On Lie point symmetries in mechanics

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Summary: We present some remarks on the existence and the properties of Lie point symmetries of finite dimensional dynamical systems expressed either in Newton-Lagrange or in Hamilton form. We show that the only Lie symmetries admitted by Newton-Lagrange-type problems are essentially linear symmetries, and construct the most general problem admitting such a symmetry. In the case of Hamilton problems, we discuss the differences and the relationships between the existence of time-independent Lie point symmetries and the invariance properties of the Hamiltonian under coordinate transformations.

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Introduction

In this paper we want to present some considerations on the symmetry properties of the dynamical problems which are expressed as systems either of second-order ordinary differential equations in Newton-Lagrange form, or of first-order equations in Hamilton form.

According to the old idea due to S. Lie, and recently reconsidered by various authors (see e.g. [1-8] and references therein, see also e.g. [9-11] for more specific applications to dynamical problems), one may introduce the very general notion of *Lie point symmetry* of a dynamical system, which is any transformation of the dynamical variables $t \to t' = t'(q,t),\ q \to q' = q'(q,t)$, where $t'(q,t),\ q'(q,t)$ are smooth (e.g. C^∞) functions on the whole domain of definition $\big[$respectively $t \to t' = t'(q,p,t),\ q \to q' = q'(q,p,t),\ p \to p' = p'(q,p,t)$ in the case of Hamiltonian formulation$\big]$, with the property of transforming into itself the set of the solutions of the given problem (i.e.: if $q(t)$ is a solution, then $q'(t')$ is too: for a more complete definition of Lie symmetry see the above mentioned references). This transformation is assumed to depend smoothly on a real parameter ϵ, and its infinitesimal generator will be denoted by γ.

The interest in the notion of Lie point symmetry, apart from its intrinsic abstract - algebraic and geometrical - features, is motivated by some successful applications in the analysis of several concrete problems: we refer to the already mentioned references for a list of these applications, which cover, especially, time-evolution problems (both in the form of ordinary and of partial differential equations), bifurcation theory, fluid dynamics. Other applications deal with Lagrangian formulation of mechanics, and the connections between Lie point symmetries and Noether theorem in its most general form [3].

In this paper, we shall avoid as far as possible any abstract approach, and we concentrate our attention on the existence and the properties of the Lie point symmetries admitted by problems in Newton-Lagrange and in Hamilton mechanics. We are mainly interested in autonomous systems, and in Lie point symmetries which are time-independent (LPTI), i.e. in the "purely geometrical " symmetries. However, to obtain more generality and to allow some useful comparisons, we shall occasionally consider also nonautonomous systems and/or possibly time dependent Lie symmetries. We shall carefully state our assumptions case by case. In any case, we do not consider transformations which change the time variable t; from now on, when speaking about Lie point symmetries, we shall always refer to transformations of this type (i.e. leaving t fixed).

In sect.1 we discuss the Lie symmetries of Newton-Lagrange-type problems; we shall show - in particular - that second-order systems of the form $\ddot q = F(q,t)$ $\big($or even, more in general, $\ddot q = F(\dot q, q, t)$, with F depending linearly on the "velocities" $\dot q\big)$ either do not admit LPTI symmetries (which is the most common case), or admit merely "linear" symmetries. Some interesting remarks are suggested by the construction of the most general system having such a symmetry.

After some short considerations on LPTI symmetries of first-order differential systems (sect. 2), we shall consider in sect. 3 the dynamical problems expressed in Hamiltonian form: here, a richer geometrical structure is obviously present, and more symmetries as well - as a natural consequence of the fact that transformations of both variables q and p are allowed; these would correspond, in the Lagrangian formalism, to (first order) contact transformations [3,5,8,12] rather than purely Lie point symmetries (enlarging th e class of allowed symmetries is actually one of the main advantages of Hamiltonian formulation of mechanics). Various, quite different, notions of symmetry for Hamiltonian problems can be introduced, related to the invariance of the Hamiltonian under coordinate transformations, or the existence of constants of motion. We shall examine these notions and discuss their relationships with the notion of LPTI symmetry.

1. Newton and Newton-Lagrange problems

Let us start considering systems of second-order differential equations of the general "Newton form"

$$\ddot{q}_i = F_i(q,\dot{q},t) \qquad (i=1,\dots,n) \tag{1.1}$$

for the n (generalized) Lagrangian coordinates $q_i = q_i(t)$, where the "forces " F_i are assumed to be smooth enough (for our purposes, a C^r differentiability with $r \geq 3$ would be sufficient). Let us look for the Lie point symmetries admitted by this problem: writing the Lie generator γ of this symmetry (possibly depending on time) in the form (sum over repeated indices)

$$\gamma = \theta_i(q,t)\frac{\partial}{\partial q_i} \tag{1.2}$$

the determining equations for the functions $\theta_i(q,t)$ are, as well known (see, e.g. [5,6]),

$$\theta_j\frac{\partial F_i}{\partial q_j} + \Big(\frac{\partial\theta_j}{\partial t} + \dot{q}_k\frac{\partial\theta_j}{\partial q_k}\Big)\frac{\partial F_i}{\partial \dot{q}_j} - F_j\frac{\partial\theta_i}{\partial q_j} - \dot{q}_j\dot{q}_k\frac{\partial^2\theta_i}{\partial q_j\partial q_k} - 2\dot{q}_j\frac{\partial^2\theta_i}{\partial t\partial q_j} - \frac{\partial^2\theta_i}{\partial t^2} = 0 \tag{1.3}$$

Observing that these are identities on $t, q, \dot{q}$, and that θ_i do not depend on $\dot{q}$, these equations can be splitted into several conditions, once the explicit form of F_i is given.

Assume now that the forces F_i in (1.1) depend at most linearly on the "velocities" $\dot{q}_i$, i.e.

$$\frac{\partial^2 F_i}{\partial\dot{q}_j\partial\dot{q}_k} = 0 \tag{1.4}$$

We stress that this is also the case if (1.1) are obtained as Euler-Lagrange equations from a Lagrangian function $\mathcal{L} = \mathcal{L}(q,\dot{q},t)$: in this case, indeed, $\mathcal{L}$ must be of the form

$$\mathcal{L} = \frac{1}{2}\dot{q}_i\dot{q}_i - \mathcal{V}(q,t) + \mathcal{A}_i(q,t)\ \dot{q}_i$$

where $\mathcal{V}$ and $\mathcal{A}_i$ are respectively the scalar and the vector potentials. This form of $\mathcal{L}$ implies (1.4) and actually covers many physically relevant problems (e.g. electromagnetism and conservative mechanics; notice that (1.4) covers linear dissipation as well). ¿From (1.3), one gets using (1.4),

$$\frac{\partial^2 \theta_i}{\partial q_j \partial q_k} = 0 \tag{1.5}$$

which implies immediately that the only admitted Lie point symmetries are purely "linear" symmetries, or - more precisely - that the functions θ_i must be of the form

$$\theta_i(q,t) = A_{ij}(t)\ q_j + b_i(t) \qquad (i,j = 1,\dots,n) \tag{1.6}$$

The result that the determining equations acquire a significant structure, or at least are greatly simplified when the r.h.s. of (1.1) depend linearly on the higher-order derivatives of the variables, is not peculiar to the case at hand: a similar situation occurs in fact also in different and more general contexts, as already remarked and discussed [5,8].

Next, assume more in particular that the dependence of the F_i on $\dot{q}_i$ has the form

$$F_i = \lambda \dot{q}_i + f_i(q,t) \tag{1.7}$$

where in general $\lambda = \lambda(t)$ may be any function of time (possibly $\lambda = 0$ or a constant, of course): in this case, again from (1.3), one obtains

$$\frac{\partial^2 \theta_i}{\partial t \partial q_j} = A_{ij} = \text{const}$$

and the determining equations become

$$(A_{jk} q_k + b_j) \frac{\partial f_i}{\partial q_j} - A_{ij} f_j = \ddot{b}_i - \lambda \dot{b}_i\ . \tag{1.8}$$

Clearly, this shows - first of all - that there are relatively "few" problems (1.1), with F_i given by (1.7), which can admit Lie point symmetries: it is in fact a quite exceptional case that, given F_i, some A, b can be found to satisfy (1.8): for instance, the only 1-dimensional problems ($n = 1$), admitting a Lie point symmetry, are (this can be easily verified differentiating (1.8) with respect to q)

$$\ddot{q} = \lambda(t)\ \dot{q} + k(t)\ q + c(t) \tag{1.9}$$

where $\lambda(t),\ k(t),\ c(t)$ are arbitrary functions of time. The symmetries admitted by (1.9) are given by

$$\gamma^{(1)} = \big(q + b^{(1)}(t)\big) \frac{\partial}{\partial q} \qquad \text{and} \qquad \gamma^{(0)} = b^{(0)}(t) \frac{\partial}{\partial q}$$

where $b^{(1)}(t)$ and $b^{(0)}(t)$ satisfy respectively the equations

$$\ddot{b}^{(1)} = \lambda\ \dot{b}^{(1)} + k\ b^{(1)} - c \qquad \text{and} \qquad \ddot{b}^{(0)} = \lambda\ \dot{b}^{(0)} + k\ b^{(0)}.$$

The presence of the first symmetry asserts that if $q_0(t)$ is a solution of (1.9), then

$$q'(t) \equiv q(t,\epsilon) = q_0(t)\ \mathrm{e}^{\epsilon} + b^{(1)}(t)\ (\mathrm{e}^{\epsilon} - 1)$$

is again a solution for any ϵ, the second simply states the elementary property that

$$q'(t) = q_0(t) + \epsilon b^{(0)}(t)$$

is a solution for any $b^{(0)}(t)$ solving the homogeneous equation.

Less trivial situations, involving nonlinear $F(q)$ or $F(q,t)$ can be constructed in dimension $n > 1$, as we shall see later on.

Another simple but interesting consequence of (1.8) can be deduced in the hypothesis that F_i do not depend on time: differentiating in fact (1.8) with respect to t, one easily obtains that either b_i are constants, or $f_i(q)$ depend linearly on the q_i.

Therefore, we can state the following result.

Proposition 1. *The Lie point symmetries (1.2) admitted by Newton-Lagrange problems (1.1), with F_i depending at most linearly on $\dot q_i$, have - if they exist - the form (1.6). If in particular*

$$F_i = \lambda(t)\ \dot q_i + f_i(q,t)$$

(including $\lambda = 0$ or constant), the admitted symmetries have the form (1.6) where A_{ij} are constants and $b_i(t)$ satisfy (1.8). In the further hypothesis that F_i are independent of time - and excluding the (rather trivial) case of f_i linearly dependent on q - then also b_i are constants, and the admitted symmetries - if any - have the form

$$\gamma = \big(A_{ij} q_j + b_i\big) \frac{\partial}{\partial q_i} \tag{1.10}$$

where the constants A_{ij} and b_i have to satisfy

$$(A_{jk} q_k + b_j) \frac{\partial f_i}{\partial q_j} - A_{ij} f_j = 0\ . \tag{1.11}$$

In the remaining of this section, we restrict our analysis to problems (1.1) where the forces F_i are independent on $\dot q$, and to Lie point symmetries which are independent of time.

To examine more carefully the existence of Lie point symmetries of a given Newton-type system, we shall reverse the problem, and construct the more general Newton system admitting a given LPTI symmetry.

In order to do this, and also in view of an application in sect. 3, let us come back preliminarily to a generic LPTI symmetry generator in the form (1.2) with $\theta_i = \theta_i(q)$, and introduce the set of the real functions $\Phi(q)$ which are left invariant by the action of the symmetry γ, i.e. $\Phi(q) = \Phi(q')$. This property is verified if and only if the condition

$$\gamma \cdot \Phi \equiv \theta_i \frac{\partial \Phi}{\partial q_i} = 0 \tag{1.12}$$

holds. From a geometrical point of view, the equation $\Phi(q) = k = \text{const}$ describes an invariant manifold under γ. Solving the above equation (1.12) by means of its characteristic equations:

$$\frac{dq_1}{\theta_1(q)} = \ldots = \frac{dq_n}{\theta_n(q)} \tag{1.13}$$

one naturally introduces a set of $(n-1)$ independent functions $h_\alpha(q)$, $\alpha = 1, \ldots, n-1$, such that

$$h_\alpha(q) = k_\alpha = \text{const} \qquad (\alpha = 1, \ldots, n-1) \tag{1.14}$$

where the k_α are the "constants of integration" of (1.13). Each of these characteristic integrals $h_\alpha(q)$ of (1.13), and any function $\Phi(q) = \Phi\big(h_1(q), \ldots, h_{n-1}(q)\big)$ of them, is an invariant function under γ.

We construct now the most general field of forces F_i which admits a given LPTI symmetry (1.10). Given γ, the conditions (1.11) become a system of partial differential equations for the F_i; this is a system "with the same principal part" [13], and its characteristic equations are

$$\frac{dq_1}{A_{1i}q_i + b_1} = \ldots = \frac{dq_n}{A_{ni}q_i + b_n} = \tag{1.15a}$$

$$= \frac{dF_1}{A_{1i}F_i} = \ldots = \frac{dF_n}{A_{ni}F_i} \tag{1.15b}$$

To solve these, one first sees from (1.11,15) that the dependence of the F_i o n the variables q_i must be of the form

$$F_i = K_{ij}q_j + c_i \tag{1.16}$$

where the matrix K and the vector c satisfy

$$KA\ q + K\ b = AK\ q + A\ c\ . \tag{1.17}$$

Then K is the most general matrix such that

$$K\ A = A\ K \tag{1.18}$$

whereas the solution of the remaining condition

$$K\ b = A\ c \tag{1.19}$$

depends on whether A is invertible or not. In the first case, $c = A^{-1}K\ b = K\ A^{-1}b$, and the most general form of F is

$$F = K\ (q + A^{-1}b) = K\ q + A^{-1}K\ b \tag{1.20}$$

with K the most general nonzero matrix commuting with A. If A is singular, one has to further distinguish various cases: if K is chosen such that $K\ b \in R(A) =$ the range of A, then c is any vector such that $A\ c = K\ b$; if b is such that for no $K \neq 0$ one has $K\ b \in R(A)$, then (1.19) can be solved by

$$K = 0, \quad \text{and} \quad c \in N(A) = \text{ the kernel of } A \tag{1.21}$$

If finally Kb is "generic", one may decompose it as a sum of two terms and obtain a combination of the two above possibilities, as easily verified (a sort of Fredholm alternative theorem). In conclusion, even if A is singular, one has again (1.16), i.e. $F = K\ q + c$, with the only difference tha t also the choice $K = 0$ and $c \neq 0$ is admitted. To obtain finally the complete general solution of (1.15), one has to take into account the subsystem (1.15a), and one can conclude [13] that the general solution is obtained assuming that th e entries K_{ij} of K in (1.16) are arbitrary functions of the characteristic integrals $h_\alpha(q)$ of (1.15a). These functions $h_\alpha(q)$ are precisely the invariant functions under the symmetry γ: indeed, they solve the equation (cf. (1.12-13))

$$\gamma \cdot h \equiv (A_{ij}q_j + b_i)\frac{\partial h}{\partial q_i}\ . \tag{1.22}$$

Notice that the above result holds for both autonomous and nonautonomous systems: in fact, time t is left invariant by γ, just as the functions $h_\alpha(q)$.

Then, summarizing, we have the following conclusion.

Proposition 2. *Given a LPTI symmetry of the form (according to Prop. 1)*

$$\gamma = (A_{ij}q_j + b_i)\frac{\partial}{\partial q_i}$$

where A_{ij}, b_i are constants, the most general Newton problem (autonomous or not) admitting the symmetry generated by γ has the form

$$\ddot{q}_i = K_{ij}\big(h_\alpha(q), t\big)\ q_j + c_i(t) \tag{1.23}$$

where K_{ij} are arbitrary functions of the quantities $h_\alpha(q)$ which are left invariant under the action of γ:

$$\gamma \cdot h_\alpha = 0$$

(and possibly of the time), restricted by the commutativity condition $A_{im}K_{mj} = K_{im}A_{mj}$, and c_i are determined by A, b, K.

We give a simple example to describe this result. Let $n = 2$, $q \equiv (q_1, q_2) \equiv (x, y)$ and

$$A = \begin{pmatrix} 1 & 0 \\ 3 & -2 \end{pmatrix} \quad ; \quad b = 0$$

i.e. consider the LPTI symmetry

$$\gamma = x \frac{\partial}{\partial x} + (3x - 2y) \frac{\partial}{\partial y} \tag{1.24}$$

A function $h = h(x, y)$ invariant under γ is

$$h = x^3 - x^2 y$$

and the matrices K are

$$K = \begin{pmatrix} k_1 & 0 \\ k_1 - k_2 & k_2 \end{pmatrix}$$

Then, the most general problem admitting the LPTI symmetry γ (1.24) is

$$\begin{aligned} \ddot{x} =& k_1(x^3 - x^2 y, t)\ x \\ \ddot{y} =& \big(k_1(x^3 - x^2 y, t) - k_2(x^3 - x^2 y, t)\big)\ x + k_2(x^3 - x^2 y, t)\ y \end{aligned} \tag{1.25}$$

where k_1, k_2 are arbitrary functions of h, t. The existence of the symmetry γ means that, if $x_0(t), y_0(t)$ is any solution of (1.25), then al so

$$\begin{aligned} x'(t) &\equiv x(t, \epsilon) = x_0(t)\ \mathrm{e}^{\epsilon} \\ y'(t) &\equiv y(t, \epsilon) = y_0(t)\ \mathrm{e}^{-2\epsilon} + x_0(t)\ (\mathrm{e}^{\epsilon} - \mathrm{e}^{-2\epsilon}) \end{aligned}$$

is a solution of (1.25) for any ϵ.

2. Newton-Lagrange problems in first-order form

In order to find a richer geometrical structure (and to find other, possibly nonlinear Lie symmetries) for Newton-Lagrange problems, one could - as well known [1-8] - look for symmetries more general than Lie point ones, of the form (contact or dynamical symmetries)

$$\widehat{\gamma} = \theta_i(q, \dot{q}, t) \frac{\partial}{\partial q_i} + \theta_i^{(1)}(q, \dot{q}, t) \frac{\partial}{\partial \dot{q}_i} \tag{2.1}$$

or transform the original system into a first-order system in $2n$ variables, putting

$$\begin{aligned} \dot{q}_i &= p_i \\ \dot{p}_i &= F_i(q, p, t) \end{aligned} \tag{2.2}$$

In this second way, one falls into the more general problem of finding the Lie point symmetries

$$\gamma = \varphi_j(u,t)\frac{\partial}{\partial u_j} \tag{2.3}$$

of a first-order system

$$\dot{u}_j = f_j(u,t) \tag{2.4}$$

The determining equations for this general case (2.3-4) are [1,8,10]

$$\varphi_i \frac{\partial f_j}{\partial u_i} - f_i \frac{\partial \varphi_j}{\partial u_i} = \frac{\partial \varphi_j}{\partial t} \tag{2.5}$$

In the special case of (2.2), putting

$$\gamma = \theta_i(q,p,t)\frac{\partial}{\partial q_i} + \psi_i(q,p,t)\frac{\partial}{\partial p_i} \tag{2.6}$$

the above conditions (2.5) become explicitly

$$\begin{aligned} \psi_j &= p_i \frac{\partial \theta_j}{\partial q_i} + F_i \frac{\partial \theta_j}{\partial p_i} \\ \theta_i \frac{\partial F_j}{\partial q_i} &= p_i \frac{\partial \psi_j}{\partial q_i} + F_i \frac{\partial \psi_j}{\partial p_i} \end{aligned} \tag{2.7}$$

The general problem of finding the Lie point symmetries of first-order systems has been considered in detail especially in [1-2] and also in [8,10,11], and we do not pursue here the analysis of this point. Let us add just another remark along the lines of the previous section. Assume that in (2.2) $F_i = F_i(q)$, and let us look for those particular LPTI symmetries which have the form

$$\gamma = \theta_i(q)\frac{\partial}{\partial q_i} + \psi_i(q,p)\frac{\partial}{\partial p_i} \tag{2.8}$$

where θ_i depend only on the variables q (this restriction is suggested by the desire of preserving, as far as possible, the physical meaning of the "coordinates" q). Then, it is easy to obtain, using (2.7), the following result.

Proposition 3. *The only LPTI symmetries having the special form (2.8), which are admitted by the problem (2.2) with $F_i = F_i(q)$, are merely of the following type*

$$\gamma = (A_{ij}q_j + b_i)\frac{\partial}{\partial q_i} + A_{ij}p_j\frac{\partial}{\partial p_i}$$

where A_{ij}, b_i are constants.

Another property of LPTI symmetries of first-order systems which should be mentioned is the following. Let $\gamma_1, \gamma_2, \ldots, \gamma_r$ be a set of LPTI symmetry generators admitted by the system (2.4), and let $\{h_\alpha = h_\alpha(u)\}$ be a set of constants of motion of (2.4), i.e. of functions satisfying

$$\frac{\mathrm{d}h_\alpha}{\mathrm{d}t} = f_j \frac{\partial h_\alpha}{\partial u_j} = 0 \tag{2.9}$$

$\big($the constants of motions h_α are precisely the functions satisfying $\gamma_f \cdot h_\alpha = 0$, i.e. the invariant fuctions under the operator

$$\gamma_f \equiv f_j \frac{\partial}{\partial u_j} \tag{2.10}$$

expressing the time-evolution of the dynamical flow, which is in fact a LPTI symmetry generator for the problem (2.4)$\big)$; then also

$$\widehat{\gamma} = h_1 \gamma_1 + \ldots + h_r \gamma_r \tag{2.11}$$

is a LPTI symmetry for (2.4). This means that the set of the LPTI symmetries for first-order systems is not only an ($\infty-$dimensional) algebra, but rather a finite dimensional *module* (cf.[14]) over the set of constants of motions.

3. Hamilton problems

In this section, we consider exclusively autonomous problems and time independent symmetries.

We specialize here our discussion to the problems expressed in the Hamiltonian form

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} \qquad (i = 1, \ldots, n) \tag{3.1}$$

with $H = H(q, p)$. There are various notions related to the symmetry properties of Hamiltonian problems. In order to clearly show the differences and discuss the relationships between them, let us recall some well known definitions, writing down the following list.

A) (LPTI symmetries). The operator

$$\gamma = \theta_i(q, p) \frac{\partial}{\partial q_i} + \psi_i(q, p) \frac{\partial}{\partial p_i} \tag{3.2}$$

is a LPTI symmetry for (3.1) if it generates a transformation $q \to q', \ p \to p'$ with the property that solutions $q(t), \ p(t)$ of (3.1) are mapped into solution s $q'(t), \ p'(t)$ of (3.1).

B) (invariance of H). Given a (continuously differentiable) coordinate transformation $q \to \widetilde{q}$, $p \to \widetilde{p}$ generated by the infinitesimal operator

$$\widetilde{\gamma} = \widetilde{\theta}_i(q,p)\frac{\partial}{\partial q_i} + \widetilde{\psi}_i(q,p)\frac{\partial}{\partial p_i} \tag{3.3}$$

the Hamiltonian is invariant under $\widetilde{\gamma}$ if $H(\widetilde{q},\widetilde{p}) = H(q,p)$, or equivalently if

$$\widetilde{\gamma} \cdot H = \widetilde{\theta}_i(q,p)\frac{\partial H}{\partial q_i} + \widetilde{\psi}_i(q,p)\frac{\partial H}{\partial p_i} = 0 \tag{3.4}$$

C) (canonical transformations). If an infinitesimal canonical transformation leaves invariant the Hamiltonian $H(q,p)$, then the generatrix function $G = G(q,p)$ of this transformation has the property

$$\frac{\partial G}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial G}{\partial p_i}\frac{\partial H}{\partial q_i} = \frac{dG}{dt} = 0 \tag{3.5}$$

and (equivalently) $G(q,p)$ is a constant of motion.

Let us remark, incidentally, that another commonly mentioned symmetry transformation, i.e. the scaling $H \to \alpha H$, is not included in the present scheme, because it requires a scaling of the time variable.

Now, notice first of all that A) and B) are different. For instance, consider the following problem, with $n = 1$, and $H = q(p^2+1)$:

$$\begin{aligned} \dot{q} &= 2qp \\ \dot{p} &= -p^2 - 1 \end{aligned} \tag{3.6}$$

It is easy to verify that

$$\gamma = q\frac{\partial}{\partial q}$$

generates a LPTI symmetry for the problem (3.6), but does not satisfy B); instead, for

$$\widetilde{\gamma} = \frac{2qp}{p^2+1}\,\frac{\partial}{\partial q} - \frac{\partial}{\partial p}$$

one has $\widetilde{\gamma} \cdot H = 0$, i.e. the Hamiltonian is invariant under $\widetilde{\gamma}$, but $\widetilde{\gamma}$ does not generates a LPTI symmetry for (3.6). Notice also that neither g nor $\widetilde{\gamma}$ above generate canonical transformations.

The determining equations (2.5) for the functions $\theta_i(q,p)$, $\psi_i(q,p)$ in order that γ (3.2) be a LPTI symmetry admitted by the Hamilton problem (3.1), become

$$\begin{aligned} \frac{\partial \theta_j}{\partial p_i}\frac{\partial H}{\partial q_i} + \theta_i\frac{\partial^2 H}{\partial q_i \partial p_j} - \frac{\partial \theta_j}{\partial q_i}\frac{\partial H}{\partial p_i} + \psi_i\frac{\partial^2 H}{\partial p_i \partial p_j} = 0 \\ -\frac{\partial \psi_j}{\partial p_i}\frac{\partial H}{\partial q_i} + \theta_i\frac{\partial^2 H}{\partial q_i \partial q_j} + \frac{\partial \psi_j}{\partial q_i}\frac{\partial H}{\partial p_i} + \psi_i\frac{\partial^2 H}{\partial p_i \partial q_j} = 0 \end{aligned} \tag{3.7}$$

Differentiating with respect to q_j and to p_j the condition (3.4) expressin g the invariance of H under $\widetilde{\gamma}$, one has instead

$$\begin{aligned}
&\frac{\partial\widetilde{\theta}_i}{\partial q_j}\frac{\partial H}{\partial q_i}+\widetilde{\theta}_i\frac{\partial^2 H}{\partial q_i\partial q_j}+\frac{\partial\widetilde{\psi}_i}{\partial q_j}\frac{\partial H}{\partial p_i}+\widetilde{\psi}_i\frac{\partial^2 H}{\partial p_i\partial q_j}=0\\
&\frac{\partial\widetilde{\theta}_i}{\partial p_j}\frac{\partial H}{\partial q_i}+\widetilde{\theta}_i\frac{\partial^2 H}{\partial q_i\partial p_j}+\frac{\partial\widetilde{\psi}_i}{\partial p_j}\frac{\partial H}{\partial p_i}+\widetilde{\psi}_i\frac{\partial^2 H}{\partial p_i\partial p_j}=0
\end{aligned}\tag{3.8}$$

Comparing with (3.7), one sees that if $\widetilde{\gamma}$ satisfies B) and the following conditions

$$\begin{gathered}
\frac{\partial\widetilde{\theta}_i}{\partial p_j}=\frac{\partial\widetilde{\theta}_j}{\partial p_i}\qquad\qquad \frac{\partial\widetilde{\psi}_i}{\partial q_j}=\frac{\partial\widetilde{\psi}_j}{\partial q_i}\\
\frac{\partial\widetilde{\theta}_i}{\partial q_j}=-\frac{\partial\widetilde{\psi}_j}{\partial p_i}
\end{gathered}\tag{3.9}$$

then $\widetilde{\gamma}$ is also a LPTI symmetry. But (3.9) are just the "integrability conditions" for a infinitesimal transformation

$$\begin{aligned}
q&\to q'+\epsilon\widetilde{\theta}(q,p)\\
p&\to p'+\epsilon\widetilde{\psi}(q,p)
\end{aligned}\tag{3.10}$$

to be canonical [12]: then, in these hypotheses, $\widetilde{\gamma}$ determines also a canonical transformation leaving invariant the Hamiltonian, and there is a generatrix function $G=G(q,p)$ which is a constant of motion $\mathrm{d}G/\mathrm{d}t=0$. Let us observe that, instead, there may exist LPTI symmetries satisfying also the integrability conditions (3.9), but which do not satisfy the invariance condition B). For instance, for the above example (3.6), the operator

$$\gamma'=\frac{1}{p^2+1}\,\frac{\partial}{\partial q}$$

generates a LPTI symmetry for (3.6), satisfies (3.9) and then generates a canonical transformation, but the Hamiltonian of the problem is not invariant under this γ'.

Obviously, the evolution operator (cf. (2.10))

$$\gamma_f=\frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}-\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}\tag{3.11}$$

generating the dynamical flow of the problem (3.1), satisfies in any case all of the three above conditions A), B), C).

We can then conclude with the following statement.

Proposition 4. *Given the Hamiltonian problem (3.1), there may be LPTI symmetries admitted by (3.1) which are not canonical transformations, or which are also canonical* *transformations but do not leave invariant the Hamiltonian. Similarly, there may be infinitesimal coordinate transformations leaving invariant the Hamiltonian but which are not LPTI symmetries. If a coordinate transformation leaving invariant the Hamiltonian is canonical, then it is also a LPTI symmetry admitted by the problem; and each constant of motion $G = G(q,p)$ generates not only a canonical transformation which leaves invariant the Hamiltonian, but also a LPTI symmetry which is given by*

$$\gamma^{(G)} = \theta_i^{(G)}(q,p)\frac{\partial}{\partial q_i} + \psi_i^{(G)}(q,p)\frac{\partial}{\partial p_i} \qquad \text{where} \qquad \theta_i^{(G)} = \frac{\partial G}{\partial p_i}, \quad \psi_i^{(G)} = -\frac{\partial G}{\partial q_i} \tag{3.12}$$

A final remark on the integrability conditions (3.9) can be useful. Clearly, if $\widetilde{\gamma}$ satisfies B), then also $k(q,p)\,\widetilde{\gamma}$ does, for any function $k(q,p)$; if in particular there is an "integrating factor" $\widetilde{k} = \widetilde{k}(q,p)$ for the differential form

$$-\widetilde{\psi}_i \; \mathrm{d}q_i + \widetilde{\theta}_i \; \mathrm{d}p_i \tag{3.13}$$

then $\widetilde{k}\ \widetilde{\gamma}$ satisfies (3.9) and determines a canonical transformation. Denoting by $G_{\widetilde{k}} = G_{\widetilde{k}}(q,p)$ the generatrix function of this transformation, one has in particular the invariance property under $\widetilde{\gamma}$

$$\widetilde{\gamma} \cdot G_{\widetilde{k}} = 0 \ . \tag{3.14}$$

The situation is quite different for the LPTI symmetries: in fact, as remarked in sect. 2, if γ generates a LPTI symmetry, also $h(q,p)\gamma$ does if (and only if) $h = h(q,p)$ is a constant of motion (and no more an arbitrary function). Then, the problem of finding an integrating factor for a LPTI symmetry is restricted to the set of constants of motion. This shows another difference between the notion of Lie point symmetry and the one related to the invariance of the Hamiltonian under generic coordinate transformations.

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