INTRODUCTION TO K.A.M. THEORY*

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1. Introduction

In this lecture, we will discuss the K.A.M. theorem about persistence of quasi-periodic motions under small perturbations of the Hamiltonian.

Rather than presenting a proof of the full theorem, we will give a proof of a caricature that goes half-way in the difficulty and give a sketch of the modifications necessary to obtain the complete result. Very pedagogical proofs of K.A.M. theorem exist [10,14,15,35]. A proof that contains very precise information about differentiability properties is [29].

In our model problem we will study diffeomorphisms of the torus $T^{2d} = \{(q_i, p_i), i = 1, \ldots, d\}$ which preserve the symplectic form $\omega = \sum dp_i \wedge dq_i$. Preserving a symplectic form is one of the key features of Hamiltonian systems, and is what gives the characteristic flavor to most of the problems and techniques of Hamiltonian mechanics.

Take a translation in the torus $E_{\boldsymbol{\alpha}} : T^{2d} \to T^{2d} \quad E_{\boldsymbol{\alpha}}(\mathbf{x}) = \mathbf{x} + \boldsymbol{\alpha} (\mathrm{mod}\, 1)$. Clearly, this preserves the symplectic form ω. Now consider a perturbation of it, that is, a family of symplectic transformations $f_\varepsilon : T^{2d} \to T^{2d}$, such that it is analytic in ε and $f_0 \equiv E_\alpha$.

The question is: When can this perturbation be eliminated by a change of coordinates; in more analytic terms, when can we find another analytic family of invertible symplectic transformations such that

$$f_\varepsilon \circ g_\varepsilon = g_\varepsilon \circ E_{\boldsymbol{\alpha}} \tag{1}$$

The most effective way (even computationally) of making perturbation calculations in mechanics is probably to try to absorb as much of the perturbation as possible in a change of coordinates (see [8] for very practical computations not necessarily in Hamiltonian mechanics).

The advantage of such a method with respect to other more straightforward methods — like trying to make perturbation around each orbit — is that we can obtain global information about the whole phase portrait; we also obtain control — uniform in all initial conditions — of the range of validity of the expansions. This takes care of secular terms which are difficult to interpret, and a computational hazard (e.g., if we try to match powers in the problem $\ddot{x} = -(\omega + \varepsilon)^2 x$; $x(0) = 0$; $\dot{x}(0) = \omega + \varepsilon$ we are led to $x = \sin(\omega t) + \omega \cos(\omega t)\varepsilon t + \omega^2 \sin(\omega t)\frac{(\varepsilon t)^2}{2!} - \omega^3 \cos(\omega t)\frac{(\varepsilon t)^3}{3!}$ which is, of course, $\sin(\omega + \varepsilon)t$. However, if we try to sum the series for large values of εt, we are led to unstable computations. Notice also that the periodicity of the perturbed solution is not apparent in any order of the perturbation theory. Those terms first appeared in the computation of rotation of perihelia and they become appreciable — for Mercury — when t is of the order of centuries, hence the name secular. The classical ways of dealing with secular terms were to resume most of them in recognizable forms (Lindstet, etc.) [28]).

Remark. (Relation to commutator groups) If we rewrite $f_\varepsilon = \hat{f}_\varepsilon \circ E_{\boldsymbol{\alpha}}$, then Eq. (1) becomes

$$\begin{aligned} \hat{f}_\varepsilon E_{\boldsymbol{\alpha}} g_\varepsilon &= g_\varepsilon E_{\boldsymbol{\alpha}} \\ \hat{f}_\varepsilon &= g_\varepsilon E_{\boldsymbol{\alpha}} g_\varepsilon^{-1} E_{\boldsymbol{\alpha}}^{-1} \ . \end{aligned} \tag{2}$$

This means that we rewrite $\hat{f}_\varepsilon$ (close to the identity) as a commutator. The problem of characterizing the group of diffeomorphisms generated by commutators is very important, since it is related to geometric properties of the manifold [13]. So, the study of which diffeomorphisms are commutators has considerable interest.

Even if Eq. (1) and Eq. (2) are equivalent, from the point of view of analysis, Eq. (1) is nicer because it has "group structure," so that it can be studied by hard implicit function theorems. The idea of using this strategy for this problem of commutators appeared in [20].

If our interest were to study the problem of the commutator subgroup, it is worth remarking that the K.A.M. strategy we are going to discuss fails in C^r when $r \neq \omega, \infty$ because we will only be able to conclude that $g \in C^{r-r_0}$ $(r_0 > 0)$, that is, we cannot show that g is in the group of C^r diffeomorphisms. However, there is another strategy of proof which works even in this case [6] (see also [23,24,25]). This suggests that K.A.M. strategy is not always the finest tool — as should be expected from a technique that works for so many problems. It would be quite interesting if the analogy between the two problems could be pushed far enough to give a proof of the K.A.M. theorem not based in K.A.M. strategy.

Remark. Looking for analogues of K.A.M. theorem, it would be more natural, perhaps, to study flows rather than diffeomorphisms. The problem for flows is simpler than for diffeomorphisms and handling the latter is useful in mechanics anyway.

A. Formal considerations

Before stating the theorem that solves the problem, we will pause to develop some formalism; this will help to find some hypothesis that this theorem should have.

We will develop several formalisms in the hope of enabling the reader to understand the relationships between several results in the literature and to serve as a motivation for the last formulation — the one I personally favor. So, it will be quite possible — indeed, even sensible in a first reading — to skip quite a lot of what comes next.

The first difficulty that stands in the way of solving the problem is how to write diffeomorphisms in a way which allows effective computation. In the case of the torus, we have an obvious way: to use coordinates. However, this is a bad solution because there are many manifolds that do not admit such global coordinates; besides in coordinates the symplectic character appears as a set of non-linear non-local constraints. A technique that solves the problem of keeping the symplectic character (but does not help with the problem of requiring a global system of coordinates — it makes it worse) is the use of generating functions; this has the side advantage that rather than dealing with $2d$ functions of $2d$ variables linked by (horrid) constraints, we will only have to deal with one function of $2d$ variables. We will consider the variables as ranging in $\mathbf{R}^{2d}$ and keep track of periodicity conditions.

We observe that $E_{\boldsymbol{\alpha}}$ admits a generating function of the type $(\mathbf{q}, \mathbf{p}')$. (It does not admit one of the form $(\mathbf{q}, \mathbf{q}')$ or $(\mathbf{p}, \mathbf{p}')$ which are the most natural.

$$\mathbf{q}' = \mathbf{q} + \mathbf{a} = \frac{\partial F_0}{\partial \mathbf{p}'} \quad ; \quad \mathbf{p} = \mathbf{p}' - \mathbf{b} = \frac{\partial F_0}{\partial \mathbf{q}} \ .$$

Hence, $F_0 = \mathbf{p}'\mathbf{q} + \mathbf{a}\mathbf{p}' - \mathbf{b}\mathbf{q}$ up to a constant which we will fix as zero without loss of generality. Since the construction of a generating function out of a diffeomorphism is quite explicit, we can conclude that f_ε will have a family of generating functions $F_\varepsilon(\mathbf{q}, \mathbf{p}')$ which will be as smooth both in the coordinates and in ε as f_ε (provided that we choose the constants adequately).

Likewise, g_ε, if it exists at all, will have a family of generating functions $G_\varepsilon(\mathbf{q}, \mathbf{p}')$; $G_0(\mathbf{q}, \mathbf{p}') = \mathbf{p}', \mathbf{q}$. We systematically will write

$$G_\varepsilon = G_0 + \varepsilon G_1 + \varepsilon^2 G_2 + \cdots$$

$$G_\varepsilon^{[\leq N]} = \sum_{k=0}^{N} \varepsilon^k G_k \ .$$

We can write Eq. (1) as an equation for G_ε in terms of F_ε and, then, match powers to obtain a hierarchy of equations of the G_n's in terms of the F_n's. We will only observe that the first equation of the hierarchy is

$$G_1(\mathbf{x} + \boldsymbol{\alpha}) - G_1(\mathbf{x}) = F_1(\mathbf{x}) \tag{3$_1$}$$

and that the other equations of the hierarchy are

$$G_n(\mathbf{x} + \boldsymbol{\alpha}) - G_n(\mathbf{x}) = F_n(\mathbf{x}) + R_n \tag{3$_n$}$$

where R_n is an (awful) expression involving only $G_1, \ldots, G_{n-1}$, $F_1, \ldots, F_{n-1}$ and their derivatives.

¿From the structure of Eq. $(3)_n$, we can proceed inductively, examining recursively the equations, provided that we have a theory for equations of the form

$$G(\mathbf{x} + \boldsymbol{\alpha}) - G(\mathbf{x}) = F(\mathbf{x}) \ .$$

We should notice that since G_ε, F_ε are generating functions of a transformation in the torus, they are obtained integrating a periodic function and, hence, they are the sum of a periodic function and a linear one. We should also remember that they are defined up to constants. These restrictions on the class of functions that can appear are a translation of the geometric content of the problem.

If the linear part of F were non-null, then G would have a quadratic term and, hence, be out of the class we are interested in.

The periodic part can be studied by using Fourier components; we are led to

$$(e^{2\pi i \mathbf{k}\boldsymbol{\alpha}} - 1) G_{\mathbf{k}} = F_{\mathbf{k}}$$

so that there is a solution provided $F_{\mathbf{k}} = 0$ whenever $\mathbf{k}\,\boldsymbol{\alpha} \in \mathbf{Z}$. We will call "resonant" those $\mathbf{k}$'s such that $\mathbf{k}\,\boldsymbol{\alpha} \in \mathbf{Z}$, $\mathbf{k} \neq 0$.

The solution is not unique for the $F_{\mathbf{k}}$'s with $\mathbf{k}\cdot\boldsymbol{\alpha} = 0$. When $\mathbf{k} = 0$ this does not matter, because it is precisely the natural ambiguity of the G's — as generating functions defined up to constants. Moreover it requires some computation to become convinced, but is nevertheless true, that the ambiguities making different choices for constant terms at some stage do not affect the solvability of the equations of higher order in the hierarchy and only affect the solution up to constants.

Some $\boldsymbol{\alpha}$, enjoy the remarkable property that there are no resonant terms. This can be expressed as the components being "linearly independent over the rationals." (For example $\boldsymbol{\alpha} = (1, \sqrt{2}, \sqrt{5})\, \boldsymbol{\alpha}\, \mathbf{k} \in \mathbf{Z}$, $\mathbf{k} \in \mathbf{Z}^3 \Rightarrow \mathbf{k} = 0$.)

Summing up, the necessary conditions for the existence of a formal solution are that $F_n + R_n$ do not have either a linear part or resonant terms for any n. The absence of resonant terms is automatic if $\boldsymbol{\alpha}$ is such that there are no resonances. The absence of linear part depends on $\hat{f}_\varepsilon$ (and, even if mysterious in this formalism, it will turn out to be quite simple when interpreted geometrically).

Discussing convergence, even in the case of rationally independent $\boldsymbol{\alpha}$ seems, at first sight, to be quite hopeless because $G_{\mathbf{k}} = (1/(e^{2\pi i \mathbf{k}\boldsymbol{\alpha}} - 1))F_{\mathbf{k}}$ and the denominators become arbitrarily small, since $\mathbf{k}\cdot\boldsymbol{\alpha}$ will become arbitrarily close to integers as $\mathbf{k}$ ranges over $\mathbf{Z}^{2d}$. Moreover, this is only the linear approximation!.

For the moment, we will postpone all discussions about convergence and proceed to develop other formalisms in perturbation theories. The next method is based on the idea of writing all symplectic diffeomorphisms close to the identity as time one maps of vector fields.

The idea is very analogous to the very successful use of Lie algebras in the study of problems of conjugacy and commutators for finite dimensional Lie groups.

We think of symplectic diffeomorphisms as an infinite-dimensional Lie group (the group operation is composition) then the elements of the Lie algebras (in old fashioned language the "infinitesimal canonical transformations") are the locally Hamiltonian vector fields.

Unfortunately, the relation of infinite dimensional Lie groups and their Lie algebras is much more complicated than for the finite dimensional ones.

For example, if the manifold is not compact, there could be vector fields some of whose trajectories get out of the manifold in a finite time so that those vector fields do not define a flow (i.e., there is no exponential for all the elements of the Lie algebra). This is not a very serious difficulty because for compact manifolds, all vector fields have time one map (i.e., we can define the exponential of the Lie algebra).

Much worse is the fact that, even when the exponential is well defined, it does not need to cover a whole neighborhood of the origin. That is, there are diffeomorphisms as close to the identity as we please which cannot be expressed as time one maps. This shows there is no hope of getting a logarithm (the inverse of the exponential) which works without problems.

Example. We will show that if $f : S^1 \to S^1$ without fixed points is the time one map of a vector field, then the derivatives of f^n are bounded independently of n.

Since arbitrarily close to the identity there are diffeomorphisms with expansive periodic points $f^N(x_0) = x_0$ $|f^{N'}(x_0)| > 1$ we have $f^{nN}(x_0) = (f^{N'}(x_0))^n$ which is arbitrarily large.

In effect, if f without fixed points in the time-one map of the vector field X, X cannot have any zeros. $t_0 = \int_S \frac{d\theta}{|X(\theta)|}$ is finite and, the time-t_0 map is the identity.

So that

$$f^n = T_n = T_{t_0[n/t_0]+r(n)} = T_{r(n)}$$

but $0 \leq r(n) \leq t_0$ so that $T_{r(n)}$ is uniformly bounded in C^1.

Therefore, for the circle, the exponential of the Lie algebra of the group of diffeomorphisms does not cover any C^∞ neighborhood of the identity.

Remark. Something slightly weaker is however true: all diffeomorphisms of class C^r, $r \notin \mathbf{Z}$, $r \neq n+1$ (n dimension of the manifold) can be written as the product of a finite number of time one maps (this number, however depends on the diffeomorphism) of C^r vector fields. This is because of the results of Mather [23,24,25] and Epstein [13], who show that the group of diffeomorphisms homotopic to the identity with this regularity is simple. We observe that the set of all diffeomorphisms that can be written as a finite product of time one maps of C^r vector fields is a group.

Moreover, it is normal (if h is time-one map of the vector field X, $g^{-1}hg$ is the time-one map of the vector field obtain changing coordinates in X) so by the results of Epstein and Mather, it should be the whole component of the identity.

But, even if the machinery of finite dimensional Lie groups does not carry over, it is true that many of the calculations are done in the spirit suggested by it and it can be a very strong guide to know which is the right framework in which to perform analysis (to identity the obstructions and decide which properties should the conjugation elements have) [34,22].

But, if we do not want bona-fide convergence the problems are much easier. Formal power expansions work pretty much like finite dimensional Lie groups.

In particular, we have

Proposition. *Given a family f_ε of smooth diffeomorphisms $f_0 = \mathrm{Id}$, we can find a sequence of vector fields $\mathcal{L}_1, \ldots, \mathcal{L}_n, \ldots$ such that*

$$d\left(f_\varepsilon, \exp\left[\sum_{i=1}^{n} \varepsilon^i \mathcal{L}_i\right]\right) = O(\varepsilon^{n+1}) \ \forall\ n\ .$$

Proof. Suffices to show that we can match enough derivatives of f_ε at $\varepsilon = 0$. A simple computation shows that

$$\left(\frac{d}{d\varepsilon}\right)^k \exp\left[\sum_{i=1}^{n} \varepsilon^i L_i\right]\Big|_{\varepsilon=0} = i!\mathcal{L}_k + \mathcal{R}_k$$

where $\mathcal{R}_k$ depends complicatedly on $\mathcal{L}_1, \ldots, \mathcal{L}_{k-1}$ but *not* on any $\mathcal{L}$ of higher index.

So that recursively we can adjust the $\mathcal{L}_k$'s so that those derivatives agree with those of f_ε. The same result holds true when f_ε preserves the symplectic form and we restrict the $\mathcal{L}_i$'s to be locally Hamiltonian.

Once we have a logarithm, (with some reasonable properties with respect to derivatives, etc. which are easy in this case) there are many pieces of machinery that can be imported. Notably Campbell-Hausdorff formula. (It suffices to walk through the proof of [12] reproduced e.g., in [30].)

$$\exp \mathcal{A} \exp \mathcal{B} = \exp \mathcal{C}$$

where

$$\mathcal{C} = \mathcal{A} + \mathcal{B} + \sum_{n>0} \frac{(-1)^{n+1}}{n(n+1)} \sum \frac{\overbrace{[\mathcal{A},[,\ldots,\mathcal{A},}^{r_1}\overbrace{[\mathcal{B},[\ldots\mathcal{B}_1}^{s_1}\overbrace{[\mathcal{A}_1[\ldots\mathcal{A}\ldots\ldots}^{r_2}\overbrace{[\mathcal{A}\ldots\mathcal{A}}^{r_n},\mathcal{B}]\ldots]}{(s_1+\cdots+s_{n-1}+1)r_1!\ldots r_{n!}s_1!\ldots s_n!}$$

which will hold in the sense of asymptotic expansions when the manifolds are compact.

Taking into account that $E_{\boldsymbol{\alpha}}$ is a canonical transformation, Eq. (2) can be written as

$$\exp\left[\varepsilon\mathcal{F}_1 + \varepsilon^2\mathcal{F}_2 + \cdots\right] = \exp\left[\varepsilon\mathcal{G}_1 + \varepsilon^2\mathcal{G}_2 + \cdots\right]\exp\left[-\varepsilon E^*_{\boldsymbol{\alpha}}\mathcal{G}_1 - \varepsilon^2 E^*_{\alpha}\mathcal{G}_2 \ldots\right] \ .$$

("*" is the usual push forward)

Using the Campbell-Hausdorff formula to express the second term, we obtain matching powers

$$\begin{aligned}\mathcal{F}_1 &= \mathcal{G}_1 - E^*_{\boldsymbol{\alpha}}\mathcal{G}_1 \\ \mathcal{F}_2 &= \mathcal{G}_2 - E^*_{\boldsymbol{\alpha}}\mathcal{G}_2 - \tfrac{1}{2}\left[\mathcal{G}_1, E^*_{\boldsymbol{\alpha}}\mathcal{G}_2\right] \ .\end{aligned}$$

Notice that all the nonlinear terms in the hierarchy are going to be commutators.

Again, we obtain a perturbation theory that has all the features desirable of a perturbation expansion; that is, it can be solved step by set (and renormalizability).

Now, we can investigate a little bit more the obstructions for solution.

Lemma 1. *If $\mathcal{G}$ is a locally Hamiltonian vector field in a torus, then $\mathcal{G} - E^*_{\boldsymbol{\alpha}}\mathcal{G}$ is globally Hamiltonian.*

Lemma 2. *If $\mathcal{G}, \mathcal{H}$ are locally Hamiltonian vector fields $[\mathcal{G}, \mathcal{H}]$ is globally Hamiltonian with Hamiltonian $\omega(\mathcal{G}, \mathcal{H})$.*

Proof. Lemma 1 can be established quickly (we will see another proof later) noticing that $\mathcal{G}$ is globally Hamiltonian in $\mathbf{R}^{2d}$ and the Hamiltonian is periodic and linear. $\mathcal{G} - E^*_{\boldsymbol{\alpha}}\mathcal{G}$ will also have a Hamiltonian in $\mathbf{R}^{2d}$ and, when we compute it, the linear parts drop. Hence, the Hamiltonian in $\mathbf{R}^{2d}$ is periodic and that is equivalent to being globally Hamiltonian.

Lemma 2 is much more important than Lemma 1; it is true for all manifolds and [36] showed the converse, that is, that all globally Hamiltonian vector fields are commutators of locally Hamiltonian ones. This means that we have found the commutator subalgebra of the Lie algebra tangent to the Lie group.

Here we present a coordinate independent proof similar to the one that can be found in [2].

Given a vector field X we call $i_\omega X$ the one form defined by $(i_\omega X)(Y) = \omega(X, Y)Y$.

X is a globally Hamiltonian vector field of Hamiltonian H and can be written as $i_\omega H = dH$ and X being a locally Hamiltonian vector field can be written as $d(i_\omega X) = 0$.

We can express $\omega(Z[X,Y]) = \alpha_z([X,Y]) = X(\alpha_Z(Y)) - Y(\alpha_Z(X))$. Since $d\omega = 0$, we also have, for all vector fields X_1, X_2, Y

$$\begin{aligned}0 &= (d\omega(X_1, X_2, Y) \\ &= Y\omega(X_1 X_2) + X_2\omega(Y, X_1) + X_1\omega(Y, X_2) \\ &\qquad + \omega\big([Y, X_1], X_2\big) + \omega\big([X_2, Y], X_1\big) + \omega\big([X_1 X_2], Y\big) \ .\end{aligned}$$

If $X_1\, X_2$ are locally Hamiltonian, we can substitute the previous identity and derive

$$\begin{aligned}\omega(X_1, X_2) &= \omega\big([X_1 X_2], Y\big) \\ d\big(\omega(X_1, X_2)\big) &= \big([X_1 X_2], Y\big)\end{aligned}$$

so that indeed, $[X_1 X_2]$ is the Hamiltonian vector field associated to the function $\omega(X_1, X_2)$.

A coordinate dependent proof is also quite feasible. The theorem can be reduced to show that in any set of coordinate patches covering the manifold, the commutator and the Hamiltonian flow agree. We can take coordinate patches in such a way that $\omega = \sum_i dp_1 \wedge dq_i$ (Darboux theorem) and we can compute both objects and show that they are the same, using the identities implied by local Hamiltonian property for the derivatives of the components of $\mathcal{G}$ and $\mathcal{H}$.

A s a corollary of the Lemmas we clearly obtain

Proposition. *In order that a solution of Eq. (2) exists, it is necessary that all $\mathcal{F}_i$ be globally Hamiltonian.*

In that case, it suffices to look for $\mathcal{G}_i$ only among globally Hamiltonian vector fields and the equations of the hierarchy are

$$
\begin{aligned}
F_1(\mathbf{x}) &= G_1(\mathbf{x}) - G_1(\mathbf{x} + \boldsymbol{\alpha}) \\
F_2(\mathbf{x}) &= G_2(\mathbf{x}) - G_2(x + \boldsymbol{\alpha}) + R_2(\mathbf{x})
\end{aligned}
$$

where F_i and G_i are the Hamiltonians of $\mathcal{F}_i$ and $\mathcal{G}_i$ and the equalities are up to constants and R_n, as before, can be computed from the previous terms.

So far, we have only been working in the formal power series context and the obstructions for solvability can be quite obscure if left in this form. So, we will have to work a bit more of formalism. This formalism will also become quite useful to establish convergence, because a direct proof estimating the coefficients of ε in the expansion is quite impractical and, in the second case, even false, due to the poor properties of log in infinite dimensional Lie algebras.

The idea is to interpolate all the f_ε by a vector field

$$\frac{d}{d\varepsilon} f_\varepsilon = \mathcal{F}_\varepsilon \circ f_\varepsilon$$

and similarly for g_ε and use the vector fields to express the conjugacy equation (1). As is well known, the $\mathcal{F}_\varepsilon$ will be locally Hamiltonian vector fields.

If we take derivatives with respect to ε in Eq. (1), we obtain

$$\mathcal{F}_\varepsilon(f_\varepsilon \circ g_\varepsilon) + f_\varepsilon^* {}_\varepsilon(f_\varepsilon g_\varepsilon) = \mathcal{G}_\varepsilon(g_\varepsilon E_\varepsilon) \ . \tag{4}$$

And using Eq. (1) again

$$\mathcal{F}_\varepsilon = \mathcal{G}_\varepsilon - f_\varepsilon^* \mathcal{G}_\varepsilon \ . \tag{5}$$

This implies that F_ε should not only be locally Hamiltonian, but also globally Hamiltonian. In some particular cases, we had already established that in Lemma 1. A more geometric proof is as follows. Globally Hamiltonian means $[i_\omega(\mathcal{F}_\varepsilon)] = 0$ (that is, the result of identifying the vector field with a one-form produces an exact differential -cohomology zero-). On the other hand, we just observe that f_ε preserves the symplectic form, so that we can write

$$[i_\omega f_\varepsilon^* \mathcal{G}_\varepsilon] = [f_\varepsilon^* i_\omega \mathcal{G}_\varepsilon] = f_{\varepsilon\#}[i_\omega \mathcal{G}_\varepsilon] \ .$$

Moreover, since f_ε is connected to the identity, its action on cohomologies is the identity Hence,

$$[i_\omega \mathcal{F}_\varepsilon] = [i_\omega \mathcal{G}_\varepsilon] - f_{\varepsilon\#}[i_\omega \mathcal{G}_\varepsilon] = 0 \ .$$

The previous calculation make it reasonable to introduce the following definition.

Definition. We can say that f_ε is a globally Hamiltonian isotopy (G.H.I.) when

$$\frac{d}{d\varepsilon} f_\varepsilon = \mathcal{F}_\varepsilon \circ f_\varepsilon$$

and $\mathcal{F}_\varepsilon$ is a globally Hamiltonian vector field. That is, we obtain f_ε as a time one map of a *globally* Hamiltonian vector field which also depends on time.

Clearly, a globally Hamiltonian isotopy is determined by the initial point and F_ε the Hamiltonian function of $\mathcal{F}_\varepsilon$. Conversely, a G.H.I. determines uniquely $\mathcal{F}_\varepsilon$ and this determines F_ε up to a constant, which we will fix by requiring $\int F_\varepsilon = 0$. We will call F_ε the Hamiltonian of f_ε.

As we will discuss at the end, the G.H.I.'s enjoy quite remarkable properties. For the moment, we will only work out some formal properties.

Notice that G.H.I.'s allow to express transformations by a simple function; in that respect, they are similar to generating functions, but, however, they are natural geometrical objects and, so, they can be defined in all manifolds; besides, the equations we will have to solve are simpler.

Lemma. *Let f_ε, g_ε be G.H.I. of Hamiltonians F_ε, G_ε, then $f_\varepsilon \circ g_\varepsilon$ has Hamiltonian $F_\varepsilon + G_\varepsilon(f_\varepsilon^{-1})$.*

The proof is very simple proceeding along the same lines as in the calculations of Eq. (5). We just have to observe that, since f_ε is a symplectic transformation, we can just transform the Hamiltonian.

Notice that, in particular, this Lemma shows that all the diffeomorphisms connected to the identity by a G.H.I. form a group.

If we decide to search for the g_ε among G.H.I., Eq. (5), — which is a necessary condition for Eq. (1) — becomes

$$F_\varepsilon = G_\varepsilon - G_\varepsilon(f_\varepsilon^{-1}) \ . \tag{6}$$

However, it turns out that, in spite of the fact that we used Eq. (1) twice to derive Eq. (5), and, therefore could fail to the sufficient on Eq. (6) is equivalent to Eq. (1).

If effect, calling

$$\Delta_\varepsilon = f_\varepsilon \circ g_\varepsilon \circ f_0^{-1} \circ g_\varepsilon$$

we can show that provided Eq. (6) holds

$$\frac{d}{d\varepsilon}\Delta_\varepsilon = 0 \quad \text{when} \quad \Delta_\varepsilon = \mathrm{Id} \ .$$

Since clearly $\Delta_0 = \mathrm{Id}$ and everything is so smooth that uniqueness of O.D.E. holds, we have $\Delta_\varepsilon = \mathrm{Id}$.

B. Convergence

We now tackle the problem of actually constructing the solutions. More precisely, we want to prove the following.

Theorem. *Let f_ε be an analytic globally Hamiltonian isotopy $f_0 = E_{\boldsymbol\alpha}$, where $\boldsymbol\alpha$ satisfies*

$$|(\boldsymbol\alpha, \mathbf{k}) - n|^{-1} \le \exp \frac{A|\mathbf{k}|}{\log|\mathbf{k}| \cdot \log\log|\mathbf{k}| \times \cdots \times \underbrace{\log\ldots\log}_{\ell \text{ times}}{}^{1+\nu}|\mathbf{k}|} + B$$

$$n \in \mathbf{Z} \quad , \quad \mathbf{k} \in \mathbf{Z}^{2d}$$

for some $A, B, \nu > 0$ ($|\mathbf{k}|$ means $\sum_i |k_i|$).

Then, for sufficiently small ε, there is an analytic globally Hamiltonian isotopy g_ε satisfying Eq. (1).

For convenience, we will call

$$\exp \frac{A|\mathbf{k}|}{\log|\mathbf{k}| \cdot \log\log|\mathbf{k}| \times \cdots \log\cdots\log^{1+\nu}|\mathbf{k}|} + B \equiv \Omega(|\mathbf{k}|)$$

following a notation due to Rüssmann. We will discuss later how abundant are the $\boldsymbol\alpha$ for which these conditions are satisfied.

Remark. We will disregard what happens for $|\mathbf{k}|$ so small that one of the iterated log does not make sense. What is going to be relevant for the proof is that this inequality holds from a certain $|\mathbf{k}|$ on. If we were systematic about using that, we could do with the B, but it takes a bit of effort — a good exercise to check it — that the $|\mathbf{k}|$ big enough that we take can be uniform in all steps, so we still keep the B.

Given an analytic function in the torus, we will denote by

$$\|F\|_r = \sup_{|\operatorname{Im} z_i| \le r} |F(\mathbf{z})|$$

and given an analytic family of analytic functions we will denote by

$$|\!|\!| F_\varepsilon |\!|\!|_f = \sup_{|\varepsilon| \in [0,1]} \|F_\varepsilon\|_r \ .$$

More precisely, what we are going to show is that, given r, u, there is some $S^*(r, u, \boldsymbol\alpha)$ such that when $|\!|\!| F_\varepsilon |\!|\!|_r \le S^*$ the conclusions of the theorem hold, when G_ε analytic in a slightly smaller domain. We remark that the loss of analyticity domains can be made as small as we please by imposing smallness conditions in $|\!|\!| F_\varepsilon |\!|\!|_r$. It is also important to remark that $\boldsymbol\alpha$ only enters in S^* through Ω so that the proof holds at the same time for all the $\boldsymbol\alpha$ with the same Ω.

Even if F_ε was defined in a smaller neighborhood in the variable, we can scale ε so that it is defined for $\varepsilon \in [0,1]$. Moreover, by scaling enough, we can adjust any smallness conditions in $|\!|\!| F_\varepsilon |\!|\!|$. So that the theorem can as well be enunciated for sufficiently small

(1) can be written as

$$g_\varepsilon^{-1} \circ f_\varepsilon \circ g_\varepsilon = E_\alpha$$

and, if we apply repeatedly the formula for the Hamiltonian of a composition of G.H.I., we see that imposing that the Hamiltonian of the left hand side be equal to zero we obtain

$$F_\varepsilon \circ g_\varepsilon - G_\varepsilon \circ g_\varepsilon + G_\varepsilon \circ f_\varepsilon^{-1} \circ g_\varepsilon = 0 \tag{7}$$

(which, of course, is to hold in some appropriate complex extension of the torus).

Even if a direct solution of Eq. (7) is beyond hope, we can observe that there is a similar equation we can solve using Fourier coefficients.

$$F_\varepsilon - G_\varepsilon^0 + G_\varepsilon^0 \circ E_{\boldsymbol{\alpha}}^{-1} = 0 \; . \qquad (8)_0$$

Then, G_ε^0 will be commensurate, in some sense, to F_ε. Then, $g_\varepsilon^0 - \mathrm{Id}$ will also be controlled by F_ε and, finally, $E_{\boldsymbol{\alpha}}^{-1} - f_\varepsilon^{-1}$ will also be controlled by F_ε.

If we substitute all this in the left hand side of Eq. (7), we obtain something which, even if not exactly zero, is "quadratic in F_ε."

The crucial fact is that Eq. (7) has some structure (usually called "group structure" [38]) that allows us to take advantage of approximate solutions.

In effect, if we could solve Eq. (7) with $(g_\varepsilon^0)^{-1} \circ f_\varepsilon \circ g_\varepsilon^0 = f^1$ in place of f_ε, we could solve the original problem. But, as we argued before, $(g_\varepsilon^0)^{-1} f_\varepsilon g_\varepsilon^0$ is much more favorable.

Of course, we still do not know how to solve the exact equation, so what we will do is to solve Eq. $(8)_1$ again and find another approximation g_ε^1 and keep repeating the process, hoping that the remainders will go to zero; $g = g_\varepsilon^0 \circ g_\varepsilon^1 \circ g_\varepsilon^2 \cdots$ will, then, be a solution of the problem.

The details of the proof of how this "quadratic convergence" is enough to beat the small denominators are quite fascinating and are the basis of a whole theory useful for many problems [38].

A very suggestive way of thinking about this technique is suggested in [14]. We could think of each one of our inductive steps as recognizing that a substantial part of the perturbation expansion can be absorbed by a change in the Hamiltonian, which is then closer to "free." This is quite analogous to the perturbative application of renormalization group and, indeed, this analogy can be exploited to obtain results in statistical mechanics of quantum field theory problems by methods reminiscent of K.A.M.. The theories involved have much more structure, their treatment has to include more technique. See [17] for a formulation of renormalization group close to this idea and for references about other points of view or results in this field.

The next four Lemmas will be a systematic control of the inductive step; then, we will show we can iterate indefinitely and that the iteration converges.

Lemma. *Assume F is analytic in T^{2d} $\int F = 0$ and $\boldsymbol{\alpha}$ as before. Then, there is a unique G, analytic, $\int G = 0$, such that*

$$F(\mathbf{x}) = G(\mathbf{x}) - G(\mathbf{x} + \boldsymbol{\alpha}) \; .$$

Moreover,

$$\|G\|_{r-u} \le e^{B+3n(u)} \|F\|_r$$

r, u sufficiently small where the smallness depends only on Ω ($n(u)$ to be defined later, is an explicit function, involving only Ω).

To avoid cluttering the proof with different constants, we will denote by C several constants which only depend on $\boldsymbol{\alpha}$ and are independent of r, F, and all the other parameters of the proof.

Proof. The equation can be solved equating Fourier coefficients

$$\begin{aligned} G_{\mathbf{k}} &= (e^{2\pi i(\mathbf{k},\boldsymbol{\alpha})} - 1)^{-1} F_k \qquad \mathbf{k} \neq 0 \\ G_{\mathbf{0}} &= 0 \ . \end{aligned}$$

Now, we have

i) $$|e^{2\pi i(\mathbf{k},\boldsymbol{\alpha})} - 1|^{-1} \leq \Omega(|\mathbf{k}|)$$

because

$$|e^{2\pi i x} - 1|^{-1} \leq C d(x, \mathbf{Z})^{-1}$$

($x = \frac{1}{2}$ gives the worst constant which is $C = 1$) also, because of Cauchy inequalities,

ii) $$|F_{\mathbf{k}}| \leq e^{-2\pi r|\mathbf{k}|} \|F\|_r$$

and the triangle inequality gives

iii) $$\|G\|_{r-u} \leq \sum |G_{\mathbf{k}}| e^{2\pi(r-u)|\mathbf{k}|} \ .$$

Hence,

$$\begin{aligned} \|G\|_{r-u} &\leq \|F\|_r \sum e^{-2\pi u|k|}(|k|) \\ &\leq \|F\|_r C \sum_{n\in\mathbf{N}} e^{-2\pi u n} \Omega(n) n^{2d-1} \end{aligned}$$

where C is the area of the unit sphere with respect to $|\_|$ in 2^d dimensions.

$$\sum_{n\in\mathbf{N}} e^{-2\pi u n} n^{2d-1} \Omega(n) \leq \left(\sum_{n\in\mathbf{N}} e^{-\pi u n} \right) \left(\max_{n\in\mathbf{N}} e^{-\pi u n} \Omega(n) n^{2d-1} \right) \ .$$

The sum can be done explicitly and bounded by Cu^{-1}. The maximum will be reached at the point that the log reaches the maximum and this can be computed taking derivatives

$$\begin{aligned} 0 &= \frac{d}{dn} \left[-\pi u n + B + \frac{An}{\log n \log\log n \ldots \log \ldots \log^{1+\nu} n} + (2d-1)\log n \right] \\ &= -\pi u + \frac{A}{\log n \log\log n \ldots \log \ldots \log^{1+\nu} n} - \frac{A}{\log^2 n \log\log^2 n \ldots \log \ldots \log^{1+\nu} n} \\ &\quad - \frac{A}{\log^2 n \log\log^2 n \ldots \log \ldots \log^{1+\nu} n} \\ &\quad - \frac{A(1+\nu)}{\log^2 n \log\log^2 n \ldots \log^2 \ldots \log^{(1-\nu)2} n} + \frac{2d-1}{n} \\ &= -u + \frac{A}{\log n \ldots \log \ldots \log^{1+\nu} n} \left[1 - \frac{1}{\log n} - \frac{1}{\log\log n} - \cdots - \frac{1+\nu}{\log \ldots \log^{1+\nu} n} \right] \\ &\quad + \frac{2d-1}{n} \ . \end{aligned}$$

So, we see that for u sufficiently small we have just one solution $n(u)$; the exact form of $n(u)$ will not be terribly important for the moment, we only have to observe that $n(u) \to \infty$ as $u \to 0$ $|\ell n(u)| = o(n(u))$. The maximum of the exponent will then be bounded by

$$\begin{aligned} &- \pi u n(u) + B + \frac{An(u)}{\log n(u) \ldots \log \ldots \log^{1+\nu} n(u)} \\ &= B + n(u) \left[\frac{A}{\log n(u) \ldots \log \ldots \log^{1+\nu} n(u)} - u \right] \ . \end{aligned}$$

And using the equation satisfied by $n(u)$

$$= B + n(u)\frac{A}{\log n(u)\ldots\log\ldots\log^{1+\nu} n(u)}\Big(\frac{1}{\log n(u)} + \frac{1}{\log\log n(u)} + \cdots$$
$$+ \frac{1+}{\log\ldots\log^{1+\nu} n(u)} \le B + 2n(u)$$

provided that $n(u)$ is large enough, that is, u small enough. We also observe that $e^{n(u)} \gg u^{-1}$ as $u \to 0$ so that the u^{-1} factor can be absorbed in adding a small coefficient of $e^{n(u)}$. Since everything was uniform with respect to extra parameters, the same result with the same bounds is true for families.

Lemma.

$$\|F_\varepsilon - F_\varepsilon \circ g_\varepsilon\|_{r-u} \le Cu^{-2}\|F\|_r\|G_\varepsilon\|_r$$

provided

$$\|G_\varepsilon\|_r \le 3u^2C \ .$$

Proof. Using analyticity estimates, we have that $\|\nabla G_\varepsilon\|_{r-u/2} \le Cu^{-1}\|G_\varepsilon\|_r$. Since g_ε is obtained solving Hamilton's equations

$$\|\mathbf{x} - g_\varepsilon(\mathbf{x})\|_{r-u} \le \|\nabla G_\varepsilon\|_{r-u/2} \le Cu^{-1}\|G_\varepsilon\|_r$$

where the first inequality assumes that $g_\varepsilon(\mathbf{x})$ is always contained in $\{|\operatorname{Im}(\mathbf{z})| \le r - u/2\}$ whenever $|\operatorname{Im}(\mathbf{x})| \le r$. But this is true because of the smallness assumption on $\|G_\varepsilon\|_r$ we included in the hypothesis of the lemma, which make the last bound $\le u/3$. (We showed that provided $g_\varepsilon(x)$ is contained in $\{\operatorname{Im}(z) \le r - u/2\}$ it can not leave $\{\operatorname{Im}(\mathbf{z}) \le r - u + u/3\}$.)

We also have

$$\|F_\varepsilon - F_\varepsilon \circ g_\varepsilon\|_{r-u} \le \|\nabla F_\varepsilon\|_{r-u/2}\|\mathbf{x} - g_\varepsilon(\mathbf{x})\|_{r-u}$$

and we can bound the first factor by analyticity estimates.

Applying the previous lemma twice and the formula of composition, we obtain

Lemma.

$$\|H_\varepsilon(f_0^{-1}) - H_\varepsilon(f_\varepsilon^{-1} \circ g_\varepsilon)\|_{r-u} \le Cu^{-4}\|H_\varepsilon\|_r\big(\|F_\varepsilon\|_r + \|G_\varepsilon\|_r\big)$$

provided

$$\|F_\varepsilon\|_r \le u^2C$$
$$\|G_\varepsilon\|_r \le u^2C \ .$$

Now, we can bound the left hand side of Eq. (7) provided that G_ε is obtained from F_ε by solving Eq. (8) (we will assume that the conditions to apply the lemma hold).

$$\begin{aligned}
&\|F_\varepsilon g_\varepsilon - G_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon^{-1} g_\varepsilon\|_{r-u} \\
&\quad = \|F_\varepsilon g_\varepsilon - G_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon^{-1} g_\varepsilon - F_\varepsilon + G_\varepsilon - G_\varepsilon f^{-1}\|_{r-1} \\
&\quad \le \|F_\varepsilon \circ g_\varepsilon - F_\varepsilon\|_{r-u} + \|G_\varepsilon \circ g_\varepsilon - G_\varepsilon\|_{r-u} + \|G_\varepsilon \circ f_\varepsilon^{-1} \circ g_\varepsilon - G_\varepsilon \circ f^{-1}\|_{r-u} \le
\end{aligned}$$

Since we want, later, to bound G in terms of F, we will use only bounds of G in a slightly smaller domain. This amounts to taking $u/2$ in the previous lemma and, therefore, to multiplying some constants by 2^4

$$\leq Cu^{-2}\|F_\varepsilon\|_r\|G_\varepsilon\|_{r-u/2} + Cu^{-2}\|G_\varepsilon\|_{r-u/2} + Cu^{-4}\|G_\varepsilon\|_{r-u/2} \quad \left(\|F_\varepsilon\|_r + \|G_\varepsilon\|_{r-u/2}\right) .$$

Now (being incredibly wasteful), we bound all this by

$$Ce^{B+4n(u/2)}\|F_\varepsilon\|_r^2 .$$

We have to assume $e^{n(u/2)} > u^{-4}u^{-2}$, which is certainly true for sufficiently small u.

Now, we have to show that this step can be iterated. (The only things to check are the smallness conditions we assumed for $\|G\|, \|F\|$) and that the iterates converge. Since each of the steps have a choice for u, we will have to show that there is a choice for the u's at each step so that indefinite iteration and convergence holds.

So, we pick once and for all a fixed sequence (u_k) in such a way that $n(u_k/2) = n_0(3/2)^k$ (n_0 to be adjusted later).

Notice that these choices imply (using the equation for $n(u)$) that

$$\sum_k u_k = \sum_k \frac{2}{\pi}\, \frac{A}{\log n_0(3/2)^k \ldots \log \ldots \log^{1+\nu} n_0(3/2)^k}$$
$$\left(1 - \frac{1}{\log n_0(3/2)^k} - \cdots - \frac{1+}{\log \ldots \log^{1+\nu} n_0(3/2)^k}\right)$$

which converges by comparison with

$$\sum \frac{1}{k \log k \ldots \log \ldots \log^{1+\nu} k}$$

which converges by the integral test. (The factor in brackets clearly tends to 1 when $k \to \infty$.)

Moreover, this sum is arbitrarily small provided n_0 is sufficiently large. Since $r - \sum u_k$ is going to be the measure of analyticity domain of G, we see that, if we can pick n_0 large, the analyticity loss will be a small factor of the original domain.

Calling $S_n = \|F_\varepsilon\|_{r_n}$ after performing n inductive steps, we have shown that

$$S_{k+1} \leq e^{B+3n\left(\frac{uk}{2}\right)} S_k^2 \equiv e^B e^{3n_0} \left(\left(\frac{3}{2}\right)^k\right) S_k^2$$

(assuming that the conditions to perform a step hold) so that

$$\begin{aligned} S_{k+1} &\leq e^{B+3n_0(3/2)^k} e^{2B+2n_0(3/2)^{k-1}} S_{k-1}^{2^2} \\ &\leq e^{B(1+2+\cdots+2^{k+1})} e^{2n_0((3/2)^k}(1 + 2\left(\frac{3}{2}\right)^{k-1} + \cdots + 2^k) S_0^{2^{k+1}} . \end{aligned}$$

Now,

$$\begin{aligned} 1 + 2 + \cdots + 2^{k+1} &\leq 2^{k+1}\left(\frac{3}{2}\right)^k + 2\left(\frac{3}{2}\right)^k + \cdots + 2^{k+1} \\ &= 2^k \sum_{\ell=0}^{k} \left(\frac{3}{2}\right)^{k-\ell}\left(\frac{1}{2}\right)^{\ell-1} \\ &\leq 2^k \sum_{\ell=0}^{\infty}\left(\frac{3}{4}\right)^{\ell} \leq C2^{k+1} \end{aligned}$$

so that

$$S_{k+1} \le \left(S_0 e^{B+Cn_0}\right)^{2^{k+1}} .$$

Reading the formula for u_k we obtain

$$u_k \le \frac{C(\nu)}{(\log(n_0) + (\log \frac{3}{2})k)^2}$$

where C depends only on ν.

The conditions to iterate that we found for G are implied then by

$$e^{-n(u_k/2)} S_k \le C_k^2$$

which are, in turn, implied by

$$e^{-(3/2)^k} \left(S_0 e^{B+Cn_0}\right)^{2k+1} \le \frac{C(\nu)}{[\log(n_0) + \log(3/2)k]^2} .$$

We can see that, given a certain n_0, we can find S_0 small enough so that the conditions hold for all k. Therefore, if they hold for the first step, the iteration keeps them valid.

Analogous, even easier, considerations hold for the other condition of the inductive step.

So, the inductive step can be iterated indefinitely. The remainder converges to zero and n_0 can be taken as big as we want provided S_0 is small enough.

Convergence of $g_\varepsilon^0 \circ g_\varepsilon^1 \cdots \circ g_\varepsilon^k$ as $k \to \infty$ in the domain given by $r - \sum u_k$ can now be very easily established. At each step of the iteration, the same bounds we proved for the inductive hypothesis show that all the compositions make sense. We also have very good bounds on G_ε^k. Using them, the formula for the composition of G.H.I. and proceeding as in Lemma 6, we are done.

C. About the nonresonance conditions

The idea of introducing a function Ω to measure the non-resonance and try to determine the weakest conditions can be attributed to Russman [31].

Rather than calculate at each step of the process as he did, he left all the calculations for the end. The conditions he found are

$$\sum \frac{\Omega(2^n)}{2^{n+1}} < \infty$$

(plus monotonicity and logarithmic convexity).

The same conditions were derived by other methods by Brjuno [9]. These conditions are weaker than those of the proof we presented. For the proof we presented here, the left hand side of the inequality above becomes

$$\sum \frac{1}{n(\log n)(\log\log n)\ldots(\log\ldots\log^{1+\nu} n)}$$

which is not, of course, the slowest convergent series, but is probably the slowest which is comfortable to write.

In the next paragraph we will present some examples of $\boldsymbol{\alpha}$'s failing to satisfy the condition and for which the theorem is false.

It is interesting to remark that there is a no man's land of numbers for which it is not known either how to prove the theorem or how to construct counterexamples. Certainly this no man's land is small in any sensible sense of the word "small" for uncountable sets. (It has zero measure, the complement is residual, has zero Hausdorff dimension, etc..) But nevertheless, its existence is vexing (it is quite similar to ruling out the existence of singular continuous spectrum whose presence prevents having a well behaved scattering theory). For a similar model problem, studying the conjugacion of analytic maps to a rotation in a vicinity of a fixed point with derivative of unit modulus, there are examples [39] that show that for a number failing to satisfy the contitions used in [9] to prove linearization, the conclusions of the theorem are false. See also [40] for some extensions of the methods.

What we will do now is to study the set of $\boldsymbol{\alpha}$'s for which the conditions

$$\Omega(|k|) \le |\,\boldsymbol{\alpha}\,\mathbf{k}|^{-1}$$

are satisfied.

We want to show it is a big set (it has positive measure, as close to full as we want by taking $\Omega \to \lambda\Omega$, λ big) and also some geometric properties that will be useful in the proof of the full K.A.M. theorem.

We will introduce the set of "almost resonances of order $\mathbf{k}$"

$$R_{\mathbf{k}} = \left\{ \boldsymbol{\alpha} \;\middle|\; |\,\boldsymbol{\alpha}\,\mathbf{k}| \le \frac{1}{\Omega(|k|)} \right\} .$$

Each one of those is a strip bounded by two planes perpendicular to $\mathbf{k}$ and separated by a distance

$$\frac{2}{\Omega(|k|)(\sum^d k_i^2)^{1/2}} .$$

The set in which the non-resonance conditions hold is the complement of $\bigcup_{\mathbf{k}\in\mathbf{Z}^d} R_{\mathbf{k}}$.

If we restrict ourselves to a ball in $\boldsymbol{\alpha}$ space,

$$\begin{aligned}\mathrm{Vol}(R_{\mathbf{k}}) &\le \mathrm{diam(Ball)}\frac{2}{\Omega(|k|)(\sum_{i=1}^d k_i^2)^{1/2}} \\ &\le \frac{\mathrm{diam(Ball)}2\sqrt{d}}{\Omega(|k|)|\mathbf{k}|} .\end{aligned}$$

(By Cauchy Schwartz we have $|\mathbf{k}| \le \sqrt{d}(\sum_{i=1}^d |k_i|^2)^{1/2}$). We then see that

$$\mathrm{Vol}\left(\bigcup_{\mathbf{k}} R_{\mathbf{k}}\right) \le \sum_{\mathbf{k}\in\mathbf{Z}^d} \frac{\mathrm{diam(Ball)}2\sqrt{d}}{\Omega(|k|)|k|}$$

which, for our choice of $\Omega(|k|)$ certainly is finite. Moreover, by choosing big enough constants we can make this measure of the excluded set as small as we wish.

But remember that the proof could go through for arbitrarily large constants in Ω provided that we took small enough perturbations.

This is one of the key ingredients in the proof that, for the full K.A.M. theorem, the proportion of phase space occupied by invariant tori is close to one, if the perturbation is sufficiently small.

The actual proof is slightly more complicated and involves other slightly stronger statements about how big the set is; something more is necessary to control the changes introduced by the partial conjugacies. The "bad" set is either those of resonances or the things that can be moved into resonances by a movement of the size of the perturbations at the corresponding stage of the iterative process.

We call

$$\widetilde{R}_{\mathbf{k},u} = \left\{\boldsymbol{\alpha} \mid \operatorname{dist}(\boldsymbol{\alpha}, R_k) \le u\right\} .$$

In the ℓ^{th} step of the proof the "bad" set will turn out to be

$$\bigcup_{|\mathbf{k}| \le n(u_\ell)} \widetilde{R}_{\mathbf{k},\varepsilon_\ell} \quad \text{where} \quad \left(\varepsilon_\ell = \|F^\ell\|_{r_\ell}\right) .$$

These sets still have small measure when restricted to a ball because $\widetilde{R}_{\mathbf{k},u_\ell}$ is still bounded by two planes perpendicular to $\mathbf{k}$ and separated by a distance bounded above by $\left(\frac{2\sqrt{d}}{\Omega(|k|)|\mathbf{k}|} + 2\varepsilon_\ell\right)$ and the sum of the measure of the bad sets is the same as before plus

$$\sum_\ell \sum_{|j| \le |k| \le n_{\ell+1}} u_\ell \le C \sum_\ell n_{\ell+1}^{d-1} \varepsilon_\ell$$

which also converges and is as small as we wish provided ε_0 is small.

Notice that, given a Cantor set even of large measure, it is by no means obvious that we can enlarge the complement at each of the stages of the construction and still obtain something non-empty.

D. A counterexample

Here we construct a G.H.I. f_ε in the torus T^2 that satisfies all the conditions of the previous theorem except for non-resonance and, such that it is not topologically conjugate for any ε to f_0.

We can find an $a \in \mathbf{R}$ such that $\limsup_{k\to\infty} \exp(-Ak) \min_{n\in\mathbf{Z}} |ak - n|^{-1} \to \infty$. An explicit construction of such an a can be achieved by, e.g., taking a number with a decimal expression $0.1\overbrace{0\ldots0}^{n_1}\ldots01\overbrace{0\ldots0}^{n_2}1\ldots$ we have

$$[10^{n_1+\cdots+n_j+j}a - 1]^{-1} = 10^{(n_1+\cdots+n_j - n_{j+1}+j+1)}$$

we can recursively choose the n_{j+1} so that the limit of the resonances along this sequence is ∞. A more elegant way could be using continued fractions.

Notice that the construction can be modified to accommodate any arbitrary finite sequence of numbers before starting with the 0's and 1's. So, these counterexamples are dense (although we showed before that they are of zero Lebesgue measure).

We will be considering families

$$f_\varepsilon = \big(x + a, y + b + \varepsilon\psi(x)\big)$$

which are G.H.I.'s whenever $\int_0^1 \psi(x)\,dx = 0$. b will not play any role in what follows and we can choose it in any way we like. In order to get a counterexample as strong as possible we will choose it in such a way that the resonances of the pair (a, b) are as weak as possible. (It turns out that we can make them as weak as those of a.)

We will call $\mathcal{H}_A$ the set of functions analytic in S^1 integrating to zero, and with finite norm $\|\psi\|_A = \sup_{k\in\mathbf{Z}^2} e^{kA}|\hat{\psi}_k|$. ($\hat{\psi}_k$ is the k^{th} Fourier coefficient)

Proposition. *There is a residual set in $\mathcal{H}_A$ such that when ψ belongs to this set f_ε is not topologically conjugate to f_0 for any ε.*

Proof. We will just establish disconjugacy of $(x+\alpha, y+\beta+\psi(x))$ to $(x+\alpha, y+\beta)$ in a residual set which will also be invariant by multiplication by scalars.

We observe that the set $\{f_0^k\}$ is equicontinuous and, if f_1 were topologically conjugate to f_0 so would be $\{f_1^k\}$. We show that this is not the case.

$$f_1^k(x,y) = \left(x + k\alpha, y + k\beta + \sum_{\ell=1}^{k} \psi(x+\ell\alpha)\right)$$

and it suffices to show that $\{C_k\psi(x) \equiv \sum_{\ell=1}^k \psi(x+\ell\alpha)\}$ are not equicontinuous for a residual set of ψ.

To establish the theorem it suffices to show that for the operators $\mathcal{C}_k$, $\|\mathcal{C}_k\|_{\mathcal{H}_A \to C^0}$ is unbounded, and invoke Baire Category theorem. (See e.g., [31], p.44.)

We have

$$\begin{aligned} C_k e^{2\pi i n x} &= e^{2\pi i n x} \sum_{\ell=0}^{k-1} e^{2\pi i \ell n \alpha} \\ &= e^{2\pi i n x} \frac{3^{2\pi i k n} - 1}{e^{2\pi i n \alpha} - 1} \\ \sup_k \|\mathcal{C}_k\|_{\mathcal{H}_A \to C^0} &\geq \sup_k \sup_n e^{-An} \frac{|e^{2\pi i k n} - 1|}{|e^{2\pi i n} - 1|} \\ &= \sup_n \frac{e^{-An}}{|e^{2\pi i n \alpha} - 1|} \sup_k |e^{2\pi i k n \alpha} - 1| \\ &= \infty\ . \end{aligned}$$

It is quite easy to modify the construction of the example so that it shows disconjugacy generically in C^r when the growth of resonances is faster than k^r.

Notice also that the conjugacy would be provided by function $g(x,y) = (x, y+\eta(y))$ where $\eta(y)$ satisfies $\eta(y) - \eta(y+\alpha) = \psi(x)$ so that one can also play the game of finding resonance conditions so that there is conjugacy in one regularity class and not in another.

Some partial converses of this counterexample are possible.

Theorem. *If $f: T^d \to T^d$ is an area preserving C^k diffeomorphisms $k > 2$ and $\|f^{n(k}\|_{C^0}$ is bounded independently of n, then there exists a C^{k-1} area preserving $h^*: T^d \to T^d$ satisfying*

$$h^*\big(f(x)\big) = h^*(x) + \mathbf{a}$$

where $\mathbf{a} = \int_{T^d} f(\mathbf{x}) - \mathbf{x}\, dx$. (The integral is to be interpreted as the functions f and the vectors taking values in $\mathbf{R}^d$ — the universal cover of T^d and the integral ranging over a fundamental domain. $f^{(k}$ means now k^{th} derivative.)

We start by observing that since

$$f(\mathbf{x}) = x + \mathbf{a} + \varphi(\mathbf{x})\ , \quad \text{where} \quad \int_{T^d} \varphi(x) = 0$$

we have

$$f^2(x) = x + 2\mathbf{a} + \varphi\big(f(x)\big) + \varphi(x)$$

in general

$$f^n(x) = x + n\mathbf{a} + \sum_{\ell=0}^{n-1} \varphi\big(f^\ell(x)\big)$$

since f preserves area

$$\int_{T^d} \varphi\big(f^\ell(x)\big) = \int_{T^d} \varphi(x) = \mathbf{0} \ .$$

Hence

$$\int_{T^{2d}} f^n(\mathbf{x}) - \mathbf{x} = n\mathbf{a} \ .$$

Now, if we form

$$h_n(\mathbf{x}) = \frac{1}{n} \sum_{\ell=0}^{n-1} \big(f^\ell(\mathbf{x}) - \ell\mathbf{a}\big)$$

we see that $\|h_n^{(k}\|_{C^0}$ is also bounded independently of n. Moreover, since

$$\int_{T^d} h_n(\mathbf{x}) - \mathbf{x} = \mathbf{0}$$

it also follows that

$$\sup_{x \in T^d} |h_n(\mathbf{x}) - n\mathbf{a}|$$

is also uniformly bounded.

Then, we have a set of equicontinuous equibounded h_n and we can select a convergent subsequence

$$h_{n_i} \to h^*$$

but

$$h_n\big(f(x)\big) = h_n(x) + a + \frac{1}{n}\big[f^n(x) - n\mathbf{a} - x\big] \ .$$

Again $f^n(\mathbf{x}) - n\mathbf{a} - \mathbf{x}$ has k derivative bounded independently of n and integrate to zero.

Hence, the bracket converges to zero pointwise so that h^* satisfies the equation of the theorem.

The fact that h^* preserves area follows from the fact that h_n does, and the convergence $h_{n_i} \to h^*$ is uniform C^{k-1} convergence so that the property of conserving the area is preserved under those limits.

This argument is very useful in many contexts. It shows that in order to produce smooth solutions of conjugacy equations, it suffices to obtain uniform bounds for the iterates in a smooth norm.

(Similar arguments appear very often in scattering theory, and are very similar to the Krylov–Bogoliubov argument [33].)

Notice however that we had to use here many properties of the space, like the fact that the cover is $\mathbf{R}^d$ so that we can add. In particular, I do not know how to prove by the same method existence of solutions to

$$f\big(h^{**}(\mathbf{x})\big) = h^{**}(\mathbf{x} + \mathbf{a}) \ .$$

(This, however follows if we can guarantee h^* is invertible which can be established assuming that $\|D(f^n) - \mathrm{Id}\,\|_{C^0}$ is small.)

Some side remarks about the proof

Remark 1. We can think about the previous proof as an implicit function theorem. Eq. (7) is a particular case of a relation between F and G (which we write $\varphi(F,G) = 0$). F and G range on a linear space and φ, even if it is very complicated, has the advantage that we know that $F = 0$, $G = 0$ is a solution and that we know how to make first order perturbation theory around it. That is, the relation φ is differentiable in some sense and we know how to invert $D_2(0,0)$.

The standard way of proceeding is to produce a scheme of successive approximations and show it converges when all the differentials, etc. are taken in the appropriate sense.

For example if we took the scheme

$$\begin{aligned} G_{n+1} &= -D_2^{-1}(0,0)\ \left[\varphi(F,G_n)\right] + G_n \\ G_0 &= 0\ . \end{aligned}$$

Provided that F and G range in a Banach space, that φ is differentiable in the strong sense, that the differential is continuous and that existence of the inverse is supposed to mean existence of a bounded inverse, it is possible to show that this scheme is a contraction (see e.g., [11]).

This is very similar to what we did, but such a simple framework does not suffice. For our problem $D_2^{-1}(0,0)$ is not bounded in any space of functions, and moreover, the derivative is not continuous in almost any sense.

So, we have to use some extra structure of our problem. We had a natural scale of Banach spaces, the spaces of analytic functions on strips, and the operator $D_2\varphi(0,0)^{-1}$ was bounded from one of those spaces to the others with smaller domain; the norm of those operators gets unboundedly large if the loss of domain gets small (as it should if we are to be able to iterate indefinitely).

The saving grace for our scheme was the "quadratic convergence": The new remainder was obtained from the old by multiplying by a factor that contains not only the bad factor due to the analyticity loss, but also a term proportional to the error. This combination is less than one, and we keep on improving.

Notice that the scheme we proposed for the easy implicit function theorem does not have this quadratic convergence in Banach spaces. The new error $\|\varphi(F,G_{n+1})\|$ can be bounded only by $\|D_2^{-1}\varphi(0,0)\|\ \|D^2\varphi\|\ \|\varphi(F,G_n)\|$ which, even if a contraction in Banach space for small F, is not quadratic.

The best known scheme with quadratic convergence is Newton's method

$$G_{n+1} = -D_2^{-1}\varphi(F,G_n)\varphi(F,G_n) + G_n\ .$$

However, this was not the one we used and, indeed, is unsuitable for our problem, because it requires to invert the linearized equation away from $(0,0)$, a task that we could not accomplish by using Fourier analysis.

What we did was to use the scheme involving only the solution of $D_2\varphi(0,0)$ but we used this knowledge to improve F (make F closer to zero) so that we can get quadratic convergence even if not using Newton's scheme.

This is something that cannot be done for all equations, only for those that have some "group structure."

To summarize: we can prove the implicit function theorem in a more general framework that in Banach space, but we need some structure on the equation: either solvability in an open neighborhood or group structure. We also need some structure on the framework in which we set up the problem (the scale of Banach spaces).

Since non-linear functional equations are pervasive in applications and the implicit function theorem is a very natural tool, there has been lots of thought given to the problem of finding more and more general settings. For Dynamical Systems, probably the most useful setting is that of Zehnder [38] but there are others, probably more useful in geometry [19]. There are still other approaches (e.g., [18]).

There is however lots of room for improvement. In particular the results of [26] which are a kind of implicit function theorem do not fit in any of the schemes above.

Besides the example we already discussed it is worth keeping the following counterexample in mind. In comes from [2].

Consider the Frechet space of functions analytic in the unit disk endowed with the usual topology of uniform convergence on compact sets.

If $f(z) = \sum_{k\geq 0} f_k z^k$ define φ by $(\varphi(f))(z) = \sum_{k\geq 0} f_k^2 z^k$ (which is again a function of radius of convergence at least one).

The function $f^* = 1/1-z$ is a fixed point of $\boldsymbol{\varphi}$ and, moreover, $D\varphi(f^*) = 2$.

However, in any neighborhood, there are functions $\tilde{f}$ different from f^* so that $\varphi(\tilde{f}) = f^*$ (just take $\tilde{f}_k = 1$ if $k \neq N$, $\tilde{f}_N = 1$), one can produce even an uncountable number of them in any neighborhood (so there is no inverse function; the example can be modified to be a counterexample of the implicit function theorem).

Remark 2. Giovanni Gallavotti has repeatedly drawn attention to the analogy between this treatment and the renormalization group approach to the obtention of uniform estimates in quantum field theory. There one also uses the group structure of the problem. The perturbation theory is used to guess a transformation that substitutes our problem by one closer to "free."

Remark 3. Observe that the quadratic convergence does not get spoiled if instead of using the full solution of Eq. (8) we only use an approximate one, solving Eq. (8) up to an error comparable to the square of $\|F\|$.

In the full K.A.M. theorem, or in the standard proof of the toy theorem discussed here with a finite number of derivatives, it becomes quite useful to use a truncation

$$G^{[\leq N]} = \sum_{|\mathbf{k}|\leq N} e^{2ri\mathbf{k}\mathbf{x}} G_{\mathbf{k}} \ .$$

Using the triangle inequality

$$\|G - G^{[\leq N]}\|_{A-\delta} \leq \sum_{|\mathbf{k}|>N} |G_k| e^{(A-\delta)|\mathbf{k}|} \ .$$

By Cauchy estimates

$$\le \sum_{|\mathbf{k}|>N} e^{-\delta|\mathbf{k}|}\|G\|_A \le$$

$$\le \|G\|_A e^{-\delta N} C \sum_{\mathbf{Z}} \ell^{d-1} e^{-\delta\ell}$$

$$\le \|G\|_A C e^{-\delta N}\delta^{-(d-1)} \ .$$

We can, at each step, accommodate an extra loss of domain and an N in such a way that

$$e^{-\delta N} \le \|F\|_A \ .$$

If we fix that N grows exponentially with the number of steps this makes a δ which is comparable to the other losses of domains and the proof of infinite iteration and convergence needs no modification.

E. The full K.A.M. Theorem

The most famous of the applications of these techniques is the theorem of persistence of quasi-periodic motions in perturbations of integrable systems, that is, systems that have coordinates $\mathbf{A}$, $\boldsymbol{\varphi}$ ($\boldsymbol{\varphi}$ are angles) such that the Hamiltonian is only a function of $\mathbf{A}$. The equations of motion for an integrable system are

$$\dot{\varphi}_i = \frac{\partial H}{\partial A_i} = \boldsymbol{\omega}(\mathbf{A})$$

$$\dot{A}_i = -\frac{\partial H}{\partial \varphi_i} = 0$$

so that the solutions are

$$\begin{aligned}\boldsymbol{\varphi} &= \boldsymbol{\varphi}(0) + \boldsymbol{\omega}(\mathbf{A})t\\ \mathbf{A} &= \mathbf{A}(0)\end{aligned}$$

and all of them are quasi-periodic.

The theorem states that, provided that $\boldsymbol{\omega}$ appears in the range of $\boldsymbol{\omega}(A)$ (and some other conditions that we will discuss later) and satisfies irrationally conditions, any sufficiently small perturbation H will have an invariant torus of dimension $2d$. The motion on this torus will be conjugate to the irrational flow of angular frequencies $\boldsymbol{\omega}$.

The conditions we need can be motivated by the following example

$$H_\varepsilon = (\boldsymbol{\omega}_0 + \varepsilon\, \boldsymbol{\omega}_1)\mathbf{A} \ .$$

Hamilton's equation for this Hamiltonian can be readily solved but there are no motions with frequency $\boldsymbol{\omega}_0$. This shows that we need some conditions that "clamp" the frequency forcing it to appear in the motions of H_ε.

Conditions which are sufficient are that $\det(\frac{\partial^2 H_0}{\partial \mathbf{A}\partial \mathbf{A}}) \ne 0$. Heuristically, the range of $\boldsymbol{\omega}(\mathbf{A}) \equiv \nabla_A H$ is persistent and those are the conditions under which the theorem is proved most often.

Unfortunately these hypotheses do not apply to the most interesting case, when H_0 is the Hamiltonian of the solar system with the interactions between the planets turned off. (As is very well known the frequency of orbit does not depend on the angular momentum so that the Hessian has less rank than the dimension.) However, there are weaker hypotheses that apply to this case [4] and allow us to prove the theorem.

There are basically two strategies to prove this theorem:

The first — we will call it Kolmogorov's strategy — starts with the observation that Hamiltonians of the form

$$H(\mathbf{A},\boldsymbol{\varphi}) = \boldsymbol{\omega}\,\mathbf{A} + 0(|\mathbf{A}|^2)$$

admit quasi-periodic solutions of frequency $\boldsymbol{\omega}$

$$\begin{aligned} \varphi(t) &= \boldsymbol{\varphi}(0) + \boldsymbol{\omega}\, t \\ \dot{\mathbf{A}}(t) &= \mathbf{0}\ . \end{aligned}$$

However, if we take coordinates in which $\omega(0) = \boldsymbol{\omega}_0$, the most general perturbation of our integrable system is

$$H(\mathbf{A},\boldsymbol{\varphi}) = F(\boldsymbol{\varphi}) + \boldsymbol{\omega}_0\,\mathbf{A} + \mathbf{G}(\boldsymbol{\varphi})\big(\mathbf{A} + o(|\mathbf{A}|^2)\big)$$

(as can be readily seen by expanding in Fourier series) where F, G are assumed to be small. So, we try to make a change of variables in which the terms F, G are absent.

If we use the variables A', φ' given by $\exp\mathcal{L}$, where $\mathcal{L}$ is a globally Hamiltonian vector field of Hamiltonian L, we see that "up to higher order terms in L, F, G," the Hamiltonian in the new variables is

$$H(A',\varphi') = F(\varphi') + \boldsymbol{\omega}_0\,A' + G(\varphi')\mathbf{A}' + \big\{\boldsymbol{\omega}_0\,\mathbf{A}', L(A'\varphi')\big\}$$

where $\{\ \}$ means Poisson Bracket.

So, we try to determine L in such a way that it cancels as many as possible undesirable terms in first order.

The equations that imply this cancellation can be studied because the Poisson bracket means either derivative $\boldsymbol{\omega}\ \mathbf{A}'$ along the Hamiltonian field of L or minus derivative of L along the field of Hamiltonian $\boldsymbol{\omega}\ A'$. In this later interpretation, we can solve by taking Fourier coefficients eliminating all Fourier coefficients except the constants.

That is, we can find L so that

$$H(A',\varphi') = \left(\int F(\boldsymbol{\varphi}')d^d\varphi'\right) + \boldsymbol{\omega}\,\mathbf{A}' + \left(\int \mathbf{G}(\varphi')d^d\,\boldsymbol{\varphi}'\right)\mathbf{A} + o(|A|^2)$$

$$\text{(up to quadratic in } F,\, G)\ .$$

We can omit the constant $\int F(\varphi')$ because adding a constant to the Hamiltonian does not change the dynamics.

As it stands now, the step is impossible to iterate because $(\omega + \int \mathbf{G})$ could fail to satisfy the irrationality conditions.

But we now try to make a change of coordinates $A'' = A' + \mathbf{Y}(\boldsymbol{\varphi}')$ (which is clearly symplectic) and that restores the form we started with

$$H(\mathbf{A}'',\boldsymbol{\varphi}'') = \tilde{F}(\boldsymbol{\varphi}'') + \boldsymbol{\omega}\,\mathbf{A} + \tilde{\mathbf{G}}(\boldsymbol{\varphi}'')A'' + o(A^2)$$

but with $\tilde{F}$, $\tilde{G}$ which are "quadratic" in F, G.

We can indeed find such a $\mathbf{Y}(\boldsymbol{\varphi})$ because of the hypothesis on the Hessian. Moreover, the size of $\mathbf{Y}$ will be commensurate with that of $\int \mathbf{G}$. (This only requires the standard implicit function theorem in $\mathbf{R}^n$.)

To prove convergence, we will proceed as before, but with the extra induction hypothesis that the Hessian is uniformly bounded away from zero; again, this is no problem because controlling the change of the Hessian involves controlling only three derivatives which, in the previous notation means that controlling $\sum u_k^{-3} S_k$ which follows from estimates we had already done.

Notice that since the hypothesis of the theorem include that $\boldsymbol{\omega}(A)$ covers an open set when $\mathbf{A}$ ranges in an open set, there will be, for sufficiently small perturbations, many invariant tori. However, the proof we gave requires a different transformation for each one of them. This makes it very hard to answer questions about the nature of the sets of invariant tori.

For example, it is known that there is a set of smooth invariant tori which can be embedded in a differentiable family of tori and which occupy a fraction of phase space arbitrarily close to 1 for sufficiently small perturbations. (It is, in principle, possible that there are some invariant tori, corresponding to frequencies closely approximated by rationals, which do not fit in this smooth family, or which are not smooth, or such that the motion of them is not conjugate, to a linear flow. I will be very curious to know whether the same techniques in the counterexample provided here can be used to provide counterexamples for flows.)

To prove these results about families of tori, it is necessary to use another strategy, due to Arnold [3]. We try to reduce the Hamiltonian flow to linear simultaneously in several places of the phase space. Again, we proceed inductively.

At stage ℓ, we try to construct the conjugacy in a domain of points that satisfy

$$|\mathbf{k}\,\boldsymbol{\omega}(A)|^{-1} \leq \Omega(|\mathbf{k}|) \qquad 0 \leq |\mathbf{k}| < (3/2)^{\ell} \ ,$$

Then, we solve the linearized conjugacy equation in Fourier coefficients up to order $(3/2)^k$. As explained before, this does not affect the quadratic convergence. However, the structure of the set where we are solving is very complicated and we should worry that each of the conjugacy does not take us out of the space for the next. The idea is to use the anisochromy condition to argue that the geometry of the resonant set, in $\mathbf{A}$, is very similar to that of the resonant set in $\boldsymbol{\omega}$ which we discussed. This produces the estimates in the fraction of the volume preserved and, taking care of keeping all the estimates uniform, also the smoothness of the family interpolating the invariant tori. For the details and quantitative statements, the reader should look at [14,29].

Besides the above statements about the global appearance of a large set of invariant tori the previous argument can be modified to yield other results:

If we stop the elimination procedure at a certain stage, we can still show that in non-resonant regions (not just in tori), the motion is up to a change of coordinates, extremely similar to a linear flow. In particular, it does not move off for an extremely long time.

This can be combined with an analysis of the motion in the resonant region to show that the motion is essentially perpendicular to the resonance line to conclude that small perturbations of integrable systems do no wander off for extremely long times, much longer than what a naive dimensional argument would suggest. This is the content of a famous theorem proved by Nekhoroshev [27]. (See simpler proofs in [7].)

The fact that for even longer times, this recurrence disappears and points eventually do wander off was established in some examples [5,1], conjecturing this type of behavior is the rule.

Some people claim that they have observed and measured this Arnold diffusion in some systems. Nevertheless, the effect to be measured is several orders of magnitude smaller than the truncation error due to the numerical methods and quite comparable to round-off and the results are not quite reproducible.

F. Computational Aspects

Most of the proofs based on the previous strategy suffer the shortcoming that the smallness conditions required are much stronger than those that are relevant for practical applications (we recommend [41] for a discussion of the applicability of K.A.M. theory specifically to the solar system).

Of course, one could argue that it is possible that the method of proof is not optimized to produce larger constants. Presumably, the figure of merit that has been optimized in many of the proofs is the number of derivatives. Since the K.A.M. theorem makes statements for whole classes of functions and this is an infinite dimensional space it is not even clear what does it mean to optimize the constants in general because there are many different ways to measure distances and sizes of functions.

There are two basic numerical methods to study directly the existence of quasi-periodic orbits lying on smooth tori. One of them is to write down the equation for conjugacy and then apply a non-linear equation solver (e.g., a Newton or continuation method) and the other is to implement numerically the perturbation expansions that we discussed in the previous section.

The numerical literature about implementation of perturbation series expansions is quite extended. Let us mention, among others[42] for an implementation of perturbation series for the quasiperiodic orbit (sometimes called Lindstedt series) or [43] (and references there), [44] for a discussion of the calculation of the approximate chage of variables that integrates the whole system approximately. For numerical implementation of direct solutions of the equation, and a very lucid discussion of the difficulties, let us mention [44].

The validity of these numerical methods has been very controversial. Of course, all numerical methods are subject to round-off and truncation so that, strictly speaking, all numerical results are not mathematically conclusive and, if not accompanied by an error analysis, suspect.

For the computation of quasi-periodic orbits this has been a little bit more so because all the counterexamples point to the effect that even very small values of derivatives of relatively high order can make a difference. It is well known that many numerical discretizations of functions — specifically of several variables — do not allow numerically stable evaluation of derivatives. (The only method we know that allows stable discretization of derivatives is the storage of Fourier coefficients.) Also the examples point out that the irrationality properties of the frequencies play a significant role. Of course, all the numbers stored in a computer are rational and it is not obvious how one could attempt to discretize such problems.

In practice, the methods based on the perturbation series expansions are somewhat easier to implement since the recursion expansion is always done by solving the first order expansion around the unperturbed case. The Newton method requires one to solve a system of linear equations of dimension the number of discrete variables. Even if with todays computers solving a system of equations with 1000 variables is quite feasible, it is a computation of the order of one hour in a typical workstation. Even if shortcuts are possible, this remains a calculation much slower than one step of a recursion. An important reason why these methods based on the solving of functional equations are feasible is that there is a natural variational formulation of the problem.

It is well known that numerically, variational methods — if the variation principle is reasonable — are more reliable and faster than direct solution of the variational equations. Moreover, expansions give you information for several values of the parameter.

On the other hand perturbation expansions in powers of the parameter have several shortcomings. First notice that power series expansions always converge in circles and diverge outside of them. So that if the domain of analyticity of the function we are studying is not a circle, there are values of the parameter for which the result is true but which cannot be studied by the perturbation expansion. It could very well happen that some of these inaccessible values are real — i.e., physical — because there are some unphysical singularities for complex values very close to the origin.

Also, for many of the perturbation expansions of periodic orbits the coefficients differ greatly in size and involve large amounts of cancellations. Especially close to the domain of convergence, evaluation may be quite unstable.

The methods of successive changes of variables used frequently in the mathematical literature are not very convenient for numerical implementations. The main reason is that the difficulty of discretizing a function increases very rapidly with the number of variables. Since the tori are half the dimension of phase space, the embedding of the torus in the phase space involves many less variables than studying functions in the phase space.

¿From the point of view of numerical applications, one of the most successful methods to compute invariant tori has been its approximation by periodic orbits. This indirect method, even if it has very good empirical justification [45] and can be justified from the point of view of renormalization [46] is hard to justify mathematically (see [47], [48] for some partial results).

Recently, there has been considerable progress in the study of the validity of numerical computations of K.A.M. tori.

At the root is the use of interval arithmetic and its extension to function spaces [49]. The basic idea of interval arithmetic is very simple. Rather than represent a real quantity by a machine number that we hope approximates its value, we represent it by two machine numbers that are upper and lower bounds for its value.

One can write operations among pairs of representable numbers that bound the usual arithmetic operations. That is, the sum of two intervals is another interval that is guaranteed to contain the possible values of the sums of two numbers, whenever these two numbers belong to the summand intervals. (If it is impossible to find a pair of representable numbers with this property, the operation should just raise an exception.)

Then, given an algebraic expression, by translating these algebraic operations into calls to the corresponding interval routines one obtains a routine that produces rigorous bounds for the values of algebraic expression given bounds on the variables.

Similarly, for all the functions for which one knows rational approximations and explicit bounds on the error, one can also obtain routines that bound them by enlarging by a convenient amount the result returned by the routines that bound the polynomial approximation.

A similar circle of ideas can be extended to function spaces. We need to identify families of sets that can be finitely specified and for which it is nevertheless easy to implement operations that bound the usual operations of analysis.

There are many such families of sets that one could consider. One, introduced in [50] that has been found to be particularly useful in the type of questions discussed in these lectures is the family of sets:

$$
\begin{aligned}
B_{I_{-N}\cdots I_N;\varepsilon} = \Big\{ f : T^1 \to \mathbb{C} \mid \ & f = f_p + f_e \\
& f_p(\theta) = \sum_{k=-N}^{N} \hat f_{p,k} e^{2\pi i k\theta} \ , \ \hat f_{p,k} \in I_i \\
& f_e(\theta) = \sum_{k=-\infty}^{\infty} \hat f_{e,k} e^{2\pi i k\theta} \ , \ \sum |\hat f_{e,k}| e^{-\delta|k|} \le \varepsilon \Big\}
\end{aligned}
$$

Roughly speaking, we are fixing the main Fourier coefficients to lie in some intervals but allowing some error.

It is quite possible to bound many of the operations of analysis — including, of course, the elementary arithmetic ones — in this class of functions. Roughly, one performs the truncated operations in the Fourier coefficients and then, estimates the errors incurred either because of the uncertainties in the original data or because of the truncation process.

Notice that the norm in this particular example satisfies the Banach algebra property $\|fg\| \le \|f\| \, \|g\|$. Another important property of such discretizations is that they contain information about all derivatives and allow for reliable evaluation of them.

There are many variations that improve several practical considerations. For example, one can keep several error terms, some of which enjoy special properties (e.g., having only coefficients of high order). Or, one can keep more detailed information about the error terms than that afforded by just one norm. For example, keeping estimates both $\sum |\hat f_k| e^{\delta k}$ and $\sum |\hat f_k| \, |k| e^{\delta k}$ leads to a more efficient proof of the K.A.M. theorem (see [51]) and hence, it is a good idea to incorporate it in the algorithms to compute how good teh approximate solutions are.

There are other discretizations of spaces of functions in the literature. For example [52] discusses in detail discretizations based on pointwise bounds of the functions. These methods are much more efficient than the one above in some problems such as integral equations. They are not so useful when one requires derivatives.

Once one has a powerful enough interval arithmetic package it is possible to show that a specific trigonometric polynomial is an approximate solution of (1). One can verify that the right and left hand sides differ by a small amount and one can also obtain estimates of the properties of an analytic extension.

It is possible to prove a theorem — more or less along the same lines as the one discussed in the text — that shows that if a function satisfies (1) with an error sufficiently small with respect to other analyticity of properties, then a Newton-like method started on it will converge to a true solution.

Hence, one can verify that there is a true solution of (1) and that it is close to our initial guess.

Notice that the way that the solution is produced is irrelevant for the verification that a Newton method started on it converges. In practice, the solution is produced using a numerical method that may include as many shortcuts as are considered expedient. The verification step, guarantees that this is a solution.

The relevant theorems and implementation for some cases is described in [Ra], [53] where it is shown that this way of proceeding — given enough time — can explore as much as desired of the region of validity of the K.A.M. theorem. The actual implementation was run in some concrete problems and got verified the results for some values of ε bigger than 95% of values for which it is known that the conclusions of the theorem are false.

Another different method to take advantage of interval arithmetic is considered in [54]. There, interval arithmetic is used to obtain bounds of each of the coefficients in the perturbative expansions and a theorem is proved stating that certain properties of the perturbative expansion imply lower bounds for the radius of convergence.

As mentioned before, there are reasons why power series expansions cannot cover the whole domain of analyticity. Nevertheless, they provide information for whole ranges of parameters, which may be preferable in some practical situations where the relevant values of parameters are covered.

One should also point out that the use of interval arithmetic prevents the use of the cancellations that provide the convergence of this series so that one should expect that the method reaches smaller values than the true ones.

One improvement of the method in which the coefficients of the power series expansions are computed without using interval arithmetic is [55].

The validity of the K.A.M. theorem can be delineated with the help of counterexamples to the conclusions. For twist maps (two dimensional) there are several counterexamples to the differentiability and irrationality properties [56], [57], [58]. For the numerical values allowed in the perturbation [59], [60], [61]. The latter two involve computer assisted calculations. The situation in higher dimensions is more confusing since tori can be notably more complicated [62]. Computer assisted counterexamples to several possible statements for sufficiently large values of the parameter can be found in [Ma].

For the model problem of iteration of analytic functions near a fixed point with derivative of modulus 1, a very satisfactory description of the boundaries of validity exists [39], [40].

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