NON–QUASILOCALITY OF PROJECTIONS OF GIBBS MEASURES

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Abstact: For monotonicity–preserving local specifications we give a sufficient condition so that the projection on some sub–σ–algebras of a Gibbs measure is not quasilocal. We show that this condition is fulfilled in the Ising model.

1 Introduction

One of the main source of examples of random fields is the statistical mechanics of lattice systems, which has strongly influenced the subject. The typical random fields here are Gibbs random fields, which are defined by an interaction potential via the Boltzmann–Gibbs formula (see Georgii (1980)). Gibbs random fields verify the important property of quasilocality, which is a generalization of the Markov property. In Sullivan (1973) this property is called the almost–Markov property. Quasilocality means, in the context of this paper, that some conditional expectation values of local functions are well–behaved, in the sense that they are continuous. It was however noticed in Griffiths Pearce (1979), and clarified in Israel (1981), that quasilocality may not be valid for the restriction of a Gibbs measure on some sub–σ–algebras. (In physicist language the projection on a sub–σ–algebra is called a decimation process.) The same observation has been made later on by Schonmann (Schonmann (1989)) for a different choice of the sub–σ–algebra, but without making the connection with the earlier works Griffiths Pearce (1979) and Israel (1981). In the comprehensive work van Enter Fernández Sokal (1993) these problems are analyzed in depth.

One should however remark that the results concerning the projection of Gibbs measures are weak in the sense that the lack of quasilocality is established by showing that continuity fails at some point. It is of obvious interest to know whether the

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lack of continuity occurs only on a set of measure zero with respect to the measure which is projected. This question is addressed, but not answered, in Maes Vande Velde (1992) and Lörinczi (1993); it is the purpose of this paper to investigate such problems for a class of random fields defined by monotonicity–preserving specifications. We give a sufficient criterion (Corollary 5.1) for the lack of quasilocality to occur a.s.. On the other hand, whenever there is a unique measure compatible with the original specification, we prove that the projected measure is quasilocal a.s.. A noteworthy aspect of this paper is that we do not use the notion of interaction potential, and work directly with the probability kernels.

The plan of the paper is as follows. In section 2 we introduce the basic notations, define the class of local specifications and give the definition of quasilocality. Here our point of view is different from the usual one (see Georgii (1988)): we distinguish between *pointwise* and *uniform* quasilocality. The main problem is formulated at the end of this section. The key idea is first to construct a global specification. For the Ising model this is equivalent to prove the global Markov property. This is done in section 3. In section 4 we define a local specification for the projected measure, as a consequence of the existence of the global specification for the original measure μ. (The measure μ is an extremal measure.) Then we establish a zero-one law: either this local specification is continuous on a set of μ–measure one, or this is true only on a set of μ–measure zero. We also give conditions so that the points of discontinuity are unremovable by changing the specification on a set of μ–measure zero. Our criterion for absence of quasilocality is established in section 5. Section 6 contains the main results, Theorem 6.1, and the discussion of the consequences of this theorem for the Ising model. We are able to extend Schonmann's results in two senses. First, we prove that the "non–Gibbsianness" pointed out by Schonmann is in fact a.s. non–quasilocality. Second, we show that this phenomenon happens not only at $d = 2$, but at arbitrary dimension $d \geq 2$.

We remark that several of our results are true in a more general context, without assuming for example the monotonicity–preserving property (condition (H_4)). However, it is for the specifications verifying this condition that we have the strongest results and can treat non-trivial examples.

2 Notation, specification, quasilocality

2.1 Basic notation

Let $\mathcal{L}$ be a countable set; the complement of a subset $M \subset \mathcal{L}$ is denoted by $M^c = \mathcal{L}\backslash M$, and the cardinality of M by $|M|$. We find it useful to adopt the following convention: Λ, or Λ_1,..., always denote sets of finite cardinality. Let $E = \{-1, +1\}$ (with the discrete topology), and $\Omega := E^{\mathcal{L}}$ with the product topology and product Borel σ–algebra; the elements of Ω are functions $\omega : \mathcal{L} \to E$, $i \mapsto \omega(i)$; the restriction of ω to a subset $M \subset \mathcal{L}$ is denoted by ω_M; two elements play a special rôle, $\omega(i) \equiv 1$ and $\omega(i) \equiv -1$ which are denoted by $+$ and $-$. Let $\Lambda \subset \mathcal{L}$, $\eta \in \Omega$, and $\omega \in \Omega$; we define $\omega^\eta_\Lambda \in \Omega$ by

$$\omega^\eta_\Lambda(k) := \begin{cases} \omega(k) & k \in \Lambda \\ \eta(k) & k \notin \Lambda \end{cases} . \tag{2.1}$$

The value at ω of the evaluation-map X_i, $i \in \mathcal{L}$, is $X_i(\omega) := \omega(i)$; for any M, the σ–algebra generated by the X_i's, $i \in M$, is $\mathcal{F}_M = \sigma\{X_i; i \in M\}$; the tail-field σ–algebra on $\mathcal{L}$ is

$$\mathcal{T}_{\mathcal{L}}^{\infty} := \bigcap_{\Lambda \subset \mathcal{L}} \mathcal{F}_{\Lambda^c} \,. \tag{2.2}$$

We say that $\omega = \omega'$ *a.e.* iff $\omega(k) = \omega'(k)$ for all but a finite number of k. Given $\omega \in \Omega$, the countable set

$$\tau_\omega := \{\omega' :\ \omega' = \omega \ a.e.\} \tag{2.3}$$

is $\mathcal{T}_{\mathcal{L}}^{\infty}$–measurable. Conversely, if $\omega \in A$, a $\mathcal{T}_{\mathcal{L}}^{\infty}$–measurable set, then $\{\omega' :\ \omega' = \omega \ a.e.\} \subset A$, and hence

$$A = \bigcup_{\omega \in A} \tau_\omega \,.$$

The family of all subsets $\{\omega' :\ \omega'_\Lambda = \omega_\Lambda\}$, $\Lambda \subset \mathcal{L}$, forms a base of open neighbourhoods of ω. Thus the set $\{\omega' :\ \omega' = \omega \ a.e.\}$ is dense in Ω, and therefore all non-empty $\mathcal{T}_{\mathcal{L}}^{\infty}$–measurable sets are dense.

Remark: Any function f, which is $\mathcal{T}_{\mathcal{L}}^{\infty}$–measurable and continuous, is constant. Indeed, let ω and η belong to Ω, and let $\{\Lambda_n :\ n = 1, 2, \ldots\}$ be a sequence, such that

$$\Lambda_n \subset \Lambda_{n+1} \,,\ \lim_n \Lambda_n = \mathcal{L} \,. \tag{2.4}$$

Let $\theta_n = \omega_{\Lambda_n} \eta_{\Lambda_n^c}$ be the element of Ω,

$$\omega_{\Lambda_n} \eta_{\Lambda_n^c}(k) := \begin{cases} \omega(k) & k \in \Lambda_n \\ \eta(k) & k \notin \Lambda_n \end{cases} \,; \tag{2.5}$$

by construction $\theta_n = \eta$ a.e., and $\lim_n \theta_n = \omega$. Thus $f(\theta_n) = f(\eta)$ since f is $\mathcal{T}_{\mathcal{L}}^{\infty}$–measurable, and $\lim_n f(\theta_n) = f(\omega)$ since f is continuous. Hence, f is constant.

We say that

$$\omega \le \eta \ \text{ iff } \ \omega(k) \le \eta(k) \ \ \forall k \,; \tag{2.6}$$

a function is *increasing* iff $f(\omega) \le f(\eta)$ whenever $\omega \le \eta$. For any ω and ω'

$$\begin{aligned} (\omega \wedge \omega')(k) &= \min(\omega(k), \omega'(k)) \\ (\omega \vee \omega')(k) &= \max(\omega(k), \omega'(k)) \,. \end{aligned} \tag{2.7}$$

The topology on the set of probability measures on $(\Omega, \mathcal{F}_{\mathcal{L}})$ is that of the *narrow convergence*, a sequence of probability measures μ_n converging to μ iff for all bounded continuous functions

$$\lim_n \mu_n(f) = \mu(f) \,. \tag{2.8}$$

In our case it is sufficient to verify (2.8) for the non-negative increasing local functions (use the inclusion–exclusion principle and the density of the local functions in the algebra of the continuous functions on Ω).

2.2 Specification and quasilocality

A *local specification* Γ on $\mathcal{L}$, hereafter a specification, is a family of probability kernels $\Gamma = \{\gamma_\Lambda \,,\ \Lambda \subset \mathcal{L}\}$ on $(\Omega, \mathcal{F}_{\mathcal{L}})$, such that

(s_1) $\gamma_\Lambda(\cdot|\omega)$ is a probability measure on $(\Omega, \mathcal{F}_\mathcal{L})$, for all $\omega \in \Omega$;

(s_2) $\gamma_\Lambda(F|\cdot)$ is $\mathcal{F}_{\Lambda^c}$–measurable for all $F \in \mathcal{F}_\mathcal{L}$;

(s_3) $\gamma_\Lambda(F|\omega) = 1_F(\omega)$ if $F \in \mathcal{F}_{\Lambda^c}$;

(s_4) $\gamma_{\Lambda_2}\gamma_{\Lambda_1} = \gamma_{\Lambda_2}$ if $\Lambda_1 \subset \Lambda_2$.

Definition 2.1 *A probability measure μ is* Γ–compatible, *if for all $F \in \mathcal{F}_\mathcal{L}$ and all $\Lambda \subset \mathcal{L}$*

$$\mathbb{E}_\mu(F|\mathcal{F}_{\Lambda^c})(\omega) = \gamma_\Lambda(F|\omega) \;\; \mu \; a.s. \; . \tag{2.9}$$

The set of all Γ–compatible probability measures is a convex set $\mathcal{G}(\Gamma)$ which may be empty; each $\mu \in \mathcal{G}(\Gamma)$ has a unique extremal decomposition; the extremal elements of $\mathcal{G}(\Gamma)$ are those probability measures $\mu \in \mathcal{G}(\Gamma)$, which satisfy a zero-one law on $\mathcal{T}_\mathcal{L}^\infty$, $\mu(F) = 0$ or $\mu(F) = 1$ for all $F \in \mathcal{T}_\mathcal{L}^\infty$ (see Theorems 7.26 and 7.7 in Georgii (1988)). Moreover, if (H_3) below is satisfied, then $\mathcal{G}(\Gamma)$ is non-empty.

Definition 2.2 *A function $f : \Omega \to \mathbb{R}$ is*

(i) Local *iff it is $\mathcal{F}_\Lambda$–measurable for some finite Λ. In this case we say that f is* Λ–local.

(ii) Quasilocal at ω *iff for any $\varepsilon > 0$ there exists Λ_ε such that*

$$\sup_{\substack{\theta: \\ \theta_{\Lambda_\varepsilon} = \omega_{\Lambda_\varepsilon}}} |f(\omega) - f(\theta)| \leq \varepsilon \; . \tag{2.10}$$

(iii) Quasilocal *iff it is quasilocal at every $\omega \in \Omega$.*

Quasilocality is equivalent to continuity if we choose the discrete topology on E, as in our case. We shall therefore use both terminologies in the paper. In the literature, see e.g. Georgii (1988), a function is called quasilocal iff for any $\varepsilon > 0$ there exists Λ_ε such that

$$\sup_{\substack{\theta \, , \, \omega: \\ \theta_{\Lambda_\varepsilon} = \omega_{\Lambda_\varepsilon}}} |f(\omega) - f(\theta)| \leq \varepsilon \; . \tag{2.11}$$

Since here Ω is compact, this definition coincides with ours, but we propose (2.10) also for more general situations; that is, our point of view is that one should introduce a *pointwise* notion of quasilocality.

Definition 2.3

(i) A specification Γ is quasilocal *if the functions $\omega \to \gamma_\Lambda(f|\omega)$ are quasilocal for each finite Λ and each bounded local function f.*

(ii) A probability measure μ is quasilocal *if for every bounded local function f and every $\Lambda \subset \mathcal{L}$, there exists a version of $\mathbb{E}_\mu(f|\mathcal{F}_{\Lambda^c})$ which is quasilocal μ a.s..*

If (H_3) below is satisfied then all elements of $\mathcal{G}(\Gamma)$ are quasilocal.

Let $A \subset \mathcal{L}$, with $|A| < \infty$, and let $\sigma_A \in E^A$; we define

$$1_{\sigma_A}(\omega) := \begin{cases} 1 & \text{if } \omega_A = \sigma_A \\ 0 & \text{otherwise} \end{cases} , \tag{2.12}$$

and we set

$$\gamma_\Lambda(\sigma_A|\omega) := \int \gamma_\Lambda(d\eta|\omega) 1_{\sigma_A}(\eta) \tag{2.13}$$

and

$$\gamma_\Lambda(\sigma|\omega) := \int \gamma_\Lambda(d\eta|\omega) 1_{\sigma_\Lambda}(\eta) . \tag{2.14}$$

We list below four properties on the specification Γ; any Gibbs specification satisfies (H_1) to (H_3); (H_4) is our main technical tool.

$(\mathbf{H_1})$ Let σ_A, $|A| < \infty$; there exists a constant $c_1(A) > 0$, so that for all Λ and σ

$$\inf_\omega \gamma_\Lambda(\sigma_A|\omega) \geq c_1(A) .$$

$(\mathbf{H_2})$ Let B, $|B| < \infty$; there exist $d_1(B) > 0$ and $d_2(B) < \infty$ such that

$$d_1(B)\gamma_\Lambda(\sigma|\omega) \leq \gamma_\Lambda(\sigma|\omega') \leq d_2(B)\gamma_\Lambda(\sigma|\omega)$$

for all σ, all Λ, and all ω, ω' such that $\omega_{B^c} = \omega'_{B^c}$.

$(\mathbf{H_3})$ For any σ and any Λ the function $\omega \mapsto \gamma_\Lambda(\sigma|\omega)$ is continuous at all ω.

$(\mathbf{H_4})$ Let f be a bounded increasing function; the function

$$\omega \mapsto \gamma_\Lambda(f|\omega) := \int \gamma_\Lambda(d\eta|\omega) f(\eta)$$

is increasing.

A specification Γ is *uniformly non-null* if (H_1) holds, *quasilocal* if (H_3) holds, and *monotonicity–preserving* if (H_4) holds. (H_2) means that the measures $\gamma_\Lambda(\cdot|\omega)$ and $\gamma_\Lambda(\cdot|\omega')$ are equivalent if $\omega = \omega'$ a.e..

Let $\mathcal{L} = \mathbb{Z}^d$, $d \geq 1$; there is a natural action of $\mathbb{Z}^d$, as abelian group, on $\mathcal{L}$ which lifts to Ω, $\tau_a(\omega)(k) := \omega(k-a)$. We say that a specification Γ is $\mathbb{Z}^d$*–invariant* if for all $a \in \mathbb{Z}^d$, all Λ, ω, and bounded functions f,

$$\gamma_\Lambda(f \circ \tau_a|\omega) = \gamma_{\Lambda+a}(f|\tau_a(\omega)) . \tag{2.15}$$

Proposition 2.1 *Let Γ be a specification which satisfies (H_4). Then*

(i) For any increasing bounded function f, the nets $(\gamma_\Lambda(f|+)\,, \Lambda \subset \mathcal{L})$, resp. $(\gamma_\Lambda(f|-)\,, \Lambda \subset \mathcal{L})$, are monotone decreasing, resp. increasing.

(ii) The nets $(\gamma_\Lambda(\cdot|\omega)\,, \Lambda \subset \mathcal{L})$, with $\omega = +$ or $\omega = -$, converge to probability measures

$$\mu^+(\cdot) := \lim_\Lambda \gamma_\Lambda(\cdot|+) \tag{2.16}$$

and

$$\mu^{-}(\cdot) := \lim_{\Lambda} \gamma_{\Lambda}(\cdot|-)\,. \tag{2.17}$$

If furthermore the specification is $\mathbb{Z}^d$–invariant then the measures μ^+ and μ^- are $\mathbb{Z}^d$–invariant.

(iii) For any $\mu \in \mathcal{G}(\Gamma)$, and any bounded increasing function f

$$\mu^{-}(f) \le \mu(f) \le \mu^{+}(f)\,. \tag{2.18}$$

(iv) If furthermore (H_3) is valid, then μ^+ and μ^- are extremal elements of $\mathcal{G}(\Gamma)$ and there is a unique Γ–compatible measure iff $\mu^+ = \mu^-$.

Proof : Let f be a bounded increasing function; if $\Lambda_1 \subset \Lambda_2$, then

$$\begin{aligned} \gamma_{\Lambda_2}(f|+) &= \int \gamma_{\Lambda_2}(d\eta|+)\gamma_{\Lambda_1}(f|\eta) \\ &\le \int \gamma_{\Lambda_2}(d\eta|+)\gamma_{\Lambda_1}(f|+) \\ &= \gamma_{\Lambda_1}(f|+) \end{aligned} \tag{2.19}$$

The existence of the limits follows now easily. $\mathbb{Z}^d$–invariance follows from monotonicity. Let f be a bounded increasing function, and $\mu \in \mathcal{G}(\Gamma)$. By the backward martingale theorem,

$$\limsup_{\Lambda} \gamma_{\Lambda}(f|\omega) \tag{2.20}$$

is a version of $\mathbb{E}_{\mu}(f|\mathcal{T}_{\mathcal{L}}^{\infty})$. But (H_4) implies

$$\mu^{-}(f) \le \limsup_{\Lambda} \gamma_{\Lambda}(f|\omega) \le \mu^{+}(f)\,, \tag{2.21}$$

and therefore

$$\mu^{-}(f) \le \mu(f) \le \mu^{+}(f)\,. \tag{2.22}$$

If (H_3) holds, then μ^+ and μ^- are also Γ–compatible, since for continuous f and any Λ_1

$$\mu^{+}(f) = \lim_{\Lambda} \gamma_{\Lambda}(f|+) = \lim_{\Lambda} \int \gamma_{\Lambda}(d\omega|+)\gamma_{\Lambda_1}(f|\omega) = \mu^{+}(\gamma_{\Lambda_1}(f|\cdot))\,. \tag{2.23}$$

The extremality of μ^+ and μ^- follows from (2.22). This also shows that $|\mathcal{G}(\Gamma)| = 1$, iff $\mu^+ = \mu^-$. □

Main problem: Let Γ be a specification, which is $\mathbb{Z}^d$–*invariant* and satisfies hypothesis (H_1) to (H_4), and let μ be an extremal element of $\mathcal{G}(\Gamma)$; let T be a subgroup of $\mathbb{Z}^d$. Is the restriction of μ to the σ–algebra $\mathcal{F}_T = \sigma\{X_i; i \in T\}$ quasilocal?

3 Global specification

Let $\mathcal{L}$ be any countable infinite set. In the whole section we suppose that the specification $\Gamma = \{\gamma_\Lambda\}$ satisfies only (H_3) and (H_4). We choose as measure μ the measure μ^+ of Proposition 2.1. We construct a *global specification* Γ^+ for μ^+, i.e.

a family of probability kernels γ_S^+ indexed by the all subsets $S \subset \mathcal{L}$, so that μ^+ is Γ^+–compatible in the sense that (2.9) of definition 2.1 is valid for any $S \subset \mathcal{L}$ if $\mu = \mu^+$. For the Ising model, the existence of the global specification of μ^+ is equivalent to the validity of the global Markov property for μ^+ (Goldstein (1980), Föllmer (1980)). The same construction holds in the general case. We give here our own version of this result for the sake of completeness and because some of the lemmas below are used afterwards. A similar construction holds for the measure μ^-, giving a global specification $\Gamma^- = \{\gamma_S^-\}$. Using the global specification Γ^+ we then define in the next section a local specification Q_T^+ for the projection of μ^+ on the σ–algebra $\mathcal{F}_T$, T any infinite subset of $\mathcal{L}$.

Lemma 3.1 *Assume that Γ satisfies (H_4), and let f be a monotone bounded function. If*

$$\lim_{\Lambda} f(\omega_\Lambda^+) = f(\omega)\ ,$$

then

$$\mu^+(f) = \lim_{\Lambda} \gamma_\Lambda(f|+)\ .$$

If

$$\lim_{\Lambda} f(\omega_\Lambda^-) = f(\omega)\ ,$$

then

$$\mu^-(f) = \lim_{\Lambda} \gamma_\Lambda(f|-)\ .$$

Proof : It is sufficient to prove the lemma for monotone increasing functions. Let $M \subset \Lambda \subset N$, $|N| < \infty$; we set (for this proof)

$$f_\Lambda^+(\omega) := f(\omega_\Lambda^+)\ , \tag{3.1}$$

and suppose that f is increasing. Then $f_\Lambda^+ \leq f_M^+$, and by Proposition 2.1

$$\gamma_N(f_\Lambda^+|+) \leq \gamma_\Lambda(f_\Lambda^+|+) = \gamma_\Lambda(f|+) \leq \gamma_\Lambda(f_M^+|+)\ . \tag{3.2}$$

Since f_Λ^+ is local, we take the limit over N, and get (see Proposition 2.1)

$$\mu^+(f_\Lambda^+) = \inf_N \gamma_N(f_\Lambda^+|+) \leq \gamma_\Lambda(f|+) \leq \gamma_\Lambda(f_M^+|+)\ . \tag{3.3}$$

By the monotone convergence theorem we have

$$\mu^+(f) \leq \liminf_{\Lambda} \gamma_\Lambda(f|+) \leq \limsup_{\Lambda} \gamma_\Lambda(f|+) \leq \mu^+(f_M^+)\ , \tag{3.4}$$

and finally by taking the limit over M,

$$\mu^+(f) \leq \liminf_{\Lambda} \gamma_\Lambda(f|+) \leq \limsup_{\Lambda} \gamma_\Lambda(f|+) \leq \mu^+(f)\ . \tag{3.5}$$

A similar proof holds for the second part of the lemma. □

For any bounded function g we set

$$\overline{g}_\Lambda(\omega) := \gamma_\Lambda(g|\omega_{S^c}^+)\ . \tag{3.6}$$

Lemma 3.2 *Assume Γ satisfy (H_4). Let $\Lambda \subset S \subset \mathcal{L}$, and let g be a non-negative local increasing function. Then the function $\overline{g}_\Lambda$ is increasing in ω, and $\overline{g}_{\Lambda_1} \geq \overline{g}_{\Lambda_2}$, if $\Lambda_1 \subset \Lambda_2 \subset S$.*

Proof : The first statement is an obvious consequence of (H_4). Let $\Lambda_1 \subset \Lambda_2 \subset S$; then

$$\begin{aligned} \overline{g}_{\Lambda_2}(\omega) &= \gamma_{\Lambda_2}(g|\omega^+_{S^c}) \\ &= \int \gamma_{\Lambda_2}(d\eta|\omega^+_{S^c})\gamma_{\Lambda_1}(g|\eta) \\ &\leq \int \gamma_{\Lambda_2}(d\eta|\omega^+_{S^c})\gamma_{\Lambda_1}(g|\eta^+_{\Lambda_2^c}) \ ; \end{aligned} \tag{3.7}$$

since

$$\eta^+_{\Lambda_2^c}(k) = \begin{cases} \eta(k) & \text{if } k \notin \Lambda_2 \\ +1 & \text{if } k \in \Lambda_2 \end{cases} , \tag{3.8}$$

the function $\eta \mapsto \gamma_{\Lambda_1}(g|\eta^+_{\Lambda_2^c})$ is $\mathcal{F}_{\Lambda_2^c}$– measurable; therefore the last line of (3.7) is equal to $\overline{g}_{\Lambda_1}(\omega)$ (property (s_3) of a specification). □

Let g be a non-negative local increasing function, and let $S \subset \mathcal{L}$, $|S| = \infty$. By Lemma 3.2, and for any fixed $\omega \in \Omega$, the net $(\gamma_\Lambda(g|\omega^+_{S^c})\,,\,\Lambda \subset S)$ is monotone decreasing in $\Lambda \subset S$. Thus

$$\gamma^+_S(g|\omega) := \lim_{\Lambda \uparrow S} \gamma_\Lambda(g|\omega^+_{S^c}) = \inf_{\Lambda \subset S} \gamma_\Lambda(g|\omega^+_{S^c}) \ . \tag{3.9}$$

Therefore the net $(\gamma_\Lambda(\cdot|\omega^+_{S^c})\,,\,\Lambda \subset S)$ converges to a probability measure $\gamma^+_S(\cdot|\omega)$ on $(\Omega, \mathcal{F}_\mathcal{L})$. If F is $\mathcal{F}_{S^c}$–measurable, then $\gamma^+_S(F|\omega) = 1_F(\omega)$, and for any $F \in \mathcal{F}_\mathcal{L}$ $\gamma^+_S(F|\omega)$ is $\mathcal{F}_{S^c}$–measurable. We define a global specification $\Gamma^+ = \{\gamma^+_S\,,\, S \subset \mathcal{L}\}$ by setting

$$\gamma^+_S(\cdot|\omega) := \begin{cases} \gamma_S(\cdot|\omega) & \text{if } |S| < \infty \\ \lim_{\Lambda \uparrow S} \gamma_\Lambda(\cdot|\omega^+_{S^c}) & \text{if } |S| = \infty \end{cases} . \tag{3.10}$$

Similarly we can define

$$\gamma^-_S(\cdot|\omega) := \begin{cases} \gamma_S(\cdot|\omega) & \text{if } |S| < \infty \\ \lim_{\Lambda \uparrow S} \gamma_\Lambda(\cdot|\omega^-_{S^c}) & \text{if } |S| = \infty \end{cases} . \tag{3.11}$$

For a monotone function g we set

$$\overline{g}_S(\omega) := \gamma^+_S(g|\omega) = \lim_{\Lambda \uparrow S} \overline{g}_\Lambda(\omega) \ . \tag{3.12}$$

Lemma 3.3 *Let Γ be a specification satisfying (H_3) and (H_4). Let $S \subset \mathcal{L}$, $|S| = \infty$, and g be a local monotone function. Then*

$$\lim_\Lambda \overline{g}_S(\omega^+_\Lambda) = \overline{g}_S(\omega) \ .$$

Proof : It is sufficient to consider the case of a local non-negative monotone increasing function g. The function $\overline{g}_S(\omega)$ is increasing in ω; thus

$$\overline{g}_S(\omega^+_\Lambda) \geq \overline{g}_S(\omega) \ , \tag{3.13}$$

and therefore

$$\lim_{\Lambda\uparrow S^c} \overline{g}_S(\omega^+_\Lambda) \geq \inf_{\Lambda\subset S^c} \overline{g}_S(\omega^+_\Lambda) \geq \overline{g}_S(\omega)\ . \tag{3.14}$$

Let $\Lambda_1 \subset S$. Since the specification Γ satisfies (H_3) the function $\omega \mapsto \gamma_{\Lambda_1}(g|\omega)$ is continuous on Ω; therefore the function

$$\omega \mapsto \overline{g}_{\Lambda_1}(\omega) = \gamma_{\Lambda_1}(g|\omega^+_{S^c}) \tag{3.15}$$

is also continuous on Ω, because it is the composition of the continuous function $\omega \mapsto \gamma_{\Lambda_1}(g|\omega)$ and the continuous the map $\omega \mapsto \omega^+_{S^c}$. Thus

$$\overline{g}_{\Lambda_1}(\omega) = \lim_{\Lambda\uparrow S^c} \gamma_{\Lambda_1}(g|\omega^+_\Lambda)\ . \tag{3.16}$$

Since g is increasing, $\Lambda \subset S^c$ and $\Lambda_1 \subset S$,

$$\gamma_{\Lambda_1}(g|\omega^+_\Lambda) \geq \overline{g}_S(\omega^+_\Lambda)\ , \tag{3.17}$$

and consequently

$$\begin{aligned} \overline{g}_{\Lambda_1}(\omega) &= \lim_{\Lambda\uparrow S^c} \gamma_{\Lambda_1}(g|\omega^+_\Lambda) \\ &\geq \lim_{\Lambda\uparrow S^c} \overline{g}_S(\omega^+_\Lambda) \\ &= \inf_{\Lambda\subset S^c} \overline{g}_S(\omega^+_\Lambda)\ . \end{aligned} \tag{3.18}$$

Taking now the limit $\Lambda_1 \uparrow S$, we get

$$\overline{g}_S(\omega) \geq \inf_{\Lambda\subset S^c} \overline{g}_S(\omega^+_\Lambda)\ . \tag{3.19}$$

□

Proposition 3.1 *Let Γ be a specification on $\mathcal{L}$ which satisfies (H_3) and (H_4). Then the probability measure μ^+ is Γ^+–compatible, where Γ^+ is the global specification defined in (3.10).*

Proof : Let $S \subset \mathcal{L}$, $|S| = \infty$. Let g be a non-negative increasing function, which is Λ_1–local with $\Lambda_1 \subset S$. We must prove that

$$\mathbb{E}_{\mu^+}(g|\mathcal{F}_{S^c}) = \gamma^+_S(g|\omega)\ \ \mu^+ - a.s.\ ; \tag{3.20}$$

it is sufficient to prove that

$$\mathbb{E}_{\mu^+}(g\, f) = \mathbb{E}_{\mu^+}(\gamma^+_S(g|\cdot)\, f) \tag{3.21}$$

for any function f which is non-negative, increasing and Λ_2–local, with $\Lambda_2 \subset S^c$. By Lemma 3.2 $\overline{g}_S \leq \overline{g}_\Lambda$ if $\Lambda \subset S$, and by Lemmas 3.3 and 3.1

$$\begin{aligned} \mathbb{E}_{\mu^+}(f\,\overline{g}_S) &= \lim_{\Lambda''} \gamma_{\Lambda''}(f\,\overline{g}_S|+) \\ &\leq \gamma_{\Lambda'}(f\,\overline{g}_S|+) \\ &\leq \gamma_{\Lambda'}(f\,\overline{g}_\Lambda|+)\ . \end{aligned} \tag{3.22}$$

We choose Λ' so that $\Lambda' \cap S = \Lambda$ and Λ large enough so that g is Λ–local; then

$$\begin{aligned} \int \gamma_{\Lambda'}(d\eta|+) f(\eta)\, \overline{g}_\Lambda(\eta) &= \int \gamma_{\Lambda'}(d\eta|+) f(\eta)\, \gamma_\Lambda(g|\eta^+_{S^c}) \\ &= \int \gamma_{\Lambda'}(d\eta|+) f(\eta)\, \gamma_\Lambda(g|\eta) \\ &= \int \gamma_{\Lambda'}(d\eta|+) f(\eta)\, g(\eta) \;. \end{aligned} \tag{3.23}$$

Therefore

$$\mathbb{E}_{\mu^+}(f\, \overline{g}_S) \leq \mathbb{E}_{\mu^+}(f\; g) \;. \tag{3.24}$$

On the other hand, if $M \subset S \cap \Lambda'$,

$$\begin{aligned} \mathbb{E}_{\mu^+}(f\; g) &= \lim_{\Lambda'} \int \gamma_{\Lambda'}(d\eta|+) f(\eta)\, g(\eta) \\ &= \lim_{\Lambda'} \int \gamma_{\Lambda'}(d\eta|+) f(\eta)\, \gamma_M(g|\eta) \\ &\leq \lim_{\Lambda'} \int \gamma_{\Lambda'}(d\eta|+) f(\eta)\, \gamma_M(g|\eta^+_{S^c}) \\ &= \lim_{\Lambda'} \int \gamma_{\Lambda'}(d\eta|+) f(\eta)\, \overline{g}_M(\eta) \\ &= \mathbb{E}_{\mu^+}(f\, \overline{g}_M) \;. \end{aligned} \tag{3.25}$$

By the monotone convergence theorem

$$\mathbb{E}_{\mu^+}(f\; g) \leq \lim_{M\uparrow S} \mathbb{E}_{\mu^+}(f\, \overline{g}_M) = \mathbb{E}_{\mu^+}(f\, \overline{g}_S) \;. \tag{3.26}$$

$\square$

The conclusion of this section is that every (local) specification which is quasilocal $[(H_3)]$ and monotonicity-preserving $[(H_4)]$ gives rise to a *global* specification which is compatible with μ^+ and also monotonicity-preserving.

4 Specification for the projection of μ^+ on $\mathcal{F}_T$

Let $\mathcal{L}$ be an infinite countable set, T any infinite subset of $\mathcal{L}$ and Γ a specification on $\mathcal{L}$ which satisfies (H_1) to (H_4). To avoid trivial cases we also suppose that $|T^c| = \infty$. We use the global specification constructed in the previous section to define a probability kernel q^+_Λ, $\Lambda \subset T$, on $(\Omega, \mathcal{F}_T)$. Let f be a bounded $\mathcal{F}_T$–measurable function; we set

$$\int q^+_\Lambda(d\eta|\omega) f(\eta) := \int \gamma^+_{T^c\cup\Lambda}(d\theta|\omega) f(\theta) = \lim_{\Lambda'\uparrow T^c\cup\Lambda} \gamma_{\Lambda'}(f|\omega^+_{T\setminus\Lambda}) \;. \tag{4.1}$$

For any $F \in \mathcal{F}_T$ the function $q^+_\Lambda(F|\cdot) = \gamma^+_{T^c\cup\Lambda}(F|\cdot)$ is $\mathcal{F}_{T\setminus\Lambda}$–measurable and if $F \in \mathcal{F}_{T\setminus\Lambda}$, then $q^+_\Lambda(F|\cdot) = 1_F(\cdot)$. The family $Q^+_T = \{q^+_\Lambda\,,\ \Lambda \subset T\}$ is a local specification on T with properties (H_1), (H_2) and (H_4). Thus the first part of Proposition 2.1 is true for Q^+_T. The restriction of μ^+ to the σ–algebra $\mathcal{F}_T$ is denoted by μ^+_T when it is considered as a measure on $(E^T, \mathcal{F}_T)$. As in (2.14) we set

$$q^+_\Lambda(\sigma|\omega) := \int q^+_\Lambda(d\eta|\omega) 1_{\sigma_\Lambda}(\eta) \;. \tag{4.2}$$

Lemma 4.1 *The projection of μ^+ to $\mathcal{F}_T$ defines a probability measure μ_T^+ on $(E^T, \mathcal{F}_T)$ which is the limit of the net $(q_\Lambda^+(\cdot|+)\,,\,\Lambda \subset T)$. The measure μ_T^+ is Q_T^+–compatible. It is an extremal element of $\mathcal{G}(Q_T^+)$. The net $(q_\Lambda^+(\cdot|-)\,,\,\Lambda \subset T)$ converges to a probability measure ν_T^+.*

Proof : Let f be a bounded increasing local function in T; by (H_4) and (4.1), if $\Lambda' \cap T = \Lambda$, then

$$\begin{aligned}\gamma_{\Lambda'}(f|+) &\geq \lim_{\Lambda_1 \uparrow T^c \cup \Lambda} \gamma_{\Lambda_1}(f|+) \\ &= q_\Lambda^+(f|+) \\ &\geq \lim_{\Lambda \uparrow T} q_\Lambda^+(f|+)\ .\end{aligned} \tag{4.3}$$

Thus

$$\mu^+(f) = \lim_{\Lambda'} \gamma_{\Lambda'}(f|+) \geq \lim_{\Lambda \uparrow T} q_\Lambda^+(f|+)\ . \tag{4.4}$$

On the other hand, for any Λ_1

$$\mu^+(f) \leq \gamma_{\Lambda_1}(f|+)\ . \tag{4.5}$$

The measure μ_T^+ is Q_T^+–compatible by Proposition 3.1. It is extremal by Proposition 2.1. The existence of ν_T^+ follows also from that proposition. □

The main question addressed to in this paper is whether the specification Q_T^+ is quasilocal. This question is related to properties of the set $\mathcal{G}(\Gamma_{T^c}^\omega)$ of the measures compatible with the specification defined on T^c, $\Gamma_{T^c}^\omega = \{\gamma_{\Lambda,\omega}\,,\,\Lambda \subset T^c\}$. More generally, let $\Gamma_S^\omega = \{\gamma_{\Lambda,\omega}\,,\,\Lambda \subset S\}$, $S = T^c \cup \Lambda'$ with $\Lambda' \subset T$. The specification Γ_S^ω is defined on S by

$$\gamma_{\Lambda,\omega}(F|\eta) := \gamma_\Lambda(F|\eta_S\omega_{S^c})\ . \tag{4.6}$$

We consider Γ_S^ω as a specification on $(E^S, \mathcal{F}_S)$. Γ_S^ω satisfies (H_1) to (H_4) for any ω. By Proposition 2.1 the measures

$$\lim_{\Lambda \uparrow S} \gamma_\Lambda(\cdot|\omega_{S^c}^+) = \gamma_S^+(\cdot|\omega) \tag{4.7}$$

and

$$\lim_{\Lambda \uparrow S} \gamma_\Lambda(\cdot|\omega_{S^c}^-) = \gamma_S^-(\cdot|\omega) \tag{4.8}$$

are extremal elements of $\mathcal{G}(\Gamma_S^\omega)$.

Lemma 4.2 *Let $S \subset \boldsymbol{\mathcal{L}}$ and $\Lambda \subset S^c$; then for any ω, any $\omega' = \omega$ a.e., and any positive $\mathcal{F}_S$–measurable function f*

$$d_1(\Lambda \cup \Lambda')\gamma_S^+(f|\omega') \leq \gamma_{S\cup\Lambda}^+(f|\omega) \leq d_2(\Lambda \cup \Lambda')\gamma_S^+(f|\omega') \tag{4.9}$$

and

$$d_1(\Lambda \cup \Lambda')\gamma_S^-(f|\omega') \leq \gamma_{S\cup\Lambda}^-(f|\omega) \leq d_2(\Lambda \cup \Lambda')\gamma_{T^c}^-(f|\omega')\ ; \tag{4.10}$$

Λ' is the subset of S^c where ω and ω' are different; the constants d_1, d_2 are those appearing in condition (H_2).

Proof : We compute

$$\begin{aligned}
\gamma^+_{S\cup\Lambda}(f|\omega) &= \int \gamma^+_{S\cup\Lambda}(d\eta|\omega) f(\eta) \\
&= \int \gamma^+_{S\cup\Lambda}(d\eta|\omega) \int \gamma^+_S(d\theta|\eta) f(\theta) = \\
&= \int \gamma^+_{S\cup\Lambda}(d\eta|\omega) \int \gamma^+_S(d\theta|\eta_\Lambda\omega_{\Lambda^c}) f(\theta_S\eta_\Lambda\omega_{S^c\setminus\Lambda}) \\
&= \sum_{\eta_\Lambda} \gamma^+_{S\cup\Lambda}(1_{\eta_\Lambda}|\omega) \int \gamma^+_S(d\theta|\eta_\Lambda\omega_{\Lambda^c}) f(\theta_S\eta_\Lambda\omega_{S^c\setminus\Lambda}) \\
&= \sum_{\eta_\Lambda} \gamma^+_{S\cup\Lambda}(1_{\eta_\Lambda}|\omega)\gamma^+_S(f|\eta_\Lambda\omega_{\Lambda^c}) \; .
\end{aligned} \tag{4.11}$$

Since f does not depend explicitely on $\eta_\Lambda\omega_{S^c\setminus\Lambda}$ because it is $\mathcal{F}_S$–measurable, we have

$$d_1(\Lambda\cup\Lambda')\gamma^+_S(f|\omega') \le \gamma^+_S(f|\eta_\Lambda\omega_{\Lambda^c}) \le d_2(\Lambda\cup\Lambda')\gamma^+_S(f|\omega') \; . \tag{4.12}$$

Therefore

$$d_1(\Lambda\cup\Lambda')\gamma^+_S(f|\omega') \le \gamma^+_{S\cup\Lambda}(f|\omega) \le d_2(\Lambda\cup\Lambda')\gamma^+_S(f|\omega') \; . \tag{4.13}$$

□

The next two lemmas are essentially corollaries of Lemma 4.2.

Lemma 4.3 *Let $S=T^c$; if $|\mathcal{G}(\Gamma^\omega_{T^c})|=1$, i.e. if $\gamma^+_{T^c}(\cdot|\omega)=\gamma^-_{T^c}(\cdot|\omega)$, then the same is true for any ω', $\omega'=\omega$ a.e.. The set $\{\omega: \; |\mathcal{G}(\Gamma^\omega_{T^c})|=1\}$ belongs to the tail-field σ–algebra $\mathcal{T}^\infty_T$ which is defined as*

$$\mathcal{T}^\infty_T := \bigcap_{\Lambda\subset T} \mathcal{F}_{T\setminus\Lambda} \; .$$

Proof : Given $\omega'=\omega$ a.e., by Lemma 4.2 there exist constants b_1 and b_2 such that for all positive functions f

$$b_1\gamma^+_{T^c}(f|\omega') \le \gamma^+_{T^c}(f|\omega) \le b_2\gamma^+_{T^c}(f|\omega') \tag{4.14}$$

and

$$b_1\gamma^-_{T^c}(f|\omega') \le \gamma^-_{T^c}(f|\omega) \le b_2\gamma^-_{T^c}(f|\omega') \; . \tag{4.15}$$

Therefore the two measures $\gamma^-_{T^c}(\cdot|\omega')$ and $\gamma^+_{T^c}(\cdot|\omega')$ are equivalent. Since they are extremal elements of $\mathcal{G}(\Gamma^{\omega'}_{T^c})$, they satisfy a zero-one law on the tail-field σ–algebra

$$\mathcal{T}^\infty_{T^c} := \bigcap_{\Lambda\subset T^c} \mathcal{F}_{T^c\setminus\Lambda} \; , \tag{4.16}$$

and consequently they coincide on this σ–algebra. This implies that they are equal. From this result and the fact that $\mathcal{G}(\Gamma^\omega_{T^c})=\mathcal{G}(\Gamma^{\omega'}_{T^c})$ for any $\omega=\omega'$ a.e., we deduce that

$$\{\omega: \; |\mathcal{G}(\Gamma^\omega_{T^c})|=1\} \in \mathcal{T}^\infty_T \; . \tag{4.17}$$

□

Lemma 4.4 *Let $S=T^c$ and $\Lambda\subset T$. Then there is an affine bijection between $\mathcal{G}(\Gamma^\omega_{T^c\cup\Lambda})$ and $\mathcal{G}(\Gamma^\omega_{T^c})$. In particular if $|\mathcal{G}(\Gamma^\omega_{T^c})|=1$, then $|\mathcal{G}(\Gamma^{\omega'}_{T^c\cup\Lambda})|=1$ if $\omega'=\omega$ a.e..*

Proof : This follows from Lemma 4.2. For details see Theorem 7.33 in Georgii (1988). $\square$

Lemma 4.5 *For any* $\Lambda_1 \subset T$

$$\gamma^{+}_{T^c\cup\Lambda_1}(\cdot|\omega) = \lim_{\Lambda\uparrow T} \gamma^{+}_{T^c\cup\Lambda_1}(\cdot|\omega^{+}_{\Lambda})$$

and if $|\mathcal{G}(\Gamma^{-}_{T^c})| = 1$, *then*

$$\gamma^{-}_{T^c\cup\Lambda_1}(\cdot|\omega) = \lim_{\Lambda\uparrow T} \gamma^{+}_{T^c\cup\Lambda_1}(\cdot|\omega^{-}_{\Lambda}) \ .$$

Analogous expressions hold with "+" and "−" interchanged.

Proof : For any local increasing function f we have by (H_4)

$$\gamma^{+}_{T^c\cup\Lambda_1}(f|\omega) \leq \gamma^{+}_{T^c\cup\Lambda_1}(f|\omega^{+}_{\Lambda}) \ . \tag{4.18}$$

On the other hand, for any $\Lambda' \subset T^c \cup \Lambda_1$, (H_3) implies that

$$\gamma_{\Lambda'}(f|\omega^{+}_{T}) = \lim_{\Lambda\uparrow T} \gamma_{\Lambda'}(f|\omega^{+}_{\Lambda}) \ , \tag{4.19}$$

since $\lim_{\Lambda\uparrow T}\omega^{+}_{\Lambda} = \omega^{+}_{T}$. Thus

$$\gamma_{\Lambda'}(f|\omega^{+}_{T}) = \lim_{\Lambda\uparrow T} \gamma_{\Lambda'}(f|\omega^{+}_{\Lambda}) \geq \lim_{\Lambda\uparrow T} \gamma^{+}_{T^c\cup\Lambda_1}(f|\omega^{+}_{\Lambda}) \ . \tag{4.20}$$

The first statement follows frow (4.18) and (4.20).

For any $\Lambda \subset T$, Lemmas 4.3, 4.4 and $|\mathcal{G}(\Gamma^{-}_{T^c})| = 1$ imply that

$$\gamma^{+}_{T^c\cup\Lambda_1}(\cdot|\omega^{-}_{\Lambda}) = \gamma^{-}_{T^c\cup\Lambda_1}(\cdot|\omega^{-}_{\Lambda}) \ . \tag{4.21}$$

The proof of the second statement is now similar to that of the first statement. $\square$

Lemma 4.6 *If* $|\mathcal{G}(\Gamma^{-}_{T^c})| = 1$, *then the measure* ν^{+}_{T} *is the restriction of the measure* μ^{-} *to the* σ*–algebra* $\mathcal{F}_T$.

Proof : By Lemma 4.5 the measure ν^{+}_{T} is Q^{-}_{T}–compatible. The result then follows from Proposition 3.1 applied to the specification Q^{-}_{T}. $\square$

We come to the study of the quasilocality of the specification Q^{+}_{T}. Quasilocality or continuity of $q^{+}_{\Lambda}(\sigma|\cdot)$ at ω means that

$$\lim_{\Lambda'\uparrow T} q^{+}_{\Lambda}(\sigma|\omega^{+}_{\Lambda'}) = \lim_{\Lambda'\uparrow T} q^{+}_{\Lambda}(\sigma|\omega^{-}_{\Lambda'}) \ . \tag{4.22}$$

¿From this expression and Lemma 4.4 is not hard to conclude that $|\mathcal{G}(\Gamma^{\omega}_{T^c})| = 1$ implies the quasilocality of Q^{+}_{T} at ω (the proof is spelled out in Proposition 4.1 below). The converse, however, is not true.

Lemma 4.7 *Let* $\Lambda_1 \subset \Lambda_2 \subset T$. *If the function* $q^{+}_{\Lambda_2}(\sigma|\cdot)$ *is continuous at* ω *for any* σ, *then the function* $q^{+}_{\Lambda_1}(\sigma|\cdot)$ *is continuous for any* σ *at* ω' *such that* $\omega'_{\Lambda_2^c} = \omega_{\Lambda_2^c}$.

Proof : Let f be an increasing Λ_1–local function. By monotonicity

$$\begin{aligned} q^+_{\Lambda_2}(f|\omega^+_\Lambda) &= \int q^+_{\Lambda_2}(d\eta|\omega^+_\Lambda) q^+_{\Lambda_1}(f|\eta_{\Lambda_2}(\omega^+_\Lambda)_{\Lambda\setminus\Lambda_2}) \\ &\geq \int q^+_{\Lambda_2}(d\eta|\omega) q^+_{\Lambda_1}(f|\eta_{\Lambda_2}(\omega^+_\Lambda)_{\Lambda\setminus\Lambda_2}) \\ &\geq \int q^+_{\Lambda_2}(d\eta|\omega) q^+_{\Lambda_1}(f|\eta_{\Lambda_2}(\omega^-_\Lambda)_{\Lambda\setminus\Lambda_2}) \\ &\geq \int q^+_{\Lambda_2}(d\eta|\omega^-_\Lambda) q^+_{\Lambda_1}(f|\eta_{\Lambda_2}(\omega^-_\Lambda)_{\Lambda\setminus\Lambda_2}) \\ &= q^+_{\Lambda_2}(f|\omega^-_\Lambda)\ . \end{aligned} \tag{4.23}$$

Therefore

$$\begin{aligned} 0 &= \lim_{\Lambda\uparrow T}\left(q^+_{\Lambda_2}(f|\omega^+_\Lambda) - q^+_{\Lambda_2}(f|\omega^-_\Lambda)\right) \\ &\geq \int q^+_{\Lambda_2}(d\eta|\omega) \lim_{\Lambda\uparrow T}[q^+_{\Lambda_1}(f|\eta_{\Lambda_2}(\omega^+_\Lambda)_{\Lambda\setminus\Lambda_2}) - q^+_{\Lambda_1}(f|\eta_{\Lambda_2}(\omega^-_\Lambda)_{\Lambda\setminus\Lambda_2})]\ . \end{aligned} \tag{4.24}$$

Since

$$q^+_{\Lambda_1}(f|\omega'^+_\Lambda) \geq q^+_{\Lambda_1}(f|\omega'^-_\Lambda)\ , \tag{4.25}$$

and (H_1) holds (this is the only place where we use this condition in this section), we have

$$\lim_{\Lambda\uparrow T} q^+_{\Lambda_1}(f|\omega'^+_\Lambda) = \lim_{\Lambda\uparrow T} q^+_{\Lambda_1}(f|\omega'^-_\Lambda)\ , \tag{4.26}$$

for all increasing Λ_1–local functions and all ω' such that $\omega'_{\Lambda_2^c} = \omega_{\Lambda_2^c}$. This implies that

$$\lim_{\Lambda\uparrow T} q^+_{\Lambda_1}(\sigma|\omega'^+_\Lambda) = \lim_{\Lambda\uparrow T} q^+_{\Lambda_1}(\sigma|\omega'^-_\Lambda) \tag{4.27}$$

for any σ. □

We collect now all the results obtained so far in reference to the set of continuity points.

Proposition 4.1 *Let Γ be a specification on $\mathcal{L}$ which satisfies (H_1) to (H_4). Then the set of the continuity points of the specification Q^+_T,*

$$\Omega_q := \{\omega :\ q^+_\Lambda(\sigma|\cdot) \text{ is continuous at } \omega \text{ for all } \sigma, \Lambda \subset T\}\ ,$$

is in the tail-field σ–algebra

$$\mathcal{T}^\infty_T = \bigcap_{\Lambda\subset T} \mathcal{F}_{T\setminus\Lambda}\ .$$

Moreover,

$$\{\omega :\ |\mathcal{G}(\Gamma^\omega_{T^c})| = 1\} \subset \Omega_q\ .$$

Proof : The first part of the proposition follows directly from Lemma 4.7. Using Lemma 4.5, if f is an increasing function, then

$$q^+_\Lambda(f|\omega) = \lim_{\Lambda'\uparrow T} q^+_\Lambda(f|\omega^+_{\Lambda'}) \geq \lim_{\Lambda'\uparrow T} q^+_\Lambda(f|\omega^-_{\Lambda'}) = \lim_{\Lambda'\uparrow T} q^-_\Lambda(f|\omega^-_{\Lambda'}) = q^-_\Lambda(f|\omega)\ ; \tag{4.28}$$

if $|\mathcal{G}(\Gamma^\omega_{T^c})| = 1$, then Lemma 4.4 implies that

$$q^+_\Lambda(\cdot|\omega) = \gamma^+_{\Lambda\cup T^c}(\cdot|\omega) = \gamma^-_{\Lambda\cup T^c}(\cdot|\omega) = q^-_\Lambda(\cdot|\omega)\ , \tag{4.29}$$

and therefore $\omega \in \Omega_q$ by (4.28). $\square$

Remark: If for all $j \in \mathcal{L}$ the functions $q_j^+(\sigma|\cdot)$ are continuous, then the same is true for the functions $q_\Lambda^+(\sigma|\cdot)$ (see e.g. (5.5) below). The specification Q_T^+ is therefore quasilocal.

In the next proposition we give a sufficient condition so that the discontinuities of Q_T^+ cannot be removed by changing the specification on a set of μ^+–measure zero. Since any neighbourhood has a non-zero μ^+–measure, this condition implies that the measure μ^+ is not quasilocal, if we can prove that Q_T^+ is discontinuous on a set of positive μ^+–measure. By Proposition 4.1 Q_T^+ is automatically discontinuous on a set of μ^+–measure one.

Proposition 4.2 *Let Γ be a specification on $\mathcal{L}$ which satisfies (H_3) and (H_4). Let $\varepsilon > 0$ and $\omega \in \Omega$, such that*

$$\lim_{\Lambda'} |q_j^+(\sigma|\omega_{\Lambda'}^+) - q_j^+(\sigma|\omega_{\Lambda'}^-)| \geq \varepsilon \; .$$

If for any $\eta = +$ a.e.

$$\lim_{\Lambda \uparrow T} q_j^+(\sigma|\eta_\Lambda^-) = q_j^+(\sigma|\eta) \; ,$$

then for any neighbourhood of ω, $V_\Lambda = \{\omega' : \; \omega'_\Lambda = \omega_\Lambda\}$, we can find two neighbourhoods, $V_{\Lambda,M}^+$ and $V_{\Lambda,M}^-$, $\Lambda \subset M$, $|M| < \infty$,

$$V_{\Lambda,M}^+ = \{\omega' : \; \omega'_\Lambda = \omega_\Lambda \; , \; \omega'_{M\setminus\Lambda} = +\} \; ,$$

$$V_{\Lambda,M}^- = \{\omega' : \; \omega'_\Lambda = \omega_\Lambda \; , \; \omega'_{M\setminus\Lambda} = -\} \; ,$$

which have the following property: for any $\alpha \in V_{\Lambda,M}^+$ and $\theta \in V_{\Lambda,M}^-$,

$$\lim_{\Lambda'} |q_j^+(\sigma|\alpha_{\Lambda'}^+) - q_j^+(\sigma|\theta_{\Lambda'}^-)| \geq \frac{\varepsilon}{2} \; .$$

Proof : By hypothesis $q_j^+(\sigma|\cdot)$ is continuous at every $\omega = +$ a.e.,

$$\lim_{M \uparrow T} q_j^+(\sigma|(\omega_\Lambda^+)_M^-) = q_j^+(\sigma|\omega_\Lambda^+) \; . \tag{4.30}$$

Lemma 3.3 implies

$$\lim_{M \uparrow T} q_j^+(\sigma|(\omega_\Lambda^-)_M^+) = q_j^+(\sigma|\omega_\Lambda^-) \; . \tag{4.31}$$

We choose $M \supset \Lambda$ so that

$$|q_j^+(\sigma|(\omega_\Lambda^+)_M^-) - q_j^+(\sigma|\omega_\Lambda^+)| \leq \frac{\varepsilon}{4} \tag{4.32}$$

and

$$|q_j^+(\sigma|(\omega_\Lambda^-)_M^+) - q_j^+(\sigma|\omega_\Lambda^-)| \leq \frac{\varepsilon}{4} \; . \tag{4.33}$$

By (H_4), if $\alpha \in V_{\Lambda,M}^+$ and $\theta \in V_{\Lambda,M}^-$, then

$$\begin{aligned}
(q_j^+(+|\alpha) - q_j^+(+|\theta)) \;&\geq\; (q_j^+(+|\alpha_M^-) - q_j^+(+|\theta_M^+)) \\
&\geq\; (q_j^+(+|\omega_\Lambda^+) - q_j^+(+|\omega_\Lambda^-)) - \frac{\varepsilon}{2} \\
&\geq\; \frac{\varepsilon}{2} \; .
\end{aligned} \tag{4.34}$$

Similar inequalities hold if $\sigma(j) = -$. $\square$

The next lemma gives a sufficient condition so that the hypothesis of Proposition 4.2 on the continuity of q_j^+ at all $\eta = +$ a.e. is verified.

Lemma 4.8 *If $|\mathcal{G}(\Gamma_{T^c}^+)| = 1$, then any ω, such that $\omega(k) = +$ a.e., is a point of continuity of the $q_j^+(\sigma|\cdot)$.*

Proof : Use Lemma 4.4 and Proposition 4.1. $\square$

There is a subclass of the monotonicity–preserving specifications for which we can weaken the hypothesis of Proposition 4.2, and improve the conclusions of Proposition 4.1. Let $\Lambda \subset \mathcal{L}$ and $\eta \in \Omega$. We define

$$I_\Lambda(\eta) := \sum_{\{i,j\}\cap\Lambda\neq\emptyset} J(i,j)X_i(\eta)X_j(\eta) + \sum_{i\in\Lambda} h(i)X_i(\eta) \tag{4.35}$$

and

$$\gamma_\Lambda^I(\sigma|\omega) := \frac{\exp I_\Lambda(\sigma_\Lambda\omega_{\Lambda^c})}{\sum_{\eta_\Lambda}\exp I_\Lambda(\eta_\Lambda\omega_{\Lambda^c})}\ , \tag{4.36}$$

where $J(i,j) \geq 0$, $h(i) \geq 0$ and

$$\sup_i h(i) < \infty \ \ , \ \ \sup_i \sum_{j\in\mathcal{L}} J(i,j) < \infty\ . \tag{4.37}$$

The specification $\Gamma(I) = \{\gamma_\Lambda^I\ ,\ \Lambda \subset \mathcal{L}\}$ satisfies properties (H_1) to (H_4).

Lemma 4.9 *Under the above hypothesis, we have for any $i \in \Lambda$, any ω and $M \supset \Lambda$,*

$$(\gamma_\Lambda^I(X_i|\omega_M^+) - \gamma_\Lambda^I(X_i|\omega_M^-)) \geq (\gamma_\Lambda^I(X_i|+) - \gamma_\Lambda^I(X_i|+_M^-))\ .$$

The proof of that lemma is given in appendix; it is based on Fröhlich Pfister (1987a).

Proposition 4.3 *For the specification $\Gamma(I)$ the following affirmations are true.*

(a) If $\omega \equiv +$ is a point of continuity of $q_j^+(\sigma|\cdot)$, then any $\omega = +$ a.e. is a point of continuity of $q_j^+(\sigma|\cdot)$.

(b) If $\omega \equiv +$ is a point of discontinuity of $q_j^+(\sigma|\cdot)$, then all ω are points of discontinuity of $q_j^+(\sigma|\cdot)$,

$$\lim_{\Lambda\uparrow T}\Big(q_j^+(\sigma|\omega_\Lambda^+) - q_j^+(\sigma|\omega_\Lambda^-)\Big) \geq \lim_{\Lambda\uparrow T}\Big(q_j^+(\sigma|+) - q_j^+(\sigma|+_\Lambda^-)\Big) > 0\ .$$

Proof : If $\omega \equiv +$ is a point of continuity of $q_j^+(\sigma|+)$, then

$$q_j^+(X_j|+) = \lim_{\Lambda\uparrow T} q_j^+(X_j|+_\Lambda^-)\ . \tag{4.38}$$

(4.38) already implies the much stronger result (see Lemma 3.3 in Fröhlich Pfister (1987a))

$$\lim_{\Lambda\uparrow T}\gamma_{\{j\}\cup T^c}^+(\cdot|+_\Lambda^-) \equiv \tilde\gamma_{\{j\}\cup T^c}^+(\cdot|+) = \gamma_{\{j\}\cup T^c}^+(\cdot|+)\ . \tag{4.39}$$

(The limit exists by (H_4).) If we replace $+$ by $\omega = +$ a.e., then the measures $\gamma_{\{j\}\cup T^c}^+(\cdot|\omega)$ and $\tilde\gamma_{\{j\}\cup T^c}^+(\cdot|\omega)$ are equivalent by Lemma 4.2. Since $\gamma_{\{j\}\cup T^c}^+(\cdot|+)$ is extremal they must coincide. This proves (a). Part (b) is a direct consequence of Lemma 4.9. $\square$

5 A criterion for non–quasilocality

Let $\Gamma = \{\gamma_\Lambda\,,\,\Lambda\subset\mathcal{L}\}$ be a specification defined on $\mathcal{L}$, which is uniformly non-null and monotonicity–preserving, i.e which satisfies (H_1) and (H_4). We establish in this section our main criterion for non–quasilocality of a specification, Corollary 5.1. This is done by estimating the relative entropy

$$\frac{1}{|\Lambda|}H_\Lambda(\gamma_\Lambda(\cdot|+)|\gamma_\Lambda(\cdot|-)) := \frac{1}{|\Lambda|}\sum_{\sigma_\Lambda}\gamma_\Lambda(\sigma|+)\log\frac{\gamma_\Lambda(\sigma|+)}{\gamma_\Lambda(\sigma|-)}\,. \tag{5.1}$$

We do this in subsection 5.1. The method is inspired by Kozlov (1974) and Sullivan (1973).

5.1 Estimates of the relative entropy

We define on the set $\mathcal{L}$ a total order denoted by $\geq$. Given any $\sigma\in\Omega$ and $j\in\mathcal{L}$, we define a new element ${}_j\sigma\in\Omega$ by

$${}_j\sigma(k) := \begin{cases} - & \text{if } k<j \\ \sigma(k) & \text{if } k\geq j \end{cases}\,. \tag{5.2}$$

We write the quotient in the right-hand side of (5.1) as

$$\frac{\gamma_\Lambda(\sigma|+)}{\gamma_\Lambda(\sigma|-)} = \frac{\gamma_\Lambda(-|+)}{\gamma_\Lambda(-|-)}\cdot\frac{\gamma_\Lambda(\sigma|+)}{\gamma_\Lambda(-|+)}\cdot\frac{\gamma_\Lambda(-|-)}{\gamma_\Lambda(\sigma|-)}\,. \tag{5.3}$$

Using the identity

$$\frac{\gamma_\Lambda(\sigma_{\Lambda_1}\eta_{\Lambda_2}|\omega)}{\gamma_\Lambda(\tau_{\Lambda_1}\eta_{\Lambda_2}|\omega)} = \frac{\gamma_{\Lambda_1}(\sigma_{\Lambda_1}|\eta_{\Lambda_2}\omega_{\Lambda_2^c})}{\gamma_{\Lambda_1}(\tau_{\Lambda_1}|\eta_{\Lambda_2}\omega_{\Lambda_2^c})}\,, \tag{5.4}$$

where $\Lambda=\Lambda_1\cup\Lambda_2$, $\Lambda_1\cap\Lambda_2=\emptyset$, we have

$$\frac{\gamma_\Lambda(\sigma|+)}{\gamma_\Lambda(-|+)} = \prod_{j\in\Lambda}\frac{\gamma_j(\sigma|{}_j\sigma_\Lambda^+)}{\gamma_j(-|{}_j\sigma_\Lambda^+)} \tag{5.5}$$

and

$$\frac{\gamma_\Lambda(-|-)}{\gamma_\Lambda(\sigma|-)} = \prod_{j\in\Lambda}\frac{\gamma_j(-|{}_j\sigma_\Lambda^-)}{\gamma_j(\sigma|{}_j\sigma_\Lambda^-)}\,. \tag{5.6}$$

Using (5.2), (5.5) and (5.6) we can write (5.1) as

$$\frac{1}{|\Lambda|}H_\Lambda(\gamma_\Lambda(\cdot|+)|\gamma_\Lambda(\cdot|-)) = \tag{5.7}$$

$$\frac{1}{|\Lambda|}\log\frac{\gamma_\Lambda(-|+)}{\gamma_\Lambda(-|-)} + \frac{1}{|\Lambda|}\sum_{\sigma_\Lambda}\gamma_\Lambda(\sigma|+)\log\prod_{j\in\Lambda}\frac{\gamma_j(\sigma|{}_j\sigma_\Lambda^+)}{\gamma_j(-|{}_j\sigma_\Lambda^+)}\frac{\gamma_j(-|{}_j\sigma_\Lambda^-)}{\gamma_j(\sigma|{}_j\sigma_\Lambda^-)}\,.$$

Let us define

$$f_{j,\Lambda}(\eta) := \log\frac{\gamma_j(+|\eta_\Lambda^+)}{\gamma_j(+|\eta_\Lambda^-)}\frac{\gamma_j(-|\eta_\Lambda^-)}{\gamma_j(-|\eta_\Lambda^+)}\,. \tag{5.8}$$

The j^{th} factor of the product in (5.7) is equal to one if $\sigma(j)=-$; it is larger than one if $\sigma(j)=+$ by (H_4); thus

$$0 \leq \log\frac{\gamma_j(\sigma|{}_j\sigma_\Lambda^+)}{\gamma_j(-|{}_j\sigma_\Lambda^+)}\frac{\gamma_j(-|{}_j\sigma_\Lambda^-)}{\gamma_j(\sigma|{}_j\sigma_\Lambda^-)} \leq f_{j,\Lambda}(\sigma) \tag{5.9}$$

A similar expression can be derived using ${}^j\sigma$ instead of ${}_j\sigma$, where

$$ {}^j\sigma(k) := \begin{cases} + & \text{if } k < j \\ \sigma(k) & \text{if } k \geq j \end{cases} ; \tag{5.10} $$

we get

$$ \frac{1}{|\Lambda|} H_\Lambda(\gamma_\Lambda(\cdot|+)|\gamma_\Lambda(\cdot|-)) = \tag{5.11} $$

$$ \frac{1}{|\Lambda|} \log \frac{\gamma_\Lambda(+|+)}{\gamma_\Lambda(+|-)} + \frac{1}{|\Lambda|} \sum_{\sigma_\Lambda} \gamma_\Lambda(\sigma|+) \log \prod_{j\in\Lambda} \frac{\gamma_j(\sigma|{}^j\sigma_\Lambda^+)}{\gamma_j(+|{}^j\sigma_\Lambda^+)} \frac{\gamma_j(+|{}^j\sigma_\Lambda^-)}{\gamma_j(\sigma|{}^j\sigma_\Lambda^-)} . $$

By (H_4)

$$ \log \frac{\gamma_j(\sigma|{}^j\sigma_\Lambda^+)}{\gamma_j(+|{}^j\sigma_\Lambda^+)} \frac{\gamma_j(+|{}^j\sigma_\Lambda^-)}{\gamma_j(\sigma|{}^j\sigma_\Lambda^-)} \ \leq \ 0 . \tag{5.12} $$

Lemma 5.1 *Let the specification $\Gamma = \{\gamma_\Lambda \, , \, \Lambda \subset \mathcal{L}\}$ on $\mathcal{L}$ satisfy (H_1) and (H_4). Then $\lim_\Lambda f_{j,\Lambda} = f_j$ exists, and*

$$ \frac{1}{|\Lambda|} \log \frac{\gamma_\Lambda(-|-)}{\gamma_\Lambda(-|+)} \leq \frac{1}{|\Lambda|} \sum_{j\in\Lambda} \int \gamma_\Lambda(d\sigma|+) f_{j,\Lambda}({}_j\sigma) , \tag{5.13} $$

$$ \frac{1}{|\Lambda|} H_\Lambda(\gamma_\Lambda(\cdot|+)|\gamma_\Lambda(\cdot|-)) \leq \frac{1}{|\Lambda|} \log \frac{\gamma_\Lambda(+|+)}{\gamma_\Lambda(+|-)} . \tag{5.14} $$

Let $\Lambda = \Lambda_1 \cup \Lambda_2$, $\Lambda_1 \cap \Lambda_2 = \emptyset$; then

$$ \log \frac{\gamma_\Lambda(-|-)}{\gamma_\Lambda(-|+)} \leq \log \frac{\gamma_{\Lambda_1}(-|-)}{\gamma_{\Lambda_1}(-|+)} + \log \frac{\gamma_{\Lambda_2}(-|-)}{\gamma_{\Lambda_2}(-|+)} , \tag{5.15} $$

and

$$ \log \frac{\gamma_\Lambda(+|+)}{\gamma_\Lambda(+|-)} \leq \log \frac{\gamma_{\Lambda_1}(+|+)}{\gamma_{\Lambda_1}(+|-)} + \log \frac{\gamma_{\Lambda_2}(+|+)}{\gamma_{\Lambda_2}(+|-)} . \tag{5.16} $$

Proof : The existence of $\lim_\Lambda f_{j,\Lambda}$ follows by monotonicity. Since the relative entropy is non-negative

$$ \frac{1}{\Lambda} \log \frac{\gamma_\Lambda(-|-)}{\gamma_\Lambda(-|+)} \leq \frac{1}{\Lambda} \sum_{j\in\Lambda} \int \gamma_\Lambda(d\sigma|+) f_{j,\Lambda}({}_j\sigma) . \tag{5.17} $$

Starting from (5.11), and observing that the second term of the right-hand side of (5.11) is non-positive (see (5.12)), we get (5.14).
Let $\Lambda = \Lambda_1 \cup \Lambda_2$, so that $\Lambda_1 \cap \Lambda_2 = \emptyset$. Let χ_j be the characteristic function of the set $\{\eta : \ \eta(j) = +\}$, and

$$ \chi_\Lambda := \prod_{j\in\Lambda} \chi_j . \tag{5.18} $$

Then, since χ_Λ is increasing,

$$ \begin{aligned} \gamma_\Lambda(\chi_\Lambda|+) &= \int \gamma_\Lambda(d\eta|+) \chi_{\Lambda_1}(\eta) \chi_{\Lambda_2}(\eta) \\ &= \int \gamma_\Lambda(d\eta|+) \gamma_{\Lambda_1}(\chi_{\Lambda_1}|\eta) \chi_{\Lambda_2}(\eta) \\ &\leq \gamma_{\Lambda_1}(\chi_{\Lambda_1}|+) \gamma_\Lambda(\chi_{\Lambda_2}|+) \\ &\leq \gamma_{\Lambda_1}(\chi_{\Lambda_1}|+) \gamma_{\Lambda_2}(\chi_{\Lambda_2}|+) . \end{aligned} \tag{5.19} $$

Similarly,

$$\gamma_\Lambda(\chi_\Lambda|-) \geq \gamma_{\Lambda_1}(\chi_{\Lambda_1}|-)\gamma_{\Lambda_2}(\chi_{\Lambda_2}|-)\ . \tag{5.20}$$

Therefore

$$\frac{\gamma_\Lambda(+|+)}{\gamma_\Lambda(+|-)} \leq \frac{\gamma_{\Lambda_1}(+|+)}{\gamma_{\Lambda_1}(+|-)}\frac{\gamma_{\Lambda_2}(+|+)}{\gamma_{\Lambda_2}(+|-)}\ ; \tag{5.21}$$

we can prove analogously that

$$\frac{\gamma_\Lambda(-|-)}{\gamma_\Lambda(-|+)} \leq \frac{\gamma_{\Lambda_1}(-|-)}{\gamma_{\Lambda_1}(-|+)}\frac{\gamma_{\Lambda_2}(-|-)}{\gamma_{\Lambda_2}(-|+)}\ . \tag{5.22}$$

□

5.2 Criterion for non–quasilocality

The main property of the function f_j is that it is non-negative, and equal to zero at η iff $\gamma_j(\sigma|\cdot)$ is continuous at η for any σ. Our criterion for non–quasilocality is stated for $\mathbb{Z}^d$–invariant specifications, but it can be generalized to other situations with suitable modifications.

Proposition 5.1 *Let $\mathcal{L} = \mathbb{Z}^d$, and Γ be a $\mathbb{Z}^d$–invariant specification satisfying (H_1) and (H_4). Then*

$$\lim_{\Lambda_n} \frac{1}{|\Lambda_n|} \log \frac{\gamma_{\Lambda_n}(-|-)}{\gamma_{\Lambda_n}(-|+)} \leq \int \mu^+(d\sigma) f_j({}_j\sigma)\ , \tag{5.23}$$

where $\{\Lambda_n\}$ is a sequence tending to $\mathbb{Z}^d$ in the sense of Fisher, e.g $\{\Lambda_n\}$ is a sequence of increasing cubes with $|\Lambda_n| \to \infty$; j is any lattice point.

Proof : For η fixed the function $\Lambda \mapsto f_{j,\Lambda}(\eta)$ is decreasing in Λ. ¿From (5.13) of Lemma 5.1, if $\Lambda_n \supset \Lambda_m$,

$$\begin{aligned} \lim_{\Lambda_n} \frac{1}{|\Lambda_n|} \log \frac{\gamma_{\Lambda_n}(-|-)}{\gamma_{\Lambda_n}(-|+)} &\leq \lim_{\Lambda_n} \frac{1}{|\Lambda_n|} \sum_{j\in\Lambda_n} \int \gamma_{\Lambda_n}(d\sigma|+) f_{j,\Lambda_n}({}_j\sigma) \\ &\leq \lim_{\Lambda_n} \frac{1}{|\Lambda_n|} \sum_{j\in\Lambda_n} \int \gamma_{\Lambda_n}(d\sigma|+) f_{j,\Lambda_m}({}_j\sigma)\ . \end{aligned} \tag{5.24}$$

The function $f_{j,\Lambda}$ can be decomposed into

$$f_{j,\Lambda}(\eta) = a_{j,\Lambda}(\eta) + b_{j,\Lambda}(\eta) \tag{5.25}$$

with

$$a_{j,\Lambda}(\eta) := \log \frac{\gamma_j(+|\eta_\Lambda^+)}{\gamma_j(-|\eta_\Lambda^+)}\ \ ,\ \ b_{j,\Lambda}(\eta) := \log \frac{\gamma_j(-|\eta_\Lambda^-)}{\gamma_j(+|\eta_\Lambda^-)}\ . \tag{5.26}$$

By (H_4) the function $a_{j,\Lambda}(\cdot)$ is increasing, and the function $b_{j,\Lambda}(\cdot)$ decreasing. Therefore

$$\int \gamma_{\Lambda_n}(d\sigma|+) a_{j,\Lambda_m}({}_j\sigma) \tag{5.27}$$

is decreasing as a function of Λ_n, and

$$\int \gamma_{\Lambda_n}(d\sigma|+) b_{j,\Lambda_m}({}_j\sigma) \tag{5.28}$$

is increasing as a function of Λ_n. Given any $\varepsilon > 0$, we can find a cube Λ_ε containing the origin 0, such that if $j + \Lambda_\varepsilon \subset \Lambda_n$, then

$$|\int \gamma_{\Lambda_\varepsilon}(d\sigma|+)a_{j,\Lambda_m}({}_j\sigma) - \int \mu^+(d\sigma)a_{j,\Lambda_m}({}_j\sigma)| \leq \varepsilon \tag{5.29}$$

$$|\int \gamma_{\Lambda_\varepsilon}(d\sigma|+)b_{j,\Lambda_m}({}_j\sigma) - \int \mu^+(d\sigma)b_{j,\Lambda_m}({}_j\sigma)| \leq \varepsilon \; . \tag{5.30}$$

Using the $\mathbb{Z}^d$–invariance of μ^+ and (5.29) and (5.30),

$$\lim_{\Lambda_n} \frac{1}{|\Lambda_n|} \sum_{j \in \Lambda_n} \int \gamma_{\Lambda_n}(d\sigma|+) f_{j,\Lambda_m}({}_j\sigma) = \int \mu^+(d\sigma) f_{j,\Lambda_m}({}_j\sigma) \; . \tag{5.31}$$

We can take now the limit $\Lambda_m \uparrow \mathbb{Z}^d$. $\square$

Corollary 5.1 *Let $\mathcal{L} = \mathbb{Z}^d$, and Γ be a $\mathbb{Z}^d$–invariant specification satisfying (H_1) and (H_4). If*

$$\lim_{\Lambda_n} \frac{1}{|\Lambda_n|} \log \frac{\gamma_{\Lambda_n}(-|-)}{\gamma_{\Lambda_n}(-|+)} > 0 \; , \tag{5.32}$$

then the specification Γ is discontinuous μ^+–a.s..

Comment: The method of this section can be adapted to prove the Variational Principle (see Georgii (1988) chapter 15) for translation–invariant monotonicity–preserving quasilocal specifications, without using the notion of interaction potential. This is interesting because it is still an open question to know whether such specifications are Gibbs specifications for a *translation–invariant* absolutely summable potential.

6 Main results and examples

6.1 Main results

Let $\mathcal{L} = \mathbb{Z}^d$, $d \geq 2$, and Γ be a $\mathbb{Z}^d$–invariant specification with properties (H_1) to (H_4). Let T be a subgroup of $\mathbb{Z}^d$. By Lemma 5.1 we can define the quantity

$$\zeta_T := \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \log \frac{q^+_{\Lambda_n}(-|-)}{q^+_{\Lambda_n}(-|+)} \geq 0 \; , \tag{6.1}$$

where $\{\Lambda_n\}$ is an increasing sequence of cubes in T such that $|\Lambda_n| \to \infty$. We recall that Ω_q is the set of continuity points of the specification Q^+_T. The restriction of the Γ–compatible measures μ^+ and μ^- to the σ–algebra $\mathcal{F}_T$ are denoted by μ^+_T and μ^-_T. The measure ν^+_T is defined by

$$\nu^+_T(\cdot) = \lim_{\Lambda \uparrow T} q^+_\Lambda(\cdot|-) \; . \tag{6.2}$$

Theorem 6.1 *Let Γ be a $\mathbb{Z}^d$–invariant specification defined on $\mathbb{Z}^d$, verifying conditions (H_1) to (H_4). Let T be a subgroup of $\mathbb{Z}^d$. The following affirmations are true.*

(a) The measure μ^+_T is Q^+_T–compatible and $\mu^+_T = \lim_{\Lambda \uparrow T} q^+_\Lambda(\cdot|+)$.

(b) If $|\mathcal{G}(\Gamma^-_{T^c})| = 1$, then $\mu^-_T(\cdot) = \nu^+_T(\cdot) = \lim_{\Lambda\uparrow T} q^+_\Lambda(\cdot|-)$.

(c) The set Ω_q is an element of the tail-field σ–algebra $\mathcal{T}^\infty_T = \cap_{\Lambda\subset T}\mathcal{F}_{T\setminus\Lambda}$.

(d) If $|\mathcal{G}(\Gamma)| = 1$, i.e. if $\mu^+ = \mu^-$, then Ω_q has μ^+–measure one.

(e) If $\zeta_T > 0$, then Ω_q has μ^+–measure zero.

(f) If $\zeta_T > 0$, then $|\mathcal{G}(\Gamma^\omega_{T^c})| > 1$ μ^+–a.s..

(g) Let f be a bounded local function in T and $\lim_{\Lambda\uparrow T} q^+_j(\sigma|\omega^-_\Lambda) = q^+_j(\sigma|\omega)$ for all $\omega = +$ a.e.. If $\zeta_T > 0$, then any version of $\mathbb{E}_{\mu^+}(f|\mathcal{F}_{T\setminus\Lambda})$, $\Lambda\subset T$, is discontinuous on a set of μ^+–measure one.

(h) ν^+_T is Q^+_T–compatible iff Ω_q has ν^+_T–measure one.

(i) If μ^-_T is Q^+_T–compatible, then Ω_q has μ^-–measure one.

Proof : It remains to prove (d), (h) and (i); we prove them in that order. Let $\Lambda\subset T$; let f be an increasing local function in $T^c\cup\Lambda$. We have

$$\mu^+(f) - \mu^-(f) \;=\; \int \mu^+(d\omega)\gamma^+_{T^c\cup\Lambda}(f|\omega) - \int \mu^-(d\omega)\gamma^-_{T^c\cup\Lambda}(f|\omega)\,, \tag{6.3}$$

which, if $\mu^+ = \mu^-$, yields

$$0 \;=\; \int \mu^+(d\omega)\Big[\gamma^+_{T^c\cup\Lambda}(f|\omega) - \gamma^-_{T^c\cup\Lambda}(f|\omega)\Big]\,. \tag{6.4}$$

Since the square bracket is non-negative (monotonicity) it must be zero μ^+-a.e. We conclude as in the second part of the proof of Proposition 4.1.

The proof of (h) is an immediate consequence of the interesting identity

$$\nu^+_T(f) = \int \nu^+_T(d\omega)\lim_{\Lambda'\uparrow T} q^+_\Lambda(f|\omega^-_{\Lambda'})\,, \tag{6.5}$$

valid for any monotone local function f. To prove it, for instance for f increasing, take the limit $\Lambda'\uparrow T$ throughout the following chain of inequalities. If $\Lambda\subset\Lambda'\subset T$, then

$$\begin{aligned}
\nu^+_T(f) &= \lim_{\Lambda_1\uparrow T} q^+_{\Lambda_1}(f|-) \\
&= \lim_{\Lambda_1\uparrow T}\int q^+_{\Lambda_1}(d\omega|-)q^+_\Lambda(f|\omega) \\
&\geq \lim_{\Lambda_1\uparrow T}\int q^+_{\Lambda_1}(d\omega|-)q^+_\Lambda(f|\omega^-_{\Lambda'}) \\
&= \int \nu^+_T(d\omega)q^+_\Lambda(f|\omega^-_{\Lambda'}) \\
&\geq \int q^+_{\Lambda'}(d\omega|-)q^+_\Lambda(f|\omega^-_{\Lambda'}) \\
&= q^+_{\Lambda'}(f|-)\,.
\end{aligned} \tag{6.6}$$

Finally, we prove (i). Let f be an increasing local function in T; if μ_T^- is Q_T^+–compatible, then for any $\Lambda \subset T$

$$\begin{aligned} \mu_T^-(f) &= \int \mu_T^-(d\omega) q_\Lambda^+(f|\omega) \\ &= \int \mu_T^-(d\omega) q_\Lambda^-(f|\omega) \, . \end{aligned} \tag{6.7}$$

Therefore

$$q_\Lambda^+(f|\omega) = q_\Lambda^-(f|\omega) \quad \mu_T^- - a.s. \, . \tag{6.8}$$

This implies that $\mu_T^-(\Omega_q) = 1$. □

6.2 Ising model

Our basic example is the Ising model on $\mathbb{Z}^d$, $d \geq 2$. Let $\langle i,j \rangle$ denotes a pair of nearest neighbour points i and j in $\mathbb{Z}^d$. For any $\Lambda \subset \mathbb{Z}^d$, we define a function $I_\Lambda(\omega)$ on Ω,

$$I_\Lambda(\omega) := \sum_{\langle i,j \rangle \cap \Lambda \neq \emptyset} X_i(\omega) X_j(\omega) + h \sum_{i \in \Lambda} X_i(\omega) \, . \tag{6.9}$$

We define a specification $\Gamma(\beta)$ on $\mathbb{Z}^d$, $\beta > 0$, by the Boltzmann-Gibbs formula

$$\gamma_\Lambda(\sigma|\eta) := \frac{\exp\left(\beta I_\Lambda(\sigma_\Lambda \eta_{\Lambda^c})\right)}{\sum_{\omega_\Lambda} \exp\left(\beta I_\Lambda(\omega_\Lambda \eta_{\Lambda^c})\right)} \, . \tag{6.10}$$

This specification is $\mathbb{Z}^d$–invariant and satisfies properties (H_1) to (H_5). When $h = 0$ it is also invariant under the symmetry $\omega \mapsto \overline{\omega}$, where $\overline{\omega}(k) := -\omega(k)$. It is well-known that there exists $\beta_c(d)$ such that for any $\beta \leq \beta_c(d)$ there is a unique probability measure which is $\Gamma(\beta)$–compatible, and for any $\beta > \beta_c(d)$ the measures μ^+ and μ^- are different.

We first consider Schomann's example, where $T \cong \mathbb{Z}^{d-1}$ and $h = 0$. We state the following result , Lemma 3.5 in Fröhlich Pfister (1987b), which implies that all points of Theorem 6.1 are true.

Proposition 6.1 *For the d–dimensional Ising model, $d \geq 2$, if $T := \mathbb{Z}^{d-1}$, then $|\mathcal{G}(\Gamma_{T^c}^+)| = 1$ and $|\mathcal{G}(\Gamma_{T^c}^-)| = 1$ for any β.*

As a consequence of Proposition 6.1 and the spin-flip symmetry of the model, we can write

$$\begin{aligned} \zeta_T &= \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \log \frac{q_{\Lambda_n}^+(-|-)}{q_{\Lambda_n}^+(-|+)} \\ &= \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \log \frac{q_{\Lambda_n}^-(-|-)}{q_{\Lambda_n}^+(-|+)} \\ &= \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \log \frac{q_{\Lambda_n}^+(+|+)}{q_{\Lambda_n}^+(-|+)} \, . \end{aligned} \tag{6.11}$$

For $\beta > \beta_c$ large enough one can show directly, and easily, by a perturbative argument that ζ_T is strictly positive. However, using results from Fröhlich Pfister (1987b) we can prove

Corollary 6.1 *Let $d \geq 2$, $h = 0$ and $T \cong \mathbb{Z}^{d-1}$. For the Ising model on $\mathbb{Z}^d$ the specification Q_T^+ is quasilocal μ^+–a.s. if $\beta \leq \beta_c(d)$, and non–quasilocal μ^+–a.s. if $\beta > \beta_c(d)$.*

Proof : By (6.11)

$$\begin{aligned} \zeta_T &= \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \log \frac{q^+_{\Lambda_n}(+|+)}{q^+_{\Lambda_n}(-|+)} \\ &= \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \lim_{\Lambda' \uparrow T^c \cup \Lambda_n} \log \frac{\gamma_{\Lambda'}(+_{\Lambda_n}|+)}{\gamma_{\Lambda'}(-_{\Lambda_n}|+)} \\ &= \lim_{\Lambda_n \uparrow T} \frac{1}{|\Lambda_n|} \lim_{\Lambda' \uparrow T^c \cup \Lambda_n} \log \frac{Z^+_{\Lambda'}(+_{\Lambda_n})}{Z^+_{\Lambda'}(-_{\Lambda_n})} \,. \end{aligned} \tag{6.12}$$

In this last formula $Z^+_{\Lambda'}(+_{\Lambda_n})$ is the partition function of the Ising model in the box Λ', with $+$ boundary condition, and such that all spins at $i \in \Lambda_n$ are equal to $+$. By monotonicity in Λ' and in Λ_n, the limit is also equal to

$$\lim_{\Lambda'' \uparrow \mathbb{Z}^d} \frac{1}{|\Lambda_n|} \log \frac{Z^+_{\Lambda''}(+_{\Lambda_n})}{Z^+_{\Lambda''}(-_{\Lambda_n})} \,. \tag{6.13}$$

Here Λ'' is a cube centered at the origin, such that $\Lambda'' \cap T = \Lambda_n$. ζ_T is two times the surface tension as a consequence of d) p.54 in Fröhlich Pfister (1987b). The surface tension is strictly positive iff $\beta > \beta_c(d)$ (Lebowitz Pfister (1981)). □

Points (e) to (g) of Theorem 6.1 are true $\beta > \beta_c$. Point (f) gives a new result for the Ising model. We can interpret the measures in $\mathcal{G}^{\omega}_{T^c}$ as the equilibrium states of an Ising model defined on $\mathcal{L} = \mathbb{Z}^{d-1} \times \mathbb{Z}^+$ with a random magnetic field acting on $\mathbb{Z}^{d-1} \times \{1\}$. If this random field is chosen according to μ_T^+, then almost surely there is a phase transition.

Remarks :
(i) When $h = 0$ the model is invariant under the symmetry $\omega \mapsto \overline{\omega}$; the set Ω_q of the continuity points of the specification Q_T^+ is also invariant under the same transformation (use conditions $|\mathcal{G}(\Gamma^+_{T^c})| = 1$ and $|\mathcal{G}(\Gamma^-_{T^c})| = 1$). Thus, when $\zeta_T > 0$, this gives an example of a specification where $\lim_{\Lambda \uparrow T} q^+_\Lambda(\cdot|-) = \mu_T^-$ is not compatible with the specification $Q_T^+ = (q^+_\Lambda \, , \, \Lambda \subset T)$.
(ii) In Lörinczi Vande Velde (1993) the authors indicate that one recovers quasilocality everywhere in the case $d = 2$, if one chooses instead of $T = \mathbb{Z}^1$ a subgoup T' of $\mathbb{Z}^1$ with a lattice–spacing large enough.

We consider Griffiths–Pearce–Israel's example for $d \geq 2$. Here T is a d–dimensional subgroup of $\mathbb{Z}^d$. Let us fix β sufficiently large. There exists $h(\beta) > 0$ such that for any h, $0 \leq h \leq h(\beta)$, the measure μ_T^+ cannot be consistent with any local specification on T, which is quasilocal everywhere. In particular the measure μ_T^+ on $(T, \mathcal{F}_T)$ is not a Gibbs measure (van Enter Fernández Sokal (1993)). There are two cases to discuss.

(I) For $h > 0$ there is a single Gibbs measure for the Ising model; point (d) of Theorem 6.1 implies that quasilocality holds μ^+–almost surely. As in remark (ii) above, it has been proved in Martinelli Olivieri (1993) that one recovers quasilocality everywhere if one chooses instead of T a subgroup T' of T of the same dimension, but with a lattice–spacing of order $O(1/h)$.

(II) For $h = 0$ we have $|\mathcal{G}(\Gamma^+_{T^c})| = 1$ and $|\mathcal{G}(\Gamma^-_{T^c})| = 1$ (on T^c we have an Ising model with a magnetic field when $\omega_T = +$ or $\omega_T = -$). A simple estimation shows that $\log \frac{q^+_{\Lambda_n}(-|-)}{q^+_{\Lambda_n}(-|+)}$ is of the order of the length of the boundary of Λ_n when Λ_n is a square; therefore $\zeta_T = 0$. Thus, this example is of different nature than Schonmann's. Whether Ω_q has μ^+–measure one or not is still an open question.

7 Appendix

We prove Lemma 4.9. Let E' be a second copy of E. We consider on $\Theta_\Lambda := E^\Lambda \times E'^\Lambda$ the product measure

$$\rho^1_\Lambda(\sigma_\Lambda, \sigma'_\Lambda) \tag{7.14}$$

which is absolutely continuous with respect to the counting measure, with density

$$\exp(I_\Lambda(\sigma_\Lambda \omega^+_{M\setminus\Lambda})) \exp(I'_\Lambda(\sigma'_\Lambda \omega'^-_{M\setminus\Lambda})) \ . \tag{7.15}$$

We denote by f' a function defined on E'^Λ. Introducing new variables on $E^\Lambda \times E'^\Lambda$,

$$U_i := X_i + X'_i \ \ , \ \ V_i := X_i - X'_i \tag{7.16}$$

we write the logarithm of the density of the measure (7.14) as

$$\begin{aligned} I_\Lambda + I'_\Lambda \ &= \ \frac{1}{2} \sum_{\{i,j\}\subset\Lambda} J(i,j)(U_iU_j + V_iV_j) + \sum_{i\in\Lambda} h(i)U_i \\ &+ \ \sum_{i\in\Lambda} \sum_{j\in M\setminus\Lambda} J(i,j)U_iX_j + \sum_{i\in\Lambda} \sum_{j\in M^c} J(i,j)V_i \ . \end{aligned} \tag{7.17}$$

We compare the measure ρ^1_Λ with the following measure defined on a second copy of Θ_Λ,

$$\rho^2_\Lambda(\eta, \eta') \tag{7.18}$$

which is absolutely continuous with respect to the counting measure with density

$$\exp(I_\Lambda(\eta^+_\Lambda)) \exp(I'_\Lambda(\eta'_\Lambda +^-_{M\setminus\Lambda})) \ . \tag{7.19}$$

The logarithm of the density can be written as

$$\begin{aligned} &\tfrac{1}{2} \sum_{\{i,j\}\subset\Lambda} J(i,j)(U_iU_j + V_iV_j) + \sum_{i\in\Lambda} h(i)U_i \\ &+ \sum_{i\in\Lambda} \sum_{j\in M\setminus\Lambda} J(i,j)U_i + \sum_{i\in\Lambda} \sum_{j\in M^c} J(i,j)V_i \ . \end{aligned} \tag{7.20}$$

On $\Theta_\Lambda \times \Theta_\Lambda$ we consider the product measure $\rho^2_\Lambda \otimes \rho^1_\Lambda$. We may apply (3.41) of Fröhlich Pfister (1987a); for all $k \in \Lambda$

$$\sum_{\eta_\Lambda} \sum_{\eta'_\Lambda} \sum_{\sigma_\Lambda} \sum_{\sigma'_\Lambda} \rho^2_\Lambda(\eta_\Lambda, \eta'_\Lambda) \otimes \rho^1_\Lambda(\sigma_\Lambda, \sigma'_\Lambda)(V_k(\sigma_\Lambda, \sigma'_\Lambda) - V_k(\eta_\Lambda, \eta'_\Lambda)) \geq 0 \ . \tag{7.21}$$

¿From this inequality we get the desired result. □

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