Michel's theorem and critical sections of gauge functionals

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1. Introduction

In many physical problems, one has to minimize a G-invariant potential, i.e. a potential invariant under the action of a symmetry group. In this case a powerful tool is offered by a theorem of L. Michel [1] and generalizations thereof, see e.g. [2-11]. The purpose of this note is to discuss just another such generalization, to gauge potentials.

After introducing the appropriate notation, we will briefly recall the content of Michel theorem and the ingredients of its proof; we will then pass to discuss the infinite dimensional case, and see that for gauge symmetries one can still use tools not too different from the original Michel's ones.

Since results as strong as in the finite dimensional case hold only restricting to a subset of sections (although a structurally stable one) and can be difficult to use to obtain directly the required minima of a gauge functional, we will also look at Michel's theorem as an heurhystical tool, allowing one to restrict to a simpler problem whose solutions are, generically, in correspondence with solution of the original problem (this will be done by means of the "reduction lemma").

The scheme of the paper is as follows: the whole paper is divided into five parts, each one composed of three sections (in studying gauge symmetry breaking, we felt obliged to give some formal symmetry to the paper).

Part I recalls Michel's theorem for critical points of invariant potentials $V : M \subseteq R^n \to R$ and the appropriate mathematical tools and concepts (section 2), needed not only for the proof but also for the statement of the theorem; a short proof of Michel's theorem is also reported in section 3.

In part II we first extend Michel's theorem to sections and gauge orbits of sections of a bundle on which a zero-th order gauge invariant functional is defined (section 4); this is essentially accomplished by introducing an appropriate (and natural) metric in the space of sections. We then discuss (section 5) in some detail this extension, and how this can be used in the case of higher order gauge functionals (section 6).

Part III deals with the difficulties arising in the stratification of gauge orbit space: we first recall some results for the stratification of orbit space for the finite-dimensional action of a compact Lie group (section 7), then use these to study the gauge orbit space (section 8), and point out that some restriction is needed in order for a stratification of this to make sense. We propose such a restriction, to the set of transverse sections (defined there); this is structurally stable and dense in the set of sections. Even with this, the stratification of gauge orbit space is by far too complex to be completely described; we will limit to study the strata Michel's theorem is concerned with, i.e. maximal ones (section 9).

In Part IV (sections 10 - 12) we discuss how, similarly to the finite dimensional case, Michel's theorem opens the way to a reduction of the problem of finding critical orbits, connecting to results of Palais; and if a control parameter appears in the theory, to an equivariant branching lemma, connecting to recent results of Cicogna.

Part V is different in spirit (which breaks the symmetry of the paper, after all). In facts we first show how our approach can be extended to take into account base-space, or in physical terms space-time, symmetries (section 13). At this point we renounce to obtain rigorous results, and shortly discuss how this could provide a scenario for spontaneous pattern formation and describe situations of phase coexistence.

Parts I-IV constitute a self contained paper, containing rigorous (and, at least for the author, interesting) results, which are illustrated by repeatedly analyzing in detail a number of examples, first introduced in section 7.

The original motivation of the paper was an attempt to understand Blue Phases; although this is not pursued here, we believe this paper sets mathematical bases for an attempt to study Blue Phases [12-14] in a rigorous and seemingly original way, along the lines of Part V, and plan to pursue such an approach in the near future.

As recalled above, the paper connects to recent results of Cicogna [15] for the bifurcation case; a discussion of equivariant bifurcation theorems general enough to accomodate the case of gauge symmetries is given in [16,17]; in a related paper [18] we discuss at lenght reduction and equivariant bifurcation lemma for such a general class of symmetries, and in particular gauge functionals and nonlinear evolution PDEs.

As also recalled above, the paper re-obtains some results of Palais [19,20]. These were unknown to the author, who thanks prof. Bourguignon for pointing them out to him (unfortunately after they were re-obtained). The author believes anyway that the present treatment of the connection between them and Michel's theorem is original.

Finally, I would like to thank a number of persons for interesting discussions, first of all proff. Michel and Bourguignon; prof. Palais was so kind to give me some of his time to discuss the present work during one of his visits in Paris; the mathematical part of the paper was also discussed with proff. Gallot and Lascoux, while proff. Chakrabarti, Collet, Doelman, B. Pansu, Peliti and Testa had the patience to listen to, and discuss, my ideas about physical applications of it. I would also like to thank an unknown referee for suggesting the remark in section 7 concerning reference [54].

2. Strata in R^N

Let us first consider a (smooth) potential V defined on an N- dimensional real space,

$$V : R^N \to R^N \tag{1}$$

which is invariant under a representation $\Lambda = \{\Lambda_g \ / \ g \in G\}$ of a Lie group G acting in R^N

$$\Lambda_g : R^N \to R^N \tag{2}$$

$$V(\Lambda_g x) = V(x) \qquad \forall x \in R^N \ , \ \forall g \in G \tag{3}$$

We will be interested in the critical points of V,

$$\nabla V(x_c) = 0 \tag{4}$$

Under the representation Λ, each point $x \in R^N$ has an isotropy subgroup G_x,

$$G_x = \{g \in G \ / \ \Lambda_g x = x\} \tag{5}$$

It is immediate to see that points on the same G-orbit, i.e. points y, x such that for some $g \in G$ it is $y = \Lambda_g x$, have isotropy subgroups conjugated in G:

$$y = \Lambda_g x \Rightarrow G_y = g G_x g^{-1} \tag{6}$$

The set of points of R^N having conjugated (in G) isotropy subgroup is called a stratum [1,2,5] and will be denoted by Σ_x, with x any of its points,

$$\Sigma_x = \{y \in R^N \ / \ G_y = g G_x g^{-1} \ , \ g \in G\} \tag{7}$$

We will denote by $\omega(x)$ the G-orbit through x,

$$\omega(x) = \{y \in R^N \ / \ y = \Lambda_g x \ , \ g \in G\} \tag{8}$$

It follows from (6) that

$$\omega(x) \subseteq \Sigma_x \tag{9}$$

so that the stratification of R^N also induces a stratification of the orbit space $\Omega = R^N/G$ (the orbit $\omega(x)$ will be denoted as ω_x when thought as a point of Ω), satisfying

$$\Sigma_\omega = \{\omega' \in \Omega \ / \ G_y = g G_x g^{-1} \ , \ g \in G, \ x \in \omega, \ y \in \omega'\} \tag{10}$$

which actually does not depend on the choice of representative points x, y on ω, ω'.

It is immediate to check that belonging to the same stratum, denoted as $x \sim y$ or $\omega \sim \omega'$, is an equivalence relation.

In the space Ξ of strata of R^N one can introduce a partial ordering by

$$\Sigma_x < \Sigma_y \Leftrightarrow G_y = gHg^{-1} \ ; \ g \in G, \ H \subset G_x \tag{11}$$

where H is a proper subgroup of G_x and again the relation does not depend on the choice of the representatives x, y. Analogously, for the space Ξ_Ω of strata of Ω,

$$\Sigma_\omega < \Sigma_{\omega'} \Leftrightarrow G_y = gHg^{-1} \ ; \ g \in G, \ x \in \omega, \ y \in \omega', \ H \subset G_x \tag{12}$$

A stratum in Ω is also called an orbit type.

It should be stressed that strata, both in R^N and in Ω, are manifolds [2,5], although Ω is in general not a manifold.

All the above is still valid if instead of R^N we consider a manifold $M \subset R^N$ which is invariant under Λ:

$$\Lambda_g : M \to M \qquad \forall g \in G \tag{13}$$

For more details on the material of this section, see [2,5]; other results concerning stratification will be recalled in sect.7.

3. Michel's theorem

Let $\mathcal{V}$ be the set of C^∞ functions from R^N to R invariant under Λ, i.e. such that $V(\Lambda_g x) = V(x) \quad \forall g \in G, \forall x \in R^N$.

If there is a point x such that $dV(x) = 0$, then necessarily $dV(y) = 0 \ \forall y \in \omega(x)$, so that critical points of $V \in \mathcal{V}$ will came in G-orbits.

An orbit $\omega = \omega(x_c)$ such that $dV(x_c) = 0 \ \forall V \in \mathcal{V}$ is called a critical orbit for G [1].

To see that these exist, just consider $N = 1$ and $G = Z_2 = \{e, g\}$, with $\Lambda_e : x \to x$, $\Lambda_g : x \to -x$. In other words, every (smooth) even potential has a critical point in the origin. Analogously, for $G = SO(N)$ we have that any rotationally invariant smooth potential has a critical point in the origin.

A less trivial example is obtained by considering $M = S^2 \subset R^3$ and $G = SO(2)$ acting in R^3 as rotations around the z-axis,

$$\Lambda_g = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

Any invariant potential will have critical points at North and South poles of the sphere (i.e. at $(0, 0, \pm 1)$).

The orbit space is isomorphic to the segment $[-1, 1]$ (this can be thought as the z coordinate of the orbit); all the points in the interior of this belong to the same stratum ($G_x = \{e\}$), and the extrema $x = \pm 1$ form another stratum ($G_x = SO(2)$).

The theorem of L. Michel [1] tells that

An orbit is critical for G if and only if it is isolated in its stratum

An orbit is, roughly speaking, isolated in its stratum if one can take a neighbourhood of ω in Ω which does not contain points of Σ_ω other than ω itself.

In order to talk of neighbourhoods in Ω, one has to provide it with a topology, which will be taken to be the quotient topology: the distance of two orbits will be defined by means of the distance in R^N to be

$$d(\omega, \omega') = \min_{x \in \omega; y \in \omega'} \ d(x, y) \tag{2}$$

where $d(x, y)$ is the standard distance in R^N (or the distance corresponding to the metric defined on $M \subset R^N$ if we deal with this case).

The minimum of (2), whose existence has to be proven, can be seen also as

$$d(\omega, \omega') = \min_{x \in \omega} d(x, \omega') \tag{3}$$

where we have introduced the distance of a point from an orbit,

$$d(x, \omega') = \min_{y \in \omega'} d(x, y) \tag{4}$$

If the point $y \in \omega'$ for which $d(x, y) = d(x, \omega')$ is unique (locally), it is called the retraction of x on ω and denoted $\rho_{\omega'}(x)$.

The function $\rho_\omega(\cdot)$ is an equivariant one:

$$\rho_\omega(\Lambda_g x) = \Lambda_g \rho_\omega(x) \tag{5}$$

and is therefore also called the equivariant retraction [1,2,5,6,8,21].

Notice that if G is compact, ω, ω' are compact sets, and the minima of (2),(3),(4) do surely exists, and therefore also the equivariant retraction does exist.

We stress that for noncompact group orbits the equivariant retraction could very well not exist, and therefore the concept of an orbit "isolated in its stratum" be ill-defined (that is why we discussed the concept at some lenght). Besides this, the very existence of a stratification is not granted for noncompact groups, as it will be discussed later.

Let us now sketch, without going to details [1,2,6,8], how Michel's theorem is proved (for compact Lie groups).

At any point $x \in M$, one has a tangent and a normal space to $\omega(x)$, $T_x\omega$ and $N_x\omega$, with

$$T_x\omega \oplus N_x\omega = T_xM \tag{6}$$

These are linear spaces; in $N_x\omega$ we can consider the invariant subspace $N_x^0\omega \subseteq N_x\omega$;

$$N_x^0\omega = \{\xi \in N_x\omega \ / \ \Lambda_g\xi = \xi \ \ \forall g \in G_x\} \tag{7}$$

$N_x\omega$ is also called the slice through x, and $N_x^0\omega$ the invariant slice through x [5,21] (for nonlinear group actions the slices are manifolds, tangent in x to these linear spaces [21]).

It should be noted that the tangent space in $x \in \omega$ to the stratum Σ_x is simply, by the definitions of Σ_x and $N_x^0\omega$,

$$T_x\Sigma_x = T_x\omega \oplus N_x^0\omega$$

Now, the gradient dV of an invariant function V has to be perpendicular to the orbit $\omega(x)$ at x

$$dV(x) \in N_x\omega \quad ; \quad dV : x \to N_x\omega \subset T_xM \tag{8}$$

Moreover, dV must be tangent to Σ_x in x. To see this consider that for an equivariant function $f : M \to TM$,

$$f(\Lambda_g x) = \Lambda_g f(x) \tag{9}$$

and

$$G_x \subseteq G_{f(x)} \tag{10}$$

which also means

$$\Sigma_{f(x)} \leq \Sigma_x \quad ; \quad f(x) \in \Sigma_x \tag{11}$$

(the equality holds in (10) and (11) if and only if f is one to one [1,2,5,6]), so that

$$dV(x) \in T_x\Sigma_x \tag{12}$$

which gives immediately Michel's theorem.

By (8) and (12) we get [4,6]

$$dV(x) \in N_x^0\omega \tag{13}$$

and conversely one can prove that, given an integrity basis [5,21,22,23] for Λ, $N^0_x\omega$ is spanned by gradients of elements of the integrity basis [6]. Relation (13) will also be fundamental in establishing the "equivariant branching lemma", as discussed in the following (sect.10); see also [15,18].

4. Zero-th order gauge functionals

In gauge theories, one considers functions (gauge fields) defined on a manifold $M \subseteq R^N$ with values in the Lie algebra $\mathcal{G}$ of a Lie group G (the gauge group); if the theory is not a pure gauge one, one also considers functions (matter fields) from the same manifold M to a space F in which a representation Λ of the group G is defined [24-26]. One is then faced with the problem of minimizing a functional L (the lagrangian), usually expressed in terms of a local density $\mathcal{L}$,

$$L = \int_M \mathcal{L}(x) d^N x \tag{1}$$

This kind of problem is better set in terms of fiber bundles [26,27].

We should in particular introduce a fiber bundle of total space E, base M, projection $\pi : E \to M$, and fiber $\pi^{-1}(x) = F$, with structural group G. The matter fields will then be (smooth) sections $\varphi : M \to E$, $\varphi(x) \in \pi^{-1}(x)$, of this bundle; the space of smooth sections of E will be denoted by Φ,

$$\Phi = \{\varphi : M \to E \ / \ \pi\varphi(x) = x \ \ , \ \varphi \text{ smooth}\}$$

We should also consider another fiber bundle of total space $\tilde{E}$, base M, projection $\tilde{\pi} : \tilde{E} \to M$, and fiber and structural group G, $\tilde{\pi}^{-1}(x) = G$. The space of smooth sections of $\tilde{E}$ will be denoted by Γ,

$$\Gamma = \{\gamma : M \to \tilde{E} \ / \ \tilde{\pi}\gamma(x) = x \ \ , \ \gamma \text{ smooth}\}$$

Let us first consider the unphysical case of a zero-th order density $\mathcal{L}$, i.e. assume that the functional $\mathcal{L}$ depends only on the matter fields $\varphi(x)$,

$$\varphi : M \to F \subseteq R^s \tag{2}$$

but not on its derivatives, which will be meant by the notation

$$L = \int_M \mathcal{L}[\varphi] dx \qquad ; \quad \mathcal{L} : F \to R \tag{3}$$

and assume that M is compact.

To say that $\mathcal{L}$ has a (local) gauge symmetry described by the representation $\Lambda = \{\Lambda_g\}$ of the group G, $\Lambda_g : F \to F$, means that for any smooth function $\gamma : M \to G$,

$$\mathcal{L}[\Lambda_{\gamma(x)}\varphi(x)] = \mathcal{L}[\varphi(x)] \tag{4}$$

In other words, we can consider the orbit space (under Λ) $\Omega = F/G$. Then φ induces a $\varphi_\Omega : M \to \Omega$, by $\omega(\varphi(x)) = \varphi_\Omega(x)$, and (4) tells that L can be thought as a functional on the space Φ_Ω of the φ_Ω. (One should anyway pay attention to smoothness problems, see the remark in section 5).

This can also be seen as introducing a fiber bundle of total space $\bar{E}$, with base M, projection $\bar{\pi} : \bar{E} \to M$, and fiber $\bar{\pi}^{-1}(x) = \Omega \equiv F/G$ (notice that now G acts on the fiber as the identity). Then $\bar{\Phi}$ is the space of sections of this bundle,

$$\bar{\Phi} = \{\bar{\varphi} : M \to \bar{E} \ / \ \bar{\pi}\bar{\varphi}(x) = x\}$$

We should try to parallel the construction of sections 2-3 for the space Φ, in order to get an analogue of Michel's theorem in this case. Notice that now the space Φ on which G acts is infinite dimensional, and therefore not compact.

The Γ-orbit of a section $\sigma \in \Phi$, denoted $\vartheta(\sigma)$, will be defined as

$$\omega(\sigma) = \{\sigma' \in \Phi \ / \ \sigma'(x) = \Lambda_{\gamma(x)}\sigma(x) \ , \ \gamma(x) \in \Gamma\} \subset \Phi \tag{5}$$

where Γ is the space of smooth sections $\gamma : M \to \tilde{E}$, $\tilde{\pi} \cdot \gamma(x) = x$ of the principal fiber bundle $\tilde{E}$ introduced above.

The orbit space for sections will be denoted $\Theta \equiv \Phi/\Gamma$; the orbit $\vartheta(\sigma)$ will be denoted as ϑ_σ when thought as a point of Θ. Notice that Θ corresponds to $\bar{\Phi}$ defined above; an orbit ϑ can be seen as a section $\bar{\varphi}$ of the bundle $\bar{E}$.

Γ is better seen as a subgroup of the group of fiber-preserving (or gauge) diffeomorphisms of E,

$$\mathrm{GDiff}(E) = \{f \in \mathrm{Diff}(E) \ / \ f : \pi^{-1}(x) \to \pi^{-1}(x) \ \ \forall x \in M\} \tag{6}$$

in particular when considering higher-order functionals (as in the following section). This subgroup is simply given by

$$\Gamma \simeq \Gamma_E = \{f \in \mathrm{GDiff}(E) \ / \ f_x \equiv f|_{\pi^{-1}(x)} = \Lambda_{\gamma(x)} \ , \ \gamma \in \Gamma\} \subset \mathrm{GDiff}(E)$$

where f_x is the restriction of f to $\pi^{-1}(x)$. In the same vein, Γ can be seen as coinciding with $\mathrm{GDiff}(\tilde{E})$.

Given a section $\sigma \in \Phi$, we can define its isotropy subgroup Γ_σ as

$$\Gamma_\sigma = \{\gamma \in \Gamma \ / \ \gamma \cdot \sigma = \sigma\} \subset \mathrm{GDiff}(E) \tag{7}$$

where $\gamma \cdot \sigma$ has to be meant as

$$\gamma \cdot \sigma \equiv \sigma'(x) \equiv \Lambda_{\gamma(x)} \cdot \sigma(x) \tag{8}$$

If $\sigma' = \gamma \cdot \sigma$, it is easy to see that

$$\Gamma_{\sigma'} = \gamma \Gamma_\sigma \gamma^{-1} \tag{9}$$

We can therefore define as before a stratification of Φ, at least formally.

The reason for which this is only formal is that in the case of infinite dimensional groups one can have a group conjugated to some of his proper subgroups, so that the order relation could not be well defined. We will assume for the moment that a stratification can be defined, and defer to a later section (sect. 8) the issue of how to actually do it, and consideration of the difficulties this can present.

We have seen before that the proof of Michel's theorem relies mainly on purely geometrical concepts, which are transferred with no harm to the present infinite dimensional setting. The only exception, i.e. obstacle to an infinite dimensional extension, is represented by giving a topology to the orbit space. In our case, anyway, we can take advantage of the fibered structure of the problem, and define a distance between two sections $\sigma, \sigma' \in \Phi$ as

$$d_\Phi(\sigma, \sigma') = \frac{1}{|M|} \int_M d_F\left(\sigma(x), \sigma'(x)\right) dx \tag{10}$$

where $|M| = \int_M 1 \cdot dx$, and $d_F(.,.)$ is a distance defined in F.

In order to define a distance in Θ, $\delta : \Theta \times \Theta \to R_+$ (here R_+ is the set of nonnegative reals), we can make use of the distance d_Ω defined in Ω, see (3.2), i.e. of the equivariant retraction ρ_ω, by

$$\delta(\vartheta_1, \vartheta_2) = \frac{1}{|M|} \int_M d_\Omega\left(\vartheta_1(x), \vartheta_2(x)\right) dx$$

where $\vartheta_i : M \to \Omega$. In other words, we are defining a distance between sections of $\bar{E}$ along the lines of what we have done for sections of E, i.e. by

$$\delta_\Omega(\bar{\sigma}_1, \bar{\sigma}_2) = \frac{1}{|M|} \int_M d_\Omega\left(\bar{\sigma}_1(x), \bar{\sigma}_2(x)\right) dx \equiv \frac{1}{|M|} \int_M d_\Omega\left(\omega(\bar{\sigma}_1(x)), \omega(\bar{\sigma}_2(x))\right) dx$$

and use the isomorphism of Θ with $\bar{\Phi}$.

At this point, we can just repeat the proof of finite dimensional Michel's theorem to obtain its extension to gauge functionals.

We will call a Γ-orbit $\omega(\sigma) \subset \Phi$ a critical gauge orbit for Γ if for every Γ-invariant functional $L = \int_M \mathcal{L}[\sigma] dx$, $L : \Phi \to R$, $\omega(\sigma)$ is a critical orbit for L. This means that $\forall \sigma \in \omega(\sigma)$, $\delta L[\sigma] = 0$; or, $\mathcal{L}(\sigma + \epsilon \delta\sigma) = \mathcal{L}(\sigma) + O(\epsilon^2)$.

5. Discussion

We think it can be useful to present some remarks.

The first is that to a smooth section $\varphi \in \Phi$ of E can correspond a nonsmooth $\bar{\varphi} \in \bar{\Phi}$. To see an example, consider a trivial bundle of base $M = S^1$ and fiber $F = R^1$; let x be the coordinate on M and y the one on F; let the group $G = Z_2$ act on F as $y \to -y$, so that $\Omega = R_+$. The section $\varphi(x) = \cos(x)$ is smooth, but to it corresponds $\bar{\varphi}(x) \equiv \omega(\varphi(x)) = |\cos(x)|$ which is not such. If wishing to consider continuous groups, one can instead e.g. consider $F = R^2$ and $G = SO(2)$.

A little thinking shows also that singular points of sections $\bar{\varphi} \in \bar{\Phi}$ lie in non-maximal (i.e. nongeneric, see [2,5] or sect.7) strata of Ω [5].

The second remark is quite closely related to this previous one: the careful reader will have noticed that we defined a distance in the gauge orbit space $\Theta = \Phi/\Gamma$ without defining an equivariant retraction in Φ. This is not only due to the fact that what we actually need is a distance in Θ, but actually to an impossibility, as we now shortly discuss.

Given a section $\varphi \in \Phi$ and a gauge orbit $\vartheta \in \Theta$, a distance of φ from ϑ can be defined using the analogous finite dimensional (i.e. on F) concept, see eq. (3.4),

$$\delta(\varphi, \vartheta) = \frac{1}{|M|} \int_M d(\varphi(x), \vartheta(x)) dx$$

which also reads, in terms of the equivariant retraction on F,

$$\delta(\varphi, \vartheta) = \frac{1}{|M|} \int_M d(\varphi(x), \rho_{\vartheta(x)}(\varphi(x))) dx$$

Now, an equivariant retraction of φ to ϑ can be defined point-like at any point $x \in M$, i.e. on any fiber $\pi^{-1}(x)$, as

$$\tau_\vartheta(\varphi, x) = \rho_{\vartheta(x)}(\varphi(x))$$

but the section $\varphi_{\tau\vartheta}$ defined as

$$\varphi_{\tau\vartheta}(x) = \tau_\vartheta(\varphi, x)$$

can well fail to be smooth even if φ is.

In order to see an example, consider once again $M = S^1$, $F = R^1$, $G = Z_2$ as before. The orbits in F are made of two points, $\omega(y) = \{y, -y\}$, except the singular orbit $\omega(0) = \{0\}$. Consider in Θ the gauge orbit ϑ_1 represented by the section $\sigma(x) = 1$. The retraction is simply

$$\tau_{\vartheta_1}(\varphi, x) = \begin{matrix} 1 & \text{if} & \varphi(x) > 0 \\ -1 & \text{if} & \varphi(x) < 0 \end{matrix}$$

(notice that $\tau_{\vartheta_1}(\varphi, x)$ is not defined if $\varphi(x) \in \omega(0)$) so that for e.g. the section $\varphi(x) = \cos(x)$, the retraction would be

$$\varphi_{\tau\vartheta_1}(x) = \begin{array}{cc} 1 & x < \pi/2\ ,\ x > 3\pi/2 \\ -1 & \pi/2 < x < 3\pi/2 \end{array}$$

No smooth section $\sigma \in \Phi$ exists such that

$$\delta(\varphi, \sigma) = \delta(\varphi, \vartheta_1) = \min_{\sigma' \in \vartheta_1} \delta(\varphi, \sigma')$$

Let us recall the main result obtained: the basics facts valid for finite dimensional compact group action, i.e.

i) The gradients dV of invariant functions $V(x)$ are in $T_x\Sigma_x$ and orthogonal to $\omega(x)$ at x; they lie therefore in $N_x^0\omega \subset T_x\Sigma_x$.

ii) One can define, by means of the distance defined in the x-space X, a distance in the orbit space $\Omega = X/G$; therefore the concept of neighbourhood of a point $\omega \in \Omega$ in orbit space is well defined, as well as that of an orbit isolated in its stratum.

iii) A G-orbit is critical if and only if it is isolated in its stratum (Michel's theorem; it follows from *i*) and *ii*) above).

can be extended to gauge compact group action.

If Γ is the set (group) of smooth functions $\gamma : M \to G$ and Φ the space of smooth sections $\varphi : M \to E$, $\varphi(x) \in \pi^{-1}(x)$, a stratification of Φ under the action of Γ through the representation Λ is well defined, and we have for *i*) – *iii*) above, the corresponding:

i) The variations $\delta\mathcal{L}$ of invariant functional densities $\mathcal{L}[\varphi]$ are in $T_\varphi\Sigma_\varphi$ and orthogonal to $\vartheta(\varphi)$ at φ; they lie therefore in $N_\varphi^0\vartheta \subset T_\varphi\Sigma_\varphi$.

ii) One can define, by means of the distance defined on the finite dimensional fiber F, a distance in the space of G-orbits in F, $\Omega = F/G$; by means of this one can define a distance in the gauge orbit space (Γ-orbits) $\Theta = \Phi/\Gamma$. The concept of neighbourhood of a point $\vartheta \in \Theta$ in gauge orbit space is well defined, as well as that of an orbit isolated in its stratum.

iii) A Γ-orbit is critical if and only if it is isolated in its stratum (Michel's theorem; it follows from *i*) and *ii*) above).

It should be noticed that the deduction of *iii*) from *i*) and *ii*), once the concept of critical orbit has been defined, is immediate.

Point *i*) is of geometrical nature, and once a stratification has been defined, it does not make any difference if it refers to a finite or infinite dimensional space as far as this point is concerned.

Therefore, as remarked earlier, the extension of Michel's theorem actually consists only in defining a distance in gauge orbit space; it should be stressed once again that we used in a crucial way the gauge (fibered) structure of orbits.

6. First order gauge functionals

In most physical cases, one is faced with a first order gauge theory [25-27]; this means that the functional L and its local density $\mathcal{L}$ depend not only on $\sigma(x)$, but on its first derivatives

$$\partial_\mu \sigma(x) \equiv \frac{\partial \sigma(x)}{\partial x^\mu} \tag{1}$$

as well, which is what is meant by the notation

$$L = \int_M \mathcal{L}[\varphi, \partial\varphi] dx \equiv \int_M \mathcal{L}[\varphi^{(1)}] dx \tag{2}$$

Notice that under the transformation

$$\sigma \to \gamma \cdot \sigma \equiv \Lambda_{\gamma(x)} \sigma(x) \tag{3}$$

the $\partial_\mu \sigma$ do not transform covariantly (i.e. in the same way as σ does): in facts,

$$\partial_\mu \sigma \to \partial_\mu (\gamma \cdot \sigma) = (\partial_\mu \gamma) \cdot \sigma + \gamma \cdot (\partial_\mu \sigma) \equiv (\partial_\mu \Lambda_{\gamma(x)}) \sigma(x) + \Lambda_{\gamma(x)} (\partial_\mu \sigma(x)) \tag{4}$$

One can introduce a covariant derivative ∇_μ by

$$\nabla_\mu = \partial_\mu + \Lambda_{A_\mu} \tag{5}$$

$$A_\mu : M \to \mathcal{G} \tag{6}$$

where the A_μ's are called gauge fields (in physical notation) or connection forms (in mathematical one). Here $\mathcal{G}$ is the Lie algebra of the Lie group G, and Λ_η the infinitesimal generator corresponding to the element η of $\mathcal{G}$ in the representation Λ.

One can check that if

$$A \to \gamma A \gamma^{-1} + (\partial_\mu \gamma) \gamma^{-1} \tag{7}$$

(which means $\Lambda_{A_\mu(x)} \to \Lambda_{\gamma(x)} \Lambda_{A_\mu(x)} \Lambda^{-1}_{\gamma(x)} + (\partial_\mu \Lambda_{\gamma(x)}) \Lambda^{-1}_{\gamma(x)}$), then $\nabla \sigma$ transforms as

$$\nabla \sigma \to \gamma \cdot (\nabla \sigma) \equiv \Lambda_{\gamma(x)} (\nabla \sigma(x)) \tag{8}$$

i.e. covariantly.

Usually, in physical problems one starts from a density $\mathcal{L}(\sigma, \partial_\mu \sigma)$ which is invariant under a global (i.e. rigid, $\gamma(x) = \gamma_0 = const.$) gauge transformation Λ_{γ_0}, and transform it into a local gauge one (i.e. a density invariant under local gauge transformations) [25] by considering $\tilde{\mathcal{L}}(\sigma, \nabla\sigma, A) = \mathcal{L}(\sigma, \nabla\sigma) + \mathcal{L}_G(A)$, i.e. by substituting $\nabla_\mu \sigma^i$ for $\partial_\mu \sigma^i$ in $\mathcal{L}$ and adding the "pure gauge" density

$$\mathcal{L}_G[A^{(1)}] = F_{\mu\nu} F^{\mu\nu} \tag{9}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{10}$$

As already remarked, a gauge theory is better seen, in mathematical language, in terms of fiber bundles [26,27].

To this purpose, we introduce a new fiber bundle of total space $\widehat{E}$, with base M, projection $\widehat{\pi} : \widehat{E} \to M$ and fiber $\widehat{\pi}^{-1}(x) = \mathcal{G}$ (this is a vector bundle since a Lie algebra is a vector space). The gauge fields will then be (smooth) sections $A_\mu : M \to \widehat{E}$, $A_\mu(x) \in \mathcal{G}$, of this bundle; the space of smooth sections of $\widehat{E}$ will be denoted as

$$\mathcal{A} = \{ A : M \to \widehat{E} \ / \ \widehat{\pi} A(x) = x \ , \ A \text{ smooth} \}$$

Γ can be seen as a subgroup of $\mathrm{GDiff}(\widehat{E})$,

$$\Gamma \simeq \Gamma_{\widehat{E}} = \{ \widehat{f} \in \mathrm{GDiff}(\widehat{E}) \ / \ \widehat{f}_x \cdot A(x) = \gamma(x) A(x) \gamma^{-1}(x) + (\partial\gamma(x))\gamma^{-1}(x) \}$$

where $\widehat{f}_x \equiv \widehat{f}|_{\widehat{\pi}^{-1}(x)}$ and we have used the notation of (7). We can also define the isotropy subgroup of $A \in \mathcal{A}$ as

$$\Gamma_A = \{ \gamma \in \Gamma \ / \ \widehat{g} \cdot A = A \}$$

The setting discussed above requires then to consider (sections of) a sum bundle $E_+ = E \oplus \widehat{E}$, with base M, fiber $F \oplus \mathcal{G}$ and projection $\pi \oplus \widehat{\pi}$ (and structural group G).

For our purposes, it is actually more convenient to consider the bundle

$$E_* = E^{(1)} \oplus \widehat{E}^{(1)} = E_+^{(1)}$$

with base M, fiber $F_* = JF \oplus J\mathcal{G}$ and projection $\pi_* = \pi^{(1)} \oplus \widehat{\pi}^{(1)} \equiv d\pi \oplus d\widehat{\pi}$. Here JU is the (first) jet space of U [28,33]; therefore the bundle E_* is naturally equipped with a contact structure [28-32].

The space of smooth sections $\chi_* : M \to E_*$, $\chi_*(x) \in \pi_*^{-1}(x)$, of E_* compatible with the contact structure of E_* will be denoted Φ_*.

The sections $\chi = (\sigma, A) : M \to E_+$ of E_+ induce sections $\chi^{(1)} = (\sigma^{(1)}, A^{(1)})$ of E_* obtained by prolongation [28,33]; conversely any section χ_* of E_* compatible with the contact structure defined in E_* is the prolongation of a section χ of E_+, $\chi_* = \chi^{(1)}$, and therefore the sections $\chi_* \in \Phi_*$ allow to recover the corresponding sections $\chi \in \Phi \times \mathcal{A}$ [33].

The group $\mathrm{GDiff}(E \oplus \widehat{E})$ extends (actually, is isomorphic) to the group $\mathrm{GDiff}^{(1)}(E \oplus \widehat{E}) \subset \mathrm{GDiff}(E_*)$ of prolongations [28,33] of elements of $\mathrm{GDiff}(E \oplus \widehat{E})$; this can also be seen as the subgroup of $\mathrm{GDiff}(E_*)$ which preserves the contact structure of E_*, as it will be discussed in the following.

In terms of explicit formulas, this setting gives back the previous ones: if $\gamma \in \Gamma \subset \mathrm{GDiff}(E \oplus \widehat{E})$ transforms σ into $\sigma' = \gamma\sigma$, then the prolongation $\gamma^{(1)} \in \Gamma^{(1)} \subset \mathrm{GDiff}^{(1)}(E \oplus \widehat{E})$ transforms $\sigma,\ \nabla\sigma,\ \partial\sigma,\ A$ as

$$\begin{aligned} \sigma &\to \sigma' = \gamma \cdot \sigma \equiv \Lambda_{\gamma(x)}\sigma(x) \\ \nabla\sigma &\to (\nabla\sigma)' = \gamma \cdot \nabla\sigma \equiv \Lambda_{\gamma(x)}(\nabla\sigma(x)) \\ A &\to A' = \gamma A \gamma^{-1} + (d\gamma)\gamma \\ \partial\sigma &\to \partial\sigma' = \gamma(\partial\sigma) + (d\gamma)\sigma \end{aligned} \tag{12}$$

The advantage of the present setting lies in that L is a zero-th order functional of sections of E_*, so that we recover the situation of the previous section.

We would like to stress that when considering the variations $\delta\mathcal{L}/\delta\chi$, only variations $\delta\chi$ preserving the contact structure should be allowed. This is satisfied if (and only if) $\delta\chi = \big(\delta\sigma^{(1)}, \delta\gamma^{(1)}\big)$, with $\delta\sigma^{(1)}, \delta\gamma^{(1)}$ prolongations of $\delta\sigma, \delta\gamma$ [33,34] (see also [48]).

We can now define the isotropy subgroup of a section of $\widehat{E}$, and therefore of one of $E \oplus \widehat{E}$. Let Γ be the group of smooth functions from M into G:

$$\Gamma = \{\gamma : M \to G\} \tag{13}$$

which will be seen as a subgroup of $\mathrm{GDiff}(E \oplus \widehat{E})$, by

$$\Gamma \simeq \Gamma_{E_+} = \{\gamma_+ : \sigma(x) \to \Lambda_{\gamma(x)}\sigma \quad , \quad A \to \gamma A \gamma^{-1} + (\partial\gamma)\gamma^{-1}\} \subset \mathrm{GDiff}(E \oplus \widehat{E}) \tag{14}$$

(using again the notation of (7))

For a section (σ, A) of $E \oplus \widehat{E}$, we define

$$\Gamma_{(\sigma,A)} = \{\gamma \in \Gamma \ / \ \gamma \cdot \sigma = \sigma \ , \ \gamma A \gamma^{-1} + (\partial\gamma)\gamma^{-1} = A\} = \Gamma_\sigma \cap \Gamma_A \tag{15}$$

(where the meaning of Γ_σ and Γ_A is obvious).

It is immediate to check that, if

$$\sigma' = \alpha \cdot \sigma \quad ; \quad A' = \alpha A \alpha^{-1} + (\partial\alpha)\alpha^{-1} \tag{16}$$

then

$$\Gamma_{(\sigma',A')} = \alpha\Gamma_{(\sigma,A)}\alpha^{-1} \tag{17}$$

In facts, let $\Gamma' \equiv \Gamma_{A'}$; $\Gamma \equiv \Gamma_A$. By definition,

$$\Gamma A\Gamma^{-1} + (\partial\Gamma)\Gamma^{-1} = A \tag{18}$$

and one has

$$\begin{aligned}
&(\alpha\Gamma\alpha^{-1})A'\alpha\Gamma^{-1}\alpha^{-1} + [\partial(\alpha\Gamma\alpha^{-1})]\alpha\Gamma^{-1}\alpha^{-1} = \\
=&\alpha\Gamma A\Gamma^{-1}\alpha^{-1} + \alpha\Gamma\alpha^{-1}(\partial\alpha)\Gamma^{-1}\alpha^{-1} + (\partial\alpha)\alpha^{-1} + \alpha(\partial\Gamma)\Gamma^{-1}\alpha^{-1} - \alpha\Gamma\alpha^{-1}(\partial\alpha)\Gamma^{-1}\alpha^{-1} = \\
=&\alpha[\Gamma A\Gamma^{-1} + (\partial\Gamma)\Gamma^{-1}]\alpha^{-1} + (\partial\alpha)\alpha^{-1} = \alpha A\alpha^{-1} + (\partial\alpha)\alpha^{-1} = A'
\end{aligned} \tag{19}$$

where we have used (16), (18) and the identity

$$\partial\alpha^{-1} = -\alpha^{-1}(\partial\alpha)\alpha^{-1} \tag{20}$$

It is trivial that

$$\Gamma_{\sigma'} = \gamma\Gamma_\sigma\gamma^{-1} \tag{21}$$

so that (17) is satisfied.

One can therefore define a stratification in $(\Phi \times \mathcal{A})$, and any stratum will be the union of Γ-orbits. By this, a stratification will also be defined in the space of Γ orbits $\Theta_+ \equiv (\Phi/\Gamma \times \mathcal{A}/\Gamma)$.

It is also possible to define a distance in Θ_+ in the same way (and with the same remarks) as done for Θ: we just have to define, given $\chi = (\sigma, A)$, $\chi' = (\sigma', A')$,

$$\delta(\chi, \chi') = \delta(\sigma, \sigma') + \delta(A, A')$$

and similarly for gauge orbits $\vartheta_+, \vartheta'_+ \in \Theta_+$, $\vartheta_+ = \vartheta \times \widehat{\vartheta}$ (the $\widehat{\vartheta}$ are orbits in the gauge orbit space $\mathcal{A}/\Gamma$) we define

$$\delta(\vartheta_+, \vartheta'_+) = \delta(\vartheta, \vartheta') + \delta(\widehat{\vartheta}, \widehat{\vartheta}')$$

With this, a distance is defined in Θ_+. It is clear that we can proceed in the same way for an n-sum vector bundle.

We have, anyway, to deal with sections $\chi_* \in \Phi_*$ of E_*; to this purpose we will take advantage of the contact structure in E_*.

In order to deal with sections of E_*, we just use the fact that $JU = U \oplus U_{[1]}$ (where $U_{[1]}$ can be seen as the space of first derivatives of functions $f(x) : M \to U$ [28,33]; if M is one dimensional the jet space JU is nothing else than the familiar tangent bundle TU), so that E_* is a 4-sum bundle equipped with an additional structure (the contact structure).

Any section $\chi_* = \chi^{(1)} = (\sigma^{(1)}, A^{(1)}) \in \Phi_*$ can be decomposed as $\chi_* = (\sigma, \nabla\sigma, A, \partial A)$, with $\sigma(x) : M \to F$, $(\nabla\sigma)(x) : M \to F_{[1]}$, $A(x) : M \to \mathcal{G}$, $(\partial A)(x) : M \to \mathcal{G}_{[1]}$. Since $F_{[1]}, \mathcal{G}_{[1]}$ are themselves vector spaces [28,33], we can proceed as before in order to define the distance $\delta(\chi_*, \chi'_*)$ of two sections in Φ_*: with an obvious notation, this will be

$$\delta(\chi_*, \chi'_*) = \delta(\sigma, \sigma') + \delta(\nabla\sigma, \nabla\sigma') + \delta(A, A') + \delta(\partial A, \partial A')$$

When considering the action of Γ on Φ_*, we should use the fact that only sections of E_* which are compatible with the contact structure are allowed; this means that $\gamma \in \Gamma$ acts on Φ_* by its (first) prolongation $\gamma^{(1)}$ [28,33].

The subgroup of $\mathrm{GDiff}(E_*)$ which preserves, in addition to the fibered structure, the contact structure of E_* will be denoted $\mathrm{CGDiff}(E_*)$; it is clear that

$$\mathrm{CGDiff}(E_*) = \{f^{(1)} \ / \ f \in \mathrm{GDiff}(E_+)\} = \mathrm{GDiff}^{(1)}(E_+)$$

(where as usual the superscript denotes first prolongation); we know by this that [33]

$$\mathrm{CGDiff}(E_*) \simeq \mathrm{GDiff}(E_+)$$

Γ can be seen as a subgroup of $\mathrm{CGDiff}(E_*)$, by

$$\Gamma \simeq \Gamma_* = \{f_*^{(1)} \subset \mathrm{GDiff}(E_*) \ / \ f_* = f \oplus \widehat{f} \in \Gamma_E \oplus \Gamma_{\widehat{E}} \equiv \Gamma_{E_+}\}$$

We can now define the isotropy subgroup of prolonged sections in the natural way:

$$\Gamma_{\sigma^{(1)}} = \{\gamma \in \Gamma \ / \ \gamma^{(1)} \cdot \sigma^{(1)} = \sigma^{(1)}\} \ \ ; \ \ \Gamma_{A^{(1)}} = \{\gamma \in \Gamma \ / \ \widehat{\gamma}^{(1)} \cdot A^{(1)} = A^{(1)}\}$$

$$\Gamma_{\chi^{(1)}} = \Gamma_{\sigma^{(1)}} \cap \Gamma_{A^{(1)}}$$

It should be noticed that the action of γ both on $\nabla\sigma$ and ∂A depends on the action of γ on A.

Clearly, $\Gamma_{\chi^{(1)}} \subseteq \Gamma_\chi$, as $\gamma^{(1)} \cdot \chi^{(1)} = \chi^{(1)}$ requires in particular $\gamma \cdot \chi = \chi$ [33], but it is also true that if a function is not changed, its derivatives are neither, so that $\Gamma_{\sigma^{(1)}} = \Gamma_\sigma$, $\Gamma_{A^{(1)}} = \Gamma_A$ and

$$\Gamma_{\chi^{(1)}} = \Gamma_\chi$$

We stress that in general $\Gamma_{\nabla\sigma} \neq \Gamma_{\sigma^{(1)}}$, since $\nabla\sigma = \partial\sigma + A\sigma$, so that $\Gamma_{\sigma^{(1)}} \cap \Gamma_\sigma = \Gamma_\sigma \cap \Gamma_A = \Gamma_\chi$.

Finally, the above discussion allows to reconduce the stratification of Φ_* under Γ (acting via Λ) to that of $\Phi \times \mathcal{A}$. As usual, a stratum in Φ_* will be the union of Γ-orbits, so that a stratification in the Γ-orbit space Θ_* is defined as well.

We still have to define a distance in Θ_*, but in order to do this we can just proceed as for Θ_+; with obvious notation,

$$\vartheta_*(\chi_*) = \vartheta(\sigma) \times \vartheta^{(1)}(\nabla\sigma) \times \widehat{\vartheta}(A) \times \widehat{\vartheta}^{(1)}(\partial A)$$

and the distance is defined as

$$\delta(\vartheta_*, \vartheta_*') = \delta(\vartheta, \vartheta') + \delta(\vartheta^{(1)}, \vartheta^{(1)'}) + \delta(\widehat{\vartheta}, \widehat{\vartheta}') + \delta(\widehat{\vartheta}^{(1)}, \widehat{\vartheta}^{(1)\prime})$$

With this, we have defined a distance in the Γ-orbit space Θ_* of interest for first order gauge theories, which we see as zero order theories with an assignement of a contact structure.

The discussion of sections 4 and 5 does therefore apply, and in particular points $i) - iii)$ of section 5 continue to hold; quite clearly, we could in the same way deal with gauge theories of any given finite order. This means that Michel theorem for gauge orbits holds as well for first order gauge theories, as those of physical interest.

We will consider some detailed examples in the following sections.

7. Geometry and stratification of Ω

In order to understand the stratification of the space Φ of sections of E under the action of Γ, one should first get a better knowledge of the stratification of F and $\Omega \equiv F/G$ under the action of G.

This is actually a classical subject [2,5,21,22], but we will now recall some relevant facts for the convenience of the reader, following [5]. We remind that F is a finite dimensional manifold, and G a compact (actually, in physical applications an orthogonal or unitary) Lie group acting on F by a linear representation Λ. (It should actually be remarked that most of these results could be extended to the case of smooth action of noncompact Lie groups provided they have compact stabilizers, see [54]).

A theorem by Hilbert [5,22] states that there is a finite set of invariant polynomials $\{\theta_0(x), \theta_1(x), \ldots, \theta_k(x)\}$ (here $x \in F$, and we take $\theta_0(x) \equiv 1$), $\theta_i(\Lambda_g x) = \theta_i(x)\ \forall g \in G$, such that any invariant polynomial $P(\Lambda_g x) = P(x)\ \forall g \in G$ can be written as a polynomial in the θ's, $P(x) = \widehat{P}(\theta(x))$. The $\{\theta_i(x)\}$ form an integrity basis.

This theorem was also extended to smooth functions [23,37]: any smooth invariant function $f(\Lambda_g x) = f(x)\ \forall g \in G$ can be written as a smooth function in the θ's, $P(x) = \widehat{P}(\theta(x))$.

Now, invariant functions separate orbits, i.e. given two distinct orbits $\omega, \omega' \in \Omega$, there is at least one function f (invariant, i.e. $f(\Lambda_g x) = f(x)$) such that $f(x) \neq f(y)$ for $x \in \omega,\ y \in \omega'$.

Weierstrass approximation theorem tells that any C^∞ function can be locally written as the limit of a uniformly converging series of polynomials.

These two facts tell that the integrity basis separates orbits; in other words we have that Ω is a semialgebraic variety in R^k (a subset of R^k defined by equalities and inequalities of polynomials); R^k can be thought as the space of values assumed by the polynomials $\vartheta_1(x), \ldots, \vartheta_k(x)$.

A semialgebraic variety Ω in R^k has a natural primary stratification [37], i.e. can be seen as the disjoint union of open manifolds of dimensions from k down to 0 (as an example, a square S is the union of interior points S_i and of border ∂S; the latter is the union of points on edges E_i and border of edges ∂E_i, which are the vertices V_i), so that

$$\Omega = \cup_{\alpha,i} E_i^{(\alpha)} \qquad \mathrm{dim} E_i^{(\alpha)} = \alpha \tag{1}$$

where $\cup$ denotes disjoint union, and

$$E_j^\beta \in \partial E_i^\alpha \qquad \beta < \alpha \tag{2}$$

The latter relation introduces a partial order in the set $\{E_i^\alpha\}$ of primary strata, i.e. bordering.

It can be proven that the stratification defined in sect. 2, i.e. based on symmetry of orbits $\omega \in \Omega$ under G (those strata will sometimes be called isotropy strata) follows this primary stratification, in the sense that connected components of isotropy strata correspond to union of primary strata, i.e. $\Sigma_\omega = \cup_{\alpha \in A; i \in I} E_i^\alpha$, for some index sets A, I.

¿From this it follow in particular two consequences: first, that only orbits in primary strata of dimension 0 can be isolated in their stratum; second, that nearly all the $\omega \in \Omega$ belong to the maximal dimensional primary stratum, and therefore to the principal stratum (some care should be taken if Ω is not connected).

It can also be seen that more peripheral primary strata have higher symmetry; i.e., the partial ordering given by the bordering relation coincides with the partial ordering given by symmetry relations (see sect. 2).

Let us now consider some examples of stratification in orbit spaces (these same examples will be considered again in the following to illustrate next steps).

Example 1: $F = R^1$, with coordinate $u \in R^1$; $G = Z_2 = \{e, g\}$ acting by $e : u \to u$; $g : u \to -u$. Then we have $\omega(u) = \{u, -u\}$, and

$$\omega(u) \simeq Z_2 \text{ for } u \neq 0 \ \ ; \ \ \omega(u) = \{0\} \simeq \{e\} \text{ for } u = 0$$

$$G_u = \{e\} \text{ for } u \neq 0 \ \ ; \ \ G_u = Z_2 \text{ for } u = 0$$

(clearly, it must be $\omega(u) = G/G_u$). The orbit space is $\Omega = R_+ = \{u \geq 0\}$ and we have two strata:

$$\Omega_1 = \{\vartheta > 0\} \ \ ; \ \ \Omega_0 = \{0\} = \partial\Omega_1$$

Example 2: $F = R^2$, $G = SO(2)$ acting by the standard representation. We have $\omega(u) = \{u' \ / \ |u'| = |u|\}$, so that

$$\omega(u) \simeq SO(2) \text{ for } u \neq 0 \ \ ; \ \ \omega(0) = \{0\} \simeq \{e\}$$

$$G_u = \{e\} \text{ for } u \neq 0 \ \ ; \ \ G_0 = SO(2)$$

The orbit space is $\Omega = R_+$, with strata

$$\Omega_1 = \{\vartheta > 0\} \ \ ; \ \ \Omega_0 = \{0\} = \partial\Omega_1$$

Example 3: $F = R^3$, $G = SO(2)$ acting as rotations around the third axis, generated by $\tau = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then, with coordinates $\{u_1, u_2, u_3\}$ and $\rho(u) = u_1^2 + u_2^2$, we have $\omega(u) = \{u' \ / \ \rho(u') = \rho(u) \ , \ u_3' = u_3\}$. i.e. a minimal integrity basis (MIB) is given by $\vartheta_1 = u_1^2 + u_2^2 \ , \ \vartheta_2 = u_3$. We also have

$$\omega(u) \simeq SO(2) \text{ for } \vartheta_1(u) \neq 0 \ \ ; \ \ \omega(u) = \{u\} \text{ for } \vartheta_1(u) = 0$$

$$G_u = \{e\} \text{ for } \vartheta_1(u) \neq 0 \ \ ; \ \ G_u = SO(2) \text{ for } \vartheta_1(u) = 0$$

The orbit space is $\Omega = \{\vartheta_1, \vartheta_2 \ / \ \vartheta_1 \geq 0\} \subset R^2$; there are two strata,

$$\Omega_1 = \{\vartheta_1, \vartheta_2 \ / \ \vartheta_1 > 0\} \ \ ; \ \ \Omega_0 = \{\vartheta_1, \vartheta_2 \ / \ \vartheta_1 = 0\} = \partial\Omega_1$$

Example 4: $F = S^2 \subset R^3$, $G = SO(2)$ acting as before. Then choose coordinates (α, u_3), $\alpha \in [0, 2\pi]$, $u_3 \in [0, 1]$, $(\alpha, \pm 1) = (0, \pm 1) \ \forall \alpha$. Then $\omega(u) = \{u' \ / \ u_3' = u_3\}$ (a minimal integrity basis is given by u_3 alone);

$$\omega(u) \simeq S^1 = SO(2) \text{ for } u_3 \neq \pm 1 \ \ ; \ \ \omega(u) = \{u\} \text{ for } u_3 = \pm 1$$

The orbit space is $\Omega = [-1, 1] \subset R^1$; there are two strata,

$$\Omega_1 = \{\vartheta \neq \pm 1\} \ \ ; \ \ \Omega_0 = \{+1, -1\} = \partial\Omega_1$$

Notice that here there are two orbits isolated in their stratum, while in example 3 there are none.

Example 5: $F = R^3$, $G = SO(2) \times Z_2$, where $SO(2)$ acts as in examples 3 and 4, and Z_2 by $h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; a MIB is given by $\{\vartheta_1 = u_1^2 + u_2^2 \ , \ \vartheta_2 = u_3^2\}$. The orbit space is $\Omega = \{\vartheta_1, \vartheta_2 \ / \ \vartheta_1 \geq 0 \ , \ \vartheta_2 \geq 0\} = R_+ \times R_+$; $\omega(u) = \{u' \ / \ \vartheta_1(u') = \vartheta_1(u), \ \vartheta_2(u') = \vartheta_2(u)\}$, so that

$$\begin{aligned} &\omega(u) \simeq S^1 \times Z_2 \text{ for } \vartheta_1 \neq 0; \ \vartheta_2 \neq 0 \\ &\omega(u) \simeq Z_2 \text{ for } \vartheta_1 = 0; \ \vartheta_2 \neq 0 \\ &\omega(u) \simeq SO(2) \text{ for } \vartheta_1 \neq 0; \ \vartheta_2 = 0 \\ &\omega(u) \simeq \{e\} \text{ for } \vartheta_1 = 0; \ \vartheta_2 = 0 \ (u = 0) \end{aligned}$$

Correspondingly, the isotropy subgroup is, in the four cases,

$$G_u = \{e\} \quad ; \quad G_u = SO(2) \quad ; \quad G_u = Z_2 \quad ; \quad G_u = G$$

We have therefore four strata:

$$\begin{aligned} \Omega_0 &= \{\vartheta_1 \neq 0; \ \vartheta_2 \neq 0\} \equiv E_1^2 \\ \Omega_1 &= \{\vartheta_1 = 0; \ \vartheta_2 \neq 0\} \equiv E_1^1 \\ \Omega_2 &= \{\vartheta_1 \neq 0; \ \vartheta_2 = 0\} \equiv E_1^1 \\ \Omega_3 &= \{\vartheta_1 = 0; \ \vartheta_2 = 0\} \equiv E_1^0 \end{aligned}$$

One can check that actually

$$\partial\Omega_0 = \Omega_1 \cup \Omega_2 \cup \Omega_3 \quad ; \quad \partial\Omega_1 = \partial\Omega_2 = \Omega_3$$

Notice that to these relations, also written

$$\begin{tikzcd} & \Omega_1 \arrow[dr] & \\ \Omega_3 \arrow[ur] \arrow[dr] & & \Omega_0 \\ & \Omega_2 \arrow[ur] & \end{tikzcd}$$

in terms of bordering relations, correspond the inclusion relations among isotropy subgroups

$$\begin{tikzcd} & SO(2) \arrow[dr] & \\ \{e\} \arrow[ur] \arrow[dr] & & SO(2) \times Z_2 \\ & Z_2 \arrow[ur] & \end{tikzcd}$$

so that the diagram of bordering relations can also be seen as relative to symmetry relations among strata.

Example 6: $F = S^2 \subset R^3$; $G = SO(2) \times Z_2$ acting as in example 5. Choosing as coordinates (α, u_3), $\alpha \in [0, 2\pi]$, $u_3 \in [-1, 1]$, $(\alpha, \pm 1) = (0, \pm 1)\ \forall \alpha$, a MIB is given by $\vartheta_1 = u_3^2$, so that $\Omega = \{\vartheta_1\} = [0, 1] \subset R^1$. as for orbits, we have

$$\begin{array}{lcr} \omega(u) \simeq & SO(2) \times Z_2 & \text{for } \vartheta_1 \neq 0,\ \vartheta_1 \neq 1 \\ \omega(u) \simeq & SO(2) & \text{for } \vartheta_1 = 0 \\ \omega(u) \simeq & Z_2 & \text{for } \vartheta_1 = 1 \end{array}$$

and correspondingly for isotropy subgroups

$$G_u = \{e\} \quad ; \quad G_u = Z_2 \quad ; \quad G_u = SO(2)$$

There are three strata,

$$\begin{array}{lcl} \Omega_0 = & \{\vartheta \neq 0,\ \vartheta \neq 1\} & \equiv E^2 \\ \Omega_1 = & \{\vartheta = 0\} & \equiv E^1 \\ \Omega_2 = & \{\vartheta = 1\} & \equiv E^0 \end{array}$$

8. Stratification of gauge orbit space

We can now discuss the stratification of the space Φ and of the Γ-orbit space $\Theta = \Phi/\Gamma$; from now on, subgroups of G conjugated in G will simply be identified.

We will actually discuss a subclass $\Phi_T \subset \Phi$ of sections of E, that of transversal ones (this name will be defined and explained in a moment). Given a section $\sigma(x) : M \to F$, we can consider the set of values it takes in F,

$$F_\sigma = \{\sigma(x)\ ,\ x \in M\} \subset F \tag{1}$$

and in an obvious way

$$\Omega_\sigma = \{\omega_{\sigma(x)}\ ,\ x \in M\} \subseteq \Omega \tag{2}$$

Now, let us consider the primary stratification of Ω,

$$\Omega = \cup . E_i^\alpha \quad , \quad \text{dim} E^\alpha = \alpha \tag{3}$$

introduced in the previous section. The primary index $\alpha(\sigma)$ of the section σ is the greater α for which there is an E_i^α with

$$\Omega_\sigma \cap E_i^\alpha \neq \{\oslash\} \tag{4}$$

A section is transversal if it meets primary strata of dimension β strictly less than its primary index transversally.

We do also define, for the sake of completeness, the set of accessible strata for σ, $[E]_\sigma$:

$$[E]_\sigma = \{E_i^\alpha \ / \ \Omega_\sigma \cap E_i^\alpha \neq \oslash\} \tag{5}$$

Clearly,

$$E_\sigma \equiv \cup . E_i^\alpha \quad , \quad E_i^\alpha \in [E]_\sigma \tag{6}$$

is a semialgebraic variety; the locus of points $\omega_{\sigma(x)}$, $x \in M$ determines a curve $e(\sigma)$ in E_σ; it is immediate to see that for a smooth section σ, $e(\sigma)$ is smooth at points belonging to strata of dimension $\alpha(\sigma)$; it is also immediate to see that for transversal sections, $e(\sigma)$ is singular (C^0 but not C^1) at points on strata of dimension strictly less than $\alpha(\sigma)$.

It should be remarked that transversality is a structurally stable property, and that transversal sections are dense in Φ; we stress that this follows from the results recalled in the previous section about the geometry of Ω.

This restriction to transverse sections could seem quite misterious, so let us discuss it.

As briefly remarked in sect. 4 above, due to the fact that the gauge group is infinite dimensional, the very concept of stratification is quite delicate. Let us show by a simple example the kind of troubles one is faced with.

In the setting of example 2 of section 7, let the base space be $I = [0, 1]$, and consider two sections given by, with v_0 a given unit vector in F,

$$\sigma_i(x) = e^{-1/(x-x_i)^2} v_0 \quad , \quad x \leq x_i \quad ; \quad \sigma_i(x) = 0 \quad , \quad x \geq x_i \quad ; \quad i = 1, 2$$

Clearly, the isotropy subgroups $\Gamma_i \subset \Gamma$ of these are

$$\Gamma_i = \{\gamma \in \Gamma \ / \ \gamma(x) = \{e\} \ \ x \geq x_i\} \equiv \{\gamma : [0, x_i] \to SO(2) \ / \ \gamma(x_i) = 0\}$$

Now, Γ_1 and Γ_2 are isomorphic but if, say, $x_2 > x_1$, then Γ_1 is a proper subgroup of Γ_2.

It is quite clear that this "patology" does not occurr when restricting to transversal sections. It *seems* to us that this restriction is sufficent for a stratification to be properly defined, but we stress that this is not being proved. We will anyway see that in our examples this restriction does actually suffice to properly define a stratification.

Actually, to the best of author's knowledge, very little is known about the geometry of orbit space for infinite dimensional, e.g. gauge, groups (see e.g. the problem of Gribov ambiguity; even the proof that the Gribov region contains representatives of every gauge orbit is very recent [49]). We will formalize our reasonable guess into the

Assumption: In the space $\Phi_T \subset \Phi$ of transversal sections, and therefore in $\Theta_T = \Phi_T/\Gamma$, a stratification is properly defined.

Actually, we will mainly use only the

Weaker assumption: For the action of Γ on Φ_T, there is a set (possibly infinite or even continuous) of subgroups Γ_μ which are: *i)* isotropy subgroups for some $\sigma \in \Phi_T$; *ii)* not contained in any other subgroup satisfying i); *iii)* such that no proper subgroup $\Gamma_\mu^{(\alpha)} \subset \Gamma_\mu$ is conjugated to any $\Gamma_{\mu'}$. In other words, the concept of maximal isotropy subgroup of Γ is properly defined.

In the following we will consider only Φ_T; since no confusion will be possible, we will write - for ease of notation - Φ for Φ_T and Θ for Θ_T. Should our assumptions be wrong, our discussion would be purely formal; we believe that anyway it would remain of eurysthical value, as will be seen in the following. Moreover, as already remarked, they apply at least to the examples considered here.

Let us first consider a section σ such that

$$[E]_\sigma = E_i^\alpha \tag{7}$$

i.e. such that $\sigma(x)$ belongs to one and the same stratum for all $x \in M$; then, if G_i^α is the isotropy subgroup corresponding to this stratum E_i^α, Γ_σ is the set of functions

$$\Gamma_\sigma = \{\gamma(x) : \ M \to G_\alpha\} \subseteq \Gamma \tag{8}$$

(we recall this is a group by $(\gamma_1 \cdot \gamma_2)(x) = \gamma_1(x) \cdot \gamma_2(x)$).

Let us now consider a σ such that

$$\sigma(x) \in E_i^\alpha \quad x \in D_1 \ \ ; \ \ \sigma(x) \in E_j^\alpha \quad x \in D_2 \tag{9}$$

for D_1 and D_2 domains of M, $\dim(D_1) = \dim(D_2) = \dim(M)$, with border $\partial D_1 = \partial D_2 = B$, $\dim B < \dim D_i$. Then, called G_1 and G_2 the isotropy subgroups relative to the two strata E_i^α and E_j^α, we have

$$\Gamma_\sigma \subset \Gamma_\sigma^* = \{\gamma : M \to G \ / \ \gamma : D_1 \to G_1 \ ; \ \gamma : D_2 \to G_2\} \subset \Gamma \tag{10}$$

One still has to impose boundary conditions on B; smoothness of γ requires that

$$\gamma : B \to (G_1 \cap G_2) \tag{11}$$

(the above intersection is not empty since it contains at least $\{e\}$).

Notice that, a priori, the section $\sigma(x)$ maps B to more peripherical strata, so that one could think to have $\gamma(x) \in G_k^\beta$ for $x \in B$, $\sigma(x) \in E_k^\beta$, since $E_k^\beta \in \partial E_i^\alpha$, $E_k^\beta \in \partial E_j^\alpha$ and given the discussion of the previous section; on the other side, $\dim B < \dim D_i$ amounts to the transversality condition, and this together with continuity of γ gives (11). Finally, for such a σ we get

$$\Gamma_\sigma = \{\gamma : M \to G \ / \ \gamma : D_1 \to G_1 \ ; \ \gamma : D_2 \to G_2 \ ; \ \gamma : B \to G_1 \cap G_2\} \tag{12}$$

Remark that for $G_1 = G_2 = G_\alpha$, this amounts to (8); in other words, the transversality condition implies that "only higher dimensional strata in $[E]_\sigma$ matter", in the sense that the isotropy subgroup Γ_σ does not depend on the strata of lower dimension, i.e. is insensible to the values taken by $\sigma(x)$ on the lower dimensional set $B \subset M$.

With the previous discussion in mind, it is easy to understand the general situation for transversal sections. We will discuss the case of compact connected M and connected orbit space Ω, of interest here.

Given a transversal section $\sigma(x)$ of primary index $\alpha(\sigma) \equiv \alpha$, let $\{E_i^\alpha \ , \ i = 1, \ldots, k\}$ be the set of α-dimensional strata in $[E]_\sigma$, that is, such that

$$\exists x \in M \ / \ \sigma(x) \in E_i^\alpha \tag{13}$$

Then, let D_i be the anti-image of E_i^α by σ,

$$D_i = \{x \in M \ / \ \sigma(x) \in E_i^\alpha\} \tag{14}$$

The transversality condition implies that

$$B \equiv M \backslash \{D_1 \cup \ldots \cup D_k\} = \partial D_1 \cup \ldots \cup \partial D_k \tag{15}$$

and therefore

$$\mathrm{dim} B < \mathrm{dim} D_i = \mathrm{dim} M \tag{16}$$

Then, if G_i is the isotropy subgroup of E_i^α, we have

$$\Gamma_\sigma = \{\gamma : M \to G \ / \ \gamma : D_i \to G_i\} \subset \Gamma \tag{17}$$

where the smoothness of γ implies

$$\gamma : (\partial D_i) \cap (\partial D_j) \to G_i \cap G_j \tag{18}$$

It should be stressed that Γ_σ does depend in a crucial way on the geometry of the D_i's, as this does fix the "boundary conditions" (i.e. the conditions on B) which the $\gamma \in \Gamma_\sigma$ have to satisfy.

As an example of this, consider the setting of example 5 in the previous section.

Let $M = S^1$, and $\sigma_1(x) = (0, 0, \sin(x))$. Then $\alpha(\sigma) = 1$, $[E]_s = \{E_1^1, E_1^0\}$; as for D_1, this is $D_1 = \{(0, \pi) \cup (\pi, 2\pi)\}$, $B = \{0, 2\pi\}$. We have therefore

$$\Gamma_1 \equiv \Gamma_{\sigma_1} = \{\gamma : S^1 \to SO(2) \ / \ \gamma(0) = \gamma(\pi) = e\} \tag{19}$$

(since $SO(2) \cap Z_2 = \{e\}$ in this setting), so that G_1 is parametrized by pairs of functions (γ_1, γ_2) from the interval I to $SO(2)$ which satisfy boundary conditions $\gamma_i(\partial I) = e$.

Let us now consider the section $\sigma_2 = (0, 0, \sin(2x))$. Now $\alpha(\sigma)$ and $[E]_\sigma$ are as before, but $D_1 = \{(0, \pi/2) \cup (\pi/2, \pi) \cup (\pi, 3\pi/2) \cup (3\pi/2, 2\pi)\}$, $B = \{0, \pi/2, \pi, 3\pi/2\}$, and

$$\Gamma_2 \equiv \Gamma_{\sigma_2} = \{\gamma : S^1 \to SO(2) \ / \ \gamma(0) = \gamma(\pi/2) = \gamma(\pi) = \gamma(3\pi/2) = e\} \tag{20}$$

so that Γ_2 is parametrized by quadruples $(\gamma_1, \gamma_2, \gamma_3, \gamma_4)$ of functions from the interval I to $SO(2)$ which satisfy boundary conditions $\gamma_i(\partial I) = e$. Therefore, a priori (i.e. without considering the requirement of smoothness of γ), we have $\Gamma_1 \subseteq \Gamma_2$.

Notice also that there is a "basis space" S^1 symmetry associated to this problem; this is broken to Z_2 for σ_1 and to Z_4 for σ_2; basis space symmetry will be shortly discussed in the last part of this paper.

By looking at this setting, with M an higher dimensional manifold, one gets easily convinced that an explicit stratification of $\Theta = \Phi/\Gamma$ is extremely complicate and difficult to describe. We will not attempt such a description here, but will instead concentrate on the description of most singular strata, i.e. those corresponding to maximal isotropy subgroups. These are also the strata on which the extension of Michel's theorem given above can be applied.

It should be stressed that maximal isotropy subgroups are not always the only ones corresponding to most singular strata: the hypotheses that they indeed are is known as the maximal isotropy subgroup conjecture, and is now known to be in general not true. A complete discussion of it, including identification of the cases (i.e. of the groups) in which it holds true, has been given recently by Field and Richardson for compact Lie groups [50-53].

9. Maximal strata in gauge orbit space

Let us consider again the action of G on F by the representation $\Lambda = \{\Lambda_g subgroups (MIS) \mathrm{G}_\mu$, $\mu = 1, \ldots, s$, i.e. of subgroups $G_\mu \subseteq G$ such that $\exists z \in F \ / \ \Lambda_g z = z \ \forall g \in G_\mu$, and there is no subgroup $G_\mu^* \subseteq G$ such that $G_\mu \subset G_\mu^*$ and G_μ^* is an isotropy subgroup. (We stress that the concept of MIS depends on both F and Λ, for given G).

The set $\Omega_\mu = \{\omega \in \Omega \ / \ \Lambda_g z = z \ \forall z \in \omega, \ \forall g \in G_\mu\}$ will correspond to a maximal (i.e. minimal dimensional) stratum.

A section σ such that $\sigma(x) \in \Omega_\mu \ \forall x \in M$ (i.e. $\Omega_\sigma \subseteq \Omega_\mu$) will admit as symmetry group

$$\Gamma_\mu = \{\gamma : M \to G_\mu\} \tag{1}$$

The groups $\Gamma_\mu \subset \Gamma$, for G_μ a MIS of G, are MIS of Γ. In facts, to have $\Gamma_\mu \subset \Gamma'$, Γ' must contain γ's such that for some $x \in M$, $\gamma(x) \in G \backslash G_\mu$. But we know that every g such that $\exists z \ / \ \Lambda_g z = z$ must belong to some $G_\mu, \ \mu = 1, \dots, s$. Therefore $\gamma \in \Gamma$ can belong to the isotropy group of some section only if

$$\gamma(x) \in \cup_{\mu=1,\dots,s} G_\mu \tag{2}$$

Suppose now that for $x \in M$ $\gamma(x)$ belongs to at least two different G_μ's, $\gamma(x_i) \in G_i$, $G_i \neq G_j$ for $i \neq j$, and let $M_i = \{x \in M \ / \ \gamma(x) \in G_i\}$. Then necessarily there are points $x \in M_i \cap M_j$; due to smoothness of γ, in these $\gamma(x) \in G_i \cap G_j$.

Now, the functions $\gamma : M \to G \ / \ \gamma : M_i \to G_i$ can be seen as n-ples of functions γ_i defined on M_i with vales in G_i, each of them subject to appropriate boundary conditions: on $M_{ij} \equiv M_i \cap M_j \in \partial M_i$, $\gamma : M_{ij} \to G_i \cap G_j$. Clearly for no distincts i, j one can have $G_i \subset G_i \cap G_j$, for the G_μ are MIS. This also means that it is not possible to find an isotropy group $\Gamma' \subset \Gamma$ such that $\Gamma_\mu \subset \Gamma'$.

We stress that the above argument shows that all the Γ_μ of the form (1), with G_μ a MIS of G, are maximal isotropy subgroups of Γ, but in general not all the MIS (even on the set of transverse sections) need to be of the form (1).

In the same way, one can see that given a stratum $\Sigma_\omega \subset \Omega$, with isotropy subgroup G_0, the sections σ such that

$$\omega_{\sigma(x)} \in \Sigma_\omega \quad \forall x \in M \tag{3}$$

have isotropy subgroup

$$\Gamma_0 = \{\gamma \in \Gamma \ / \ \gamma : M \to G_0\} \tag{4}$$

Let us go back to MIS: if in the stratification of Ω by G $\Sigma_\omega^{(\mu)} \equiv \Omega_\mu$ is a maximal stratum (i.e. minimal dimensional, corresponding to a MIS), we have just seen that

$$\Phi_\mu = \{\sigma \in \Phi \ / \ \sigma : M \to \Omega_\mu\} \tag{5}$$

form a maximal stratum with isotropy Γ_μ. In other words,

$$\Theta_\mu = \Phi_\mu / \Gamma \tag{6}$$

is a maximal stratum in the stratification of Θ by Γ. (Remark that actually $\Theta_\mu = \Phi_\mu/(\Gamma/\Gamma_\mu)$, as Γ_μ is the identity on Φ_μ).

Now, let us assume that $\omega \in \Omega$ is isolated in its stratum; it is immediate that $\vartheta \in \Theta$, where ϑ is the gauge orbit of sections $\sigma \ / \ \sigma(x) \in \omega \ \forall x \in M$, is isolated in its stratum (recall that Θ is equipped with the topology induced by the topology of Ω). This gives a constructive way for determining some (not all, in general !) of the critical gauge orbits.

We will summarize our discussion as follows:

Theorem: Let $G_1 \subseteq G$ be a MIS for the action of G on F by the representation Λ, and let $\Omega = F/G$. Then

i) $\Gamma_1 = \{\gamma : M \to G_1\}$ is a MIS of Γ;

ii) The set $\Theta_1 = \{\vartheta \ / \ \omega \cdot \sigma(x) : M \to \Omega_1 \ \forall x \in M \ \forall \sigma \in \vartheta\}$, where $\Omega_1 = \{\omega \ / \ \Lambda_g z = z \ \forall z \in \omega \ \forall g \in G_1\}$, is a maximal stratum of Θ;

iii) If ω_0 is isolated in its stratum Ω_1, then ϑ_0, the gauge orbit such that any section $\sigma(x) \in \omega_0 \ \forall x \in M$ belongs to ϑ_0, is isolated in its stratum Θ_1;

iv) As a consequence of iii), for every critical orbit $\omega_0 \in \Omega$ there is a critical gauge orbit $\vartheta_0 \in \Theta$.

Let us now consider some examples, following (also in the numerotation) those given in sect. 4.

Example 1: The section $\sigma_0(x) = 0$ constitutes a stratum $\Sigma_0 \subset \Theta$, with isotropy subgroup $\Gamma_0 = \{\gamma : M \to G\} = \Gamma$ (notice that, due to the discreteness of G, these are actually constant functions). All the other transverse sections form the principal stratum $\Sigma_1 \subset \Theta$, with isotropy $\Gamma_1 = \{\gamma(x) \equiv e\}$.

Example 2: As in example 1, $\sigma_0(x) = 0$ is the only section in the stratum Σ_0, with isotropy $\Gamma_0 = \Gamma$; all the other transverse sections are in the principal stratum Σ_1 with isotropy $\Gamma_1 = \{\gamma(x) \equiv e\}$.

Example 3: The section $\sigma(x) : M \to \Omega_0$ have isotropy $\Gamma_0 = \Gamma = \{\gamma : M \to SO(2)\}$; these form a stratum $\Sigma_0 \subset \Phi$, to which corresponds $\Sigma_0^\vartheta \subset \Theta$, isomorphic to $\{\vartheta : M \to R\} \simeq \Sigma_0$. The other transverse section have $\alpha(\sigma) = 2$ and form the generic stratum Σ_1 with isotropy $\Gamma_1 = \{\gamma(x) \equiv e\}$. Notice that for $\sigma \in \Sigma_0$, $\alpha(\sigma) = 1$ and in facts there are no sections isolated in their stratum.

Example 4: The sections $\sigma_\pm(x) = (0, 0, \pm 1)$ form a stratum Σ_0 with isotropy $\Gamma_0 = \Gamma = \{\gamma : M \to G\}$. All the other transverse sections belong to the principal stratum Σ_1 with isotropy $\Gamma_1 = \{\gamma(x) \equiv e\}$. Notice that now $\sigma_\pm$ have $\alpha(\sigma) = 0$, and are indeed isolated in their stratum.

Example 5: The section $\sigma_0(x) = (0,0,0)$ form a stratum Σ_0 with isotropy $\Gamma_0 = \Gamma = \{\gamma : M \to G = SO(2) \times Z_2\}$; notice that $\alpha(\sigma_0) = 0$ and σ_0 is in facts isolated in its stratum.

The transverse sections with $\alpha(\sigma) = 2$, i.e. such that $E_1^2 \in [E]_\sigma$, have isotropy $\Gamma_2 = \{\gamma(x) \equiv e\}$.

There are two one-dimensional strata, E_1^1 and E_2^1, in Ω; therefore we have several possibilities for σ's with $\alpha(\sigma) = 1$. Two strata Σ_1^1, Σ_2^1 are made of sections such that only one of the E_i^1 is in $[E]_\sigma$; they have isotropy $\Gamma_1^1 = \{\gamma(x) : M \to Z_2\}$ and $\Gamma_1^2 = \{\gamma(x) : M \to SO(2)\}$.

As for the sections such that both E_1^1 and E_2^1 are in $[E]_\sigma$, notice that since $SO(2) \cap Z_2 = \{e\}$, necessarily on the domain $D_1 \subset M$, $D_1 = \{x \ / \ \omega(\sigma(x)) \in E_1^1\}$ we have $\gamma(x) = e \ \forall \gamma \in \Gamma_\sigma$, so that if $D_2 = \{x \ / \ \omega(\sigma(x)) \in E_2^1\} \subset M$ is the disjoint union of subdomains D_2^k, $k = 1, \ldots, K$, Γ_σ is made up of K-ples of functions $\gamma_k : D_2^k \to SO(2)$ with boundary conditions $\gamma_k : \partial D_2^k \to \{e\}$.

Example 6: Here, the sections $\sigma_\pm = (0,0,\pm 1)$ form a stratum Σ_1^0 with isotropy $\Gamma_0^1 = \{\gamma : M \to SO(2)\}$. The sections $\sigma(x) = (z(x), w(x), 0)$ form a stratum Σ_2^0 with isotropy $\Gamma_0^2 = \{\gamma : M \to Z_2\}$; the other transverse sections form a stratum Σ_1 with isotropy $\Gamma_1 = \{\gamma(x) = e\}$.

10. The equivariant branching lemma

One of the most fruitful applications of Michel's theorem in finite dimensional case is to symmetric bifurcation theory [38,8,10,11]; this is done by means of the so called *equivariant branching lemma* (EBL), first proved for bifurcation of stationary solutions [9,39] and extended later to other types of bifurcations [40,41,11,15,17]. We will consider it only in the stationary and variational settings. A related full discussion is given in [18]

We have seen before (sect.3) that

$$\nabla V(x) \in N_x^0 \omega(x) \tag{1}$$

If we denote by W_x the space on which G_x acts trivially,

$$W_x = \{y \in M \ / \ \Lambda_g y = y \quad \forall g \in G_x\} = \text{Ker } G_x \subseteq M \tag{2}$$

we have immediately

$$\nabla V(x) \in T_x W_x \quad \forall x \in M \tag{3}$$

Notice also that for linear representations the linearity of Λ implies that if M is embedded in R^m, W_x is the intersection with M of a linear subspace om R^m, $W_x = R^{d(x)} \cap M$.

When considering gradient dynamical systems, i.e. of the form (V an invariant potential)

$$\dot{x}^i = F^i(x) = \frac{\partial V(x)}{\partial x^i} \quad x \in R^N \tag{4}$$
$$\dot{x} = F(x) = \nabla V(x)$$

equation (3) means that W_x is invariant under the flow of (4), that is

$$x(t) \in W_{x_0} \ \ \forall t \geq t_0 \ \ , \ \ x(t_0) = x_0 \tag{5}$$

This is therefore called the reduction lemma (RL):

Reduction lemma: The solutions of (4) with initial datum x_0 are also solutions of the restriction of (4) to the linear subspace $W_0 \equiv W_{x_0}$.

In other words, one is authorized to study the reduced system $\dot{x} = \widehat{F}(x)$,

In the variational setting, the reduction lemma is most concisely expressed by eq. (3); in other words, one has the

Reduction lemma (variational case): The point $x_0 \in M$ is critical for the invariant potential $V(x)$ if and only if the projection of $\nabla V(x_0)$ on W_0 vanishes; equivalently, if and only if $(y, \nabla V(x)) = 0 \ \forall y \in T_x W_0$, where $(.,.)$ is the scalar product in R^m.

¿From the RL, it follows at once the EBL; in this case, the variational formulation is - in a sense which will be clear in the following - much more powerful than the general evolution one.

In the generic case, we will consider

$$\dot{x} = F(\ell, x) \quad x \in M = R^m \tag{6}$$

where ℓ is a parameter, $\ell \in R$, and $F : R \times M \to TM$ is an equivariant vector field (i.e. $F(\ell, \Lambda_g x) = \Lambda_g F(\ell, x) \ \ \forall g \in G$), smooth in both arguments. F will also assumed to be confining, i.e. it exists a compact B of the same dimension as M such that, if $n(x) \in T_x M \ , \ x \in \partial B$ is the unit tangent vector pointing outward of B, then $(n(x), F(\ell, x)) \leq 0 \ \forall x \in \partial B \ , \ \forall \ell$. We have then

Equivariant bifurcation lemma: Let G admit an isotropy subgroup G_0 with $\mathrm{dimFix}(G_0) = 1$, where $\mathrm{Fix}(G_0) \equiv W_0 = \{x \in M stationarypoints \mathrm{x}_0(\ell) \in W_0$ under (6).

Remark that the assumption about $(x, F(\ell, x))$ on ∂B could be replaced by an analogous assumption on points x_-, x_+ on ∂I, with I a nonempty interval in W_0.

Remark also that we assumed $M = R^m$ in order to conform to the usual bifurcation-theoretic setting: there (6) represents the bifurcation equation, living by construction in (a neighbourhood of the origin in) R^m. One could as well consider M a general manifold; in this case, anyway, a one-dimensional W_0 could be diffeomorphic to S^1 rather than to R^1, and no stationary solution exist. In this case, it is opportune to drop the confining condition, and use instead an analogue of the above mentioned one: i.e. we ask that the one dimensional submanifold $W_0 \subset M$ contains an invariant (under $\dot{x} = F(\ell, x)$) interval $I_0 \subset W_0$.

Let us now discuss the variational case; i.e., we look for critical points of the invariant potential V:

$$\nabla_x V(\ell, x) = 0 \qquad x \in M \tag{7}$$

or, equivalently, stationary solutions of

$$\dot{x} = F(\ell, x) \equiv \nabla_x V(\ell, x) \tag{8}$$

Here again $\ell \in R^1$ is a real parameter, V is assumed to be smooth in both arguments, and invariance means

$$V(\ell, \Lambda_g x) = V(\ell, x) \qquad \forall g \in G \tag{9}$$

We can now allow M to be of infinite (but numerable) dimension. We have then that:

Equivariant bifurcation lemma (variational case): Let G admit an isotropy subgroup G_0 such that $\dim W_0 = d < \infty$, where $W_0 \equiv \mathrm{Fix}(G_0) = \{x \in M \ / \ \Lambda_g x = x \ \forall x \in G_0\}$. Then, if it exists a d-dimensional compact set $B \subset W_0$ (topologically, a ball), such that $\nabla V(\ell, x)$ points outward of B for all $x \in \partial B$, $\forall \ell$, then it exists a smooth family of critical G-orbits $\omega(\ell) \subset B \subset W_0$ for the potential V; each point $x \in \omega$ is a minimum for V.

Obviously, we could as well consider ∇V pointing inward on ∂B, and grant the existence of a maximum.

The proof of the lemma is immediate: the reduction lemma allows to reduce to W_0, but now we can further reduce to B; this is a compact set in a finite dimensional space, and V is a confining potential for B (due to the assumption ∇V points outward of B on ∂B). Therefore V has a minimum in B.

Remark that the restriction $\widehat{V}$ of V to W_0 will exhibit some invariance: in particular, it will be symmetric under $N(G_0) = \{g \in G \ / \ g G_0 g^{-1} = G_0\} \equiv \{g \in G \ / \ \Lambda_g : W_0 \to W_0\}$ (see e.g. [11]), the normalizer of G_0 in G. Actually, G_0 acts trivially in W_0, and is normal by definition in $N(G_0)$, so that the symmetry of $\widehat{V}$ inherited by the G-invariance of V corresponds to the group

$$D_0 \equiv D(G_0) = N(G_0)/G_0 \tag{10}$$

Therefore, critical points of $\widehat{V}$ appear in D_0-orbits, and that is why in the statement of the lemma we have a critical set - which is actually a D_0-orbit - rather than a critical point.

Remark also that $\widehat{V}$ could possess other symmetries beside those described by D_0, as a result of the reduction process. For a discussion of this matter, we refer the reader to [10,11,17,42].

11. A reduction lemma for gauge invariant functionals

The EBL has recently been generalized by Cicogna [15] (see also [43]) to the case of symmetry under general (i.e. not necessarily linear) diffeomorphism groups; his approach can be readily applied to the case ay hand, i.e. to gauge symmetries.

We stress that the gauge group G is assumed to be a compact finite dimensional Lie group, the fiber F to be finite dimensional and the action of G on F to be linear; we will not repeat this caveat in our statements, but these statements are false if we drop these conditions, i.e. if extended to more general cases (see also [19,20]).

Let $\{u_1, ..., u_N\}$ be a basis of F, and $\{x_1, ..., x_m\}$ coordinates in $M \subseteq R^m$. One can consider the Lie algebra $\mathcal{G}$ of G; let $\{L_1, ..., L_k\}$ be a basis of $\mathcal{G}$: the L_i can be represented as linear differential operators of the form

$$L_i = A_j^{(i)}(u)\partial_j \tag{1}$$

where $\partial_j \equiv \partial/\partial u_j$ and $A_j^{(i)}(u)$ is linear in the u; i.e. $A^{(i)}$ can be represented by a matrix $A_{jk}^{(i)}$, and

$$L_i = \left(A_{jk}^{(i)} x_k \right) \partial_j \tag{2}$$

An arbitrary element η of $\mathcal{G}$ can be written as a linear combination of the L_i:

$$\eta = a_k L_k \qquad a_k \in R \tag{3}$$

Now, an arbitrary function $f : M \to \mathcal{G}$ can also be written as a vector function $\varphi : M \to R^k$, by

$$f(x) = \sum_{j=1}^{k} \varphi_j(x) L_j \equiv \varphi(x) \cdot L \tag{4}$$

The above representation shows that Γ is a module [44] (over the ring of functions on M), and is generated by the constant functions $f_i(x) = L_i$, $i = 1, ..., k$.

In facts, we have just seen that $\alpha(x)\eta_1 + \beta(x)\eta_2 \in \Gamma$ if $\eta_1, \eta_2 \in \Gamma$, and that any $\gamma \in \Gamma$ can be written as $\gamma = \varphi^i \cdot L_i$. Moreover,

$$[\gamma_1, \gamma_2] = [\varphi^i L_i, \psi^j L_j] = (\varphi^i(x)\psi^j(x))[L_i, L_j] = \varphi^i(x)\psi^j(x) c_{ij}^k L_k \equiv \chi^k(x) L_k \tag{5}$$

where the c_{ij}^k are the structure constants of the algebra $\mathcal{G}$.

Now we will consider a function $\mathcal{L}(x,u)$, $\mathcal{L} : M \times F \to R$, such that

$$\gamma \cdot \mathcal{L} = 0 \quad \forall \gamma \in \Gamma \tag{6}$$

This means that at fixed x, $\mathcal{L}(x_0,u)$ is invariant under G, i.e. $\mathcal{L}(x_0,\Lambda_g u) = \mathcal{L}(x_0,u)$ $\forall g \in G$, $\forall x \in M$.

We recognize the setting of the previous section: x can be thought as a multidimensional parameter and (6) just reflects the G-invariance of the "potential" $\mathcal{L}$. One just has to remark that the discussion of the previous section survives the extension from one-dimensional to m-dimensional parameter space; then we conclude it holds the

Reduction lemma (variational gauge case): Let $L[\varphi] = \int_M \mathcal{L}[\varphi(x)]dx$ be a gauge invariant functional with gauge group G. Let the group G admit a Lie subgroup G_0 with $W_0 = \mathrm{Fix}(G_0) = \{u \in F \ / \ \Lambda_g u = u \ \forall g \in G_0\}$. Let $\Phi_0 \subset \Phi$ be the set of sections such that $\sigma(x) \in W_0 \subseteq F \ \ \forall x \in M$, and let $L_0[\varphi] = \int_M \mathcal{L}_0[\varphi(x)]dx$ be the restriction of $L : \Phi \to R$ to Φ_0. Then, a section $\sigma \in \Phi_0$ is critical for $\mathcal{L}_0$ if and only if it is critical for L.

It also follows immediately the generalization of the EBL:

Corollary (existence lemma): Let L, $\mathcal{L}$, G, G_0, W_0, F be as above, $\mathrm{dim}W_0 = d < \infty$, and let W_0 contain a nonempty compact subset $B \subset W_0$ of dimension d such that the vector $\nabla_u \mathcal{L}(x,u)$ points outward of B on ∂B $\forall x \in M$. Then there is a local section $\sigma : M \to E$, $\sigma(x) \in F_x$ (F_x is the fiber through x) which is entirely contained in W_0, i.e. $\sigma(x) \in W_0 \subset F_x$ $\forall x \in M$, and which is critical for the functional $L : \Phi \to R$.

We remark that the section whose existence is ensured by this corollary could happen to be trivial, $\sigma(x) = 0$. If the trivial section is a local maximum for L, the same reasoning leads to affirm the existence of a local section $\sigma_0(x) \in W_0$ which is a minimum.

We stress that the existence of global nonzero sections (and a fortiori of global sections if the fiber is not a linear space, e.g. if the fiber is the group itself) cannot, quite obviously, be granted on purely algebraic terms and would require a topological discussion, out of the scope of this paper.

We remark also that one could introduce in $\mathcal{L}(x,u)$ a dependence on a control parameter ℓ and consider bifurcations of critical sections of $L_\ell[\varphi] = \int_M \mathcal{L}_\ell(x,\varphi(x))dx$, but this is beyond the scope of the present paper; the problem will be dealt with in [18].

12. Some examples of reduction

We would now like to briefly see some examples of applications of the above reduction lemma for some of the group actions already considered in sections 7 and 9; we will follow the notation and numerotation employed there.

For physical reasons, one is mainly interested in the case $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1$, where

$$\mathcal{L}_1[\varphi] = \frac{1}{2}(\nabla\varphi)^2 = \frac{1}{2}\sum_i (\nabla\varphi_i)(\nabla\varphi_i) \tag{1}$$

and $\mathcal{L}_0$ is a polynomial of degree ≤ 4 in the φ_i's, and independent of $x \in M$; we also denote $\mathcal{L}_0[\varphi]$ by $V(\varphi)$. We will always tacitely assume that $\mathcal{L}_0[\varphi] \to \infty$ for $|\varphi| \to \infty$.

Example 1: Here $\varphi(x) \in F = R^1$; the invariance of $\mathcal{L}$ under $G = Z_2$ implies $V = V(\varphi^2)$. The space W_0 invariant under $G_0 = G$ is $W_0 = \{0\}$, so that the RL and its corollary just say that $\varphi(x) = 0$ is a critical section.

Example 2: Here $\varphi(x) \in F = R^2$; the invariance of $\mathcal{L}$ under $G = SO(2)$ implies $V = V(\varphi^2)$. The space W_0 for the MIS $G_0 = G$ is again $W_0 = \{0\}$, and $\varphi(x) = 0$ is a critical section.

Example 3: Here $\varphi(x) \in F = R^3$; invariance under $G = SO(2)$ implies that, with obvious notation, $V = V(\rho^2, \varphi_3)$, where $\rho^2 = \varphi_1^2 + \varphi_2^2$. The space W_0 for $G_0 = G$ is $W_0 = \{(0,0,\varphi_3)\}$, and it is easy to see that on W_0 indeed $\nabla V(x) \in T_x W_0$. We can therefore consider the restrictions $V_0(\varphi_3) = V(0,\varphi_3)$, $\widehat{\mathcal{L}} = (1/2)(\nabla\varphi_3)^2 + V_0(\varphi_3)$ and $L_0[\varphi_3] = \int_M \widehat{\mathcal{L}}[\varphi_3]dx$, whose critical points are also critical points of L; conversely, critical sections φ of $\mathcal{L}$ such that $\varphi(x) \in W_0 \; \forall x \in M$ are clearly also critical sections of $\widehat{\mathcal{L}}$.

Example 4: Here $\varphi(x) \in F = R^3$; now there is a relation among φ_3 and ρ^2, i.e. $\varphi_3^2 + \rho^2 = 1$. Now for $G_0 = G = SO(2)$, we have $W_0 = \{(0,0,\pm 1)\}$, and the RL tells that the sections $\sigma_\pm(x) = (0,0,\pm 1)$ are critical.

Example 5: Here $\varphi(x) \in F = R^3$; invariance under $G = SO(2) \times Z_2$ implies that $V(\varphi) = V(\theta_1, \theta_2)$ (see sect. 9); apart from $G_0 = G$ with $W_0 = \{(0,0,0)\}$, the MIS are $G_1 = SO(2)$ and $G_2 = Z_2$. These have $W_1 = \{(0,0,\varphi_3)\}$, to which it corresponds $\theta_1 = 0$, and $W_2 = \{(\varphi_1, \varphi_2, 0)\}$, to which it corresponds $\theta_2 = 0$. We stress that, a priori, the critical sections $\sigma_i(x) \in W_i \; \forall x \in M$ whose existence is ensured by the RL could be trivial, $\sigma_i(x) = 0$, since $W_1 \subset W_0, \; W_2 \subset W_0$.

We will consider this example in some lenght. Let us consider the general density $\mathcal{L}$ satisfying the constraints stated in the beginning of this section, i.e.

$$\mathcal{L}[\varphi] = \mathcal{L}_1[\varphi] + \mathcal{L}_0[\varphi] \tag{2}$$

with

$$\mathcal{L}_1[\varphi] = \frac{1}{2}\sum_{i=1}^{3}(\nabla\varphi_i)^2 \tag{3}$$

$$\mathcal{L}_0[\varphi] = -\frac{1}{2}(\ell\theta_1 + \mu\theta_2) + \frac{1}{4}(a\theta_1^2 + b\theta_2^2 + c\theta_1\theta_2) \tag{4}$$

where $a > 0,\ b > 0,\ c^2 < 4ab$.

A section $\varphi \in W_1$ satisfies $\varphi_3 = \nabla\varphi_3 = 0$, and we get

$$\widehat{\mathcal{L}}_1 = \frac{1}{2}\sum_{i=1}^{2}(\nabla\varphi_i)^2 - \frac{1}{2}\ell\theta_1 + \frac{1}{4}a\theta_1^2 \tag{4}$$

while for $\varphi \in W_2$, $\varphi_1 = \varphi_2 = 0$ and

$$\widehat{\mathcal{L}}_2 = \frac{1}{2}(\nabla\varphi_3)^2 - \frac{1}{2}\mu\varphi_3^2 + \frac{1}{4}b\varphi_3^4 \tag{5}$$

The Euler-Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\varphi_i} = \nabla_\alpha \frac{\partial\mathcal{L}}{\partial(\nabla_\alpha\varphi_i)} \tag{6}$$

give then in the two cases

$$\begin{aligned} \triangle\varphi_1 &= -\,\ell\varphi_1 + a(\varphi_1^2 + \varphi_2^2)\varphi_1 \\ \triangle\varphi_2 &= -\,\ell\varphi_2 + a(\varphi_1^2 + \varphi_2^2)\varphi_2 \end{aligned} \tag{7}$$

and

$$\triangle\varphi_3 = -\mu\varphi_3 + b\varphi_3^3 \tag{8}$$

where $\triangle = (\nabla_\alpha \cdot \nabla_\alpha)$ is the Laplacian.

The existence of nontrivial solutions to (7), (8) depends on the base space M and on boundary conditions (i.e. on the global structure of the bundle E of base M and fiber F), but it is easy to imagine M's such that these exist (e.g. $M = S^1,\ S^2,\ T^2, \ldots$).

Example 6: Here $\varphi(x) \in F = R^3$; now there is a relation among θ_1 and θ_2, i.e. $\theta_1 + \theta_2 = 1$. With the notations of the previous example, W_0 is now empty (i.e. $G_0 = G$ is not an isotropy subgroup), $W_1 = \{(0,0,\pm 1)\}$, $W_2 = \{(\varphi_1, \varphi_2, 0)\ ;\ \varphi_1^2 + \varphi_2^2 = 1\}$, so that when applied to $G_1 = SO(2)$ the RL just tells that the sections $\varphi_\pm = (0,0,\pm 1)$ are critical; applied to $G_2 = Z_2$, the RL ensures the existence of a local critical section lying entirely on the S^1 circle W_2.

We stress once again that what the RL tells is just that if a critical section exists for the restriction $\widehat{\mathcal{L}}_i$ of $\mathcal{L}$ to W_i, then it is also a critical section for $\mathcal{L}$. The global existence of sections depends on the global structure of the fiber bundle, and infacts the corollary of the RL just ensures the existence of local critical sections. When the space W_i has the structure of a linear space the local existence can be extended to global one (but the section will not be nonzero, in general), but this depends on this additional structure.

13. Base space symmetries

Up to now, we have considered pure gauge symmetries; anyway, other kind of symmetries can be present for the functional $L[\varphi] = \int_M \mathcal{L}[\varphi(x)]dx$, i.e. *base space symmetries.* In physical gauge theories, these are typically space-time symmetries. Also, motivated by physical considerations, we will limit to consider the semidirect product of base space symmetries by gauge ones (i.e., the transformation of space time does not depend on the values of the fields φ).

We have briefly dealt with the representation of the Lie algebra of the gauge group as an algebra of differential operators ; in this language, if x are coordinates on the base space M and u are coordinates along the fiber F, and

$$L_i = \ell_j^{(i)}(u)\frac{\partial}{\partial u_j} \equiv \ell_j^{(i)}(u)\partial_j \tag{1}$$

are the generators of the gauge group G, then the pure gauge transformations can be written as

$$\gamma = \sum_i \alpha_i(x)L_i = \tilde{\alpha}_j(x,u)\partial_j \tag{2}$$

The infinitesimal base space symmetries can be written as

$$\chi = \beta_a(x)\frac{\partial}{\partial x_\alpha} \tag{3}$$

and if the Lie algebra of (the group R of) base space symmetries of our functional is X with generators

$$X_i = \nu_j^i(x)\frac{\partial}{\partial x_j} \tag{4}$$

we can rewrite (3) as

$$\chi = \sum_i \beta_i X_i = \beta_i\nu_j^i\frac{\partial}{\partial x_j} = \tilde{\beta}_j(x)\frac{\partial}{\partial x_j} \tag{5}$$

The semidirect sum $X \oplus_\rightarrow L$ will be made of vectors fields of the form

$$\eta = \xi(x)\partial_x + \zeta(x,u)\partial_u \tag{6}$$

where we have used the shortcut notation

$$\xi\partial_x \equiv \sum_i \xi^i\frac{\partial}{\partial x^i} \quad ; \quad \zeta\partial_u \equiv \sum_j \zeta^j\frac{\partial}{\partial u^j} \tag{7}$$

Vector fields of the form (6), and the corresponding algebras, are for obvious reasons also called projectable [33].

A typical case of occurrence of base space symmetries is that of a density $\mathcal{L}$ of the functional

$$L[\varphi] = \int_M \mathcal{L}[\varphi(x), x] dx \equiv \int_M \mathcal{L}[\varphi(x)] dx \tag{8}$$

which does not depend explicitely on x (we also say that the base manifold M is homogeneous: all its points are equivalent). In this case, the group of rigid transformations of M leaves L invariant; notice that a general - i.e. not rigid - diffeomorphism of M does not leave L invariant: even in the case $\mathcal{L}$ depends only on φ and not on its derivatives $\nabla\varphi$, such a transformation would modify the integration measure.

In this paper we always assume that $\mathcal{L}$ does not depend explicitely on x (this is also expressed by saying that $\mathcal{L}$ is autonomous).

When $M = R^m$, the group of corresponding rigid transformations is the group $E(m) = R^m \times O(m)$ of euclidean transformations, with Lie algebra generated by the m traslations $\partial/\partial x_1, ..., \partial/\partial x_m$ and by the Lie algebra of rotations in R^m, $so(m)$ (whose generators are of the form $x_i \partial/\partial x_j - x_j \partial/\partial x_i$); for $M = S^{(m-1)} \subset R^m$, we just have the group $O(m)$ (with Lie algebra $so(m)$) of rotations in R^m.

If now we consider also these base space symmetries, the full symmetry group of the functional L is

$$\tilde{\Gamma} = B \otimes_{\rightarrow} \Gamma \tag{9}$$

where Γ is, as before, the group of gauge diffeomorphism, $\Gamma = \{\gamma : M \to G\}$. Clearly, B acts on Γ but Γ does not act on B.

In the analysis of symmetries of sections $\sigma \in \Phi$, one should now consider $\tilde{\Gamma}$ rather than Γ. On one side, this leads to consider the orbit space $\tilde{\Theta} = \Phi/\tilde{\Gamma}$ rather than $\Theta = \Phi/\Gamma$, which means that sections differing only by a rotation and/or a change in the origin of the coordinate system in M are identified; this is indeed quite a reasonable and desirable feature.

On the other side, base space symmetries will also enter in the isotropy group of the sections $\sigma \in \Phi$ and of the gauge orbits $\tilde{\vartheta} \in \tilde{\Theta}$; this feature is capable, as we will readily discuss, to favour pattern formation or, in other words, the selection of sections with some spatial (in the sense of the base space M; it can well be a spatiotemporal one in physical terms) regularity.

We would like to stress that the (structurally stable) set of transversal sections is also invariant under rigid transformations of M.

We will now specifically discuss the case of $M = S^m \subset R^{m+1}$ or $M = R^m$; in the last case we will also consider two compactifications (we recall that in sects. 4-9 we needed to assume M compact): one is the one-point compactification to S^m, which is the natural one if we ask our sections to vanish sufficently quickly at infinity and which comes back to the previous case; the other amounts to ask periodicity to the sections, $\sigma(x) = \sigma(x + K \cdot \Lambda_0)$, where $K \in Z^m$ and $\Lambda_0 \in R^m$ represents the basis vectors of a lattice in R^m, so that R^m is compactified to the (standard) m-torus $T^m = \left(S^1\right)^m$; in this case we can still discuss in the full R^m setting, but integration will be over a compact region M corresponding to a (simple, or finite) covering of T^m or, in physical terminology, to a unit cell of the lattice generated by Λ_0.

For M compact, also the group B of rigid diffeomorphisms of M is a compact Lie group; we stress anyway that its action on the infinite dimensional space Φ (or even Θ) gives an infinite dimensional representation.

In the case $M = S^m$, B is actually the rotation group in $(m+1)$ dimensions, $R = O(m+1)$, $X = so(m+1)$, and the situation is well known, amounting to an analysis in spherical harmonics.

In the case $M = R^m$, i.e. $M = T^m$; B is the euclidean group $E(m)$ (modulo the compactification $R^m \to T^m$); this is generated by rotations, i.e. $O(m)$ again, and by m traslations, i.e. R^m. With the torus compactification, the latter becomes just rotations along the m fundamental cycles of the torus, i.e. $(SO(2))^m$; we are therefore in a familiar situation, corresponding to multidimensional Fourier analysis.

We stress that we could have chosen different compact manifolds M, giving again familiar situations: e.g., if M is (flat) space-time, $M = R^d \times R$, we can pretend sections to go fast enough to zero at spatial infinity and consider only time periodic sections. This amount to one-point compactification of R^d, and torus compactification of R, i.e. to considering $M = S^d \times S^1$; this is again a well known situation.

14. A scenario for pattern formation

If we now consider the full symmetry algebra of the functional

$$L[\varphi] = \int_M \mathcal{L}(\varphi) dx \tag{1}$$

i.e., as discussed in the previous section,

$$\tilde{\Gamma} = X \oplus_{\to} \Gamma \tag{2}$$

(the symbol $\oplus_{\to}$ denotes semidirect sum) the maximal subalgebras of $\tilde{\Gamma}$ will be of the form

$$\tilde{\Gamma}_0 = X_0 \oplus_{\to} \Gamma \quad \text{or} \quad \tilde{\Gamma}_0 = X_0 \Gamma_0 \tag{3}$$

with X_0 a maximal subalgebra of X, Γ_0 a maximal subalgebra of Γ.

Our aim would be to obtain informations about the existence of solutions (or bifurcating solutions) with maximal symmetry. Clearly, the section $\sigma \equiv 0$ has the full $\tilde{\Gamma}_0$ symmetry, so we will consider only nontrivial solutions (we notice explicitely that these could vanish at some points; anyway, they will not be identically zero).

Notice that, since X generates the rigid transformations of M (and we want to consider $M = S^n$ or $M = T^n$) the only sections having the full X symmetry are constant ones, $\sigma(x) = \varphi_0$. In this case we have

$$L[\sigma] = \int_M \mathcal{L}(\varphi_0) dx = \mathcal{L}(\varphi_0) \cdot \mu \tag{4}$$

where $\mu = \int_M dx$ is the volume of the manifold M, and we reduce to the finite dimensional problem of finding minima of $\mathcal{L} : F \to R$, i.e. to the problem solved by Michel's theorem. In particular, there will be solutions corresponding to maximal Γ_0, as discussed in sect.9.

Let us now consider non constant sections. It follows from the above discussion that for these, the symmetry algebra is of the form

$$\tilde{\Gamma}_\sigma = X_0 \oplus_{\to} \Gamma_0 \qquad\qquad X_0 \subset X \ ; \ \Gamma_0 \subseteq \Gamma \tag{5}$$

Now we have to distinguish two cases: either there are nonzero $\varphi \in F$ such that $\mathcal{G}_\varphi = \mathcal{G}$, either not. In the first case, one can have sections σ such that

$$\sigma(x) \in W_0 \quad , \quad \tilde{\Gamma}_\sigma = X_0 \times \Gamma \tag{6}$$

where $W_0 = \{\varphi \in F \ / \ \eta\varphi = 0 \ \ \forall \eta \in \mathcal{G}\} \simeq R^s \subseteq F$, $s \geq 1$. In the second case this is impossible, since the only section having the whole $\mathcal{G}$ and therefore Γ symmetry is the trivial one. Therefore, in this case

$$\sigma \not\equiv 0 \Longrightarrow \tilde{\Gamma}_0 = X_0 \oplus_{\to} \Gamma_0 \ ; \qquad X_0 \subset X, \ \Gamma_0 \subset \Gamma \tag{7}$$

In both cases, what we want to stress is that the symmetry X is broken to X_0 (or, at group level, B is broken to B_0); the discussion given in previous parts of this paper shows that, under some additional conditions in order to precise the setting, maximal isotropy subgroups (MIS) are favoured if the trivial solution σ_0 becomes unstable and bifurcates to a new solution $\sigma_b(x)$.

The facts that MISs of R^n (except those containing traslations) correspond to regular lattices (with the same holding for T^n, S^n) suggests that this could be a mechanism implied in *pattern formation* [45,46].

Notice that, if $\mathrm{Fix}(G) = \{0\}$, in such a bifurcation both the X and Γ symmetries must break, and the maximal symmetries of nontrivial solutions correspond to both a *pattern* *formation*, i.e. a breaking of the symmetry B to a regular lattice, and a *gauge-symmetry breaking.*

More precisely, given a MIS $\tilde{\Gamma}_0 = B_0 \times_\to \Gamma_0 \subset B \times \Gamma_0 \equiv \tilde{\Gamma}$, we restrict $L : \Phi \to R$ (recall Φ is the space of sections of $M \times F$) to the subspace Φ_0 of sections invariant under $\tilde{\Gamma}_0$, and get the functional $L_0 : \Phi_0 \to R$. As we have proved before, sections $\sigma_1 \in \Phi_0$ which are critical for L_0 are also critical for L.

Consider now the case that $\mathcal{L}$, and through it L, depend (smoothly) on a real parameter ℓ, and for $\ell < \ell_0$ the trivial section σ_0 is a minimum for L, while at $\ell = \ell_0$ it loses stability (for precise conditions of this losing of stability, see e.g. [10,11,47]), and let us look for minima of L for $\ell > \ell_0$.

We can go over the set of MISs $B_i \subset B$, $\Gamma_\alpha \subseteq \Gamma$, and study critical sections for $L_{i,\alpha} : \Phi_{i,\alpha} \to R$, where $\Phi_{i,\alpha} \subset \Phi$ is the set of sections admitting $\tilde{\Gamma}_{i,\alpha} = B_i \times_\to \Gamma_\alpha$ as symmetry. The minima for $L_{i,\alpha}$ are also critical sections (not necessarily minima !) for L. This gives a constructive algorithm: we first determine sections $\sigma_{i,\alpha} \in \Phi_{i,\alpha}$ which are minima for $L_{i,\alpha}$, and then check their stability against general perturbations.

15. A scenario for phase coexistence

In previous parts of this paper we pointed out that our results concerning stratification and critical gauge orbits need, to be rigorous, to restrict to the set of transverse sections, defined in sect.8. It was also remarked that in some special cases a transverse section could connect different strata of the G-orbit space Ω. In physical terms, this suggests that such solutions correspond to *phase coexistence*

Again, we could consider a bifurcation from the trivial solution; it was pointed out in sect.9 that the isotropy group Γ_0 of the solution connecting strata corresponding to non-conjugated subgroups $G_1, G_2 \subset G$ is not a subgroup of Γ_1, neither of Γ_2, where $\Gamma_i = \{g : X \to G_i\}$. This means in particular that for G_1, G_2 MISs of G, and therefore Γ_1, Γ_2 MISs of Γ (for transverse sections), also Γ_0 would be a MIS of Γ. By the same argument as in the case of "pattern formation" dealt with in the previous section, one can explain the appearance of these "phase-coexistence" solutions.

We remark that in our qualitative discussion pattern formation is an independent phenomenon, but phase coexistence seems to be intimately tied with pattern formation, as it is indeed the case in a number of experimental observations.

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