Reduction and Equivariant Branching Lemma:
Dynamical Systems, Evolution PDEs, and Gauge Theories

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1. Introduction

One of the simplest yet most useful tools in equivariant bifurcation theory is the so called "Equivariant Branching Lemma" (EBL in the following).

This was first proved by Cicogna [1] and Vanderbauwhede [2] in the context of bifurcation of stationary solutions; Golubitsky and Stewart [3] provided then an extension to the case of Hopf bifurcation (of periodic solutions). The EBL is also effective in the context of the so called "quaternionic bifurcation" [4,5,6], but this latter case seems to be of no use in applications.

All the above mentioned results deal with bifurcation problem which are equivariant under the action of a linear representation of a (compact) Lie group [7-12], and have proved to be of great use in application, see e.g. [10].

The EBL was recently extended by Cicogna [13] to the case of bifurcation problems symmetric under general - i.e. not necessarily linear - groups of Lie-point transformations. For a treatment of these transformations in the context of differential equations, we refer to [14-18].

The purpose of the present note is to extend the EBL to gauge symmetries and gauge-symmetric problems; this will be done by looking at gauge symmetries as a specific class of Lie-point symmetries in an appropriate space. We will also obtain a weaker result (reduction lemma) for evolution PDEs.

In the preparation of this work, after obtaining our results, we became aware that they - as well as the previous similar ones - can be also seen as deriving from the "Symmetric Criticality Principle" of Palais [19,20] (we thank prof. Bourguignon for pointing out this fact and the references). This principle seems to have passed unnoticed in the context of bifurcation theory (probably due to the fact it was originally meant to deal just with gauge theories), but it is actually a very powerful tool and worth being widely known.

On the other side, the powerful results obtained by equivariant bifurcation theory over the last few years are probably not well known to gauge theorists; one good reason for this is that equivariant bifurcation theory deals usually with finite-dimensional linear group actions, but recently it has been shown that the basic results of the theory extend to Lie-point, and therefore gauge, symmetries as well [21,22]. A related discussion, focused on gauge theories, is presented in [23].

2. Symmetry of differential equations

Despite the simplicity of the EBL, we will best understand it by looking at it as composed of two parts, a "reduction lemma" and a "branching lemma"; the reduction part is in facts completely general and does not depend on any bifurcation phenomena or assumption.

Let us consider a time-evolution ordinary differential equation (ODE), which could also be the bifurcation equation relative to some bifurcation problem,

$$\dot{u} = F(u) \qquad\qquad F(0) = 0 \tag{1}$$

where u belongs to a smooth manifold $\mathcal{U}$ which we will consider as embedded in R^N with coordinates $\{u^1, ..., u^N\}$; $F(u)$ will then be a C^∞ tangent vector field on $\mathcal{U}$,

$$F : \mathcal{U} \to T\mathcal{U} \tag{2}$$

Next, assume that eq. (1), which will also be written as

$$\Delta(u, \dot{u}) \equiv \dot{u} - F(u) = 0 \tag{3}$$

has a Lie-point time-independent group of (evolutionary) symmetries G_Δ, with Lie algebra $\mathcal{G}_\Delta$, whose elements will be denoted by γ.

This means the following (for details we refer to e.g. [14] or to the other texts quoted above): a generic vector field (VF) η on $\mathcal{U}$ will be written as

$$\eta = \varphi^i(u) \frac{\partial}{\partial u^i} \tag{4}$$

It determines a VF $\eta^{(1)}$ (its first prolongation) in the jet space $J_1\mathcal{U}$ (actually, in this case $J_1\mathcal{U} = T\mathcal{U}$), which can be thought with coordinates $\{\dot{u}^1, ..., \dot{u}^N; u^1, ..., u^N; t)\}$. The equation $\Delta = 0$ determines a manifold S_Δ in $J_1\mathcal{U}$,

$$S_\Delta = \{(\dot{u}, u; t) \ / \ u \in \mathcal{U} \ , \ \dot{u}^i = F^i(u, t)\} \tag{5}$$

The VF $\gamma \in \mathcal{G}_\Delta$ are those such that their first prolongation leaves invariant S_Δ, or

$$\gamma^{(1)} : S_\Delta \to TS_\Delta \tag{6}$$

Remark: As the notation of eq. (1) suggests, we want actually to deal with autonomous ODEs. In this case, S_Δ will actually be of the form

$$S_\Delta = S_\Delta^0 \times R$$

with S^0_Δ belonging to the reduced jet space $J^0_1\mathcal{U} = J_1\mathcal{U}/R$ which can be thought as the subspace of $J_1\mathcal{U}$ with coordinates $\{(\dot{u}^1, ..., \dot{u}^N; u^1, ..., u^N)\}$.

Remark: Obviously, any nonautonomous ODE can be seen as autonomous by adding a new variable $u^{N+1} = t$; in this way one can consider VFs on $R \times \mathcal{U}$, i.e. of the general form $\eta = \varphi^i(u,t)\partial/\partial u^i + \tau(u,t)\partial/\partial t$; we will keep to (1), (4) for notational simplicity.

Now, to any subgroup $G_i \subset G_\Delta$ we can associate a pointwise invariant manifold, $\mathrm{Fix}(G_i) \subseteq \mathcal{U}$:

$$\mathrm{Fix}(G_i) = \{u \in \mathcal{U} \ / \ gu = u \ \ \forall g \in G_i\} \subseteq \mathcal{U} \tag{7}$$

(this could be empty); at the Lie algebra level, if $\mathcal{G}_i$ is the Lie algebra of G_i, we have

$$\mathrm{Fix}(G_i) \subseteq W_i \equiv \mathrm{Ker}(\mathcal{G}_i) = \{u \in \mathcal{U} \ / \ J\gamma \cdot u = 0 \ \ \forall \gamma \in \mathcal{G}_i\} \subseteq \mathcal{U} \tag{8}$$

The equality sign between $\mathrm{Fix}(G_i)$ and W_i holds for G_i a connected Lie group; if this is not the case, $\mathrm{Fix}(G_i)$ can in general be a proper subspace of W_i.

Example: Let us consider a linear example: let $\mathcal{U} = R^3$, and G_1 be given by $SO(2) \times Z_2$, where $SO(2)$ represents rotations around the axis u^3, and Z_2 reflections across the (u^1, u^2) plane. Then W_1 is the whole u^3 axis, while $\mathrm{Fix}(G_1)$ is just the origin. Remark that for $G_2 = SO(2) \times Z_2$ with $SO(2)$ as before and Z_2 a reflection in the plane (u^1, u^3) (or any plane including the axis u^3) one has $\mathrm{Fix}(G_2) = W_2$.

To any point $u \in \mathcal{U}$ are associated an isotropy subgroup $G_u \subseteq G_\Delta$ and an isotropy subalgebra $\mathcal{G}_u \subseteq \mathcal{G}_\Delta$ (which is the Lie algebra of G_u):

$$G_u = \{g \in G_\Delta : \ gu = u\} \subseteq G_\Delta \tag{9}$$

$$\mathcal{G}_u = \{\gamma \in \mathcal{G}_\Delta : \ \gamma u = 0\} \subseteq \mathcal{G}_\Delta \tag{10}$$

3. The reduction lemma

We can now state the reduction lemma:

Reduction Lemma: A solution $u(t)$ of (1) with initial datum $u(t_0) = u_0$ will satisfy $u(t) \in W_0 \equiv \mathrm{Ker}(\mathcal{G}_{u_0})$ $\forall t \geq t_0$.

This can also be stated as

Reduction Lemma: The manifolds $W_i \equiv \mathrm{Ker}(\mathcal{G}_i)$ are invariant under the flow of Δ for any $\mathcal{G}_i \subseteq \mathcal{G}_\Delta$.

Remark: Clearly, if $\mathcal{G}_\alpha \subset \mathcal{G}_\beta$, then $W_\beta \subseteq W_\alpha$; in other words an order relation among subalgebras of $\mathcal{G}_\Delta$ implies an inclusion relation among the invariant manifolds W_i; the invariant manifolds corresponding to maximal isotropy subalgebras will not contain any other invariant manifold.

Remark: If, as before, $\mathcal{G}_\alpha \subset \mathcal{G}_\beta$ and $W_\beta \subset W_\alpha$, a solution with $u_0 \in W_\alpha \backslash W_\beta$ can flow into the invariant submanifold W_β. We stress that this can be a limit process, but it can also be a process which takes place in a finite time. As an example (again, linear), let $\mathcal{U} = R^2$ and consider the equation $\dot{u}_i = -k^2 u_i^\alpha$, $k \neq 0$, $u \in R^2$, so that $G_1 = SO(2) \in G_\Delta$, with $G_2 = \{e\} \subset G_1$ and $W_1 = \{0\}$, $W_2 = R^2$. Then for $u(t_0) \neq 0$, we have that if $\alpha = 1$ the solution $u(t)$ tends to W_1 as a limit point, while for $0 < \alpha < 1$ this is reached in a finite time. Anyway, it should be remarked that in this ODE case $F(u)$ fails to be C^∞ (or even C^1) just on W_1, which indeed could be not invariant, as e.g. for $k = 3/2$, $\alpha = 1/2$.

Proof: Let us now prove the reduction lemma. First of all, we notice that, writing the evolution VF $\varepsilon : \mathcal{U} \to T\mathcal{U}$, i.e. the VF such that $\dot{u} = \varepsilon \cdot u$,

$$\varepsilon = F^i(u) \frac{\partial}{\partial u^i} \tag{11}$$

the condition for $\eta \in \mathcal{G}_\Delta$ is precisely (see [22])

$$[\eta, \varepsilon] = 0 \tag{15}$$

¿From this the lemma follows at once: in facts, let $\gamma \in \mathcal{G}_0 \subset \mathcal{G}_\Delta$. Then by definition $\gamma \cdot u = 0 \ \forall u \in W_0$, and $\varepsilon \cdot 0 = 0$ since $\varepsilon = F(u)\partial_u$ and $F(0) = 0$. But $\gamma \in \mathcal{G}_\Delta \Longrightarrow [\varepsilon, \gamma] = 0$, so that $\gamma \varepsilon u - \varepsilon \gamma u = 0$, i.e. $\varepsilon : W_0 \to TW_0$ •

The reduction lemma implies a useful consequence: let $F_\alpha(u)$ be the restriction of $F(u)$ to $W_\alpha \subset \mathcal{U}$; then $F_\alpha : W_\alpha \to TW_\alpha$, and one has the

Corollary: Let $u(t)$ be a solution of $\dot{u} = F(u)$ and $v(t)$ a solution of $\dot{v} = F_\alpha(v)$; let $u(t_0) = v(t_0) \in W_\alpha$. Then $u(t) = v(t) \ \ \forall t \geq t_0$.

In other words, in order to study solutions of (1) one can consider the simplest (or not more difficult) equation $\dot{x} = F_0(x)$, where $F_0(x) = F(u)|_{W_0}$.

Remark: If one has to find critical points of a potential $V(x)$ defined on a manifold $\mathcal{M} \subseteq R^N$, i.e. solutions of the equation $\nabla V(x) = 0$, these can be seen as stationary solutions of the equation $\dot{x}^i = F^i(x) = -\nabla^i V(x)$

In this way, we see that the reduction lemma can be recast in the following form in variational case:

Reduction Lemma (variational case): If $V : \mathcal{M} \to R$ and $V(x)$ is invariant under the action of the Lie group $G : \mathcal{M} \to \mathcal{M}$, and $G_x = G_\alpha \subseteq G$ is the isotropy group of the point $x \in \mathcal{M}$, then $\nabla V(x) \in T_x\mathrm{Fix}(G_x) \subseteq T_x W_\alpha$. Therefore $\nabla V(x_0) = 0$ if and only if $(y, \nabla V(x)) = 0$, $\forall y \in T_x W_\alpha$, where $(.,.)$ is the standard scalar product induced in $T_x W_\alpha$ by the scalar product in R^N.

Remark: This is actually at the basis of Michel's theorem [21], which inspired the first version of the EBL. I.e., we have followed a path which is just opposite to the historical one.

Remark: It should be noticed that the argument used to prove the reduction lemma are purely geometrical; in particular, nothing changes if the manifold $\mathcal{U}$ happens to be of countably infinite dimension, or if $\mathcal{M}$ is an Hilbert space or an infinite dimensional Riemannian manifold (so that a scalar product is well defined).

4. The equivariant branching lemma

We can now introduce the bifurcation setting. We consider the case in which one of the u^i's in (1) can be regarded as a parameter, also denoted λ. For the sake of simplicity, we will assume $\mathcal{U} = \mathcal{U}_e \times \Lambda$, where $\Lambda = (-\mu, \mu) \subset R$ is the parameter space, and $\mathcal{U}_e \subseteq R^{N-1}$ with coordinates $(u^1, ..., u^{N-1})$.

It was shown in [22] that in considering symmetries of the form (4) it suffices then to consider those with $\varphi^N = 0$, i.e. those which do not act on λ; in this case the invariant manifolds W_α will be foliated as $W_\alpha = \Lambda \times W_\alpha^{(\lambda)}$. If moreover

$$\frac{\partial \varphi^i(u)}{\partial u^N} \equiv \frac{\partial \varphi^i}{\partial \lambda} = 0 \tag{16}$$

we have a trivial fibration

$$W_\alpha = \Lambda \times W_\alpha^0 \tag{18}$$

We can restate the bifurcation lemma in the bifurcation setting as follows:

Reduction Lemma (bifurcation setting): Let us consider the equation $\dot{u} = F(\lambda, u)$, where $\lambda \in \Lambda,\ u \in \mathcal{U};\ F(\lambda, u_0) = 0\ \forall \lambda$,with F a C^∞ vector field, $F : \Lambda \times \mathcal{U} \to T\mathcal{U}$, and let G_Δ^0 be its λ-independent symmetry group which stabilizes u_0 (i.e. $gu_0 = 0$), with Lie algebra $\mathcal{G}_\Delta^0$. Then for any subalgebra $\mathcal{G}_\alpha^0 \subset \mathcal{G}_\Delta^0$, the manifold $W_\alpha^0 = \{u \in \mathcal{U}\ /\ \gamma u = 0\ \forall \gamma \in \mathcal{G}_\Delta^0\}$ is invariant under the flow of the equation, and so is therefore $\Lambda \times W_\alpha^0$.

¿From this the EBL follows at once. Let us first introduce *standard bifurcation assumptions*: *i)* $F(\lambda, u_0) = 0\ \forall \lambda \in \Lambda$; *ii)* if $L(\lambda) = F_u(\lambda, u_0)$, $\mathrm{Ker}L(\lambda) = 0$ for $\lambda \neq 0$, and $L(0)$ is a Fredholm operator of index zero; *iii)* if $\sigma_i(\lambda)$ are the critical eigenvalues of $L(\lambda)$, $d\sigma_i(\lambda)/d\lambda > 0$ for $\lambda = 0$. Let us moreover assume a *stability condition*: there is an open compact set $\mathcal{K} \subset \mathcal{U}$, of the same dimension m as $\mathcal{U}$ and whose border $\partial\mathcal{K}$ is $m-1$ dimensional (a disk in $\mathcal{U}$) such that $F(\lambda, u)$ points inward on $\partial\mathcal{K}$ and $u_0 \in \mathcal{K}$. Then we have the

Equivariant Branching Lemma (stationary case): If $\mathcal{G}_\Delta^0$ admits a subalgebra $\mathcal{G}_\alpha^0 \subset \mathcal{G}_\Delta^0$ such that W_α^0 is one-dimensional (with $T_0 W_\alpha^0 \subset T_0 \mathrm{Ker}L(0)$), then there is a branch of stationary solutions $u_\alpha(\lambda)$ bifurcating from u_0 and such that $u_\alpha(\lambda) \in W_\alpha^0\ \forall \lambda \in \Lambda$.

Remark: Clearly, $T_0 W_\alpha^0$ is the tangent space to W_α^0 in $u = u_0$. Also, $\mathrm{Ker}L(0)$ is obviously a linear space.

Proof: By the reduction lemma, we are authorized to consider the reduction $F_\alpha(\lambda, u)$ of $F(\lambda, u)$ to the manifold W_α^0; this manifold contains u_0 by definition of $\mathcal{G}_\Delta^0$. Let $\mathcal{K}_\alpha = \mathcal{K} \cap W_\alpha^0$; this is an open interval containing u_0, so we can give an orientation to it, and $\partial\mathcal{K} = K_- \cup K_+$. By the stability condition $F_\alpha(\lambda, K_-) > 0$, $F_\alpha(\lambda, K_+) < 0$. It follows at once from the bifurcation assumptions that $\dot{u} = F_\alpha(\lambda, u)$ undergoes a bifurcation at $\lambda = \lambda_0$, and there is a branch of stationary solutions; the reduction lemma ensures that if $F_\alpha(\lambda, u_\alpha(\lambda)) = 0$, then also $F(\lambda, u_\alpha(\lambda)) = 0$. •

In the Hopf case, one adopts Hopf bifurcation assumptions, i.e. in *ii*) above we ask for the spectrum of $L(\lambda)$ not to touch the line $\mathrm{Re}\sigma(\lambda) = 0$ for $\lambda \neq \lambda_0$, and that only a finite number of eigenvalues cross the imaginary axis of the complex plane for $\lambda = \lambda_0$ (the linear space spanned by the corresponding eigenvectors will be called $\mathcal{N}$), with $d\mathrm{Re}\sigma(\lambda)/d\lambda > 0$ and the remaining of the spectrum at finite distance from the imaginary axis.

Equivariant Branching Lemma (periodic case): If $\mathcal{G}_\Delta^0$ admits a subalgebra $\mathcal{G}_\alpha^0 \subset \mathcal{G}_\Delta^0$ such that W_α^0 is two-dimensional (with $T_0 W_\alpha^0 \subset T_0 \mathcal{N}$), then there is a branch of periodic solutions $u_\alpha(\lambda)$ bifurcating from u_0 and such that $u_\alpha(\lambda) \in W_\alpha^0 \ \forall \lambda \in \Lambda$.

Proof: As before, we can reduce to W_α^0 by the reduction lemma, and consider in it the invariant compact set $\mathcal{K}_\alpha = \mathcal{K} \cap W_\alpha^0$. It suffices then to invoke Poincare'-Bendixson theorem [26] to ensure the existence of periodic solutions. •

Remark: The above case is nongeneric, but can occurr, for $\mathcal{G}_\alpha^0$ a maximal isotropy subalgebra of $\mathcal{G}_\Delta^0$; notice that it can also occurr if $\exists \mathcal{G}_\beta^0$ such that $\mathcal{G}_\alpha^0 \subset \mathcal{G}_\beta^0 \subseteq \mathcal{G}_\Delta^0$, and which meets the conditions for the stationary case EBL.

Remark: The stability assumption could be substituted, in both stationary and periodic case, by a weaker one: i.e., it suffices to ask $F_\alpha(\lambda, u)$ points inward of $\mathcal{K}_\alpha$ on $\partial\mathcal{K}_\alpha$.

For higher dimensional W_α, we have a result due to Cicogna [27] (his proof for the linear case is purely topological and so applies to the Lie-point case as well):

Equivariant Branching Lemma (stationary case, bis): If $\mathcal{G}_\Delta^0$ admits a subalgebra $\mathcal{G}_\alpha^0 \subset \mathcal{G}_\Delta^0$ such that W_α^0 is $(2m+1)$-dimensional (with $T_0 W_\alpha^0 \subset T_0 \mathrm{Ker} L(0)$), then there is a branch of stationary solutions $u_\alpha(\lambda)$ bifurcating from u_0 and such that $u_\alpha(\lambda) \in W_\alpha^0 \ \forall \lambda \in \Lambda$.

Proof: It is well known that any VF on D^{2m+1} has at least a zero [28,29]. The reduction lemma and the stability assumptions allows actually to reduce to a disk $\mathcal{K}_\alpha = D^{2m+1} \subset W_\alpha^0$, and hence the lemma. •

Remark: Such an extension is not possible for the case of stationary solutions: there is no equivalent of the fixed point theorem for invariant circles, so that we can not go beyond the classical Poincare'-Bendixson theorem.

Equivariant Branching Lemma (variational case, bis): Consider the case Δ is issued from a variational problem, i.e. $\Delta \equiv \dot{u}_i - \partial V(u)/\partial u_i$, so that stable stationary solutions of Δ corresponds to minima of the potential. If $\mathcal{G}_\Delta^0$ admits a subalgebra $\mathcal{G}_\alpha^0 \subset \mathcal{G}_\Delta^0$ such that W_α^0 is of any finite dimension (with $T_0 W_\alpha^0 \subset T_0 \mathrm{Ker} L(0)$), then there is a branch of minima $u_\alpha(\lambda)$ of $V(u)$ bifurcating from u_0 and such that $u_\alpha(\lambda) \in W_\alpha^0 \ \forall \lambda \in \Lambda$.

Remark: If in the previous cases $\mathcal{K}_\alpha$ is not D^1, D^2 but S^1, S^2, the lemma continues to hold in a weaker form. This is due to the assumptions on $L(\lambda)$: infacts, $S^1 \backslash u_0 \simeq D^1$, and now u_0 corresponds to ∂D^1; in the same way, $S^2 \backslash u_0 \simeq D^2$, and u_0 corresponds to ∂D^2 (e.g. by stereographic projection, see fig.1). In this way one can also deal with the case $\mathcal{K}_\alpha = S^{2m+1}$: since $S^n \backslash u_0 \simeq D^n$, we are reconducted to the above mentioned case (EBL, stationary case bis). We still have stationary or periodic solutions in the appropriate subspaces, but these will not in general form a smooth branch bifurcating from u_0, see the example.

Example: Let $\mathcal{U}$ be the torus $T^2 = S^1 \times S^1$, with coordinates $\theta, \psi \in I \times I$, $I = [-\pi, \pi]$. Consider the equation

$$\begin{aligned} \dot{\theta} =& \lambda \sin\theta - \sin^3\theta \\ \dot{\psi} =& - \sin\psi \end{aligned} \tag{19}$$

Clearly, $(\theta, \psi) = (0,0)$ is a stationary solution for any λ, and at $\lambda = 0$ this undergoes a bifurcation ($L(\lambda) = \mathrm{diag}(\lambda, -1)$). Also, $\sin\psi \partial_\psi \in \mathcal{G}_\Delta$, and it leaves pointwise invariant the circle $\psi = 0$ (and the one $\psi = \pm\pi$), so $WP/a^0 = S^1$; it is clear also that we have a bifurcation of stationary solutions. Notice that in the absence of the term $\sin^3\theta$ one would have stationary solutions in W_α^0 as well, but now they would not bifurcate from $(0,0)$; instead, one would have a discontinuous transition. In physical terms, one would have a first order transition instead than a second order one.

We have then a weaker form of the EBL, which we will state as

Equivariant "Branching" Lemma (discontinuous case): Let us consider the equation $\dot{u} = F(\lambda, u)$, in which $F(\lambda, u_0) = 0$, with λ, u, F as before, and G_Δ its λ-independent symmetry group; let $G_\Delta^0 = G_\Delta \cap G_{u_0}$, with Lie algebra $\mathcal{G}_\Delta^0$. Let the bifurcation assumptions hold for $L(\lambda) = F_u(\lambda, u)$ and let there be a compact set $\mathcal{K} \subset \mathcal{U}$, $u_0 \in \mathcal{K}$, invariant under the flow of F. Then, let $\mathcal{G}_\Delta^0$ admit an isotropy subalgebra $\mathcal{G}_\alpha$ such that $\mathcal{K}_\alpha = \mathcal{K} \cap W_\alpha = S^n$ and such that $T_0 W_\alpha \subset \mathcal{N}$. If $n = 2m+1$, for $\lambda > \lambda_0$ there exists in $\mathcal{K}_\alpha$ a new stable stationary solution $\widetilde{u}_0(\lambda)$; if $n = 2$, for $\lambda > \lambda_0$ there exists in $\mathcal{K}_\alpha$ a periodic solution $\widetilde{u}(t)$

distinct from u_0.

Proof: Immediate from the reduction lemma and the above remarks.

Remark: If no additional assumption is made, we do not know if $\lim_{\lambda \to 0} \widetilde{u}_0(\lambda) = u_0$ or not.

Remark: Again, one can consider stationary solutions of a problem $\dot{x} = \nabla V(\lambda, x)$ and reach the case of G-invariant potentials. In this context, in the language of Landau theory [30] the above lemma guarantees the existence of a (first or second order) phase transition.

Remark: All the above results hold true if we consider G_Δ^0, G_α instead of $\mathcal{G}_\Delta^0, \mathcal{G}_\alpha$, and $\mathrm{Fix}(G_\alpha)$ instead of W_α, etc. We have preferred the Lie algebra setting because one has algorithmic ways to compute the Lie algebra of Lie-point symmetries of differential equations [31], while no systematic procedure exists for discrete symmetries. In the context of first order transitions, anyway, it becomes more relevant to consider groups than algebras, in view of physical applications [30].

5. Symmetries of PDEs

We want now to make contact with gauge symmetries; to this purpose, let us see how the above discussion extends to PDEs and gauge theories.

In the case of PDEs, we will denote as x, t the independent variables, and as u the dependent ones; the independent spatial variables x will belong to a manifold X which we will see as a d-dimensional manifold embedded in R^m, with coordinates $\{u^1, ..., u^m\}$; (the time t, when appearing, will belong to a manifold $\Theta \subseteq R$, i.e. $\Theta = R$ or R_+ or still S^1 if we look for periodic solutions). The dependent variables will belong to a manifold $\mathcal{U}$, which again we will consider as embedded in R^N with coordinates $u^1, ..., u^N$ as in the ODE case.

We have therefore a natural setting in terms of a fiber bundle $\pi : E \to \mathcal{B}$, where $\mathcal{B} = X \times \Theta$ or $\mathcal{B} = X$ as appropriate (i.e., $\mathcal{B}$ is the base space of independent coordinates; from now on we consider $\mathcal{B} = X$ unless discussing evolution equations), with fiber $\mathcal{U}$ and projection $\pi(u, x) = x$

A generic Lie point VF would then be written as

$$\eta = \varphi \partial_u + \xi \partial_x \equiv \varphi^\alpha(x,u) \frac{\partial}{\partial u^\alpha} + \xi^i(x,u) \frac{\partial}{\partial x^i} \tag{1}$$

We will consider only projectable VFs, i.e. those of the form

$$\eta = \varphi^\alpha(x,u) \frac{\partial}{\partial u^\alpha} + \xi^i(x) \frac{\partial}{\partial x^i} \tag{2}$$

In physical terms, eq. (2) means that the transformations of space-time do not depend on the fields $u(x)$, and is therefore a natural assumption.

In the class of projectable VFs, we will single out the evolutionary VFs, i.e. those of the form

$$\eta = \varphi^\alpha(x,u) \frac{\partial}{\partial u^\alpha} \tag{3}$$

which are those which do not affect space-time, but only the fields u. These are in facts local gauge symmetry generators, as we will point out in a moment; the case $\varphi = \varphi(u)$ would correspond to global gauge symmetries.

Remark on terminology: We will denote local gauge symmetries simply as gauge ones, and reserve for global gauge symmetries the term "rigid gauge symmetries".

A PDE of n-th order will be written as

$$\Delta(x, u^{(n)}) = 0 \tag{4}$$

or also, when in the following we want to point out its evolutionary character (and writing then t for the time coordinate)

$$\Delta(x,t;u^{(n)}) \equiv u_t - F(x,t;u^{[n]}) = 0 \tag{5}$$

In this case it will be understood that in F do not appear time derivatives of the u's. If the x, t do not appear explicitely in F, the equation is autonomous.

As before, the equation $\Delta = 0$ determines a manifold S_Δ in $J^n\mathcal{U}$,

$$S_\Delta = \{(x,t;u^n) \ / \ Ju^{(k)} \in J^k\mathcal{U}, \ k \le n \ ; \ \Delta(x,t;u^n) = 0\} \tag{6}$$

and if the equation is autonomous, this is naturally fibered as

$$S_\Delta = S_\Delta^0 \times \mathcal{B} \tag{7}$$

with S_Δ^0 belonging to the reduced jet space $J^n\mathcal{U}/\mathcal{B} \equiv \mathcal{U}^n$.

Now, as it is well known, [14-18], a solution $u(x,t)$ to (4) is a section of $\pi : E \to B$ such that the n-lift of its graph (which is the same as the graph of its n-th prolongation) $\Gamma_{u^n} \equiv \Gamma_u^n \subset J^n\mathcal{U}$ belongs entirely to S_Δ, and the lift of a Lie-point symmetry of (4) is a differentiable transformation of $J^n\mathcal{U}$ which leaves invariant the manifold $S_\Delta \subset J^n\mathcal{U}$; the Lie point algebra of symmetries of $\Delta = 0$, $\mathcal{G}_\Delta$, is made up of those VF γ as in (1) which satisfy

$$\gamma^n : S_\Delta \to TS_\Delta \tag{8}$$

We will denote by $\mathcal{G}_\Delta^{(g)} \subset \mathcal{G}_\Delta$ the algebra of evolutionary, or gauge, symmetries (see (3)), by $\mathcal{G}_\Delta^{(r)} \subset \mathcal{G}_\Delta$ that of rigid gauge symmetries (4), and by $\mathcal{G}_\Delta^{(p)} \subset \mathcal{G}_\Delta$ that of projectable symmetries (2). Clearly, one has

$$\mathcal{G}_\Delta^{(r)} \subset \mathcal{G}_\Delta^{(g)} \subset \mathcal{G}_\Delta^{(p)} \subset \mathcal{G}_\Delta \tag{9}$$

6. Gauge symmetries and Lie point vector fields

We want now to point out that, as claimed above, gauge symmetries are a special case of evolutionary Lie-point symmetries. We will now denote all the independent variables as $x \in X$, i.e. we suppose space and time play equivalent roles and do not distinguish between them.

In a gauge theory [32,33,34] the physical fields are sections of a fiber bundle $\pi : E \to X$ with typical fiber $F = \pi^{-1}(x)$ (F is a linear space for matter fields; the Lie gauge group itself for gauge fields); there is an action of the Lie group G, the gauge group, defined in F by the representation T, with $T_g : F \to F$ the operator corresponding to the group element $g \in G$. One is specially interested, for physical reasons [32], in the case of G a compact Lie group and T a unitary linear representation.

A gauge transformation is then a function $g : X \to G$, and this acts on a section $\sigma(x)$, $\sigma : X \to F$, of the bundle as

$$(g \cdot \sigma)(x) = T_{g(x)}\sigma(x) \tag{10}$$

Obviously, one can also look at gauge transformations at the Lie algebra level: let $\mathcal{G}$ be the Lie algebra of the group G, and L the representation of $\mathcal{G}$ corresponding to the representation T of G (i.e., L gives the infinitesimal generators for T). Then, an (infinitesimal) gauge transformation will be a function $\gamma : X \to \mathcal{G}$, and this will act on a section $\sigma(x)$, $\sigma : X \to F$ of the bundle as

$$(e + \varepsilon\gamma) \cdot \sigma = \sigma + \varepsilon\delta\sigma \tag{11}$$

$$(\delta\sigma)(x) \equiv (\gamma \cdot \sigma)(x) = L_{\gamma(x)}\sigma(x) \tag{12}$$

Now, given the Lie algebra $\mathcal{G}$, we can choose a basis in it, $\{\ell_1, ..., \ell_s\}$, such that

$$[J\ell_i, \ell_j] = c^k_{ij}\ell_k \tag{13}$$

and that $\mathcal{G}$ is the linear span of $\{\ell_1, ..., \ell_s\}$, i.e. $\forall \ell_0 \in \mathcal{G}$, $\exists a = (a_1, ..., a_s) \in R^s$ such that

$$\ell_0 = \sum_{i=1}^{s} a_i\ell_i \tag{14}$$

Clearly, to such ℓ_i's correspond L_i's with the same properties, i.e.

$$[L_i, L_j] = c^k_{ij}L_k \tag{15}$$

and $\forall L_0 \in L$, $\exists a \in R^s$ such that

$$L_0 = \sum_{i=1}^{s} a_iL_i \tag{16}$$

Now, any function $\gamma : X \to \mathcal{G}$ induces a function $\alpha : X \to R^s$ by

$$\gamma(x) = \sum_{i=1}^{s} \alpha_i(x)L_i \tag{17}$$

where $\alpha(x) = (\alpha_1(x), ..., \alpha_s(x))$, $\alpha_i(x) \in R$.

Remark: In algebraic language, the set Γ of infinitesimal gauge transformations $\gamma : X \to \mathcal{G}$ is a module over $\mathcal{G}$ [35].

To connect with the formalism used to discuss Lie-point symmetries, let us just look at the action of $\mathcal{G}$ on F: if we choose in F a basis $\{u_1, ..., u_n\}$, and $\ell_0 \in \mathcal{G}$ is represented in L by the matrix $L_0 = L_{ik}$, then it is also possible to consider a representation $\widehat{L}$, equivalent to L, in terms of differential operators or, which is the same, of tangent vector fields on F; now to the matrix $L_0 = L_{ik}$ corresponds the operator

$$\widehat{L}_0 = (L_{ik}^+ u_k) \frac{\partial}{\partial u_i} \tag{18}$$

so that $L_{ik} u_k \equiv -\widehat{L}_0 u$ (using $L^+ = -L$, since we have a unitary representation of a compact Lie group, as recalled above).

Therefore, to the infinitesimal gauge transformation $\gamma : X \to \mathcal{G}$ is naturally associated a Lie point VF

$$\widehat{\gamma} = -\sum_{i=1}^{s} a_i(x) \widehat{L}_i \equiv \alpha_i(x) [L_i]_{mn} u_n \frac{\partial}{\partial u_m} \equiv \varphi^m(x,u) \frac{\partial}{\partial u_m} \equiv \varphi(x,u) \partial_u \tag{19}$$

Remark: The linearity of the representation L is reflected in the fact that φ is linear in the u's,

$$\frac{\partial^2 \varphi^m}{\partial u_i \partial u_j} = 0 \tag{20}$$

Also, $L^+ = -L$ ensures $\partial \varphi^m / \partial u_m = 0$.

Example: It can be worth considering shortly a very simple example, just to fix ideas. Let $\mathcal{U} = R^2$, $G = SO(2) \simeq U(1)$ and $T_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, so that $s = 1$ and the Lie algebra $\mathcal{G}$ is generated by

$$L_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -L_1^+ \tag{21}$$

To this corresponds the VF

$$\widehat{L}_1 = -u_1 \frac{\partial}{\partial u_2} + u_2 \frac{\partial}{\partial u_1} \tag{22}$$

In facts, $(I + \varepsilon L_1) u = u + \varepsilon \delta u$; $\delta u = \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix} = -\widehat{L}_1 u$. To any $\gamma : X \to \mathcal{G}$ corresponds then, via the α introduced above, a VF

$$\widehat{\gamma} = -\alpha(x) \widehat{L}_1 = (\alpha(x) u_1) \frac{\partial}{\partial u_2} - (\alpha(x) u_2) \frac{\partial}{\partial u_1} \tag{23}$$

Remark: In (18) one has L^+ instead than L in order to ensure that $[\widehat{L}_i, \widehat{L}_j] = c_{ij}^k \widehat{L}_k$ with the same c_{ij}^k as in (15); if using L, one would have a minus sign in the c_{ij}^k.

7. Reduction Lemma for gauge theories

We do now aim at extending the EBL to critical section of gauge invariant functionals, or gauge functionals for short.

Let $u = u(x)$ with $x \in X \subseteq R^d$ and $u \in U \subseteq R^n$, with X a smooth manifold of dimension q embedded in R^d (with coordinates $\{x_1, ..., x_d\}$) and $\mathcal{U}$ a smooth manifold of dimension p embedded in R^n (with coordinates $\{u_1, ..., u_n\}$), and let $L[u]$ be a smooth functional defined by a smooth density $\mathcal{L} : U \times X \to R$, i.e.

$$L[u] = \int_M \mathcal{L}(u(x), x) dx \qquad\qquad M \subseteq X \tag{1}$$

Let us now consider a compact Lie group G (with Lie algebra $\mathcal{G}$) acting on U by the representation Λ. If for any smooth function $g : X \to G$ one has $L[\Lambda_g u] = L[u]$ then we say that L is gauge-invariant for G. In facts, this is equivalent to

$$\mathcal{L}(\Lambda_{g(x)} u(x), x) = \mathcal{L}(u(x), x) \tag{2}$$

due the arbitrarity of $g : X \to G$.

At the Lie algebra level, we have that under a function $\gamma : X \to \mathcal{G}$, the functional (1) becomes

$$\begin{aligned} (I + \varepsilon\gamma) L[u] = & \int_M [\mathcal{L}(u(x) + \varepsilon(\gamma \cdot u)(x), x)]\, dx = \\ & = \int_M [\mathcal{L}(u(x), x) + \varepsilon \delta\mathcal{L}(u(x), x)]\, dx = L[u] + \varepsilon(\delta L)[u] \end{aligned} \tag{3}$$

where the variation δL is given by

$$\delta L[u] \equiv \int_M \left[\frac{\partial \mathcal{L}}{\partial u}(u(x), x) \cdot (\gamma \cdot u)(x) \right] dx \tag{4}$$

Writing γ as a vector field, $\gamma = \varphi(u, x)\partial_u$, this becomes

$$\delta L[u] = \int_M \left[\frac{\partial \mathcal{L}}{\partial u_i}(u(x), x) \cdot \varphi^i(u(x), x) \right] dx \tag{5}$$

Remark: The above notation suggests that $\mathcal{L}$ depends on u but not on its derivatives, as in facts we will assume in the following. Anyway, the case in which $\mathcal{L}$ depends e.g. on first derivatives of u is readily seen to be equivalent to the above, enlarging the space U to $U \times R^{pq}$, where the new variables are $\partial_i u_j \equiv \partial u_j / \partial x_i$, $i = 1, ..., d$, $j = 1, ..., n$, with constraints to take into account the relations existing between the u's and the ∂u's (i.e. we assign a *contact structure* [36,37]), and others to ensure that in facts $x \in X$, $u \in U$. Also, in this case $\mathcal{G}$ will act not by γ but rather by its first prolongation $\gamma^{(1)}$, see [14] for details; see also [23].

For the sake of semplicity, in the following we will also assume that $\partial \mathcal{L} / \partial x = 0$, i.e. the density $\mathcal{L}$ is autonomous (or homogeneous, as all the points on M are equivalent).

The group of gauge transformations has a Lie algebra Γ,

$$\Gamma = \{ \eta = \varphi(x, u) \partial_u \ / \ \varphi(x, u) = \alpha_i(x) L_i \} \tag{6}$$

where α, L_i are as in the previous section; given a section σ of the bundle $\pi : E \to X$, $E = X \times U$, its symmetry (isotropy) algebra is

$$\Gamma_\sigma = \{ \eta \in \Gamma \ / \ \eta \cdot \sigma = 0 \} \tag{7}$$

Given a subalgebra Γ_0 of Γ, we can define the space of Γ_0-invariant sections , $\mathcal{W}_0$, by

$$\mathcal{W}_0 = \{ \sigma \in \Sigma(E, X) \ / \ \eta \cdot \sigma = 0 \quad J \forall \eta \in \Gamma_0 \} \tag{8}$$

where $\Sigma(E, X)$ is the set of sections of $\pi : E \to X$.

Remark: If the action of $\mathcal{G}$ on the fiber U is linear (we have seen that this amounts to have φ linear in u), then $\mathcal{W}_0$ is a linear space [19].

Remark: In general, it can be proven that if the action of $\mathcal{G}$ is smooth with compact orbits, $\mathcal{W}_0$ is a smooth submanifold of Σ.

The variation of δL under $\sigma \to \sigma + \varepsilon \tau$, with $\tau \in \Sigma$, can be written in the notation used for (5) as

$$\delta L[\sigma] = \varepsilon \int_M \left[\tau^i(x) \frac{\partial}{\partial u^i} \mathcal{L}(\sigma(x)) \right] dx \equiv \varepsilon \int_M \left[J(\tau \cdot \nabla) \mathcal{L} \right] dx \tag{9}$$

This shows that for the variation of $L[\sigma]$ to vanish for every τ, it is needed to have

$$\nabla \mathcal{L}(\sigma(x)) = 0 \qquad\qquad \forall x \in M \tag{10}$$

A section $\sigma(x)$ is said to be critical for L if $\delta L[\sigma] = 0$ under $\sigma \to \sigma + \tau$, for every $\tau \in \Sigma$. The above discussion means that we have the following *criticality criterion:* σ is critical for L as in (1) if and only if (10) is satisfied.

We can now state the RL for the case of gauge functionals

Reduction Lemma (gauge functionals): If $\mathcal{L} : U \to R$ and $\mathcal{L}$ is invariant under the action of the group of gauge diffeomorphisms generated by Γ, and $\Gamma_\sigma = \Gamma_0 \subseteq \Gamma$ is the isotropy algebra of the section $\sigma \in \Sigma$, then $\nabla\mathcal{L}(\sigma(x)) \in T_\sigma \mathcal{W}_0$. Therefore, $\delta L[\sigma] = 0$ if and only if $\delta L = 0$ under $\sigma \to \sigma + \varepsilon\tau$ for all $\tau \in \mathcal{W}_0$.

Proof: The gradient appearing in (10) is, by definition (see (9)), in TU_x, where $U_x \equiv \pi^{-1}(x)$ is the fiber through x. More precisely, $(\nabla\mathcal{L})(\sigma(x)) \in T_{\sigma(x)}U_x$. It suffices therefore to consider the finite dimensional restriction of our setting to U_x. The algebra Γ_σ can also be characterized as

$$\Gamma_\sigma = \{\gamma : X \to \mathcal{G} \ / \ J\gamma(x) \in \mathcal{G}_{\sigma(x)} \quad \forall x \in X\} \tag{11}$$

where $\mathcal{G}_{\sigma(x)} \subseteq \mathcal{G}$ is the isotropy subalgebra of $\sigma(x) \in U$. Analogously, $\mathcal{W}_\sigma$ can be characterized as

$$\mathcal{W}_\sigma = \{\theta \in \Sigma(E, X) \ / \ \theta(x) \in \mathcal{W}_{\sigma(x)} \quad \forall x \in X\} \tag{12}$$

Now, we know by the discussion of the RL in variational case and for finite dimension that, if $\mathcal{L} : U \to R$ is invariant under $\mathcal{G}_\alpha$, then $\nabla\mathcal{L} \in TW_\alpha$; in the present case $\mathcal{G}_\alpha \equiv \mathcal{G}_{\sigma(x)}$ (in facts by definition, or by (11), $\gamma(x) \in \mathcal{G}_{\sigma(x)}$ $\forall\gamma \in \Gamma_\sigma$). Therefore, for any x we have $\nabla\mathcal{L}(\sigma(x)) \in T_{\sigma(x)}W_{\sigma(x)} \subseteq T_{\sigma(x)}U_x$. By the definition (12) we have that $\nabla\mathcal{L}[\sigma] \in T_{\sigma(x)}\mathcal{W}_0$: indeed, for any $\tau \in \Sigma(E, X)$, $(\tau(x) \cdot \nabla)\mathcal{L}(\sigma(x)) \in \mathcal{W}_\sigma$, since $\nabla\mathcal{L}(\sigma(x)) \in TW_{\sigma(x)}$.

¿From the above discussion it also follows that given $\tau(x) = \tau_0(x) + \tau_1(x)$, with $\tau_0 \in \mathcal{W}_0$ and $\tau_1 \in W_0^c \equiv \Sigma(E, X)\backslash\mathcal{W}_0$, the variation of $L[\sigma]$, for $\sigma \in \mathcal{W}_0$, under $\sigma \to \sigma + \varepsilon\tau$ is the same as the one under $\sigma \to \sigma + \varepsilon\tau_0$, i.e. what is affirmed by the lemma. •

Remark: the above lemma can be restated as follows: the sections which are critical among sections with prescribed symmetry are also critical *tout court.* For this reason, an analogue of this lemma [19,20] was also called "Principle of Symmetric Criticality".

As remarked in the introduction, the result of Palais just mentioned seems to be not so widely known - especially in the bifurcation community - as it would deserve to be. Therefore, we quote it here (adapting the notation to that of the present paper) from [19]:

Symmetric Criticality Theorem (Palais): Let G be a compact Lie group, X a smooth manifold, $\pi : E \to X$ a smooth G-fiber bundle over X, and Σ a Banach manifold of sections of E. Let G act on Σ by $(g\sigma)(x) = g(\sigma(x))$ and let $L : \Sigma \to R$ be a smooth G-invariant function on Σ. Then the set $\mathcal{W}$ of G- equivariant sections in Σ is a smooth submanifold of Σ, and if $\sigma \in \mathcal{W}$ is a critical point of $L|_{\mathcal{W}}$ then σ is in fact a critical point of L.

Remark: Palais does actually also consider the case of G acting nontrivially on the base manifold X; we have excluded this case from his statement as well as from our discussion just for simplicity, but it does not present new difficulties.

Remark: For the proof of the Symmetric Criticality Theorem, see [19], as well as for extensions of it, examples, and counterexamples to naive generalizations. A number of applications to mathematical physics are given in [20].

Remark: Strictly speaking, the Symmetric Criticality Theorem concerns only sections σ such that $\Gamma_\sigma = \{\gamma : X \to \mathcal{G}_0\}$ for some subalgebra $\mathcal{G}_0 \subseteq \mathcal{G}$, but the generalization to $\Gamma_\sigma = \Gamma_0 \subseteq \Gamma$ is straightforward.

8. Symmetric critical sections of gauge functionals

We can now use the RL in order to get informations about critical sections of gauge functionals. From now on, an isotropy subalgebra will be a subalgebra which is the isotropy algebra of a point different than the origin. We will need, as in the finite dimensional case, a *stability assumption*: there is an open compact set $K \subset U$, topologically a disk, containing the origin, such that $\nabla\mathcal{L}(u)$ points outward of K on ∂K. We have then that:

Symmetric existence lemma: Given a maximal isotropy subalgebra $\Gamma_0 \subseteq \Gamma$, there is a critical section $\sigma \in \mathcal{W}_0$ for L.

Proof: Let us consider L_0, the restriction of L to $\mathcal{W}_0$. If we consider $W_0 = \{u \in U \ / \ \mathcal{G}_0 \subseteq \mathcal{G}_u\}$, then $K_0 = K \cap W_0$ is a compact set, and $\nabla\mathcal{L}$ points outward of K_0 on ∂K_0. It is easy to see that there are critical sections $\sigma_c \in \mathcal{W}_0$ of L_0, which satisfy $\sigma_c(x) \in K_0$. The RL ensures these are also critical for L. •

Remark: Notice that this holds for any finite dimension of W_0.

Remark: For a linear action of $\mathcal{G}$ on U, the trivial section $\sigma_0(x) = 0$ has always full symmetry, $\Gamma_{\sigma_0} = \Gamma$; the critical section whose existence is granted by the lemma could just be σ_0. This is also the reason for considering Γ_0 a maximal isotropy subalgebra: for nonmaximal ones $\Gamma_1 \subset \Gamma_0$, the lemma continues to hold, but it could happen that for every critical section σ_1 in $\mathcal{W}_1$, $\Gamma_{\sigma_1} = \Gamma_0$, i.e. actually $\sigma_1 \in \mathcal{W}_0$; i.e., repeatedly applying the lemma over a chain of subgroups does not give any new information.

Remark: Obviously, the lemma continues to hold if $\nabla\mathcal{L}$ points inward of K_0 on ∂K_0; one prefers to look for minima of L due to clear physical reasons.

The above remarks suggest to consider the case of σ_0 a local (non degenerate) maximum for L. This guarantees the existence of a small disk $B \subset U$ such that $\nabla\mathcal{L}$ points outward of B on ∂B. Our stability assumption will then also require that $\bar{B} \subset K$. We have then:

Symmetric existence lemma (bis): Given the above assumptions and a maximal isotropy subalgebra $\Gamma_0 \subseteq \Gamma$, there is a critical section $\sigma \in \mathcal{W}_0$ for L, with $\sigma \neq \sigma_0$.

Proof: Just as before, considering the compact set $K \backslash B$ in the place of K. Now, the critical section will satisfy $\sigma_c(x) \in K_0 \backslash B_0$, where $K_0 = X \cap W_0$ as before, and $B_0 = B \cap W_0$. •

Remark: The property $\sigma_c(x) \in K_0 \backslash B_0$ depends crucially on the triviality of the bundle $E = X \times U$. For nontrivial E, one can just say $\sigma_c(x) \in K_0$. In other words, for nontrivial E, the theorem holds only locally.

Remark: We stress that the above results guarantee criticality of $\sigma \in \mathcal{W}_0$ also "in directions transverse to $\mathcal{W}_0$", but no statement can be made about stability: a section which is a minimum for L_0 could be a saddle point for L.

Remark: This is also an appropriate point to stress that in our discussion we avoid topological matters, which are of a different nature than those discussed here. In other words, our discussion deals with (and holds only for) trivial bundles or, equivalently, local sections.

9. Equivariant Branching Lemma for gauge functionals

Finally, let us introduce in $\mathcal{L}$, and therefore in L, a $\mathcal{C}^\infty$ dependence on a real parameter $\lambda \in \Lambda = [-a, a] \subset R$, so to have

$$L_\lambda[\sigma] = \int_M \mathcal{L}(\lambda, \sigma(x))dx \tag{13}$$

Define the $n \times n$ real symmetric matrix $H(\lambda)$ as

$$H_{ij}(\lambda) = \frac{\partial^2 \mathcal{L}(\lambda, 0)}{\partial u_i \partial u_j} J \tag{14}$$

and denote its eigenvalues as $h_i(\lambda)$, $i = 1, ..., n$.

We want to consider the situation of $\sigma_0(x) \equiv 0$ a critical section $\forall \lambda \in \Lambda$, stable for $\lambda < \lambda_0 = 0$ and unstable for $\lambda > \lambda_0$. This means that $u = 0$ is a minimum of $\mathcal{L}(\lambda, u)$ for $\lambda < 0$, and a saddle point or a maximum for $\lambda \geq 0$. For simplicity of notation, we will just consider the case in which $u = 0$ loses stability in all directions at the same time, the degeneracy being entirely due to the symmetry; i.e. $G(u)$ is not contained in any proper linear subspace of U for $u \neq 0$.

We have therefore, for what concerns the eigenvalues $h_i(\lambda)$, the following *bifurcation assumptions*:

$$h_i(\lambda_0) = 0 \qquad ; \qquad \frac{\partial h_i(\lambda_0)}{\partial \lambda} J > 0 \tag{15}$$

(the matrix $H(\lambda)$ is playing here the role played by L_0 in the finite dimensional ODE case, see sect.4)

We have then the

Equivariant Branching Lemma (gauge): If the above bifurcation and stability assumptions are verified, and $\mathcal{G}_0 \subseteq \mathcal{G}$ is a maximal isotropy subalgebra of $\mathcal{G}$ for the action of $\mathcal{G}$ on U, then there is a branch of critical sections $\sigma_\lambda(x)$ for $L_\lambda[\sigma]$, bifurcating from σ_0 at $\lambda = \lambda_0$ and such that $\sigma_\lambda \in \mathcal{W}_0 \;\; \forall \lambda \geq \lambda_0$.

Remark: As already pointed out, $\sigma_\lambda \in \mathcal{W}_0$ is equivalent to $\sigma_\lambda(x) \in W_0 \;\; J \forall x \in X$.

Proof: We do actually use both the fact that $\mathcal{L}$ does not depend on derivatives of σ neither on x, and the trivial structure of the bundle $E = X \times U$. In facts, if we just consider a fiber $\pi^{-1}(x)$, and $\mathcal{L} : \Lambda \times \pi^{-1}(x) \to R$, we recover the finite dimensional situation studied above, and we are granted there is a branch of subsets $U_\lambda \subset U$ bifurcating from the origin $U_0 = \{0\}$ at $\lambda = \lambda_0$ and such that $u \in U_\lambda$ is a (degenerate) minimum of $\mathcal{L}(\lambda, u)$ (in general, $U_\lambda = G(u)\ \forall u \in U_\lambda$, i.e. the degeneracy is entirely due to the symmetry and U_λ is just a group orbit; we also recall that $G(u) \simeq G/G_u$). Moreover, the finite dimensional EBL ensures that there is such a branch U_λ^0 satisfying $U_\lambda^0 \in W_0 \ \ J \forall \lambda$.

Now, if we consider the bundle

$$C_\lambda^0 = M \times U_\lambda^0 \subset E$$

with $\pi : C_\lambda^0 \to U_\lambda^0 \subset W_0 \subset U$, sections τ of this bundle are transversally critical sections of E, i.e. for any variation $\tau \to \tau + \varepsilon \delta\tau$ with $\delta\tau(x) \in N_{\tau(x)} U_\lambda^0$ (where NU_λ^0 is the normal bundle to U_λ^0), we have $L_\lambda[\tau + \varepsilon\delta\tau] = L_\lambda[\tau] + O(\varepsilon^2)$.

In other words, not only critical sections for L_λ restricted to $\mathcal{W}_0$ are critical sections for unrestricted L_λ, but in turn critical sections for L_λ restricted to U_λ^0 are also critical for L_λ on $\mathcal{W}_0$, and therefore for unrestricted L_λ.

We can now just consider that by definition of U_λ, $L_\lambda[\tau + \varepsilon\delta\tau] = L_\lambda[\tau] + O(\varepsilon^2)$ for $\delta\tau(x) \in T_{\tau(x)} U_\lambda$, so that every section $\tau : M \to U_\lambda^0$ is critical for L. Such a degeneracy should not be surprising: as recalled above, $U_\lambda^0 \simeq G(u)$, $u \in U_\lambda^0$, and $G(u) \simeq G/G_0$ (indeed, $G_u = G_0$ for $u \in W_0$ and G_0 a maximal isotropy subgroup), so that the degeneracy does actually just correspond to the action of the gauge symmetry. $\bullet$

Remark: This seems an appropriate point to stress that our proofs are generalizable to x-dependent $\mathcal{L}$, or to higher order $\mathcal{L}$ (i.e. $\mathcal{L}$ depending on higher derivatives of u), but they rely essentially on the fibered structure of E and the fact the gauge symmetry preserves this structure. I.e., they are not generalizable to general Lie-point (i.e. not projectable) symmetries. See also [13].

10. Evolution PDEs

Let us now consider evolution PDEs. Among the independent variables, we single out the time $t \in \Theta$, where usually $\Theta = R$ or R_+ (but if we want to consider only time-periodic solutions, we should set $\Theta = S^1$), and denote by $x \in X \subseteq R^d$ the spatial ones. In the following discussion, we will usually set $X = R^1$ for ease of notation, but it will be quite clear that our results continue to hold for any smooth finite dimensional manifold X. As for the dependent variable $u = u(x,t)$, we set $u \in \mathcal{U} \subseteq R^n$, where $\mathcal{U}$ is a smooth finite dimensional manifold, which we will think as embedded in R^n, i.e. with coordinates $\{u_1, ..., u_n\}$. Therefore, a function $u(x,t)$ can also be thought as a section σ_u of the trivial fiber bundle $E = \mathcal{U} \times (X \times \Theta)$, with projection $\pi : E \to \mathcal{U} \times (X \times \Theta)$, $\pi\sigma(x,t) = (x,t)$, and fiber $\mathcal{U}$. The base space $(X \times \Theta)$ will also be denoted by $\mathcal{B}$; with this, $E = \mathcal{U} \times \mathcal{B}$, $\pi : E \to \mathcal{B}$.

We will write an evolution PDE in the form

$$\Delta(u^{(N)}) \equiv u_t - F[u] = 0 \tag{1}$$

where F depends smoothly on u and its x-derivatives up to a finite order N. The space spanned by u and its x-derivatives of order up to N will be denoted in the following as $\mathcal{M}$. Therefore, in (1)

$$F : \mathcal{M} \to T\mathcal{U} \tag{2}$$

We will also write (1) in a slightly different way, introducing the evolutionary VF

$$f = F[u]\partial_u \equiv F^i[u]\frac{\partial}{\partial u^i}J \tag{3}$$

Notice that f is not a vector field on $\mathcal{U}$, since F depends on derivatives of u and not only on u itself. It can instead be seen as a vector field on $\mathcal{M}$ (such that the components in the directions corresponding to x-derivatives of u are zero), i.e.

$$f : \mathcal{M} \to T\mathcal{M} \tag{4}$$

It should be quite clear that this f is *not* the prolongation of any VF on $\mathcal{U}$.

In the following, we will find it useful to denote the set of smooth VFs on a manifold M by $V(M)$. Therefore,

$$f \in V(\mathcal{M}) \tag{4'}$$

With this notation, the equation Δ (by this we will mean $\Delta = 0$) reads also

$$u_t = f \cdot u \tag{5}$$

Now, the solution manifold $S_\Delta = \{u^{(N)} \ / \ \Delta(u^{(N)}) = 0\} \subseteq J^N\mathcal{U}$ is actually a fibered one: let us denote by $\widetilde{\mathcal{U}}$ the space of first time derivatives of u (so that $\widetilde{\mathcal{U}} \simeq T_x\mathcal{U}$), and by $\mathcal{R}$

that of higher t-derivatives and of mixed (i.e., involving both x and t) derivatives. Then, it is clear that actually one has

$$S_\Delta = S^e_\Delta \times \mathcal{R} \qquad ; \qquad S^e_\Delta \subset \widetilde{\mathcal{U}} \times \mathcal{M} \times \mathcal{B} \equiv \widehat{E} \tag{6}$$

and it will be convenient to consider only S^e_Δ, from now on denoted S_Δ tout court, which is a smooth submanifold of the bundle $\widehat{\pi} : \widehat{E} \to \mathcal{B}$. With this notation,

$$S_\Delta \subseteq \widehat{E} \tag{7}$$

We have remarked before that a function $u(x,t)$ can be naturally seen as a section σ_u of the bundle E. This is naturally lifted to a section $\widehat{\sigma}_u$ of the bundle $\widehat{E}$; this corresponds to the function $\widehat{u}(x,t)$, $\widehat{u} : \mathcal{B} \to \widetilde{\mathcal{U}} \times \mathcal{M}$ which associates to a point $(x,t) \in \mathcal{B}$ the values of $u_t(x,t)$, of $u(x,t)$, and of x-derivatives of u of order up to N (i.e., of $D_J u(x,t)$, where J is a multiindex in x, $|J| \le N$). The space $\widehat{\mathcal{U}} \times \mathcal{M}$, which is the fiber of $\widehat{E}$, will also be denoted by $\widetilde{E}$, so that $\widehat{u} : \mathcal{B} \to \widetilde{E}$.

With this notation, a function $u(x,t)$ is a solution to (1) if and only if the corresponding section $\widehat{\sigma}_u$ lies entirely in S_Δ:

$$\Delta[u] = 0 \Leftrightarrow \widehat{\sigma}_u \subseteq S_\Delta \subseteq \widehat{E} \tag{8}$$

Remark: As the notation used in (1), (4) suggests, we are actually interested in considering autonomous (in both x and t) equations, i.e. $\partial_t F = \partial_x F = 0$. In this case, we have

$$S_\Delta = \widetilde{S}_\Delta \times \mathcal{B} \qquad ; \qquad \widetilde{S}_\Delta \subseteq \widetilde{\mathcal{U}} \times \mathcal{M} \equiv \widetilde{E} \tag{9}$$

Remark: We can still use the particular form of eq.(1): in facts, this is solved for u_t, so that S_Δ can be seen as a section of the bundle $\widehat{E}'$ (with total space $\widehat{E}' = \widehat{E}$) over $\mathcal{M} \times \mathcal{B}$; for autonomous equations, also $\widetilde{S}_\Delta$ can be seen as a section of the bundle $\widetilde{E}$ over $\mathcal{M}$. In both cases, the fiber of the bundle is $\widetilde{\mathcal{U}}$.

11. Symmetries of evolution PDEs

Let us now consider a VF on $\mathcal{U}$, $\eta \in V(\mathcal{U})$, which we write as

$$\eta = \varphi(u)\partial_u \equiv \varphi^i \frac{\partial}{\partial u^i} \tag{1}$$

Remark: In general, we could take $\eta = \varphi(x,u)\partial_u + \xi(x)\partial_x$, which is a projectable VF on $\mathcal{U} \times X$, and the discussion to follow would give similar results (we stick to (1) in order to keep the notation as simple as possible). It is instead essential that $\varphi_t = 0$, $\xi_t = 0$, and even more that η has zero component in the t direction. We will shortly discuss this later on.

Now, $\eta \in V(\mathcal{U})$ induces a VF $\widehat{\eta} \in V(\mathcal{M})$ by

$$\widehat{\eta} = \widehat{\varphi} \cdot \widehat{\nabla} \tag{2}$$

where $\widehat{\varphi}$, $\widehat{\nabla}$ are vectors of components

$$\begin{aligned} \widehat{\varphi} &= \left(\varphi, D_x\varphi, ..., D_x^N\varphi\right) \\ \widehat{\nabla} &= \left(\frac{\partial}{\partial u}, \frac{\partial}{\partial(D_x u)}, ..., \frac{\partial}{\partial(D_x^N u)}\right) \end{aligned} \tag{3}$$

with D_x the total x derivative [14], so that $\widehat{\eta}$ is simply

$$\widehat{\eta} = \sum_{m=0}^{N} (D_x^m\varphi) \frac{\partial}{\partial(D_x^m u)} \tag{4}$$

(i.e. the Lie derivative along η, or φ, acting on functions defined on $\mathcal{M}$).

In the case of multidimensional X, this is better rewritten in terms of the multiindex J [14] as

$$\widehat{\eta} = \sum_{|J|\le N} (D_J\varphi) \cdot \nabla_J \qquad ; \qquad \nabla_J = \frac{\partial}{\partial(D_J u)} \qquad (\nabla \equiv \nabla_0) \tag{5}$$

Remark: In geometrical terms, $\widehat{\eta}$ is the projection along $T\mathcal{M}$ of η^N, the N-th prolongation of η, which is in $V(J^N\mathcal{U})$.

As for the action of η on $\widetilde{\mathcal{U}}$, by standard prolongation formula again, we get

$$\eta^{(t)} = (D_t\varphi) \frac{\partial}{\partial u_t} \equiv \varphi_u u_t \frac{\partial}{\partial u_t} \tag{6}$$

(in the second equality we have used $\partial_t\varphi = 0$).

We have therefore

$$\eta^N \cdot \Delta \equiv \left(\eta^{(t)} + \widehat{\eta}\right) \cdot \Delta \tag{7}$$

and the condition [14] for η as in (1) to be a symmetry of Δ, i.e. for $\eta \in \mathcal{G}_\Delta$, is

$$[\varphi_u u_t - \widehat{\eta} F]_{S_\Delta} \equiv \left[\varphi_u u_t - (\widehat{\varphi} \cdot \widehat{\nabla}) F \right]_{S_\Delta} = 0 \tag{8}$$

and since $u_t|_{S_\Delta} \equiv F[u]$, this also reads

$$(F \cdot \nabla)\varphi - (\widehat{\varphi} \cdot \widehat{\nabla}) F = 0 \tag{9}$$

By recalling the f introduced above, one sees that (9) is nothing else than the commutation relation

$$[f, \widehat{\eta}] = 0 \tag{10}$$

among vector fields in $V(\mathcal{M})$.

Remark: Condition (10) is formally analogous to the one obtained for symmetries of ODEs (this latter involving elements of $V(\mathcal{U})$ alone). In facts, the idea behind our treatment is that of looking at an evolution PDE as an infinite dimensional dynamical system. Condition (19) will indeed permit to parallel the discussion conducted in the ODE case, with of course some new features and difficulties.

The formula (19) suggests we can also define a Poisson bracket $\{.,.\} : V(\mathcal{M}) \times V(\mathcal{M}) \to V(\mathcal{M})$. Given $A, B \in V(\mathcal{M})$,

$$A = \alpha^J \nabla_J \qquad ; \qquad B = \beta^J \nabla_J \tag{11}$$

their Poisson bracket will be

$$\{\alpha, \beta\} = (\alpha \cdot \widehat{\nabla})\beta - (\beta \cdot \widehat{\nabla})\alpha \tag{12}$$

or, in component notation,

$$\{\alpha, \beta\}^j = (\alpha^k \cdot \widehat{\nabla}_k)\beta^j - (\beta^k \cdot \widehat{\nabla}_k)\alpha^j \tag{13}$$

The obvious relation among $[.,.]$ and $\{.,.\}$ is given by

$$[A, B] = \{J\alpha, \beta\}^J \nabla_J \tag{14}$$

With this, (19) reads also

$$\{\widetilde{F}, \widetilde{\varphi}\} = 0 \tag{15}$$

where $\widetilde{F} = (F, 0, ..., 0)$ in the notation (3).

Remark: One can check that $\{.,.\}$ is actually a Poisson bracket by verifying the Jacobi identity holds. All the usual properties of Poisson brackets are then granted; in particular, given two VFs $\eta_1 = \varphi(u)\partial_u$ and $\eta_2 = \psi(u)\partial_u$ in $\mathcal{G}_\Delta$, i.e. such that $\{J\widetilde{F}, \widetilde{\varphi}\} = \{J\widetilde{F}, \widetilde{\psi}\} = 0$, their bracket $\widetilde{\zeta} = \{\widetilde{\varphi}, \widetilde{\psi}\}$ does also satisfy $\{J\widetilde{F}, \widetilde{\zeta}\} = 0$. This is no surprise, since $\widehat{\eta}$ is the (projection of the) N-prolongation of η, and the above relation then follows from [14] $\left[pr^{(N)}\eta_1, pr^{(N)}\eta_2\right] = pr^{(N)}\left[\eta_1, \eta_2\right]$, so that actually $\widetilde{\zeta} = (\zeta, D_x\zeta, ...)$ where $\zeta = \{\varphi, \psi\}$.

To summarize our discussion, we have the

Symmetry criterion (evolution PDEs): Let $\Delta = u_t - F[u]$ be an evolution PDE, and $\mathcal{G}_\Delta$ its Lie point symmetry algebra. Then, with the above notations,

$$\eta = \varphi(u)\partial_u \in \mathcal{G}_\Delta \Leftrightarrow [f, \widehat{\eta}] = 0 \Leftrightarrow \{\widetilde{F}, \widehat{\varphi}\} = 0$$

Aside: Let us briefly discuss what happens for η of more general form than in (1). We consider indeed

$$\eta = \varphi(u, x, t)\partial_u + \xi(x)\partial_x + \tau(t)\partial_t$$

i.e. a projectable VF, which in turn keeps separate the space and time variables (this guarantees that $\eta^{(N)}$ transforms evolution equations into evolution equations).

Then (7) continues to hold, with

$$\eta^{(t)} = \Phi^{(t)}\frac{\partial}{\partial u_t} \qquad ; \qquad \Phi^t = D_t(\varphi - \tau u_t) + \tau u_{tt}$$

and in (3),(4) we have

$$\widehat{\eta} = \widehat{\Phi} \cdot \widehat{\nabla} \qquad ; \qquad \widehat{\Phi} = \left(\varphi, \Phi^{(1)}, ..., \Phi^{(N)}\right)$$

where

$$\Phi^{(m)} = D_x^m(\varphi - \xi u_x) + \xi D_x^m u_x$$

or, in multiindex notation,

$$\widehat{\eta} = \Phi^{(J)} \cdot \nabla_J$$

These follows from the general prolongation formula [14] and from $\xi_u = \tau_u = \xi_t = \tau_x = 0$. Now, (7) reads

$$\left[\Phi^{(t)} - (\Phi^{(J)}\nabla_J)F\right]_{S_\Delta} \equiv \left[\Phi^{(t)} - (\widehat{\Phi} \cdot \widehat{\nabla})F\right]_{S_\Delta} = 0$$

which gives

$$(\varphi_u u_t + \varphi_t - F\tau_t)_{S_\Delta} - (\widehat{\Phi} \cdot \widehat{\nabla})F = 0$$

or, in the notation (9),

$$\varphi_t + \Big[(F \cdot \nabla)\varphi - (\widehat{\Phi} \cdot \widehat{\nabla})F\Big] = F\tau_t$$

We see that for $\tau_t = 0$ (notice that for F autonomous, ∂_t is always a symmetry of Δ, so we can take it out from η, and assume directly $\tau = 0$) this is analogous to (9), i.e. we get $\varphi_t(u)\partial_u = [\widetilde{\eta}, f]$ and in particular

$$\varphi_t = \{\widetilde{\varphi}, \widetilde{F}\} \qquad \bullet$$

We do now return to consider η as in (1).

Now, let $\mathcal{G}_0$ be a subalgebra of $\mathcal{G}_\Delta$. To it, we associate the spaces

$$\begin{aligned} W_0 =& \{u \in \mathcal{U} \ / \ \eta u = 0 \ \ \forall \eta \in \mathcal{G}_0\} \subseteq \mathcal{U} \\ \mathcal{W}_0 =& \{m \in \mathcal{M} \ / \ \widehat{\eta} m = 0 \ \ \forall \eta \in \mathcal{G}_0\} \subseteq \mathcal{M} \end{aligned} \tag{16}$$

For completeness of notation, we also consider the isotropy subalgebra $\mathcal{G}_u$ of $u \in \mathcal{U}$ and $\mathcal{G}_\sigma$ of the section σ of $\mathcal{M} \times \mathcal{B}$, and the fixed space conjugated to u and to σ. For $u \in \mathcal{U}$, we have

$$\begin{aligned} \mathcal{G}_u =& \{\eta \in \mathcal{G}_\Delta \ / \ \eta u = 0\} \subseteq \mathcal{G}_\Delta \\ W_u =& \{v \in \mathcal{U} \ / \ \eta v = 0 \ \forall \eta \in \mathcal{G}_u\} = \{v \in \mathcal{U} \ / \ \mathcal{G}_u \subseteq \mathcal{G}_v\} \subseteq \mathcal{U} \end{aligned} \tag{17}$$

while for $\sigma \in \Sigma(E)$, after noticing that η as in (10) acts on σ in such a way that $\eta : \sigma \to T\sigma \Leftrightarrow \eta \cdot \sigma = 0$ (notice this is not true if η has a component in the ∂_x direction) these are

$$\begin{aligned} \mathcal{G}_\sigma =& \{\eta \in \mathcal{G}_\Delta \ / \ \eta \cdot \sigma = 0\} \\ W_\sigma =& \{\sigma' \in \Sigma(E) \ / \ \eta \cdot \sigma' = 0 \ \ \forall \eta \in \mathcal{G}_\sigma\} \end{aligned} \tag{18}$$

12. Reduction Lemma for evolution PDEs

With the notation $\widehat{u}(x,t) = \big(u(x,t), D_x u(x,t), ..., D_x^N u(x,t)\big)$, we can now state the RL for evolution PDEs as follows:

Reduction Lemma (evolution PDEs): Let $\mathcal{G}_0$ be a subalgebra of $\mathcal{G}_\Delta$, the symmetry algebra of the evolution PDE $\Delta[u] = u_t - F[u]$, and let $\mathcal{W}_0 \subseteq \mathcal{M}$ be the corresponding fixed space.

Then, for an initial datum $u(x,0) = u_0(x)$ such that $\widehat{u}(x,0) \in \mathcal{W}_0 otherwords, if \widehat{u}(x,0) : X \to \mathcal{W}_0$, then $\widehat{u}(x,t) : \mathcal{B} \to \mathcal{W}_0$.

Proof: Just repeat the proof of RL in ODE case. Indeed, that was purely geometrical and did not depend on the dimension of $\mathcal{U}$. We have that by definition the flow of $\widehat{\eta}$ on $\mathcal{W}_0$ is trivial for $\eta \in \mathcal{G}_0$, so that if $\Psi(m;t)$ is the evolute of $m \in \mathcal{M}$ under the flow induced by F after time t, by $[f, \widehat{\eta}] = 0$ it follows $\widehat{\eta}\Psi(m;t) = \Psi(m;t)\ \forall m \in \mathcal{W}_0,\ \forall t \geq 0$; this yields $\Psi(m;t) \in \mathcal{W}_0$, and therefore the lemma. •

Corollary: $F : \mathcal{W}_0 \to TW_0$

Proof: $F : \mathcal{M} \to T\mathcal{M}$ and leaves invariant $\mathcal{W}_0$, i.e. $F : \mathcal{W}_0 \to T\mathcal{W}_0$. But actually $F : \mathcal{M} \to T\mathcal{U} \subset T\mathcal{M}$, and $T\mathcal{W}_0 \cap T\mathcal{U} = TW_0$. Hence $F : \mathcal{W}_0 \to TW_0$. •

One would obviously like to have a statement simply in terms of $u(x,t)$ and not of $\widehat{u}(x,t)$; to this aim we notice that in the same way as $\mathcal{M}$ is fibered as $\pi : \mathcal{M} \to \mathcal{U}$, also $\mathcal{W}_0$ is fibered, by the same projection π, as $\pi : \mathcal{W}_0 \to W_0$. In other words, we have the

Lemma: $\widehat{u}(x,t) \in \mathcal{W}_0\ \ \forall x \Leftrightarrow u(x,t) \in W_0\ \ \forall x$

Proof: First of all, notice that the dependence on t is inessential here, so that we can just consider $u_0(x) \equiv u(x,t_0)$ and $\widehat{u}_0(x) = \widehat{u}(x,t_0)$. Then, by definition of $\mathcal{W}_0$,

$$\widehat{u}_0(x) \in \mathcal{W}_0 \Rightarrow D_J\varphi(u) \equiv \widehat{\varphi}^J(\widehat{u}) = 0 \qquad , \ |J| \leq N \tag{1}$$

so that in particular we must have $\varphi(u) = 0$, i.e.

$$\widehat{u}_0(x) \in \mathcal{W}_0 \Rightarrow u_0(x) \in W_0 \tag{2}$$

To see that the converse is also true, let $u_0(x) \in W_0\ \forall x$; by definition $\varphi(u)$ vanishes along W_0. On the other side, if $u(x)$ is in $W_0\ \forall x$, then $\partial_x u_0(x) \in TW_0\ \forall x$. Now just notice that $D_x\varphi = \varphi_u u_x = (u_x \cdot \nabla)\varphi$, i.e. is the gradient of φ in a direction lying in TW_0, and by the above discussion $D_x\varphi = 0$. This argument is readily generalized to higher orders, yielding that $\left[\partial^k\varphi/\partial u^k\right]_{W_0} = 0\ \forall k > 0$. Therefore,

$$u_0(x) \in W_0\ \ \forall x \Rightarrow \widehat{u}_0(x) \in \mathcal{W}_0 \tag{3}$$

and the lemma is proved. •

The above discussion proves that we can restate our RL as

Reduction Lemma (evolution PDE): Let $\mathcal{G}_0$ be a subalgebra of $\mathcal{G}_\Delta$, the symmetry algebra of the evolution PDE $\Delta[u] = u_t - F[u]$, and let $\mathcal{W}_0 \subseteq \mathcal{M}$ be the corresponding fixed space. Then, for an initial datum $u(x,0) = u_0(x)$ such that $u_0(x) \in W_0 \ \forall x$, we have $u(x,t) \in W_0 \ \forall x, \ \forall t \geq 0$.

Remark: One could have obtained the RL directly in this form by noticing earlier the relation among W_0 and $\mathcal{W}_0$. We have preferred this two-steps path in order to stress the analogy with the ODE case.

As always with RL, we will therefore consider the restriction F_0 of F to $\mathcal{W}_0 \subseteq \mathcal{M}$. this gives the restriction Δ_0 of Δ,

$$\Delta_0[v] \equiv v_t - F[v] \tag{4}$$

$$v \in W_0 \subseteq \mathcal{U} \qquad ; \qquad F_0 : \mathcal{W}_0 \to TW_0$$

The RL has then the usual

Corollary: Let $u(x,t)$ be a solution of Δ, and $v(x,t)$ a solution of Δ_0, with $u(x,0) = v(x,0) \in W_0 \ \forall x$. Then $u(x,t) = v(x,t) \in W_0 \ \forall t \geq 0$.

As usual, given $u_0(x) \in W_0 \ \forall x$, we can study the simpler equation Δ_0 instead than Δ.

Remark: It is a trivial observation that, given an arbitrary smooth $u_0(x)$, $u_0 : X \to \mathcal{U}$, we can consider $\mathcal{G}_0^{(x)}$ as defined by

$$\mathcal{G}_0^{(x)} = \{\eta \in \mathcal{G}_\Delta \ / \ \eta u_0(x) = 0\} = \mathcal{G}_{u_0(x)}$$

and $\mathcal{G}_0 = \cap_{x \in X} \mathcal{G}_0^{(x)}$ is such that, by definition, $u_0(x)$ lies entirely in the corresponding W_0. This tells how to use the RL for arbitrary $u_0(x)$; clearly, it can very well happen in this case that $\mathcal{G}_0 = \{e\}$, $W_0 = \mathcal{U}$, $\mathcal{W}_0 = \mathcal{M}$, $\Delta_0 = \Delta$, i.e. no reduction arises.

Remark: We notice that, as a consequence of the corollary, given an evolution equation Δ with symmetry algebra $\mathcal{G}_\Delta$, the existence of solutions to Δ invariant under $\mathcal{G}_0 \subseteq \mathcal{G}$ can be studied by means of (is equivalent to the existence of solutions to) Δ_0.

Remark: The above remark does also suggest a possible way to attack the problem of which kind of conditional symmetries [38,39,40] can appear in the solutions of Δ (these correspond in short to invariance of solutions under Lie-point transformations which are not in $\mathcal{G}_\Delta$). We will not pursue this point here.

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