No More Than Mechanics. I*

Plain Mechanics:
Classical and Quantum Mechanics as well

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Abstract

This is the written version of a short talk on 10^{th} *Conference on Problems and Methods in Mathematical Physics* (September 13 - 17, 1993 in Chemnitz, Germany). A new scheme of the quantization is presented. A realization of the scheme for a particle in n-dimensional space by two-sided convolutions on the Heisenberg group is constructed.

Contents

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1 Introduction

The main goal of this paper is to present a new approach to relationships between classical and quantum mechanics. I think, there is a large discrepancy between the mathematical beauty of quantum mechanics and its absurdity from the common point of view. This discrepancy does not prevent us to make our computations with a large preciosity, but it induces future investigations of our basic assumptions. The problem of a quantization is still under the serious investigation (see [**?**, **?**, **?**, **?**, **?**, **?**, **?**]) beyond more than a half century after the creation of quantum mechanics.

This paper makes principal steps in such direction (but does not achieve the final point, of course). The usual "quantization" means some (more or less complete) set of rules for the construction of a quantum algebra from the classical description of a physical system. Our main suggestion is to replace such quantization by the constitution of an operator algebra from which both the classical and the "usual" quantum descriptions may be derived. The paper gives the more precise formulation for this approach and illustrates it on the simplest example: the quantization for a particle in n-dimensional space. Future papers in this series will present a more concluded description of *Plain Mechanics*. For example, application of plain mechanics to the quantum field theory requires consideration of Clifford valued convolutions on the Heisenberg group (see [**?**]).

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2 Mathematical Background

Our scheme is based on the properties of convolutions on the Heisenberg group[1]. This subject is well known and there are many suitable sources on it [**?**, **?**]. So we can introduce a few definitions only. The Heisenberg group is a step 2 nilpotent Lie group. As a $C^\infty-$manifold it coincides with $\mathbb{R}^{2n+1}$. If an element of it is given in the form $g = (u, v) \in \mathbb{H}^n$, where $u \in \mathbb{R}$ and $v = (v_1, \ldots, v_n) \in \mathbb{C}^n$, then the group law on $\mathbb{H}^n$ can be written as

$$(u, v) * (u', v') = \left(u + u' - 2\mathbf{I}m \sum_1^n v'_k \bar{v}_k,\ v_1 + v'_1, \ldots, v_n + v'_n \right).$$

We single out on $\mathbb{H}^n$ the group of nonisotropic dilations $\{\delta_\tau\}$, $\tau \in \mathbb{R}_+$:

$$\delta_\tau(u, v) = (\tau^2 u, \tau v).$$

Functions with the property

$$(f \circ \delta_\tau)(g) = \tau^k f(g)$$

will be called *δ_τ-homogeneous functions of degree k*. The class of such functions having continuous restriction to the nonisotropic unit sphere $\Omega^{2n} := \{(u, v) \in \mathbb{H}^n \mid u^4 + |v|^2 = 1\}$ is denoted by $H^k_\delta(C(\Omega^{2n}))$.

The left and right Haar measure[2] on the Heisenberg group coincides with the Lebesgue measure. The operators of right, left, and two-sided convolution on the Heisenberg group with kernel $k_{l,r}(g)$ or $k(g_1, g_2)$ are introduced as the integrals of the shift operators $\pi_l(g)$ and $\pi_r(g)$ giving rise to the regular representation of the Heisenberg group $\mathbb{H}^n$ on the space $L^2(\mathbb{H}^n)$:

$$K_{l,r} = (2\pi)^{-N/2} \int_{\mathbb{H}^n} k_{l,r}(g) \pi_{l,r}(g)\, dg, \tag{2.1}$$

$$K = (2\pi)^{-N} \int_{\mathbb{H}^n} \int_{\mathbb{H}^n} k(g_1, g_2) \pi_l(g_1) \pi_r(g_2)\, dg_1\, dg_2. \tag{2.2}$$

[1]More general case of arbitrary step 2 nilpotent Lie group may be considered also, but we do not touch this theme here. For corresponding results see [**?**, **?**].

[2]The left (right) Haar measure on a group is a measure that is invariant under the left (right) action of group.

where $N = 2n + 1$.

The Heisenberg group is the simplest non-commutative nilpotent Lie group. It is well known [**?**], convolution operators on a step 2 nilpotent Lie group with kernel $k(g)$ are pseudodifferential operators (PDO, see [**?**, **?**, **?**]) having the following form:

$$\begin{aligned} a(h, D)u(h) &= (2\pi)^{-N/2} \int_{\mathbb{R}^N} e^{i<h,\nu>} a(h,\nu)\widehat{u}(\nu) d\nu = \\ &= (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{i<h-g,\nu>} a(h,\nu)\, u(g)\, dg\, d\nu, \qquad (2.3) \end{aligned}$$

where

$$a(h, \nu) = \widehat{k}(\widetilde{L}_h(\nu)), \qquad (2.4)$$

and $\widetilde{L}_h(\cdot) = {}^tL_h^{-1}(\cdot)$ is the linear operator, which is inverse and transpose to the operator $L_h(\cdot) = -I - \frac{1}{2}[h, \cdot]$.

In [**?**] the more general PDOs containing τ-symbol were defined:

$$a_\tau(h, D)u(h) = (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{i<h-g,\nu>} a(\tau h+(1-\tau)g, \nu)\, u(g)\, dg\, d\nu. \qquad (2.5)$$

If $\tau = 1$, then this formula gives the same result as (2.3) and such operator is called PDO with the right symbol. If $\tau = 0$, then this operator is called PDO with the left symbol, if $\tau = \frac{1}{2}$ — PDO with the Weyl (symmetric) symbol. The connection between the τ-calculus of PDO and the problem of a quantization (in the usual sense) was discussed in [**?**].

It is easy to calculate by the formula (2.4) not only the right symbol of convolution but also any τ-symbol. Indeed the obvious equalities (here $(\mathbf{ad}h)g = [h, g]$):

$$(\mathbf{ad}h)h = 0, \;\; (\mathbf{ad}h)g = -(\mathbf{ad}g)h,$$

and (2.4) imply:

$$L_h(h - g) = L_{\tau h+(1-\tau)g}(h - g).$$

Substituting this to (2.5) one can obtain:

$$Ku(h) = (2\pi)^{-N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} e^{i<h-g,\nu>}\, \widehat{k}({}^tL^{-1}_{\tau h+(1-\tau)g}\nu)\, u(g)\, dg\, d\nu.$$

Thus we have

Proposition 2.1 *[?] A $\tau-$symbol of a PDO corresponding to a convolution on a step 2 nilpotent Lie group does not depend on τ and has the form:*

$$a_\tau(h,\nu) = \widehat{k}({}^t L_h^{-1}(\nu)) \tag{2.6}$$

REMARK 2.2 In the language of quantum mechanics this result means that the description of a physical system, which symmetry group is a step 2 nilpotent Lie group, does not depend on a method of the quantization (right, left or Weyl-symmetric) we use [**?**]. Nilpotent Lie group has special meaning in quantum mechanics, in particular, the Lie algebra of the Heisenberg group realizes the famous Heisenberg commutator relations for coordinates and impulses.

For our purpose we need all irreducible representations of the group $\mathbb{H}^n$. They are given by the Stone-von Neumann theorem [**?**, **?**, **?**] up to unitary equivalence. For any $\lambda \in (0,\infty)$ the irreducible noncommutative unitary representations on $L_2(\mathbb{R}^n)$ are given by

$$\pi_{\pm\lambda}(t,x,y) = e^{i(\pm\lambda t I \pm \lambda^{1/2} yM + \lambda^{1/2} xD)}, \tag{2.7}$$

where yM and xD are such operators on $L_2(\mathbb{R}^n)$:

$$\begin{aligned}(yM)u(v) &= \sum y_j v_j u(v),\\ (xD)u(v) &= (\frac{1}{i})\sum x_j \frac{\partial u}{\partial v_j}.\end{aligned}$$

For $(q,p) \in \mathbb{R}^{2n}$, there are also commutative one-dimensional representations on $\mathbb{C}$:

$$\pi_{(q,p)}(t,x,y)u = e^{i(qx+py)}u, \;\; u \in \mathbb{C}. \tag{2.8}$$

Then relative to (2.7) – (2.8) representations of convolution algebra are expressed by formulas [**?**]:

$$\begin{aligned}\pi_{\pm\lambda}[k(t,x,y)] &= \widehat{k}(\pm\lambda, \pm\lambda^{1/2}X, \lambda^{1/2}D), &(2.9)\\ \pi_{(q,p)}[k(t,x,y)] &= \widehat{k}(0,q,p). &(2.10)\end{aligned}$$

The right side of (2.9) specifies a PDO with the Weyl symbol $\widehat{k}(\pm\lambda, \pm\lambda^{1/2}x, \lambda^{1/2}\xi)$ accordingly to (2.5) with $\tau = \frac{1}{2}$. In the right side of (2.10) one can find just a constant from $\mathbb{C}$.

Quantum Mechanics		**Classical Mechanics**
Observables: Operators on $L_2(\mathbb{R}^n)$	$\xrightarrow[\hbar \to 0]{}$	*Observables:* Functions on $\mathbb{R}^{2n}$
Motion Equation: (Heisenberg Equation) $\dfrac{\partial K(\tau)}{\partial \tau} = \dfrac{i}{\hbar}[K(\tau), H]$	$\xrightarrow[\hbar \to 0]{}$	*Motion Equation:* (Hamilton Equation) $\dfrac{\partial k(\tau)}{\partial \tau} = \{k(\tau), H\}$

Figure 1: The quantization in the usual sense and the correspondence principle (case of a particle).

3 The Correspondence Principle between Classical and Quantum Mechanics

The more recent approaches to the quantization problem may be found in [?, ?, ?, ?]. Let us remind the sketch of this scheme for the future references (see Figure 1).

They say that there is a quantization, if

1. There is a family of operator algebras $\{\mathfrak{Q}_\hbar \mid \hbar \in \mathbb{R}_+ \cup \{0\}\}$, where
   (a) algebras $\mathfrak{Q}_\hbar$ for $\hbar \neq 0$ are non-commutative algebras of operators on some Hilbert spaces;
   (b) the algebra $\mathfrak{Q}_\hbar$ for $\hbar = 0$ is a commutative algebra of functions on $\mathbb{R}^{2n}$.
2. There is a topology on $\mathbb{R}_+$ such that:
   (a) there are limits[3] $\lim\limits_{\hbar \to 0} A_\hbar = A_0$, for $A_\hbar \in \mathfrak{Q}_\hbar$;

[3] The more exact meaning of these limits is following: $\lim\limits_{\epsilon \to 0} a_\epsilon = b$ iff b belong to the closure of the set $\cup_{t \leq \epsilon} a_t$ for all ϵ.

(b) for any $A_\hbar$ and $B_\hbar$ the following equalities hold:

$$\lim_{\hbar\to 0} A_\hbar \circ_\hbar B_\hbar = A_0 \cdot B_0 \tag{3.1}$$

$$\lim_{\hbar\to 0} \frac{i}{\hbar}[A_\hbar, B_\hbar]_\hbar = \{A_0, B_0\}. \tag{3.2}$$

Here $\circ_\hbar$ and $\cdot$ in (3.1) denote the operator of composition in $\mathfrak{Q}_\hbar$ and the ordinary product in $L_2(\mathbb{R}^{2n})$ correspondingly. In the regular way we denote in (3.2) the commutator of two operators by $[\cdot,\cdot]_\hbar$, and $\{\cdot,\cdot\}$ is the Poisson brackets of two functions.

Of course, the accordance between the operator commutator and the Poisson brackets in (3.2) produce the agreement between the Heisenberg and Hamilton motion equations when $\hbar \to 0$ (see Figure 1).

So we have two very different description for one system with only a thin bridge among them: the limit by $\hbar \to 0$. Please, do not ask the question: *Why should we consider such limits by $\hbar$, if $\hbar$ is the Planck **constant**?* I think, such questions cannot be answered within mentioned scheme.

4 Joining of Classical and Quantum Mechanics

Now we can discuss a new approach to our question.

We will speak that there is a *plain mechanical description* of a system if the following requirements hold:

1. There is an operator algebra $\mathfrak{P}$ which has

   (a) the family of all infinite-dimensional noncommutative irreducible representations parametrized by points of a set P:

   $$\pi_\hbar : \mathfrak{P} \to \mathfrak{P}_\hbar,\ \hbar \in P,\ \mathbf{dim}\,(\mathfrak{P}_\hbar) = \infty. \tag{4.1}$$

   We will call P by the *set of the Planck constants*;

   (b) the family of all (commutative) one-dimensional representations parametrized by points of a set M. This family gives the mapping of $\mathfrak{P}$ into algebra of functions on M by the obvious rule:

   $$\pi_x : P \to p(x) \in \mathbb{C},\ P \in \mathfrak{P},\ x \in M \tag{4.2}$$

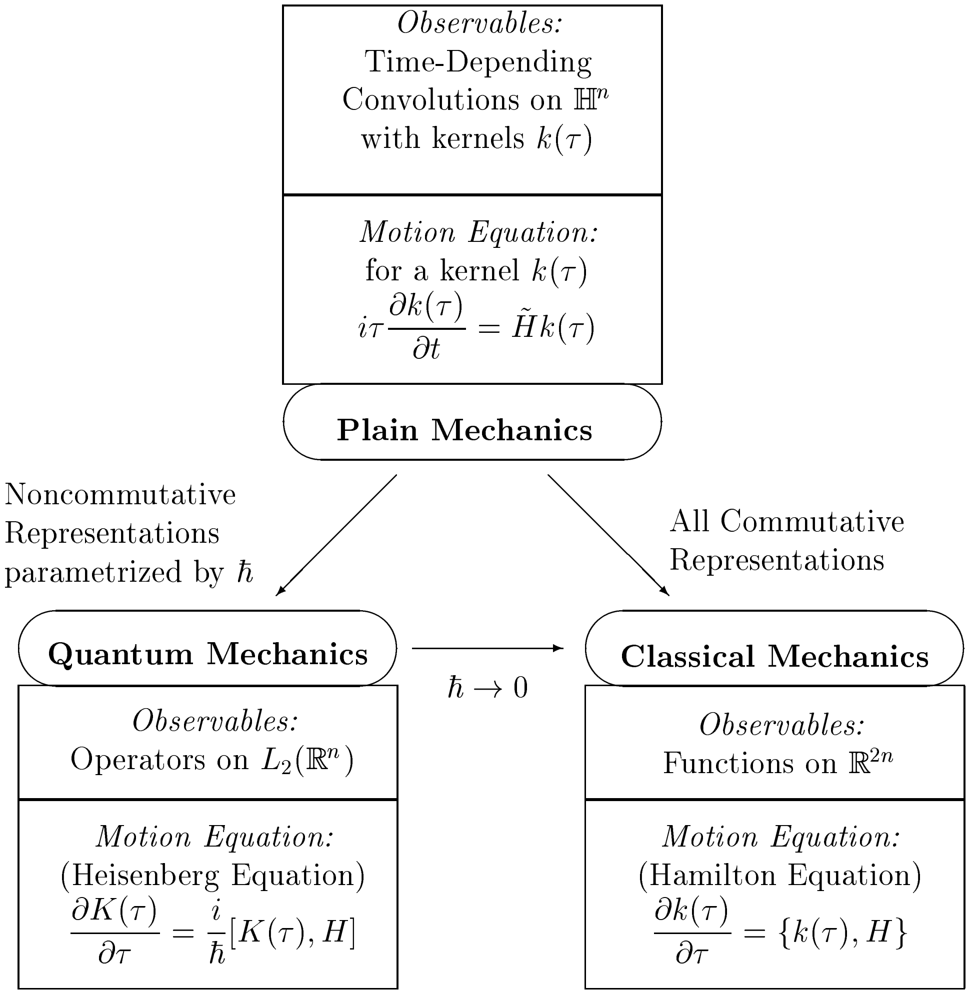

Figure 2: Plain Mechanics: Superstructure for the usual scheme (case of a particle).

The set M will be called as *the phase space of the classical system*, and the mapping defined by (4.2) from $\mathfrak{P}$ to an algebra of functions on M will be denoted by π_0;

(c) the topology[4] T on the set $P \cup M$ of all its representations.

2. The algebra $\mathfrak{P}$ is equipped by the operation $[\cdot, \cdot]$ of the commutation such that its image under mapping π_0 should coincide with the Poisson brackets[5] on M.

3. The diagram on Figure 2 should be commutative. The left down-going arrow denotes the set of all infinite-dimensional representations of $\mathfrak{P}$, the right arrow indicates the mapping π_0 and the horizontal arrow means the limit in topology T.

Comparing the Figure 1 and Figure 2 it is easy to see that plain mechanics is a superstructure on the usual scheme of a quantization.

Remark 4.1 To give a short philosophical interpretation of plain mechanics I would like to stress the following. By my opinion, plain mechanics correspond to the inner structure of the world. But we cannot see this inner structure. During the process of an observation (measurement) a representation of the world is selected from all possible ones. It may be either the classic representation or any from different (for different $\hbar$) quantum ones. Which representation was selected depends on the observer and his equipment (apparatus).

5 Example: a Particle in n-dimensional Space

In this section we give an illustration of the given abstract schemes on the simplest example of a particle in n-dimensional space. The Weyl and the Wick–Berezin quantization (Subsections 5.1 and 5.2) represent the usual methodology. In Subsection 5.3 a realization of the abstract scheme from Section 4 by two-sided convolutions on the Heisenberg group is done.

[4]This topology should be naturally generated by the structure of the algebra $\mathfrak{P}$, for example it may be the Jacobson topology [?] or the *-bundle topology [?, ?].

[5]The Poisson brackets assumes that M should have the structure of a manyfold. We don't discuss now how it is happened in the general case. Luckily, this structure will appear very natural in Subsection 5.3.

There is a simple reminding of the classical case for our problem. States (q, p) of a particle in $\mathbb{R}^n$ form the manifold (phase space) $\mathbb{R}^{2n}$. The observables are real functions on the phase space $\mathbb{R}^{2n}$ and the value of measurement an observable k in a state $(q,p) \in \mathbb{R}^{2n}$ is just $k(q,p)$. The change of an observable $k(\tau)$ during the time[6] defined by the Hamilton equation:

$$\frac{\partial k(\tau)}{\partial \tau} = \{k(\tau), H\}, \tag{5.1}$$

where $H(q,p)$ is the Hamilton function of the full energy of the particle.

5.1 The Weyl Quantization: PDO Calculus on $\mathbb{R}^n$

The realization is based on the well-known symbolic calculus of PDO [**?**, **?**, **?**]. States are the functions $f(x)$ from $L_2(\mathbb{R}^n)$ and observables are operators on $L_2(\mathbb{R}^n)$. The mathematical expectation of an observable K on a state $f(x)$ is equal to $\langle Kf, f\rangle$[7]. The Heisenberg equation

$$\frac{\partial K(\tau)}{\partial \tau} = \frac{i}{\hbar}[H, K(\tau)] \tag{5.2}$$

defines the motion of observables. The classic observables of a coordinate q_i and an impulse p_i correspond to the operators of multiplication by x_i and derivative $\frac{\hbar}{i}\frac{\partial}{\partial x_i}$ correspondingly.

The plainness of this description is suspended by the question: *Which operator* $(x_i\frac{\partial}{\partial x_i}, \ \frac{\partial}{\partial x_i}x_i$ *or* $\frac{1}{2}(x_i\frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i}x_i))$ *corresponds to the classical observable* $p_i q_i$*?* Different answers on this question generate different correspondences between symbols and operators in the calculus of PDO (see (2.5)). Remark 2.2 defines an important class of symbols without this obstacle.

5.2 The Berezin (Anti-Wick) Quantization: Toeplitz Operators in the Fock Space

Again we give only short summary of this topic, the relevant information may be found in [**?**, **?**, **?**, **?**, **?**]. Let $L_2(\mathbb{C}^n, d\mu_n)$ be a space of all square-integrable

[6] We employ the symbol τ for notation of time-parameter because the letter t have been already used for the first coordinate on the Heisenberg group.

[7] This means the scalar product on $L_2(\mathbb{R}^n)$.

functions on $\mathbb{C}^n$ with respect to the Gaussian measure

$$d\mu_n(z) = \pi^{-n} e^{-z\cdot\bar{z}} dv(z),$$

where $dv(z) = dxdy$ is the usual Euclidean volume measure on $\mathbb{C}^n = \mathbb{R}^{2n}$. Denote by P_n the orthogonal Bargmann projector [**?**] of $L_2(\mathbb{C}^n, d\mu_n)$ onto the Fock space $F_2(\mathbb{C}^n)$, the subspace of $L_2(\mathbb{C}^n, d\mu_n)$ consisting of all entire functions. Then the formula

$$k(q,p) \to T_{k(q+ip)} = P_n k(p+iq)I \tag{5.3}$$

defines an another (anti-Wick or Berezin) quantization, which maps a function $k(q,p)$ on $\mathbb{R}^{2n}$ to the Toeplitz operator T_k with the pre-symbol $k(q+ip)$ on $\mathbb{C}^n$. There is an identification between the Berezin quantization and the Weyl quantization [**?**, **?**, **?**].

5.3 Plain Mechanics: Two-Sided Convolutions on the Heisenberg Group

Now we illustrate the scheme from Section 4. Let us take a convolution operator algebra on the Heisenberg group. The kernels of convolutions may be taken from $L_1(\mathbb{H}^n)$ for example. Using the formulas (2.9) – (2.10) we conclude that the set of the Planck constants P coincides with $\mathbb{R} \setminus 0$. The phase space in the sense of Section 4 (the set of all one-dimensional representations) complies with the classical phase space $\mathbb{R}^{2n}$ and we can transfer the manifold structure of $\mathbb{R}^{2n}$ to M (with the associated Poisson brackets). The Jacobson topology on P is induced by the usual topology of the real line and any interval $(-\alpha, 0)$ or $(0, \alpha) \subset P$, $\alpha > 0$ is everywhere dense in M. This mean that the limit

$$\widehat{k}(\pm\lambda, \pm\lambda^{1/2}X, \lambda^{1/2}D) \to \widehat{k}(0,q,p) \tag{5.4}$$

while $\lambda \to 0$ is well defined[8] in the Jacobson topology.

Now we check the commutator property (see item 2 from page 9):

Proposition 5.1 *The limit of* $\dfrac{i}{\lambda}\pi_\lambda([K_1, K_2])$ *by* $\lambda \to 0$ *is equal to* $\{\widehat{k}_1(0,q,p), \widehat{k}_2(0,q,p)\}$. *Here* π_λ *is defined by* (2.9), K_1 *and* K_2 *are convolutions with kernels* k_1, k_2 *respectively.*

[8]See footnote 3 for the exact definition of this limit.

PROOF. Using the standard PDO calculus one can calculate that the symbol of commutator image under representation π_λ is equal to

$$\begin{aligned}\mathbf{sym}\,(\pi_\lambda([K_1,K_2]))(q,p) &= -i\frac{\partial\widehat{k}_1(\lambda,\lambda^{1/2}q,\lambda^{1/2}p)}{\partial q}\frac{\partial\widehat{k}_2(\lambda,\lambda^{1/2}q,\lambda^{1/2}p)}{\partial p}\\ &\quad +i\frac{\partial\widehat{k}_1(\lambda,\lambda^{1/2}q,\lambda^{1/2}p)}{\partial p}\frac{\partial\widehat{k}_2(\lambda,\lambda^{1/2}q,\lambda^{1/2}p)}{\partial q}\\ &\quad + (\text{derivatives of orders} > 2)\end{aligned}$$

Note that in this expansion any derivative of order m has the vanishing order $\lambda^{m/2}$ when $\lambda \to 0$. Thus if we multiply the image of commutator $\mathbf{sym}\,(\pi_\lambda([K_1,K_2]))$ by $\frac{i}{\lambda}$ and take the limit accordingly to (5.4) then we obtain the assertion. $\square$

REMARK 5.2 It is easy to see that many essentially different observables may have the same classical representation. Really, if

$$\widehat{k}_1(0,q,p) = \widehat{k}_2(0,q,p) \tag{5.5}$$

then observables the K_1 and K_2 are identical from the classical point of view. A differentness among them can be found only on a quantum level after selection a tool for the observation (see Remark 4.1). This note gives another dimension to the old dispute on the existence of hidden variables.

REMARK 5.3 If we take the algebra of observables with δ_τ-homogeneous kernels $\widehat{k}(t,x,y) \in H_\delta^0(C(\Omega^{2n}))$ (see Section 2) then different quantum representations will not depend from $\hbar$. Nevertheless the classical limit of these "constant" quantum observables by $\hbar \to 0$ is still defined by (5.4).

Let H be a convolution on the Heisenberg group corresponding to the Hamiltonians H_λ by representations $\pi_\lambda : H \to H_\lambda$. Let us calculate the kernel $c(h)$ of its commutator $[H,K]$ with a convolution K. We denote the

ernels of operators H and K by $j(h)$ and $k(h)$ correspondingly. It is easy to verify that:

$$\begin{aligned}c(h) &= \int_{\mathbb{H}^n} (j(g)k(h*g) - k(g)j(h*g))\,dg\\ &= \int_{\mathbb{H}^n} j(g)k(h*g)\,dg - \int_{\mathbb{H}^n} k(g)j(h*g)\,dg\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{H}^n} j(g)k(h*g)\,dg - \int_{\mathbb{H}^n} k(g)j(h*g)\,dg \\
&= \int_{\mathbb{H}^n} j(g)k(h*g)\,dg - \int_{\mathbb{H}^n} k(g_1^{-1}*h)j(h*g_1^{-1}*h)\,dg_1 \\
&= \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} j(g)\delta(g_1)k(g_1^{-1}*h*g)\,dg dg_1 \\
&\quad -\int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \delta(g)j(h*g_1^{-1}*h)k(g_1^{-1}*h*g)\,dg dg_1 \\
&= \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} (j(g)\delta(g_1)-\delta(g)j(h*g_1^{-1}*h))k(g_1^{-1}*h*g)\,dg dg_1.
\end{aligned}
$$

Here $\delta(g)$ is the Dirac function. Thus

$$
[H,K]=\widetilde{H}K, \tag{5.6}
$$

where $\widetilde{H}$ is an operator of two-sided convolution with the kernel depended from a point of $h\in\mathbb{H}^n$:

$$
\widetilde{j}(h,g,g_1)=(j(g)\delta(g_1)-\delta(g)j(h*g_1^{-1}*h)). \tag{5.7}
$$

Such operator $\widetilde{H}$ belongs to an algebra generated by two-sided convolutions (2.2) and operators of multiplication by functions on $\mathbb{H}^n$ [?].

Proposition 5.4 ***(The Main Equation of Plain Mechanics)*** *The equation*

$$
i\tau\frac{\partial K(\tau)}{\partial t}=-\widetilde{H}K(\tau), \tag{5.8}
$$

where $\widetilde{H}$ is an operator defined by the kernel (5.7) turns into the Heisenberg equation under a mapping π_λ and turns into the Hamilton equation under the mapping π_0.

PROOF. Let convolution $K(\tau)$ have the kernel $k(\tau,t,x,y)$. Then convolution $i\tau\frac{\partial K(\tau)}{\partial t}$ has the kernel $i\tau\frac{\partial k}{\partial t}(\tau,t,x,y)$. Under representations (2.9) – (2.10) the convolution $i\tau\frac{\partial K(\tau)}{\partial t}$ has the images

$$
i\lambda\frac{\partial\widehat{k}}{\partial\tau}(\tau,\pm\lambda,\pm\lambda^{1/2}X,\lambda^{1/2}D),\qquad i\frac{\partial\widehat{k}}{\partial\tau}(\tau,0,q,p)
$$

correspondingly. We take a liberty to denote by τ both the time-parameter and its dual in the Fourier transform sense. Note please that λ have the meaning of the Planck constant here.

The right side of (5.8) in accordance with (5.6) is equal to commutator $[H, K]$. Thus taking in account Proposition 5.1 we have

$$\begin{aligned} \pi_\lambda \quad : \quad & i\tau\frac{\partial K(\tau)}{\partial t} = \widetilde{H}K(\tau) \\ & \to i\lambda\frac{\partial\widehat{k}}{\partial\tau}(\tau, \pm\lambda, \pm\lambda^{1/2}X, \lambda^{1/2}D) = [H_\lambda, \widehat{k}(\tau, \pm\lambda, \pm\lambda^{1/2}X, \lambda^{1/2}D)] \end{aligned} \tag{5.9}$$

$$\begin{aligned} \pi_0 \quad : \quad & i\tau\frac{\partial K(\tau)}{\partial t} = \widetilde{H}K(\tau) \\ & \to i\frac{\partial\widehat{k}}{\partial\tau}(\tau, 0, q, p) = i\left(\frac{\partial\widehat{k}}{\partial q}\frac{\partial\widehat{j}}{\partial p} - \frac{\partial\widehat{j}}{\partial q}\frac{\partial\widehat{k}}{\partial p}\right)(\tau, 0, q, p). \end{aligned} \tag{5.10}$$

Here H_λ is the image of the convolution H under the representation π_λ and j is the kernel of the convolution H. It is clear that equations (5.9) – (5.10) coincide with the Heisenberg equation (5.2) and the Hamilton equation (5.1) correspondingly. □

Remark 5.5 The Heisenberg equation (equation for observables) (5.8) looks like the Schrödinger equation (equation for states) in quantum mechanics. The left side of (5.8) contains the partial derivative by t, so it looks like t is the time

parameter in plain mechanics.