Zero-curvature condition in two dimensions. Relativistic particle models and finite $\mathcal{W}$-transformations

J. Gomis,[†*] J. Herrero,[♯] K. Kamimura[♭] and J. Roca[♯]

[†] *Theory Group, Department of Physics*
The Univ. of Texas at Austin
RLM 5208, Austin, TEXAS

[♯] *Departament d'Estructura i Constituents de la Matèria*
Universitat de Barcelona
Diagonal, 647
E-08028 BARCELONA

[♭] *Department of Physics, Toho University*
Funabashi
274 Japan

e-mails: GOMIS@UTAPHY, HERRERO@EBUBECM1,
KAMIMURA@JPNYITP, ROCA@EBUBECM1

Abstract

A relation between an $Sp(2M)$ gauge particle model and the zero-curvature condition in a two-dimensional gauge theory is presented. For the $Sp(4)$ case we construct finite $\mathcal{W}$-transformations.

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[*] On leave of absence from Dept. d'Estructura i Constituents de la Matèria, U. Barcelona.

1 Introduction

In the last few years a lot of attention has been devoted to the study of $\mathcal{W}$-algebras [1]. For recent update reviews see [2], [3] where extensive lists of references can be found.

An interesting way to construct classical $\mathcal{W}$-algebras is by the zero-curvature method [4], [5], [6],[7] [8]. If one constraints an A_z gauge potential the residual gauge transformations can be obtained as a zero-curvature condition $F_{z\bar{z}} = 0$. This zero-curvature condition is the integrability condition of a linear system of partial differential equations. As we will see this system can be related to the transformation properties and equations of motion of matter coupled to the gauge fields.

In this letter we consider a relativistic model of M particles with an $Sp(2M)$ gauge group, the matter variables being the coordinates and momenta of the particles and the gauge variables being the Lagrange multipliers. We find that under some formal identifications between $2d$ gauge theories and $1d$ particle models, the equations of motion and transformation properties of the matter variables can be written as a system of partial differential equations whose integrability condition is precisely the zero-curvature condition $F_{z\bar{z}} = 0$. This condition is equivalent to the transformation properties of the Lagrange multipliers. These relations continue to hold when we fix the gauge partially. This fact explains why a model of relativistic particles exhibits, after a partial gauge-fixing, invariance under non-linear $\mathcal{W}$-symmetry tranformations. In a sense, it can be understood as a coupling of matter to (world-line) $\mathcal{W}$-gravity.

The particle model is also useful for the construction of finite $\mathcal{W}$-transformations. Finite transformations are necessary in order to understand completely the $\mathcal{W}$-geometry [6], [8], [9], [10], [11], [12], [13], [14], [15]. The strategy is the following: we first construct the finite linear transformations of the $Sp(2M)$ model and then, by a partial gauge-fixing at the finite level, we find residual finite $\mathcal{W}$-transformations. In this way one avoids the direct integration of non-linear infinitesimal $\mathcal{W}$-transformations. We will explicitly construct in this paper finite $\mathcal{W}$-transformations obtained from the $Sp(4)$ gauge group.

2 $Sp(2M)$ model

Let us consider a reparametrization-invariant model of M relativistic particles with an $Sp(2M)$ gauge group living in a flat non-euclidean d-dimensional space-time. The dimension d is large enough so the constraints do not trivialize the model. The canonical action is given by

$$S = \int \mathrm{d}t \left(p_i \dot{x}_i - \lambda_{A_{ij}} \phi_{A_{ij}} \right), \qquad i,j = 1,\ldots,M, \quad A = 1,2,3. \tag{2.1}$$

The variable $x_i^\mu(t)$ is the world-line coordinate of the i-th particle and p_i^μ is its corresponding momentum. The Lagrange multipliers $\lambda_{A_{ij}}$ implement the constraints $\phi_{A_{ij}} = 0$ and satisfy

$$\lambda_{1_{ji}} = \lambda_{1_{ij}}, \qquad \lambda_{3_{ji}} = \lambda_{3_{ij}}.$$

The explicit form of $\phi_{A_{ij}}$ is

$$\phi_{1_{ij}} = \frac{1}{2} p_i p_j, \qquad \phi_{2_{ij}} = p_i x_j \qquad \text{and} \qquad \phi_{3_{ij}} = \frac{1}{2} x_i x_j. \tag{2.2}$$

These $2M^2 + M$ constraints close under Poisson bracket giving a realization of the $Sp(2M)$ algebra.

It is useful to introduce a matrix notation for the coordinates and momenta of the particles

$$R = \begin{pmatrix} r \\ p \end{pmatrix}, \qquad \bar{R} = \left(p^\top, \, -r^\top\right) \tag{2.3}$$

with

$$r = \begin{pmatrix} x_1 \\ \vdots \\ x_M \end{pmatrix}, \qquad p = \begin{pmatrix} p_1 \\ \vdots \\ p_M \end{pmatrix}.$$

We can put the Lagrange multipliers in a $2M \times 2M$ symplectic matrix

$$\Lambda = \begin{pmatrix} B & A \\ -C & -B^\top \end{pmatrix} \tag{2.4}$$

where the components of the $M \times M$ matrices A, B, C are the Lagrange multipliers $\lambda_{1_{ij}}$, $\lambda_{2_{ij}}$, $\lambda_{3_{ij}}$ respectively.

The canonical action (2.1) can be written in a matrix form as

$$S = \int \mathrm{d}t \, \frac{1}{2} \left(\bar{R}\dot{R} - \bar{R}\Lambda R \right). \tag{2.5}$$

In this formulation the gauge invariance of the action is given by ordinary Yang-Mills type transformations [1] with the gauge group $Sp(2M)$:

$$\delta R = \beta R, \tag{2.6}$$

$$\delta \Lambda = \dot{\beta} - [\Lambda, \beta], \tag{2.7}$$

where β is the $2M \times 2M$ matrix of gauge parameters

$$\beta = \begin{pmatrix} B_\beta & A_\beta \\ -C_\beta & -B_\beta^\top \end{pmatrix} \tag{2.8}$$

[1] For a previous discussion of geometrical models and Yang-Mills gauge theories see [16].

and $A_\beta, B_\beta, C_\beta$ are the $M \times M$ matrices gauge parameters associated to the constraints $\phi_{1_{ij}}, \phi_{2_{ij}}, \phi_{3_{ij}}$.

The equations of motion of the matter fields are

$$\dot{R} - \Lambda R = 0. \tag{2.9}$$

If we make the following identifications

$$\delta \to \bar{\partial}, \quad \beta \to A_{\bar{z}}, \quad \Lambda \to A_z, \quad \frac{\mathrm{d}}{\mathrm{d}t} \to \partial, \tag{2.10}$$

the equations of motion (2.9) and transformation properties of the matter fields (2.6) can be written as

$$(\bar{\partial} - A_{\bar{z}})R = 0, \quad (\partial - A_z)R = 0. \tag{2.11}$$

This linear system of partial differential equations has an integrability condition $F_{z\bar{z}} = 0$, which is equivalent to the transformation law of the Lagrange multipliers (2.7) under the identifications (2.10). These relations will continue to hold when we fix the gauge partially. The above discussion explains why a relativistic particle model becomes after partial gauge-fixing a model of matter with a non-linear $\mathcal{W}$-symmetry.

As we will see below it is useful to express the model in terms of lagrangian variables in order to construct the finite transformations of the model. If we write the momenta p in terms of the lagrangian variables

$$p = A^{-1}(\dot{r} - Br) \equiv K, \tag{2.12}$$

the action is now rewritten as

$$S = \int \mathrm{d}t \; \frac{1}{2} \left(K^\top A K - r^\top C r \right). \tag{2.13}$$

The gauge transformations are

$$\delta r = A_\beta K + B_\beta r, \qquad \delta\Lambda = \dot{\beta} - [\Lambda, \beta]. \tag{2.14}$$

A characteristic feature of these lagrangian transformations is that the algebra is open, except for $Sp(2)$,

$$[\delta_1, \delta_2]\; r = \delta_{\beta^*} r + (A_{\beta_2} A^{-1} A_{\beta_1} - A_{\beta_1} A^{-1} A_{\beta_2})[L]_r,$$
$$[\delta_1, \delta_2]\; \Lambda = \delta_{\beta^*} \Lambda, \tag{2.15}$$

where $\beta^* = [\beta_2, \beta_1]$ and $[L]_r$ are the Euler-Lagrange equations of motion of r. There are two reasons for the appearance of an open algebra: 1) the transformations of the momenta at the lagrangian and hamiltonian level do not generally coincide, 2) there are more than one first-class constraints quadratic in the momenta.

In order to close the gauge algebra we introduce M auxiliary vectors $(F_1, ...F_M)$ and modify the transformation law of the coordinates r as

$$\delta r = A_\beta (K + F) + B_\beta r. \tag{2.16}$$

The transformation of F is determined by the condition that $K + F$ transforms as p in the hamiltonian formalism. Explicitly we get

$$\delta F = -A^{-1} \left[A_\beta \partial_t (K + F) + A_\beta B^\top (K + F) + (\delta A - B_\beta A) F + A_\beta C r \right], \tag{2.17}$$

while the transformation of Λ remains unchanged

$$\delta \Lambda = \dot{\beta} - [\Lambda, \beta]. \tag{2.18}$$

The new algebra closes off-shell.

The invariant action under the modified gauge transformations is

$$S = \int \mathrm{d}t \, \frac{1}{2} \left(K^\top A K - r^\top C r - F^\top A F \right). \tag{2.19}$$

The redundancy of the auxiliary variables F is guaranteed by the action itself. This action is also invariant under ordinary diffeomorphisms (Diff).

$$\delta_D r = \epsilon \dot{r}, \quad \delta_D F = \epsilon \dot{F}, \quad \delta_D \lambda_j = \partial_t (\epsilon \lambda_j). \tag{2.20}$$

Diffeomorphisms are not an independent symmetry on-shell. In fact they can be obtained from β transformations by setting $\beta_{A_{ij}} = \epsilon \lambda_{A_{ij}}$ up to some terms vanishing on the equations of motion. Diffeomorphisms and β transformations form a closed algebra

$$[\delta_\beta, \delta_D] = \delta_{\beta'}, \qquad \text{with} \quad \beta' = \epsilon \beta. \tag{2.21}$$

As we want to obtain $\mathcal{W}$-transformations after a partial gauge-fixing and these transformations contain ordinary Diff it is natural to consider Diff and $(M(2M+1)-1)$ of β transformations as the independent symmetry transformations.

3 Finite transformations of $Sp(2M)$ model

Now we want to find the finite linear transformations of this model.

Finite transformations [2] can be obtained by exponentiating the infinitesimal ones as $X^{i'} = \exp\{\theta^\alpha \Gamma_\alpha\} X^i$, where the generators $\Gamma_\alpha = R^i_\alpha \frac{\partial}{\partial X^i}$ satisfy $[\Gamma_\alpha, \Gamma_\beta] = f^\gamma_{\alpha\beta} \Gamma_\gamma$ and X^i represents any of the variables. The coefficients $f^\gamma_{\alpha\beta}$ are the structure functions of the $Sp(2M)$ gauge algebra.

[2] For a recent discussion on finite gauge transformations see [17].

It is useful to perform the integration using the matrix notation. The explicit form of the finite gauge transformations is considered in the following four sets of transformations. Any finite transformation may be obtained by the composition of them.

- Transformations generated by A_β:

$$\begin{gathered} A' = A + \{\dot{A}_\beta - A_\beta B^\top - B A_\beta\} + A_\beta C A_\beta, \\ B' = B - A_\beta C, \qquad\qquad C' = C, \\ r' = r + A_\beta (K+F), \\ F' = {A'}^{-1}\left[AF - A_\beta\{\partial_t(K+F) + B^\top (K+F) + Cr\}\right]. \end{gathered} \tag{3.1}$$

- Transformations generated by B_β:

$$\begin{gathered} A' = e^{B_\beta} A e^{B_\beta^\top}, \quad B' = e^{B_\beta}(B - \partial_t) e^{-B_\beta}, \quad C' = e^{-B_\beta^\top} C e^{-B_\beta}, \\ r' = e^{B_\beta} r, \quad F' = e^{-B_\beta^\top} F. \end{gathered} \tag{3.2}$$

- Transformations generated by C_β:

$$\begin{gathered} A' = A, \qquad\qquad r' = r, \qquad\qquad F' = F, \\ B' = B + A C_\beta, \\ C' = C + (\dot{C}_\beta + C_\beta B + B^\top C_\beta) + C_\beta A C_\beta. \end{gathered} \tag{3.3}$$

- The diffeomorphism transformations:

$$\lambda_i'(t) = \dot{f}(t)\lambda_i(f(t)), \qquad r'(t) = r(f(t)), \qquad F'(t) = F(f(t)). \tag{3.4}$$

4 Finite $\mathcal{W}$-transformations

Now we will concentrate on the $Sp(4)$ model in order to obtain $\mathcal{W}$-transformations associated to a partial gauge-fixing of the model. The corresponding restricted model describes a coupling of the matter variables x_i to a world-line version of chiral $\mathcal{W}$-gravity.

The partial gauge-fixing we consider is

$$\Lambda_r = \begin{pmatrix} H & 0 & 0 & 1 \\ 0 & -H & 1 & 0 \\ C & T & -H & 0 \\ T & F & 0 & H \end{pmatrix}. \tag{4.1}$$

In this gauge the action (2.1) becomes

$$S=\int \mathrm{d}t\left[(\dot{x}_1-Hx_1)(\dot{x}_2+Hx_2)+\frac{1}{2}\left(Cx_1^2+2Tx_1x_2+Fx_2^2\right)-F_1F_2\right]. \tag{4.2}$$

The equations of motion arising from this action are:

$$\begin{aligned}
[L]_{x_1} &= -(\tfrac{\mathrm{d}}{\mathrm{d}t}+H)(\dot{x}_2+Hx_2)+Cx_1+Tx_2,\\
[L]_{x_2} &= -(\tfrac{\mathrm{d}}{\mathrm{d}t}-H)(\dot{x}_1-Hx_1)+Fx_2+Tx_1,\\
[L]_H &= \dot{x}_1x_2-x_1\dot{x}_2-2Hx_1x_2,\\
[L]_C = \tfrac{1}{2}x_1^2,&\quad [L]_F=\tfrac{1}{2}x_2^2,\quad [L]_T=x_1x_2,\\
[L]_{F_1} = -F_2,&\qquad [L]_{F_2}=-F_1.
\end{aligned} \tag{4.3}$$

In order to find the gauge transformations of this action we can use the infinitesimal or finite linear transformations of the $Sp(4)$ model. We start by using the infinitesimal transformations found in the last section. The matrix of gauge parameters is now parametrized as

$$A_\beta=\begin{pmatrix}\beta_2 & \beta_{10}\\ \beta_{10} & \beta_5\end{pmatrix},\qquad B_\beta=\begin{pmatrix}\beta_3 & \beta_9\\ \beta_8 & \beta_6\end{pmatrix},\qquad C_\beta=\begin{pmatrix}\beta_1 & \beta_7\\ \beta_7 & \beta_4\end{pmatrix}. \tag{4.4}$$

The residual transformations in the gauge (4.1) are parametrized by β_{10}, β_5, β_2 and β_6. The relations between the dependent and the independent parameters are given by

$$\begin{aligned}
\beta_1 &= -C\beta_{10}+\beta_5(-T+2H^2+\dot{H})+2\dot{\beta}_5H+\frac{1}{2}\ddot{\beta}_5,\\
\beta_4 &= -F\beta_{10}+\beta_2(-T+2H^2-\dot{H})-2\dot{\beta}_2H+\frac{1}{2}\ddot{\beta}_2,\\
\beta_7 &= -T\beta_{10}+\frac{1}{2}\ddot{\beta}_{10}-\frac{1}{2}\beta_2C-\frac{1}{2}F\beta_5,\\
\beta_3 = -\dot{\beta}_{10}-\beta_6,&\qquad \beta_8=-\beta_5H-\frac{1}{2}\dot{\beta}_5,\qquad \beta_9=\beta_2H-\frac{1}{2}\dot{\beta}_2.
\end{aligned} \tag{4.5}$$

Let us consider the following redefinition of the gauge parameters:

$$\epsilon=\beta_{10},\qquad \alpha=\beta_6+H\beta_{10}+\frac{1}{2}k\dot{\beta}_{10}, \tag{4.6}$$

where k is an arbitrary constant parameter. The non-trivial expression of the parameter α is related to two facts: 1) the field H can not be a primary field under reparametrizations unless a term $H\beta_{10}$ is included, 2) The non-empty intersection between $\ker ad\bar{\Lambda}_r$ (where $\bar{\Lambda}_r$ is equal to Λ_r with all remnants gauge fields put to zero) and the Cartan subalgebra of $Sp(4)$ algebra allows us to introduce in a natural way the parameter k.

There are four infinitesimal remnant transformations corresponding to the independent parameters $\epsilon, \alpha, \beta_2$ and β_5.

•ϵ-sector.

$$\hat{\delta}H = \epsilon\dot{H} + H\dot{\epsilon} + \tfrac{1}{2}(k-1)\ddot{\epsilon},$$
$$\hat{\delta}T = \epsilon\dot{T} + 2\dot{\epsilon}T - \tfrac{1}{2}\dddot{\epsilon},$$
$$\hat{\delta}C = \epsilon\dot{C} + (3-k)C\dot{\epsilon},$$
$$\hat{\delta}F = \epsilon\dot{F} + (1+k)F\dot{\epsilon},$$
$$\hat{\delta}x_1 = \epsilon(\dot{x}_1 + F_2) + \tfrac{1}{2}(k-2)x_1\dot{\epsilon},$$
$$\hat{\delta}x_2 = \epsilon(\dot{x}_2 + F_1) - \tfrac{1}{2}kx_2\dot{\epsilon},$$
$$\hat{\delta}F_1 = \epsilon(-\dot{F}_1 - 2HF_1 + [L]_{x_1}) - \tfrac{1}{2}kF_1\dot{\epsilon},$$
$$\hat{\delta}F_2 = \epsilon(-\dot{F}_2 + 2HF_2 + [L]_{x_2}) - \frac{1}{2}(2-k)F_2\dot{\epsilon}. \tag{4.7}$$

Notice that the gauge fields transform as quasi-primary fields under Diff; instead, the matter fields do not have nice transformation properties under Diff. We can obtain the standard diffeomorphism transformations for matter and auxiliary variables by introducing in the corresponding transformation an anti-symmetric combination of their equations of motion:

$$\delta H = \hat{\delta}H, \quad \delta T = \hat{\delta}T, \quad \delta C = \hat{\delta}C, \quad \delta F = \hat{\delta}F,$$
$$\delta q^i(t) = \hat{\delta}q^i(t) + \int dt'\, M^{ij}(t,t')\,[L]_{q^j}(t'), \tag{4.8}$$

where

$$q^1 = x_1, \quad q^2 = x_2, \quad q^3 = F_1, \quad q^4 = F_2. \tag{4.9}$$

The only non-zero elements of M^{ij} are:

$$M^{13}(t,t') = M^{24}(t,t') = -M^{31}(t,t') = -M^{42}(t,t') = \epsilon(t)\delta(t-t'),$$
$$M^{34}(t,t') = -\left(\dot{\epsilon}(t) + 2\epsilon(t)H(t)\right)\delta(t-t') + 2\epsilon(t)\frac{\mathrm{d}}{\mathrm{d}t'}\delta(t'-t),$$
$$M^{43}(t,t') = -\left(\dot{\epsilon}(t) - 2\epsilon(t)H(t)\right)\delta(t-t') + 2\epsilon(t)\frac{\mathrm{d}}{\mathrm{d}t'}\delta(t'-t). \tag{4.10}$$

M is an anti-symmetric matrix: $M^{ij}(t,t') = -M^{ji}(t',t)$. Hence, δ can replace $\tilde{\delta}$ as the symmetry transformation associated with ϵ. The matter and auxiliary variables transform as primary fields under the new transformation,

$$\delta x_1 = \epsilon\dot{x}_1 + \frac{1}{2}(k-2)x_1\dot{\epsilon},$$
$$\delta x_2 = \epsilon\dot{x}_2 - \frac{1}{2}kx_2\dot{\epsilon},$$
$$\delta F_1 = \epsilon\dot{F}_1 - \frac{1}{2}(k-2)F_1\dot{\epsilon},$$

$$\delta F_2 = \epsilon \dot{F}_2 + \frac{1}{2} k F_2 \dot{\epsilon}. \tag{4.11}$$

•α-sector.

$$\delta H = -\dot{\alpha}, \qquad \delta T = 0, \quad \delta C = 2\alpha C, \qquad \delta F = -2\alpha F,$$
$$\delta x_1 = -\alpha x_1, \qquad \delta x_2 = \alpha x_2, \qquad \delta F_1 = \alpha F_1, \qquad \delta F_2 = -\alpha F_2. \tag{4.12}$$

•β_2-sector.

$$\begin{aligned}
\delta H = \frac{1}{2} C \beta_2, \quad \delta T = \frac{1}{2} \beta_2 (\dot{C} - 2CH) + \dot{\beta}_2 C, \quad \delta C = 0,\\
\delta F = \beta_2 (4H^3 - 4HT - 6H\dot{H} + \dot{T} + \ddot{H})\\
-\dot{\beta}_2 (6H^2 - 2T - 3\dot{H}) + 3H\ddot{\beta}_2 - \frac{1}{2} \dddot{\beta}_2,\\
\delta x_1 = \beta_2 (2Hx_2 + \dot{x}_2) - \frac{1}{2} x_2 \dot{\beta}_2 + \beta_2 F_1, \quad \delta x_2 = 0,\\
\delta F_1 = 0, \qquad \delta F_2 = -\beta_2 \left(\dot{F}_1 - [L]_{x_1} \right) - \frac{1}{2} \dot{\beta}_2 F_1.
\end{aligned} \tag{4.13}$$

•β_5-sector.

$$\begin{aligned}
\delta H = -\frac{1}{2} F \beta_5, \quad \delta T = \frac{1}{2} \beta_5 (\dot{F} + 2FH) + \dot{\beta}_5 F, \quad \delta F = 0,\\
\delta C = -\beta_5 (4H^3 - 4HT + 6H\dot{H} - \dot{T} + \ddot{H})\\
-\dot{\beta}_5 (6H^2 - 2T + 3\dot{H}) - 3H\ddot{\beta}_5 - \frac{1}{2} \dddot{\beta}_5,\\
\delta x_1 = 0, \quad \delta x_2 = -\beta_5 (2Hx_1 - \dot{x}_1) - \frac{1}{2} x_1 \dot{\beta}_5 + \beta_5 F_2,\\
\delta F_1 = -\beta_5 \left(\dot{F}_2 - [L]_{x_2} \right) - \frac{1}{2} \dot{\beta}_5 F_2, \quad \delta F_2 = 0.
\end{aligned} \tag{4.14}$$

The algebra of these residual transformations is:

$$\begin{aligned}
[\ \delta_\epsilon\ ,\ \delta_{\epsilon'}\ ] \ =\ \delta_{\epsilon''}; \quad \epsilon'' \ =\ \epsilon'\ \dot{\epsilon}\ - \epsilon\ \dot{\epsilon}',\\
[\ \delta_\epsilon\ ,\ \delta_\alpha\ ] \ =\ \delta_{\alpha'}; \quad \alpha' =\ -\epsilon\ \dot{\alpha}\\
[\ \delta_\epsilon\ ,\ \delta_{\beta_2}\ ] \ =\ \delta_{\beta_2{}'}; \quad \beta_2{}' =\ (2-k)\ \beta_2\ \dot{\epsilon} - \epsilon \dot{\beta}_2,\\
[\ \delta_\epsilon\ ,\ \delta_{\beta_5}\ ] \ =\ \delta_{\beta_5{}'}; \quad \beta_5{}' =\ k\ \beta_5\ \dot{\epsilon} - \epsilon \dot{\beta}_5,
\end{aligned}$$

$$\begin{aligned}
[\ \delta_\alpha\ ,\ \delta_{\beta_2}\ ] \ =\ \delta_{\beta_2{}'}; \quad \beta_2{}' =\ 2\ \alpha\ \beta_2,\\
[\ \delta_\alpha\ ,\ \delta_{\beta_5}\ ] \ =\ \delta_{\beta_5{}'}; \quad \beta_5{}' =\ -2\ \alpha\ \beta_5,
\end{aligned}$$

$$
\begin{aligned}
&[\ \delta_\alpha\ ,\ \delta_{\alpha'}\ ] = [\ \delta_\alpha\ ,\ \delta_{\beta_{10}}\ ] = 0 \\
&[\ \delta_{\beta_2}\ ,\ \delta_{\beta_5}\ ] \ = \ \delta_{\epsilon'} + \delta_{\alpha'} + \delta_{\beta'_{10}}; \\
&\epsilon' \ = \ \beta'_{10} \ = \ -\frac{1}{2}(\beta_2 \dot\beta_5 - \dot\beta_2\ \beta_5) - 2\ H\ \beta_2\beta_5, \\
&\alpha' = \frac{1}{4}(\dot\beta_2\ \dot\beta_5 - k\ \beta_2\ \ddot\beta_5 + (k-2)\beta_5\ \ddot\beta_2) - (1+k)\beta_2\ \dot\beta_5\ H + \\
&(3-k)\ \beta_5\dot\beta_2\ H + \beta_2\beta_5(T - 5H^2 + (1-k)\dot H), \\
&[\ \delta_{\beta_2}\ ,\ \delta_{\beta_2'}\ ] = [\ \delta_{\beta_5}\ ,\ \delta_{\beta_5'}\ ] = [\ \delta_{\beta_2}\ ,\ \delta_{\beta_{10}}\ ] = [\ \delta_{\beta_5}\ ,\ \delta_{\beta_{10}}\ ] \ = \ 0,
\end{aligned}
\tag{4.15}
$$

where the β_{10} transformation is a trivial transformation, i.e. it is proportional to the equations of motion. It is explicitly given by

$$
\begin{aligned}
&\delta H = \ \delta T = \ \delta C = \ \delta F = 0, \\
&\delta x_1 = \beta_{10}\ F_2, \quad \delta x_2 = \beta_{10}\ F_1, \\
&\delta F_1 = -2\beta_{10}(\dot F_1 + H\ F_1) - \dot\beta_{10}\ F_1 - \beta_{10}[L]_1, \\
&\delta F_2 = -2\beta_{10}(\dot F_2 - H\ F_2) - \dot\beta_{10}\ F_2 - \beta_{10}[L]_2.
\end{aligned}
\tag{4.16}
$$

Notice that we have an open algebra with field-dependent structure functions. The non-closure of the present algebra is due to the introduction of equations of motion in the definition of the ϵ-transformation in terms of the original β transformations.

The matter and auxiliary variables x_1, x_2, F_1 and F_2 transform as primary fields under diffeomorphisms with weights $(\frac{k}{2}-1)$, $-\frac{k}{2}$, $(1-\frac{k}{2})$ and $\frac{k}{2}$ respectively. The gauge variables C and F transform also as primary fields with weights $(3-k)$ and $(1+k)$. Instead, H and T are quasi-primary fields with weights 1 and 2 . The transformations with parameters β_2, β_5 generate non-linear $\mathcal{W}$-diffeomorphisms. The parameter α generates dilatations.

Now we are going to construct the finite form of the previous transformations. In order to do that we shall perform the partial gauge-fixing with the help of the finite $Sp(4)$ transformations.

• Diff. sector: The residual finite diffeomorphisms are obtained by the following composition of finite transformations:

$$
X \xrightarrow{\beta_7} \square \xrightarrow{\beta_3,\beta_6} \square \xrightarrow{\text{diff}} \tilde X,
$$

where X stands for any variable. We can express β_7, β_3, β_6 in terms of the Diff parameter f:

$$
\beta_7 = \frac{\ddot f}{2\dot f}, \quad \beta_3 = \frac{k-2}{2}\ \ln\ \dot f, \quad \beta_6 = -\frac{k}{2}\ \ln\ \dot f. \tag{4.17}
$$

and

$$\left\{\begin{array}{lcl} T & \rightarrow & \dot{f}^2\, T(f) - \frac{1}{2}\left(\frac{f^{(3)}}{\dot{f}} - \frac{3}{2}\frac{\ddot{f}^2}{\dot{f}^2}\right) \\ H & \rightarrow & \dot{f}\, H(f) + \frac{1}{2}(k-1)\frac{\ddot{f}}{\dot{f}} \\ C & \rightarrow & \dot{f}^{3-k}\, C(f) \\ F & \rightarrow & \dot{f}^{1+k}\, F(f) \end{array}\right.$$

$$\text{matter variables}\left\{\begin{array}{lcl} x_1 & \rightarrow & \dot{f}^{\frac{k}{2}-1}\, x_1(f) \\ x_2 & \rightarrow & \dot{f}^{-\frac{k}{2}}\, x_2(f) \end{array}\right.$$

$$\text{auxiliary variables}\left\{\begin{array}{lcl} F_1 & \rightarrow & \dot{f}^{1-\frac{k}{2}}\, F_1(f) \\ F_2 & \rightarrow & \dot{f}^{\frac{k}{2}}\, F_2(f). \end{array}\right. \tag{4.18}$$

•α-sector (dilatations): The finite transformation corresponding to the α-sector infinitesimal residual transformations is a composition of β_3 and β_6 finite transformations:

$$X \stackrel{\beta_3,\beta_6}{\longrightarrow} \tilde{X},$$

with

$$-\beta_3 \;=\; \beta_6 \;=\; \alpha \tag{4.19}$$

and

$$\left\{\begin{array}{lcl} T & \rightarrow & T \\ H & \rightarrow & H - \dot{\alpha} \\ C & \rightarrow & e^{2\alpha}\, C \\ F & \rightarrow & e^{-2\alpha}\, F \end{array}\right.$$

$$\text{matter variables}\left\{\begin{array}{lcl} x_1 & \rightarrow & e^{-\alpha}\, x_1 \\ x_2 & \rightarrow & e^{\alpha}\, x_2 \end{array}\right.$$

$$\text{auxiliary variables}\left\{\begin{array}{lcl} F_1 & \rightarrow & e^{\alpha}\, F_1 \\ F_2 & \rightarrow & e^{-\alpha}\, F_2. \end{array}\right. \tag{4.20}$$

•β_2-sector: The residual finite β_2-diffeomorphisms are obtained by the following composition of finite transformations:

$$X \stackrel{\beta_4,\beta_7}{\longrightarrow} \Box \stackrel{\beta_9}{\longrightarrow} \Box \stackrel{\beta_2}{\longrightarrow} \tilde{X},$$

with

$$\beta_9 = \; H\beta_2 + \frac{1}{2}C\beta_2^2 - \frac{1}{2}\dot{\beta}_2, \qquad \beta_7 = -\frac{1}{2}C\beta_2,$$

$$\beta_4 = (2H^2 - T - \dot H)\beta_2 - 2H\dot\beta_2 + \tfrac{1}{2}\ddot\beta_2 + (2HC - \tfrac{1}{2}\dot C)\beta_2^2$$
$$-\tfrac{3}{2}C\beta_2\dot\beta_2 + \tfrac{1}{2}C^2\beta_2^3. \qquad (4.21)$$

and

$$\left\{\begin{array}{lll} T & \rightarrow & T + \frac{1}{2}\beta_2\,\dot C + C\,\dot\beta_2 - \beta_2\,C\,H - \frac{1}{4}{\beta_2}^2\,C^2 \\ C & \rightarrow & C \\ H & \rightarrow & H + \frac{1}{2}\beta_2\,C \\ F & \rightarrow & F + \beta_2\,\left(\dot T - 6\,H\,\dot H + \ddot H - 4\,H\,T + 4\,H^3\right) + \\ & & \dot\beta_2\,\left(2\,T - 6\,H^2 + 3\,\dot H\right) + 3\,\ddot\beta_2\,H - \frac{1}{2}{\beta_2}^{(3)} + \\ & & {\beta_2}^2\,\left(5\,C\,H^2 - C\,T - 2\,C\,\dot H - 3\,\dot C\,H + \frac{1}{2}\ddot C\right) + \\ & & \beta_2\,\dot\beta_2\,\left(\frac{5}{2}\dot C - 8\,C\,H\right) + \frac{7}{4}\dot\beta_2^2\,C + \frac{3}{2}\beta_2\,\ddot\beta_2\,C + \\ & & {\beta_2}^3\left(2\,H\,C^2 - C\,\dot C\right) - 2\,{\beta_2}^2\,\dot\beta_2\,C^2 + \frac{1}{4}{\beta_2}^4\,C^3 \end{array}\right.$$

$$\text{matter variables}\left\{\begin{array}{lll} x_1 & \rightarrow & x_1 + \beta_2\,\left(\dot x_2 + F_1 + 2\,H\,x_2\right) - \frac{1}{2}\dot\beta_2\,x_2 + \frac{1}{2}{\beta_2}^2\,C\,x_2 \\ x_2 & \rightarrow & x_2 \end{array}\right.$$

$$\text{auxiliary variables}\left\{\begin{array}{lll} F_1 & \rightarrow & F_1 \\ F_2 & \rightarrow & F_2 - \beta_2\,(\dot F_1 - [L]_{x_1}) - \frac{1}{2}\dot\beta_2\,F_1 + \frac{1}{2}{\beta_2}^2\,C\,F_1. \end{array}\right. \qquad (4.22)$$

•β_5-sector: The residual finite β_5-diffeomorphisms are obtained by the following composition of finite transformations:

$$X \xrightarrow{\beta_1,\beta_7} \Box \xrightarrow{\beta_8} \Box \xrightarrow{\beta_5} \tilde X,$$

with

$$\beta_8 = -H\beta_5 + \frac{1}{2}F\beta_5^2 - \frac{1}{2}\dot\beta_5, \qquad \beta_7 = -\frac{1}{2}F\beta_5,$$

$$\beta_1 = (2H^2 - T + \dot H)\beta_5 + 2H\dot\beta_5 + \tfrac{1}{2}\ddot\beta_5 - (2HF + \tfrac{1}{2}\dot F)\beta_5^2$$
$$-\tfrac{3}{2}F\beta_5\dot\beta_5 + \tfrac{1}{2}F^2\beta_5^3. \qquad (4.23)$$

and

$$\left\{\begin{array}{lll} T & \rightarrow & T + \frac{1}{2}\beta_5\,\dot F + F\,\dot\beta_5 + \beta_5\,F\,H - \frac{1}{4}{\beta_5}^2\,F^2 \\ H & \rightarrow & H - \frac{1}{2}\beta_5\,F \\ C & \rightarrow & C + \beta_5\,\left(\dot T - 6\,H\,\dot H - \ddot H + 4\,H\,T - 4\,H^3\right) \\ & & +\dot\beta_5\,\left(2\,T - 6\,H^2 - 3\,\dot H\right) - 3\,\ddot\beta_5\,H - \frac{1}{2}{\beta_5}^{(3)} + \\ & & {\beta_5}^2\,\left(5\,F\,H^2 - F\,T + 2\,F\,\dot H + 3\,\dot F\,H + \frac{1}{2}\ddot F\right) + \\ & & \beta_5\,\dot\beta_5\,\left(\frac{5}{2}\dot F + 8\,F\,H\right) + \frac{7}{4}{\dot\beta_5}^2\,F + \frac{3}{2}\beta_5\,\ddot\beta_5\,F - \\ & & {\beta_5}^3\left(2\,H\,F^2 + F\,\dot F\right) - 2\,{\beta_5}^2\,\dot\beta_5\,F^2 + \frac{1}{4}{\beta_5}^4\,F^3 \\ F & \rightarrow & F \end{array}\right.$$

$$
\text{matter variables}\left\{\begin{array}{lcl} x_1 & \to & x_1 \\ x_2 & \to & x_2 + \beta_5\ (\dot{x_1} + F_2 - 2\,H\,x_1) - \frac{1}{2}\dot{\beta}_5\,x_1 + \frac{1}{2}{\beta_5}^2\,F\,x_1 \end{array}\right.
$$

$$
\text{auxiliary variables}\left\{\begin{array}{lcl} F_1 & \to & F_1 - \beta_5\,(\dot{F}_2 - [L]_{x_2}) - \frac{1}{2}\dot{\beta}_5\,F_2 + \frac{1}{2}{\beta_5}^2\,F\,F_2 \\ F_2 & \to & F_2 . \end{array}\right. \tag{4.24}
$$

In this way we have integrated the non-linear $\mathcal{W}$-transformations from the knowledge of the finite linear $Sp(4)$ transformations. Notice the appearence of the schwarzian derivative in the transformation of T. We believe the procedure followed in this paper may be useful for the study of finite $\mathcal{W}$-transformations corresponding to general $\mathcal{W}$-algebras. The possible difficulties of this approach arise in the election of the independent parameters and gauge fields.

5 Conclusions

We have established a connection between zero-curvature condition of $2d$ gauge theories and relativistic particle models with an $Sp(2M)$ gauge group. The zero-curvature condition encodes a particular coupling between matter and $\mathcal{W}$-gravity.

We construct the finite linear gauge transformations of the $Sp(4)$ particle and perform a partial gauge-fixing at the level of finite transformations. We obtain its finite remnant gauge transformations. They can be formally considered as classical chiral finite $\mathcal{W}$-transformations of matter coupled to $Sp(4)$ $\mathcal{W}$-gravity. A peculiarity of these transformations is the non-closure of the algebra if we require the matter variables to transform as primary fields under Diff. A natural geometric interpretation of these transformations can be given in the framework of the flag bundle approach to zero-curvature condition [10] [6]. The supersymmetric extension of this approach has been studied in [18]. Further details will appear elsewhere [19].

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