial roles in the thermodynamical formalism.

Definition. A Borel probability measure $\mu = \mu_\phi$ on Q is called an **equilibrium measure** for the potential ϕ if

$$P(\phi) = h_\mu(\sigma) + \int_{\Sigma_A} \phi d\mu. \tag{21}$$

Since the shift map on a general symbolic system is expansive, the supremum in the variational principle is attained by some measure [W]. This measure need not be unique.

Another important class of measures are Gibbs measures:

Definition. Let $\phi \in C^0(Q)$. A Borel probability measure $\mu = \mu_\phi$ on Q is called a **Gibbs measure** for the potential ϕ if there exist constants $D_1, D_2 > 0$ such that

$$D_1 \le \frac{\mu\{y : y_i = x_i,\ i = 0, \cdots, n-1\}}{\exp(-nP(\phi) + \sum_{k=0}^{n-1} \phi(\sigma^k x))} \le D_2 \tag{22}$$

for all $x = (x_1 x_2 \cdots) \in \Sigma_A^+$ and $n \geq 0$.

For subshifts of finite type, Gibbs measures exist for any Holdër continuous potential ϕ, are unique, and coincide with the equilibrium measure for ϕ. Little is known about the existence of Gibbs measures for general symbolic systems.

The main tool used in constructing and studying Gibbs measures for subshifts of finite type is the (linear bounded) transfer operator $L_f : C^\alpha(\Sigma_A^+) \to C^\alpha(\Sigma_A^+)$ defined by

$$(L_f \phi)(x) = \sum_{y \in \sigma^{-1}(x)} \exp(f(y))\phi(y) = \sum_k \exp(f(kx))\phi(kx).$$

along with its dual operator $L_f^* : M(\Sigma_A^+) \to M(\Sigma_A^+)$, where $M(\Sigma_A^+)$ denotes the space of Borel probability measures on Σ_A^+. The following theorem of Ruelle constructs Gibbs measures using the operator L_f:

Proposition [PP]. *Let (Σ_A^+, σ) be mixing subshift of finite type. There exists $\lambda = \exp(P(\phi)) \geq 1$, $h \in C^0(\Sigma_A^+, \mathbb{R})$ with $h > 0$ and $\nu \in M(\Sigma_A^+)$ for which $L_f h = \lambda h, L_f^* \nu = \lambda \nu$, and $\nu(h) = 1$. Then $\mu = h\nu$ is a $\sigma-$invariant probability measure on Σ_A^+ and is a Gibbs measure for ϕ.*

In this paper we deal exclusively with a special class of potentials that depend only on the first coordinate, i.e., $\phi(x) = \phi(x_1)$. In this case, the measures ν and μ are Markov and can be described explicitly: the eigenfunction $h = h(x_1)$ satisfies

$$\sum_i h(i) A(i,j) \exp(\phi(i)) = \lambda h(j)$$

and the measure ν is defined on cylinder sets by

$$\nu[i_0, \cdots, i_n] = P(i_0, i_1) \cdots P(i_{n-1}, i_n) p(i_n) \quad \text{where} \quad P(i,j) = \frac{A(i,j)h(i)\exp(\phi(i))}{\lambda h(j)}$$

and $Pp = p$ with $\sum_i p(i) = 1$. Finally, the Gibbs measure μ is defined by $d\mu = h(x_1) d\nu$.

The Gibbs measure μ is unique provided the potential f is Hölder continuous.

Proposition [PP]. *Given numbers λ_i, $i = 1, \cdots, p$, define the function*

$$\phi : \Sigma_A^+ \to \Sigma_A^+ \quad \text{by } \phi(x) = \phi(x_1, x_2, \cdots) = \log \lambda_{x_1}.$$

Then ϕ is Hölder continuous with respect to the standard metric. Furthermore, there exists a unique s, $0 \leq s \leq 1$ such that $P(s\phi) = 0$.

Appendix 4: Gauge Functions

One can generalize the definition of Hausdorff measure to give more refined information about a set whose Hausdorff measure at the dimension is zero.

Suppose $h(t)$ is a continuous increasing function defined on $(0,\epsilon)$ with $\epsilon > 0$ such that $h(t) \to 0$ as $t \to 0$. Using the notation in Appendix 1, we define

$$\mathrm{m}_{\mathrm{H}}^{h}(Z) = \lim_{\delta \to 0} \inf_{\{U_i\}} \left\{ \sum_i h(\mathrm{diam}(U_i)) : \{\delta\text{-cover of } Z\} \right\}.$$

We call $\mathrm{m}_{\mathrm{H}}^{h}(Z)$ the *s-dimensional Hausdorff measure of Z with respect to the gauge function h.* Clearly letting $h(t) = t^s$ gives the $s-$dimensional Hausdorff measure defined in Appendix 1.

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