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Differential equation for a correlation function of the spin-$\frac{1}{2}$ Heisenberg chain

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ABSTRACT

We consider the probability to find a string of x adjacent parallel spins in the antiferromagnetic ground state of the model (in a magnetic field). We derive a system of integro-difference equations which define this probability. This system is completely integrable, it has Lax representation and a corresponding Riemann-Hilbert problem. The quantum correlation funtion is a τ-function of this system.

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1 Introduction

In this paper we continue to investigate a special correlation function in the XXX Heisenberg model. The model is defined in terms of the hamiltonian

$$\mathcal{H} = \sum_j \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \sigma_j^z \sigma_{j+1}^z - h\sigma_j^z \ . \tag{1.1}$$

where the sum is over all integers j, σ^α are Pauli matrices and h is an external magnetic field. This model was solved by Bethe [1] by explicit construction of eigenfunctions of the model. The corresponding Bethe Ansatz equations define the allowed spectrum of excitations over the ferromagnetic vacuum $|\uparrow\uparrow\uparrow\ldots\rangle$ which are characterized by complex valued rapidities λ. The *antiferromagnetic* ground state $|0\rangle$ is obtained by filling the sea of these excitations with real λ. The case of finite magnetic field h leading to a nonzero magnetization of the ground state was first studied by Griffiths [2]. The energy of the excitiations above this ground state is given in terms of the solution of the following integral equation:

$$\varepsilon(\lambda) + \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} d\mu K(\lambda,\mu)\varepsilon(\mu) d\mu = 2h - \frac{2}{\lambda^2 + \frac{1}{4}} \ , \qquad \varepsilon(\pm\Lambda) = 0 \ . \tag{1.2}$$

The kernel of this integral equation is $K(\lambda,\mu) = 2/\left(1 + (\lambda-\mu)^2\right)$. The boundaries Λ of integration are a function of the magnetic field h through the requirement of vanishing ε at these points. 2Λ is the size of the Fermi sphere in this model (in relativistic field theory it can be compared with cut-off). While the spectrum and thermodynamic properties of the model have been extracted from the Bethe Ansatz integral equations such as (1.2) correlation functions have not been computed: The reason for this is the complicated form of Bethe's original wave functions. However, within the Quantum Inverse Scattering method the Bethe Ansatz states are constructed within an algebraic framework which has been used successfully to compute correlation functions in systems such as the nonlinear Schrödinger model [3]. Within this approach three main steps have to be performed:

- First, starting from the solution of the model by means of the algebraic Bethe Ansatz, the correlation functions have to be expressed as determinants of Fredholm integral operators, see formula (1.5).

- Then a description of these determinants in terms of the solutions of certain integrable integro-difference equations has to be found. These equations determine completely the correlation functions in question, see (2.6), (2.7). Fredholm determinant can be identified as a fredholm determinant of the Gelfand-Levitan-Marchenko integral operator of these equations evaluated on special solution (for definition of Gelfand-Levitan-Marchenko integral operator see [4, 5]).

- The long distance asymptotics of the correlatiors can be obtained from the solution of a matrix Riemann-Hilbert problem related to these integro-difference equations. The Riemann-Hilbert problem also provides an adequate parametrization of the space of the solutions for integro-difference equations. It sould be noted that just this type of parametrization is most suitable for the asymptotic analysis of the integrable systems (see review paper [6]).

This means that quantum correlation function is a τ function (in a sense of [7]) of these integro-difference equations. In order to define the correlation function that we are interested in we introduce the operators

$$P_j = \frac{1}{2}(\sigma_j^z + 1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} , \tag{1.3}$$

projecting onto the state with spin up at site number j. In terms of these operators the correlation function that we study here is

$$P(x) = \langle 0 | \prod_{j=1}^{x} P_j \; |0\rangle. \tag{1.4}$$

The physical meaning of $P(x)$ is the probability of finding a string of x (adjacent) parallel spins up (i.e. a piece of the *ferro*magnetic ground state) in the *antiferro*magnetic ground state $|0\rangle$ of the model (1.1) for a given value of the magnetic field h. From a mathematical point of view this correlator turns out the simplest one to be considered (see [3]).

Recently, the first step in the program outlined above has been solved, i.e. an expression of the function $P(x)$ in terms of a Fredholm determinant of an integral operator [8] has been constructed:

$$P(x) = \frac{\left(0 \left| \det\left(1 + \widehat{V}^{(x)}\right) \right| 0\right)}{\det\left(1 + \widehat{K}/2\pi\right)} . \tag{1.5}$$

Here $\widehat{K}$ is the integral operator appearing in (1.2), $\widehat{V}^{(x)}$ is another integral operator acting on a function $f(\lambda)$ as

$$\left(\widehat{V}^{(x)} f\right)(\lambda) = \int_{-\Lambda}^{\Lambda} d\mu \; V^{(x)}(\lambda|\mu) f(\mu) . \tag{1.6}$$

Its kernel V is given by

$$V^{(x)}(\lambda|\mu) = \frac{1}{2\pi}\left(\frac{e_+^{(x)}(\lambda)e_-^{(x)}(\mu)}{(\lambda-\mu)(\lambda-\mu+i)} + \frac{e_-^{(x)}(\lambda)e_+^{(x)}(\mu)}{(\mu-\lambda)(\mu-\lambda+i)}\right) \tag{1.7}$$

with functions $e_\pm^{(x)}$

$$e_\pm^{(x)}(\lambda) = \left(\left(\frac{2\lambda+i}{2\lambda-i}\right)^x e^{\phi(\lambda)}\right)^{\pm\frac{1}{2}} \quad . \tag{1.8}$$

The appearence of the dual quantum fields $\phi(\lambda)$ is a consequence of the interacting nature of the Heisenberg model (see Section XI.1 of [3]). They are acting in the bosonic Fock space with vacuum $|0)$. This means that $\det(1+\widehat{V})$ will be a complicated operator in this Fock space. However, by construction the operators $\phi(\lambda)$ commute among each other:

$$[\phi(\lambda), \phi(\mu)] = 0 \ .$$

In fact they are related to canonical Bose fields a and $a^\dagger$ which standard commutation relations

$$\begin{aligned} \left[a(\lambda), a^\dagger(\mu)\right] &= \delta(\lambda-\mu) \\ \left[a(\lambda), a(\mu)\right] &= \left[a^\dagger(\lambda), a^\dagger(\mu)\right] = 0 \ . \end{aligned} \tag{1.9}$$

These fields annihilate the Fock vacuum and its dual, resp.: $a(\lambda)|0) = 0 = (0|a^\dagger(\lambda)$. The dual fields ϕ are given by the expressions:

$$\phi(\lambda) = a(\lambda) - \int_{-\infty}^{\infty} d\nu \ln\left[1+(\lambda-\nu)^2\right] a^\dagger(\nu) \ . \tag{1.10}$$

The plan of this paper is as follows: in the following section we shall present the main results, namely the complete system of integro-difference equations that drive the Fredholm determinant of the operator $\widehat{V}$ in the correlation function $P(x)$ (1.5). In Section 3 we will give the derivation of these results and in Section 4 the Riemann-Hilbert problem associated with the equations is studied.

We also want to mention that quantum correlation function were first described by differential equation [Painlevé] by Barouch, McCoy, Wu and Tracy and McCoy [9, 10].

2 Main results

We obtain a complete set of integro-difference equations for eight functions $F_\pm^{(x)}(s)$, $G_\pm^{(x)}(s)$ and the kernels of integral operators $B_{\pm,\pm}^{(x)}(s,t)$ as well as the Riemann-Hilbert problem machinery (see e.g. [4]) to integrate these equations. In terms of these quantities we can compute the Fredholm determinant of $1+\widehat{V}$ using the relations

$$\frac{\det\left(1+\widehat{V}^{(x+1)}\right)}{\det\left(1+\widehat{V}^{(x)}\right)} = \det\left(1+B_{-+}^{(x)}\right) \tag{2.1}$$

(which is the lattice equivalent of the logarithmic derivative of $\det\left(1+\widehat{V}^{(x)}\right)$) and

$$\frac{\det\left(1+\widehat{V}^{(x+1)}\right)}{\det\left(1+\widehat{V}^{(x)}\right)}\,\frac{\det\left(1+\widehat{V}^{(x-1)}\right)}{\det\left(1+\widehat{V}^{(x)}\right)}=\det\left(1-\left(B_{++}^{(x)}\right)^{\top}B_{--}^{(x)}\right)\ . \tag{2.2}$$

($(B)^{\top}$ is the transpose of B). In addition we have

$$\begin{aligned}-i\partial_{\psi}\ln\det\left(1+\widehat{V}\right) &= \frac{1}{2\pi}\int_{s}\Big\{F_{+}(s)\partial_{\psi}F_{-}(s)-F_{-}(s)\partial_{\psi}F_{+}(s)\\ &\qquad +G_{-}(s)\partial_{\psi}G_{+}(s)-G_{+}(s)\partial_{\psi}G_{-}(s)\Big\}\\ &\quad +\frac{1}{4\pi^{2}}\,\frac{\xi+\bar{\xi}}{\xi-\bar{\xi}}\,\left(\int_{s}F_{+}(s)G_{-}(s)-F_{-}(s)G_{+}(s)\right)^{2}\ .\end{aligned} \tag{2.3}$$

Here the integration with respect to s is to be performed with the measure

$$\int_{s}f(s)=\int_{0}^{\infty}dse^{-s}f(s)\ . \tag{2.4}$$

The variable ψ is related to the cut-off Λ which is a function of the magnetic field:

$$e^{i\psi}\equiv\xi=\frac{2\Lambda-i}{2\Lambda+i}\ . \tag{2.5}$$

(Note that $\psi<0$).

The integro-difference equations for $f_{\pm}$ and $B_{\pm,\pm}$ are

$$\begin{aligned}\frac{1}{\sqrt{\xi}}F_{+}^{(x+1)} &= \frac{1}{\xi}\left\{F_{+}^{(x)}-\left(1-B_{+-}^{(x+1)}\right)\left(B_{++}^{(x)}\right)^{\top}F_{-}^{(x)}\right\}\\ \frac{1}{\sqrt{\xi}}F_{-}^{(x+1)} &= \frac{1}{\xi}\left\{\left(\xi+B_{--}^{(x+1)}\left(B_{++}^{(x)}\right)^{\top}\right)F_{-}^{(x)}-B_{--}^{(x+1)}\left(1+\left(B_{+-}^{(x)}\right)^{\top}\right)F_{+}^{(x)}\right\}\\ \frac{1}{\sqrt{\bar{\xi}}}G_{+}^{(x+1)} &= \frac{1}{\bar{\xi}}\left\{G_{+}^{(x)}-\left(1-B_{+-}^{(x+1)}\right)\left(B_{++}^{(x)}\right)^{\top}G_{-}^{(x)}\right\}\\ \frac{1}{\sqrt{\bar{\xi}}}G_{-}^{(x+1)} &= \frac{1}{\bar{\xi}}\left\{\left(\bar{\xi}+B_{--}^{(x+1)}\left(B_{++}^{(x)}\right)^{\top}\right)G_{-}^{(x)}-B_{--}^{(x+1)}\left(1+\left(B_{+-}^{(x)}\right)^{\top}\right)G_{+}^{(x)}\right\}\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}-i\frac{\partial}{\partial\psi}B_{ab}(s,t) &= \frac{i}{2\pi}\Big\{F_{a}(s)\left[F_{b}(t)+\left(F_{+}B_{-b}-F_{-}B_{+b}\right)(t)\right]\\ &\qquad +G_{a}(s)\left[G_{b}(t)+\left(G_{+}B_{-b}-G_{-}B_{+b}\right)(t)\right]\Big\}\end{aligned} \tag{2.7}$$

where $a,b=\pm$. In these equations the action of the integral operators B_{ab} is given by

$$\left(B_{ab}f\right)(s)=\int_{s}B_{ab}(s,t)f(t),\qquad\left((B_{ab})^{\top}f\right)(s)=\int_{s}B_{ab}(t,s)f(t)\ . \tag{2.8}$$

These equations are completely integrable, they have Lax representation, see Eqs. (3.20), (3.24) and (4.26).

The additional restrictions necessary to solve the equations uniquely are provided by the requirements on analyticity and the asymptotic behaviour of the solutions of the corresponding linear system (Eqs. (3.20), (3.24)), i.e. by specification of the data in the corresponding Riemann-Hilbert problem (see Section 4).

3 Derivation

The plan of the Section is the following: (i) We shall introduce functions $f_\pm^{(x)}(z|s)$ as solutions of linear integral equation (3.9). (ii) We shall prove that these funtions satisfy linear difference equation (3.20) and linear differential equation (3.24). Compatibility of these two equations will lead to the system of nonlinear eqautions (2.6), (2.7). The unknown functions which enter these equations are defined by (3.10) and by

$$F_\pm^{(x)}(s) = f_\pm^{(x)}(\xi|s), \qquad G_\pm^{(x)}(s) = f_\pm^{(x)}(\bar\xi|s)$$

Equations (3.20) and (3.24) are Lax representation for nonlinear equations (2.6), (2.7).

First we have to rewrite the kernel of $\widehat{V}$ (1.7) such that it is of the "standard" form of integrable integral operators [3]. Using the identity

$$\int_0^\infty ds\ e^{-(1-ix)s} = \frac{1}{1-ix} \tag{3.1}$$

we obtain

$$V^{(x)}(\lambda|\mu) = -\frac{i}{2\pi}\int_s \frac{e_+^{(x)}(\lambda|s)e_-^{(x)}(\mu|s) - e_-^{(x)}(\lambda|s)e_+^{(x)}(\mu|s)}{\lambda-\mu} \tag{3.2}$$

with $e_\pm(\lambda|s) = e_\pm(\lambda)e^{\pm i\lambda s}$. Furthermore, we prefer to express V in terms of the transformed variables

$$z(\lambda) = \frac{2\lambda - i}{2\lambda + i}\ . \tag{3.3}$$

After performing a similarity transform which leaves the determinant unchanged the integral operator $\widehat{V}^{(x)}$ acts on a function $f(z)$ as

$$\left(\widehat{V}^{(x)}f\right)(z_1) = \int_C dz_2 V^{(x)}(z_1|z_2)f(z_2) \tag{3.4}$$

where the integration is to be performed along the contour $C : \alpha \to z = \exp i\alpha$ where $-\psi < \alpha < 2\pi + \psi$ (see Fig. 1) and the kernel V is given by

$$V^{(x)}(z_1|z_2) = -\frac{i}{2\pi}\int_s \frac{e_+^{(x)}(z_1|s)e_-^{(x)}(z_2|s) - e_-^{(x)}(z_1|s)e_+^{(x)}(z_2|s)}{z_1-z_2}\ . \tag{3.5}$$

Now, the functions $e_\pm^{(x)}$ in (3.5) are given by

$$e_\pm^{(x)}(z|s) = \left(z^{-x} \exp\left(\phi(z) + s\,\frac{z+1}{z-1}\right)\right)^{\pm\frac{1}{2}} \,. \tag{3.6}$$

In the following we shall drop the superscript indicating the x-dependence in equations where all quantities are to be taken with the same x. Note that V is symmetric and nonsingular at $z_1 = z_2$. The resolvent $\widehat{R}$ of $\widehat{V}$ is defined by

$$(1+\widehat{V})(1-\widehat{R}) = 1 \quad \Leftrightarrow \quad (1+\widehat{V})\widehat{R} = \widehat{V} \tag{3.7}$$

Its kernel $R(z_1|z_2)$ can be written in a form similar to Eq. (3.5), namely [3]

$$R(z_1|z_2) = -\frac{i}{2\pi}\int_s \frac{f_+(z_1|s)f_-(z_2|s) - f_-(z_1|s)f_+(z_2|s)}{z_1 - z_2} \,. \tag{3.8}$$

Here the functions $f_\pm^{(x)}$ are solutions of the linear integral equations

$$f_\pm(z_1|s) + \int_C dz_2 V(z_1|z_2) f_\pm(z_2|s) = e_\pm(z_1|s) \,. \tag{3.9}$$

In terms of these functions we introduce the integral operators B_{ab}, $a,b=\pm$ acting as $(Bf)(s) = \int_t B(s,t)f(t)$ with the kernel

$$B_{ab}(s,t) = \frac{i}{2\pi}\int_C \frac{dz}{z} f_a(z|s) e_b(z|t)\,, \qquad a,b = \pm\,. \tag{3.10}$$

3.1 The difference equations in x

Now we are ready to prove Eq. (2.1): Starting from the representation (3.5) of the kernel $V^{(x)}$ in terms of the functions $e_\pm^{(x)}$ we can write

$$z_1^{-\frac{1}{2}}\, V^{(x+1)}(z_1|z_2)\; z_2^{\frac{1}{2}} = V^{(x)}(z_1|z_2) + \frac{i}{2\pi}\int_s \frac{e_+^{(x)}(z_1|s)e_-^{(x)}(z_2|s)}{z_1} \,. \tag{3.11}$$

The modification of $V^{(x+1)}$ amounts to a similarity transform only, so it follows that

$$\begin{aligned} \det\left(1+\widehat{V}^{(x+1)}\right) &= \det\left(1 + \widehat{V}^{(x)} + \frac{i}{2\pi}\int_s \frac{e_+^{(x)}(z_1|s)e_-^{(x)}(z_2|s)}{z_1}\right) \\ &= \det\left(1+\widehat{V}^{(x)}\right)\,\det\left(1 + \frac{i}{2\pi}\int_s \frac{e_+^{(x)}(z_1|s)f_-^{(x)}(z_2|s)}{z_1}\right) \end{aligned} \tag{3.12}$$

In appendix A we show that this indeed equivalent to (2.1). Together with the identities derived in appendix B this also implies Eq. (2.2).

Next we shall derive the equations (2.6). We start from the defining integral equation (3.9) for the functions $f_\pm^{(x)}(z|s)$. Using (3.11) we obtain

$$
\begin{aligned}
&\frac{1}{\sqrt{z_1}} f_\pm^{(x+1)}(z_1|s) + \int_C dz_2 V^{(x)}(z_1|z_2) \frac{1}{\sqrt{z_2}} f_\pm^{(x+1)}(z_2|s) \\
&\quad = \frac{1}{\sqrt{z_1}} e_\pm^{(x+1)}(z_1|s) - \frac{1}{z_1} \int_t B_{\pm-}^{(x+1)}(s,t) e_+^{(x)}(z_1|t) \ . \qquad (3.13)
\end{aligned}
$$

To continue from here we need to express the solution of the integral equation

$$
\left(\left(1+\widehat{V}\right) g_\pm\right)(z|s) = \frac{1}{z} e_\pm^{(x)}(z|s) \qquad (3.14)
$$

in terms of $f_\pm^{(x)}(z|s)$. To do so, we rewrite the integral equation for $f_\pm$ as

$$
\begin{aligned}
&\frac{1}{z_1} f_\pm(z_1|s) + \int_C dz_2 V(z_1|z_2) \frac{1}{z_2} f_\pm(z_2|s) \\
&\quad = \frac{1}{z_1} e_\pm(z_1|s) - \int_C dz2 \left(\frac{1}{z_1} - \frac{1}{z_2}\right) V(z_1|z_2) f_\pm(z_2|s) \\
&\quad = \frac{e_\pm(z_1|s)}{z_1} - \int_t \left(B_{\pm-}(s,t) \frac{e_+(z_1|t)}{z_1} - B_{\pm+}(s,t) \frac{e_-(z_1|t)}{z_1} \right) \ . \qquad (3.15)
\end{aligned}
$$

These equations can be put in matrix form in the $\pm$-components giving

$$
\left(1+\widehat{V}\right) \begin{pmatrix} \frac{f_+(z)}{z} \\ \frac{f_-(z)}{z} \end{pmatrix} = \begin{pmatrix} 1-B_{+-} & B_{++} \\ -B_{--} & 1+B_{-+} \end{pmatrix} \begin{pmatrix} \frac{e_+(z)}{z} \\ \frac{e_-(z)}{z} \end{pmatrix} \ . \qquad (3.16)
$$

To proceed we introduce operators A_{ab} that satisfy

$$
\begin{pmatrix} 1-B_{+-} & B_{++} \\ -B_{--} & 1+B_{-+} \end{pmatrix}^{-1} = \begin{pmatrix} A_{+-} & A_{++} \\ A_{--} & A_{-+} \end{pmatrix} \ . \qquad (3.17)
$$

In terms of these new operators we obtain the solution of the integral equation (3.14) for $g_\pm$ in terms of $f_\pm$:

$$
\begin{aligned}
g_+(z|s) &= \frac{1}{z} \left(A_{+-} f_+ + A_{++} f_-\right)(z|s) \\
g_-(z|s) &= \frac{1}{z} \left(A_{--} f_+ + A_{-+} f_-\right)(z|s) \ . \qquad (3.18)
\end{aligned}
$$

Note that as B_{ab} the operators A_{ab} act on the s dependence only and leave z untouched. This result allows to rewrite (3.13) giving a set of integro-difference equations for the functions $f_\pm$

$$
\begin{aligned}
\frac{1}{\sqrt{z}} f_+^{(x+1)}(z|s) &= \frac{1}{z}\left(1 - B_{+-}^{(x+1)}\right)\left(A_{+-}^{(x)} f_+^{(x)} + A_{++}^{(x)} f_-^{(x)}\right)(z|s) \ , \\
\frac{1}{\sqrt{z}} f_-^{(x+1)}(z|s) &= f_-^{(x)}(z|s) - \frac{1}{z} B_{--}^{(x+1)} \left(A_{+-}^{(x)} f_+^{(x)} + A_{++}^{(x)} f_-^{(x)}\right)(z|s) \ . \qquad (3.19)
\end{aligned}
$$

Using the identities between operators A and B derived in appendix B this can be rewritten as

$$\begin{aligned}\frac{1}{\sqrt{z}}f_+^{(x+1)} &= \frac{1}{z}\left(f_+^{(x)} + \left(1 - B_{+-}^{(x+1)}\right)A_{++}^{(x)}f_-^{(x)}\right) \;, \\ \frac{1}{\sqrt{z}}f_-^{(x+1)} &= f_-^{(x)} - \frac{1}{z}B_{--}^{(x+1)}\left(A_{+-}^{(x)}f_+^{(x)} + A_{++}^{(x)}f_-^{(x)}\right) \;. \end{aligned} \tag{3.20}$$

Choosing $z = \xi$ or $z = \bar{\xi}$ here reproduces equations (2.6).

Alternatively, one can consider a shift $x + 1 \mapsto x$ of the functions $f_\pm$ which results in the following identities

$$\begin{aligned}\frac{1}{\sqrt{z}}f_+^{(x)} &= \left(1 + \frac{1}{z}B_{++}^{(x)}A_{--}^{(x+1)}\right)f_+^{(x+1)} + \frac{1}{z}B_{++}^{(x)}A_{-+}^{(x+1)}f_-^{(x+1)} \;, \\ \frac{1}{\sqrt{z}}f_-^{(x)} &= \frac{1}{z}\left(f_-^{(x+1)} + \left(1 + B_{-+}^{(x)}\right)A_{--}^{(x+1)}f_+^{(x+1)}\right) \;. \end{aligned} \tag{3.21}$$

Eqs. (3.20) and (3.21) are equivalent as is straightforward to show using the identities from appendix B.

3.2 Derivatives with respect to ψ

Next, for (2.3) and (2.7) we take the derivative of $f_\pm$ with respect to the variable ψ (related to the magnetic field through Eqs. (1.2)). With

$$\frac{1}{i}\frac{\partial}{\partial\psi}\int_C dz\,(\ldots) = \xi\,(\ldots)_{z=\xi} + \bar{\xi}\,(\ldots)_{z=\bar{\xi}} \tag{3.22}$$

we find from the integral equation (3.9)

$$\frac{1}{i}\frac{\partial}{\partial\psi}f_\pm(z|s) = -\xi R(z|\xi)f_\pm(\xi|s) - \bar{\xi}R(z|\bar{\xi})f_\pm(\bar{\xi}|s) \;. \tag{3.23}$$

Using the expression (3.8) for R in terms of $f_\pm^{(x)}$ this can be expressed as

$$\begin{aligned}\frac{\partial}{\partial\psi}f_\pm(z|s) &= \frac{1}{2\pi}\left\{\frac{\xi}{z-\xi}f_\pm(\xi|s)\int_t \left(f_+(\xi|t)f_-(z|t) - f_-(\xi|t)f_+(z|t)\right)\right. \\ &\quad \left. + \frac{\bar{\xi}}{z-\bar{\xi}}f_\pm(\bar{\xi}|s)\int_t\left(f_+(\bar{\xi}|t)f_-(z|t) - f_-(\bar{\xi}|t)f_+(z|t)\right)\right\} \;. \end{aligned} \tag{3.24}$$

At this point everything follows from the requirement of compatibility between the shifts (3.20) or (3.21) and the derivative (3.24). Shifting $\partial_\psi f^{(x)}$ and equating the result with $\partial_\psi f^{(x+1)}$ one obtains the results announced in Section 2.

On the other hand, taking the ψ-derivative of (3.10) directly we obtain

$$\begin{aligned}\frac{1}{i}\frac{\partial}{\partial\psi}B_{ab}(s,t) &= \frac{i}{2\pi}\left\{f_a(\xi|s)\left[f_b - (B_{+b})^\top f_- + (B_{-b})^\top f_+\right](\xi|t)\right. \\ &\quad \left. + f_a(\bar{\xi}|s)\left[f_b - (B_{+b})^\top f_- + (B_{-b})^\top f_+\right](\bar{\xi}|t)\right\} \end{aligned} \tag{3.25}$$

which are Eqs. (2.7) when we identify $F_\pm(s) = f_\pm(\xi|s)$ and $G_\pm(s) = f_\pm(\bar\xi|s)$.

Now we have to compute the logarithmic derivative of our determinant wrt. ψ:

$$-i\frac{\partial}{\partial\psi}\ln\det\left(1+\widehat{V}\right) = \xi R(\xi|\xi) + \bar\xi R(\bar\xi|\bar\xi)\ . \tag{3.26}$$

Starting from (3.8) we find

$$\begin{aligned}\xi R(\xi|\xi) \quad &= \quad \frac{1}{2\pi}\int_s \{f_+(\xi|s)\partial_\psi f_-(\xi|s) - f_-(\xi|s)\partial_\psi f_+(\xi|s)\} \\ &+\frac{1}{4\pi^2}\,\frac{\bar\xi}{\xi-\bar\xi}\,\left(\int_s f_+(\xi|s)f_-(\bar\xi|s) - f_-(\xi|s)f_+(\bar\xi|s)\right)^2\ .\end{aligned} \tag{3.27}$$

Together with an analogeous expression for $\bar\xi R(\bar\xi|\bar\xi)$ this completes the proof of the announced result (2.3).

4 The operator-valued Riemann-Hilbert problem for the XXX chain

Consider now the integral operator-valued function

$$M(z) = I + m(z) \tag{4.1}$$

where

$$m(z|s,t) = \begin{pmatrix} -e_+(z|s)e_-(z|t) & e_+(z|s)e_+(z|t) \\ -e_-(z|s)e_-(z|t) & e_-(z|s)e_+(z|t)\end{pmatrix} \tag{4.2}$$

We propose the operator $M(z)$ as the “conjugation matrix” for an infinite dimensional Riemann-Hilbert problem for an integral operator-valued function $\chi(z)$:

1. $\chi(z)$ is analytic outside the contour C (Fig. 1).

2. $\chi^-(z) = \chi^+(z)\cdot M(z)$ for $z\in C$, and $\chi^\pm$ are the boundary values of the function $\chi(z)$ as indicated in Fig. 1.

3. $\chi(z)\to I$ as $z\to\infty$.

In terms of the corresponding kernels the properties 1–3 can be rewritten in the following way:

P1. $\chi(z|s,t)$ is an analytic function of $z\notin C$ for all s,t.

P2. $\chi^-(z|s,t) = \chi^+(z|s,t) + \int_{s'}\chi^+(z|s,s')m(z|s',t)$

P3. $\chi(z|s,t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \delta(s-t)e^t + \frac{1}{z}\Psi_1(s,t) + \ldots$ as $z \to \infty$

(As before we suppress the dependence on x)

The usual (see Section XV.6 of [3], and [11]) relationship between the Riemann-Hilbert problem (1–3) and the basic solutions $f_\pm(z|s)$ of the initial integral operator takes place:

$$\begin{pmatrix} f_+(z|s) \\ f_-(z|s) \end{pmatrix} = \int_t \chi^+(z|s,t) \begin{pmatrix} e_+(z|t) \\ e_-(z|t) \end{pmatrix} . \tag{4.3}$$

Suppose that the solution of the Riemann-Hilbert problem 1–3 exists and is unique. Then we shall show that the function

$$\Psi(z) = \chi(z) \cdot \begin{pmatrix} z^{-x} & 0 \\ 0 & 1 \end{pmatrix} \tag{4.4}$$

satisfies the integral operator-valued linear system (3.20), (3.24).

We begin with the x-equation. Applying the standard arguments based on Liouville's theorem and on the x independence of the conjugation integral operator

$$M_0(z) = \begin{pmatrix} z^x & 0 \\ 0 & 1 \end{pmatrix} M(z) \begin{pmatrix} z^{-x} & 0 \\ 0 & 1 \end{pmatrix} \tag{4.5}$$

we have the equation

$$\Psi^{(x+1)} \left[\Psi^{(x)}\right]^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot I \; + \; \frac{1}{z}\, U_0 \tag{4.6}$$

where $I(s,t) = \delta(s-t)e^t$ and

$$U_0 = \chi^{(x+1)}(0) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \left[\chi^{(x)}(0)\right]^{-1} \tag{4.7}$$

The operator $m(z)$ is nilpotent $m^2(z) = 0$, hence we have

$$M^{-1}(z) = I - m(z) \; , \tag{4.8}$$

which in turn implies the equation

$$\chi^{-1}(z|s,t) = \begin{pmatrix} \chi_{22}(z|t,s) & -\chi_{12}(z|t,s) \\ -\chi_{21}(z|t,s) & \chi_{11}(z|t,s) \end{pmatrix} \tag{4.9}$$

for the solution of the RH-Problem 1–3. Introducing notations

$$\chi(0) = \begin{pmatrix} 1 - B_{+-} & B_{++} \\ -B_{--} & 1 + B_{-+} \end{pmatrix} \tag{4.10}$$

we conclude from (4.9) that

$$\chi^{-1}(0) = \begin{pmatrix} 1 + B^{\top}_{-+} & -B^{\top}_{++} \\ B^{\top}_{--} & 1 - B^{\top}_{+-} \end{pmatrix} \tag{4.11}$$

(this is an alternative proof of relations (B.3)!)

Substituting the expansion P3 for $\chi(z)$ as $z \to \infty$ into (4.6), we see that $(U_0)_{11} = 1$. This together with formulae (4.10) and (4.11) allow us to rewrite U_0 as

$$U_0 = \begin{pmatrix} 1 & -\left(1 - B^{(x+1)}_{+-}\right)\left(B^{(x)}_{++}\right)^{\top} \\ -B^{(x+1)}_{--}\left(1 + \left(B^{(x)}_{-+}\right)^{\top}\right) & B^{(x+1)}_{--}\left(B^{(x)}_{++}\right)^{\top} \end{pmatrix} . \tag{4.12}$$

To conclude the analysis of the x-equation (4.6), one has to show that the kernels $B_{ab}(s,t)$ from (4.10) are just the kernels introduced in the previous section. To this end, one has to consider the canonical integral representation for the solution of the problem 1–3

$$\chi(z_1) = I - \frac{1}{2\pi i}\int_C \frac{dz_2}{z_2 - z_1}\chi^{+}(z_2)m(z_2) , \tag{4.13}$$

which implies

$$\chi(0) = I + \frac{i}{2\pi}\int_C \frac{dz}{z}\chi^{+}(z)m(z) . \tag{4.14}$$

Taking into account the explicit formula for $m(z)$ and Eq. (4.3) we obtain the representations (3.10) for $B_{ab}(s,t)$ defined in (4.10).

Eq. (4.6) can be rewritten as

$$z^{-\frac{1}{2}}\chi^{(x+1)}(z)\cdot\begin{pmatrix} z^{-\frac{1}{2}} & 0 \\ 0 & z^{\frac{1}{2}} \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\cdot I \;+\; \frac{1}{z}\,U_0\right]\chi^{(x)}(z) . \tag{4.15}$$

Applying both sides of (4.15) to the vector $\begin{pmatrix} e^{(x)}_{+}(z|s) \\ e^{(x)}_{-}(z|s) \end{pmatrix}$ and taking into account (4.3) and that $e^{(x)}_{\pm} z^{\mp\frac{1}{2}} = e^{(x+1)}_{\pm}$ we obtain the basic x-equations (3.20).

<u>Remark:</u> The substitution of the expansion P3 into (4.6) yields the following identities for the potentials B_{ab} (compare with (B.4)):

$$\begin{aligned} &\left(1 - B^{(x+1)}_{+-}\right)\left(1 + \left(B^{(x)}_{-+}\right)^{\top}\right) = 1, && -\left(1 - B^{(x+1)}_{+-}\right)\left(B^{(x)}_{++}\right)^{\top} = -C^{(x+1)}_{++} \\ &-B^{(x+1)}_{--}\left(1 + \left(B^{(x)}_{-+}\right)^{\top}\right) = -C^{(x)}_{--}, && B^{(x+1)}_{--}\left(B^{(x)}_{++}\right)^{\top} = C^{(x)}_{-+} - C^{(x+1)}_{-+} , \end{aligned} \tag{4.16}$$

where the new potentials C_{ab} are defined as

$$C_{ab}(s,t) = \frac{i}{2\pi}\int_C dz f_a(z|s)e_b(z|t), \qquad a,b = \pm\ . \tag{4.17}$$

Let us now study the ξ-derivative of the function $\chi(z)$. The corresponding analysis almost literally repeats the one performed in [11] for the Bose-gas case.

We notice (infinite dimensional analogue of the corresponding relation in [7]) that in the neighbourhood of C the function $\chi(z)$ can be represented as

$$\chi(z) = \widehat{\chi}(z)\cdot\chi_0(z) \tag{4.18}$$

where $\widehat{\chi}(z)$ is singlevalued, invertable and analytic in that neighbourhood, while

$$\chi_0(z|s,t) = \left[\delta(s-t)\ e^t\ -\ \frac{i}{2\pi}\ln\frac{z-\bar{\xi}}{z-\xi}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right]\ \cdot\ \begin{pmatrix}e_-(z|t) & 0\\ -e_-(z|t) & e_+(z|t)\end{pmatrix}\ . \tag{4.19}$$

From (4.18) we conclude at once that

$$\chi_\psi(z)\cdot\chi^{-1}(z) = \frac{A_+}{z-\bar{\xi}} + \frac{A_-}{z-\xi} \tag{4.20}$$

where

$$A_\pm = \lim_{z\to z_\pm}(z-z_\pm)\,\widehat{\chi}(z)\cdot\chi_{0\psi}\cdot\chi_0^{-1}(z)\cdot\widehat{\chi}^{-1}(z)\ , \qquad z_+ = \bar{\xi}\ , \quad z_- = \xi\ . \tag{4.21}$$

It is easily verified that

$$\chi_0^{-1}(z|s,t) = \begin{pmatrix}e_+(z|s) & 0\\ e_-(z|s) & e_-(z|s)\end{pmatrix}\cdot\left[\delta(s-t)\ e^t\ +\ \frac{i}{2\pi}\ln\frac{z-\bar{\xi}}{z-\xi}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\right]\ , \tag{4.22}$$

which immediately implies

$$\left(\chi_{0\psi}\chi_0^{-1}\right)(z|s,t) = \frac{1}{2\pi}\left[\frac{\bar{\xi}}{z-\bar{\xi}} + \frac{\xi}{z-\xi}\right]\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \tag{4.23}$$

Returning to the integral operators $A_\pm$ we obtain from (4.18)–(4.23) and (4.9) that (compare with the corresponding derivation in [11]):

$$\begin{aligned}
A_\pm(s,t) &= \frac{z_\pm}{2\pi}\int_{s'}\int_{s''}\widehat{\chi}(z_\pm|s,s')\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\widehat{\chi}^{-1}(z_\pm|s'',t)\\
&= \frac{z_\pm}{2\pi}\int_{s'}\chi(z_\pm|s,s')\begin{pmatrix}0 & e_+(z_\pm|s')\\ 0 & e_-(z_\pm|s')\end{pmatrix}\cdot\int_{s'}\begin{pmatrix}0 & 0\\ -e_-(z_\pm|s') & e_+(z_\pm|s')\end{pmatrix}\chi^{-1}(z_\pm|s',t)\\
&= \frac{z_\pm}{2\pi}\begin{pmatrix}-f_+(z_\pm|s)f_-(z_\pm|t) & f_+(z_\pm|s)f_+(z_\pm|t)\\ -f_-(z_\pm|s)f_-(z_\pm|t) & f_-(z_\pm|s)f_+(z_\pm|t)\end{pmatrix}\ .
\end{aligned} \tag{4.24}$$

Remembering the definition of the potentials $F_\pm(s)$, $G_\pm(s)$, we can rewrite the final formulae for $A_\pm$ as

$$
\begin{aligned}
A_+(s,t) &= \frac{\bar{\xi}}{2\pi}\begin{pmatrix} -G_+(s)G_-(t) & G_+(s)G_+(t) \\ -G_-(s)G_-(t) & G_-(s)G_+(t) \end{pmatrix}, \\
A_-(s,t) &= \frac{\xi}{2\pi}\begin{pmatrix} -F_+(s)F_-(t) & F_+(s)F_+(t) \\ -F_-(s)F_-(t) & F_-(s)F_+(t) \end{pmatrix}.
\end{aligned}
\tag{4.25}
$$

Equations (4.20) and (4.25) imply the basic equation (3.24) for the ψ-derivative of $f_\pm(z|s)$.

Remark: Writing down the compatibility condition for the L–A pair (4.6), (4.20) explicitely

$$
\begin{aligned}
&\left(\frac{A_+^{(x+1)}}{z-\bar{\xi}} + \frac{A_-^{(x+1)}}{z-\xi}\right)\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{z}\,U_0\right) \\
&\qquad = \frac{1}{z}\,U_{0\psi} + \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{z}\,U_0\right)\left(\frac{A_+^{(x)}}{z-\bar{\xi}} + \frac{A_-^{(x)}}{z-\xi}\right)
\end{aligned}
\tag{4.26}
$$

one can obtain some useful identities for the potentials B_{ab}, $F_\pm$ and $G_\pm$ that are of course consequences of the basic system (2.6) and (2.7).

In a forthcoming publication we intend to use the Riemann-Hilbert problem 1–3 to analyze the asymptotic behaviour of the correlation function $P(x)$. The results will allow for a comparison of the approach to the computation of correlation functions used here to the symmetry based one used e.g. in [12].

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A Completion of the proof of the relation between the Fredholm determinants of $\widehat{V}$ and B_{-+}

To prove the equivalence of Eqs. (2.1) and (3.12) we shall use the identity

$$\ln\det(1+\mathcal{O}) = \sum_{n>0} \mathrm{tr}\,(\mathcal{O}^n)$$

and consider the equations order by order in B_{-+}. The first term is

$$\mathrm{tr}\left(\frac{i}{2\pi}\int_s \frac{e_+^{(x)}(z_1|s)f_-^{(x)}(z_2|s)}{z_1}\right) = \frac{i}{2\pi}\int_C dz \int_s \frac{e_+^{(x)}(z|s)f_-^{(x)}(z|s)}{z} \tag{A.1}$$

which by definition (3.10) is equal to $\mathrm{tr}\,(B_{-+})$. Similarly, to second order we find

$$\begin{aligned} &\mathrm{tr}\left\{\left(\frac{i}{2\pi}\int_s \frac{e_+^{(x)}(z_1|s)f_-^{(x)}(z_2|s)}{z_1}\right)^2\right\} \\ &\quad = \left(\frac{i}{2\pi}\right)^2 \int_C dz_1 \int_C dz_2 \int_s \frac{e_+^{(x)}(z_1|s)f_-^{(x)}(z_2|s)}{z_1}\int_t \frac{e_+^{(x)}(z_2|t)f_-^{(x)}(z_1|t)}{z_2} \end{aligned} \tag{A.2}$$

which, after reordering of terms is equal to $\mathrm{tr}\left\{(B_{-+})^2\right\}$. Turning to arbitrary n now we have $(z_{n+1} \equiv z_1)$

$$\begin{aligned} &\left(\frac{i}{2\pi}\right)^n \left(\prod_\ell \int_c dz_\ell\right) \prod_k \int_s \frac{e_+^{(x)}(z_k|s_k)f_-^{(x)}(z_{k+1}|s_k)}{z_k} \\ &\quad = \left(\prod_k \int_s\right)\left\{\prod_\ell \frac{i}{2\pi}\int_c dz_\ell \frac{e_+^{(x)}(z_\ell|s_\ell)f_-^{(x)}(z_\ell|s_{\ell-1})}{z_\ell}\right\} \end{aligned} \tag{A.3}$$

which completes the proof of (2.1).

B Identities between the operators $A_{ab}^{(x)}$ and $B_{ab}^{(x)}$

In this appendix a collection of useful identities in rewriting the integral equations in this paper is derived. First, from the definition (3.17) of A_{ab} it follows

$$\begin{aligned} &A_{+-}(1-B_{+-}) - A_{++}B_{--} = 1\ , \qquad A_{+-}B_{++} + A_{++}(1+B_{-+}) = 0\ , \\ &A_{--}(1-B_{+-}) - A_{-+}B_{--} = 0\ , \qquad A_{--}B_{++} + A_{-+}(1+B_{-+}) = 1\ . \end{aligned} \tag{B.1}$$

Next, using the explicit x-dependence (3.6) of $e_\pm^{(x)}$ and the shift equation (3.13) for f_+ we find

$$\begin{aligned} B_{+-}^{(x+1)}(s,t) &= \frac{i}{2\pi}\int_C \frac{dz}{z} f_+^{(x+1)}(z|s)e_-^{(x)}(z|t) \\ &= \left(1 - B_{+-}^{(x+1)}\right)\left(A_{+-}^{(x)}B_{+-}^{(x)} + A_{++}^{(x)}B_{--}^{(x)}\right) \end{aligned}$$

which with (B.1) leads to

$$\left(1 - B_{+-}^{(x+1)}\right) A_{+-}^{(x)} = 1 \ . \tag{B.2}$$

From the definition (3.10) we obtain with (3.14)

$$\begin{aligned} B_{ab}(s,t) &= \frac{i}{2\pi} \int_C \frac{dz}{z} f_a(z|s) e_b(z|t) \\ &= \frac{i}{2\pi} \int_C \frac{dz}{z} e_a(z|s) g_b(z|t) \ . \end{aligned}$$

Using the expression (3.18) for $g_\pm$ together with the relations (B.1) we obtain relations between B and the transpose of A:

$$\begin{aligned} B_{++} &= -\left(A_{++}\right)^\top \ , \qquad B_{+-} = 1 - \left(A_{-+}\right)^\top \ , \\ B_{-+} &= \left(A_{+-}\right)^\top - 1 \ , \qquad B_{--} = \left(A_{--}\right)^\top \ . \end{aligned} \tag{B.3}$$

Finally, we use these identities to get

$$\left(1 + B_{-+}^{(x+1)}\right) A_{-+}^{(x)} = 1 \tag{B.4}$$

from (B.2).

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Figure 1:

Contour C on the unit circle for the integration with respect to z in the integral operator $\widehat{V}$. '+' and '$-$' indicate direction in which the limit $z \to C$ has to be taken to obtain the boundary values $\chi^{\pm}(z)$ for the Riemann-Hilbert problem in Section 4.