Invariants of the length spectrum and spectral invariants of planar convex domains

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Abstract

This paper is concerned with a conjecture of V.Guillemin and R. Melrose that the length spectrum of a strictly convex bounded domain together with the spectra of the linear Poincaré maps corresponding to the periodic broken geodesics in Ω determine uniquely the billiard ball map up to a symplectic conjugation. We consider continuous deformations of bounded strictly convex domains $\Omega_s,\ s \in [0,1]$, with smooth boundaries. If the length spectrum does not change along the deformation, we prove that the invariant KAM circles of the corresponding billiard ball map B_s with rotation numbers in a suitable Cantor set of a positive Lebesgue measure as well as the restriction of $B_s,\ s \in [0,1]$, on their union are symplectically equivalent to each other. We prove as well that the KAM circles and the restriction of the billiard ball map on them are spectral invariants of the Laplacian with Dirichlet (Neumann) boundary conditions for suitable deformations of strictly convex domains.

1 Introduction

This paper is concerned with certain length spectrum invariants of a strictly convex and bounded planar domain Ω with a smooth boundary $\partial\Omega$. The motivation for studying such invariants comes from the inverse spectral problem formulated by M. Kac [12]. It is known [10], [18], that the length spectrum $\mathcal{L}(\Omega)$ of Ω is encoded in the spectrum of the Laplace operator Δ in Ω with Dirichlet (Neumann) boundary conditions, and that $\mathcal{L}(\Omega)$ can be extracted from the spectrum of Δ by means of the Poisson formula at least for generic domains. In this connection, V. Guillemin and R. Melrose [9] formulated the conjecture that the length spectrum of Ω and the spectra of the linear Poincarè maps of the periodic broken geodesics of Ω form together a complete set of symplectic invariants for the corresponding billiard ball map B. As it was mentioned in [9], this conjecture seems to be a little optimistic and the local version of it is more hopeful.

The first result in this direction was obtained by Sh. Marvizi and R. Melrose [16] who described new length spectrum invariants of a strictly convex domain Ω, studying the asymptotics of the lengths of the closed broken geodesics approaching the boundary $\partial\Omega$. Let us take l_{mn} arbitrarily in the set $\mathcal{L}(\Omega; m, n)$ of lengths of all closed broken geodesics of Ω with n vertices and winding number m. When m is fixed and n tends to infinity, l_{mn} has an asymptotic expansion in powers of n^{-2}. The corresponding coefficients $c_{mk},\ k = 1, 2, \ldots$, do not depend on the choice of l_{mn} in $\mathcal{L}(\Omega; m, n)$ and they are length spectrum as well as spectral invariants of Ω [16]. Y. Colin de Verdière [4] proved that the labeled length spectrum and the spectra of the linear Poincarè maps determine uniquely the Birkhoff invariants of a closed and elliptic broken ray in $\Omega \subset \mathbf{R}^2$. Recently this result was generated in higher dimensions as well as for contact manifolds by J. P. Francoiçe and V. Guillemin [8].

R. de la Llave, J. Marco and R. Moriyón [15] proved that there are no non-trivial deformations of exact symplectic mappings $B_s,\ s \in [0, 1]$, leaving the period spectrum fixed when B_s are Anosov's mappings on a compact symplectic manifold. One of the reasons for symplectic rigidity in [15] is that all periodic points of B_s are hyperbolic and form a dense set. Although the billiard ball map of a strictly convex domain is in the opposite situation, conjugation can still be made on a large part of the domain of B_s.

Consider the billiard ball map B corresponding to a strictly convex domain Ω with a smooth boundary. Near the boundary B is an exact symplectic map close to a completely integrable one. Using that fact V. Lazutkin [14] proved that there exists a large family of invariant KAM circles $\Lambda(\omega)$ of B with rotation numbers ω in a Cantor subset Θ of the interval [0,1/2) of positive Lebesgue measure. The corresponding caustics $C(\omega)$ are strictly convex and smooth curves in Ω accumulating at $\partial\Omega$.

There are two invariants related to any invariant curve $\Lambda(\omega)$, namely the length $\ell(\omega)$ of the caustic $C(\omega)$ and the Lazutkin parameter $t(\omega)$ [1], [14], (see also Section 2).

The main goal of this paper is to prove that the vector function

$$\Theta \ni \omega \to (\ell(\omega), t(\omega)) \tag{1.1}$$

is a length spectrum invariant for continuous deformations of the domain. The main result (see Theorem 2.1) says that the length spectrum determines uniquely the invariant circles $\Lambda(\omega)$, $\omega \in \Theta$, as well as the restriction of B on them (up to a symplectic conjugation) for continuous deformations of Ω. We prove as well that the vector function (1.1), the invariant circles $\Lambda(\omega)$, $\omega \in \Theta$, and the restriction of B on them are spectral invariants of the Laplacian with Dirichlet (Neumann) boundary conditions for suitable continuous deformations of Ω. Let Ω_1 and Ω_2 be two strictly convex and bounded domains with smooth boundaries. Suppose that the corresponding "marked" length spectra $\mathcal{L}(\Omega_j, m, n)$, $m \leq n/2$, $j = 1, 2$ coincide. We show in Section 5 that the corresponding vector functions (1.1) coincide as well which improves Theorem 3 in [13].

The plan of the paper is as follows: In Section 2 we formulate the main result about the length spectrum invariants of continuous deformations of a strictly convex domain (see Theorem 2.1). Section 3 is devoted to a symplectic KAM theorem for the billiard ball map, which is the basic tool in the proof of the main results. First we introduce action-angle coordinates $(\theta, r) \in \mathbf{T} \times \Gamma$, $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$, for the approximated interpolating Hamiltonian of B where $\Gamma = (l - \varepsilon, l)$ and $2\pi l = \ell = \ell(0)$ is the length of the boundary $\partial\Omega$. In these coordinates B is a small perturbation of the completely integrable map

$$(\theta, r) \longrightarrow (\theta + (\tau^0)'(r), r), \ \tau^0(r) = -\frac{4}{3}\,\zeta(r)^{3/2},$$

where $\zeta(l) = 0$, the first derivative $\zeta'(l) < 0$ and $\zeta > 0$ in Γ. Moreover, $\mathbf{T} \times \{l\}$ is a component of the boundary $\partial\Sigma$ of the phase space Σ of B. Applying suitable KAM theorem, we find symplectic coordinates $(\varphi, I) \in \mathbf{T} \times \Gamma$ and smooth functions $K(I)$ and $Q(\varphi, I)$ in $\mathbf{R}$ and $\mathbf{T} \times \mathbf{R}$ respectively such that

$$B(\varphi, I) \ = \ (\varphi + \tau'(I), I) \ + \ Q(\varphi, I), \ \tau(I) = -\frac{4}{3}\,K(I)^{3/2},$$

where $K(l) = 0$, $K'(l) < 0$ and $K > 0$ in Γ, the vector function Q has a zero of infinite order on $\mathbf{T} \times E$, and the Cantor set $E = \{I \in \Gamma : \ \tau'(I)/2\pi \in \Theta\}$ has a positive Lebesgue measure. In Section 4 we give a simple geometric meaning to the restriction of τ on E. We prove that the function $\mathcal{J}(\omega)$ inverse to the frequency mapping $\Gamma \ni I \to \tau'(I)/2\pi$ is equal to $\ell(\omega)/2\pi$ while $\tau(\mathcal{J}(\omega)) = t(\omega)$ for any $\omega \in \Theta$. In particular, the Legendre transform $\mathcal{I}(\omega) = \omega\mathcal{J}(\omega) - \tau(\mathcal{J}(\omega))$ of $\tau(I)/2\pi$ is given on Θ by

$$2\pi\mathcal{I}(\omega) \ = \ \omega\ell(\omega) - t(\omega), \ \forall\, \omega \in \Theta. \tag{1.2}$$

Theorem 2.1 is proved in Section 5. The main ideas here are:

1. Let $[0, b] \ni s \to \Omega_s$, $b > 0$, be a continuous deformation of strictly convex bounded domains with smooth boundaries. Following an argument due to Birkhoff, we prove that for any pair $(m, n) \in \mathbf{Z}^2_+$, $m \leq n/2$, the function

$$[0, b] \ni s \ \longrightarrow \ T_{mn}(s) \ = \ \max\{t : \ t \in \mathcal{L}(\Omega_s, m, n)\},$$

is continuous, and that $\mathcal{L}(\Omega_s)$ is a subset of $\mathbf{R}$ of Lebesgue measure zero for any s. If the length spectrum of Ω_s is independent of s along the deformation, the continuous function $T_{mn}(s)$ takes values in the set $\mathcal{L}(\Omega_s) = \mathcal{L}(\Omega_0)$ which does not contain intervals. Hence,

$$T_{mn}(s) \ = \ T_{mn}(0), \ s \in [0, b]. \tag{1.3}$$

2. Fix $\omega \in \Theta$ and choose a sequence $(m_j, n_j) \in \mathbf{Z}_+^2,\ j = 1, 2, ...,$ such that

$$| m_j/n_j - \omega | \leq n_j^{-1/2}.$$

We prove that

$$\mathcal{I}_s(\omega) = \lim_{j \to \infty} (T_{m_j n_j}(s)/n_j),\ \forall s \in [0, b], \tag{1.4}$$

and taking into account (1.3) we obtain

$$\mathcal{I}_s(\omega) \ = \ \mathcal{I}_0(\omega),\ \forall \omega \in \Theta,\ \forall s \in [0, b].$$

Since Θ has no isolated points, differentiating the last equality with respect to ω we obtain

$$\ell_s(\omega) \ = \ \ell_0(\omega),\ t_s(\omega) \ = \ t_0(\omega),\ \forall \omega \in \Theta,\ \forall s \in [0, b]. \tag{1.5}$$

Close idea has been used in [21] to study the length spectrum invariants of elliptic geodesics in Riemannian manifolds.

Equality (1.4) is a consequence of the following estimate (see Theorem 5 . 1) :

$$| \ l_{mn} \ - \ n\mathcal{I}(m/n) \ | \leq C_p n^p,\ \forall p > 0, \tag{1.6}$$

for any $l_{mn} \in \mathcal{L}(\Omega, m, n)$ and any pair $(m, n) \in \mathbf{Z}_+^2$ satisfying the inequality $\mathrm{dist}(m/n, \Theta) \leq n^{-1/2}$. Here, C_p depends only on p and Ω. In particular, if we fix m, let n go to infinity, and expend $\mathcal{I}(t)$ in Taylor series at $t = 0$, we obtain Theorem 5.15 in [16] as a consequence of (1.6) since $0 \in \Theta$ (see Corollary 5.1). The invariants c_{mk} of Sh. Marvizi and R. Melrose are explicitly given by the Taylor coefficients of $\mathcal{I}(t)$ at $t = 0$.

3. We assume that Ω_s is strictly convex only in a small interval $[0, b_0),\ b_0 > 0$. To show that Ω_s is strictly convex for any s in $[0, 1]$ along the deformation, we use (1.5) as well as the integral invariants $I^{(k+1)}(0)$ of Sh. Marvizi and R. Melrose [16]. In particular, we prove that $I^{(k+1)}(0) = 2\pi\mathcal{R}^{(k)}(0),\ k = 1, 2, \ldots,$ where $\mathcal{R}(t)$ is the function inverse to $t = K(I)$ and $\mathcal{R}^{(k)}(0)$ are the corresponding derivatives at $t = 0$.

Section 6 is devoted to spectral invariants of the Laplacian $\Delta = -\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2$ in Ω with Dirichlet (Neumann) boundary conditions. We suppose that Ω is a strictly convex bounded domain in $\mathbf{R}^2$ with a smooth boundary. Then, the spectrum of Δ consists of non-negative eigenvalues tending to infinity. V. Guillemin and R. Melrose formulated in [9] the conjecture that the spectrum of the Laplace operator in Ω with Dirichlet (Neumann) boundary conditions determines uniquely the billiard ball map. Partial affirmative answer to this conjecture is given in Section 6 (see Theorem 6.2). We prove that the vector function (1.1), the invariant circles $\Lambda(\omega),\ \omega \in \Theta$ as well as the restriction of the billiard ball map on them are spectral invariants of the Laplacian for suitable continuous deformations of a strictly convex domain. In particular, the corresponding billiard ball maps are conjugated to each other on a large subset of Σ of a positive Lebesgue measure. We investigate the singularities of the distribution

$$Z(t) = \mathrm{trace}\cos(t\Delta^{1/2}) = (1/2) \sum \exp(i\lambda t) \tag{1.7}$$

where the sum is taken over all λ with λ^2 in the spectrum of Δ counted with multiplicity. The singular support of $Z(t)$ satisfies the Poisson relation

$$\text{sing.supp.} Z(t) \ \subseteq \ \{T \in \mathbf{R} : \ \pm T \in \mathcal{L}(\Omega)\} \cup \{0\} \tag{1.8}$$

[2], [18]. The inverse relation may not always be true, because singularities created by different closed broken geodesics may cancel each other. It is known that (1.8) turns into equality in the generic case when all periodic broken geodesics are non-degenerate and of different lengths [18].

Let $(m, n) \in \mathbf{Z}_+^2$ satisfy the inequality $\text{dist}(m/n, \Theta) \leq n^{-1/2}$. In Section 6 we prove under the natural condition (6.1) that

$$T_{mn} \ \in \ \text{sing.supp.} Z$$

if $n \geq n_0(\Omega)$. For $m = 1$ and n sufficiently large this result has been proved in [16]. The main idea in [16] is to write $Z(t)$ in a neighborhood of T_{1n} as a Lagrangian distribution with a suitable phase function and then to apply a result of Soga. We use another representation on $Z(t)$ in a neighborhood of T_{mn} which is based on the KAM theorem and the results obtained in Section 5. As a consequence, we show that the vector function (1.1) is a spectral invariant of the Laplace operator with Dirichlet (Neumann) boundary conditions for suitable continuous deformations of the domain.

2 Length spectrum

Let Ω be a bounded domain in $\mathbf{R}^2$ with a smooth boundary $\partial\Omega$. The length spectrum $\mathcal{L}(\Omega)$ of Ω is defined as the set of lengths of all periodic generalized geodesics $\tilde{\gamma}$ of $\overline{\Omega}$ ($\tilde{\gamma}$ is the projection on $\overline{\Omega}$ of a closed generalized bicharacteristic of the Hamiltonian $p(x, \xi) = \mid \xi \mid^2 - 1$, (cf. [11], Def. 24.3.7)). Suppose in addition that Ω is strictly convex. Then any generalized bicharacteristic of Ω is either a broken bicharacteristic reflecting at the boundary by the usual law of the geometric optics or it is a gliding ray traveling along the boundary. Hence, $\mathcal{L}(\Omega) = \mathcal{L}_b(\Omega) \cup \mathcal{L}(\partial\Omega)$ in this case where $\mathcal{L}_b(\Omega)$ is the set of lengths of all closed broken geodesics and $\mathcal{L}(\partial\Omega) = \{n\ell : n \in \mathbf{N}\}, \ell$ being the length of the boundary.

The broken bicharacteristic flow induces a discrete dynamical system on the boundary

$$B : \Sigma \ \longrightarrow \ \Sigma, \ \Sigma \ = \ \{(x, \xi) \in T^*\partial\Omega : \mid \xi \mid \leq 1\},$$

called billiard ball map which is defined as follows: Pick $\varrho = (x, \xi)$ in $T^*\partial\Omega$ with $\mid \xi \mid < 1$ and set $\varrho^\pm = \pi^\pm(\varrho) = (x, e^\pm(\varrho))$. Here $e^\pm(\varrho) \in (\mathbf{R}^2)^*$ are unit covectors such that

$$\pm < e^\pm(x, \xi), n(x) > \ > \ 0, \ < e^\pm(x, \xi), v > = < \xi, v >, \ \forall v \in T_x\partial\Omega,$$

$n(x)$ being the inward normal to $\partial\Omega$ at x. Via the canonical inner product in $\mathbf{R}^2$ we identify $e^\pm(\varrho)$ with a vector $\tilde{e}^\pm(\varrho)$ in $\mathbf{R}^2$. The bicharacteristic

$$\mathbf{R} \ni t \ \longrightarrow \ (x + t\tilde{e}^+, e^+)$$

of $S^*\mathbf{R}^2 = \{(y,\eta) \in T^*\mathbf{R}^2 : |\ \eta\ |= 1\}$ passing through $\pi^+(\varrho)$ intersects $(T^*\mathbf{R}^2)_{|\partial\Omega}$ at a second point (y, e^+). Define $\eta \in T_y^*\partial\Omega$ by the equality

$$< e^+, v >=< \eta, v >,\ \forall v \in T_y\partial\Omega.$$

Then $|\ \eta\ |< 1$ and B sends (x,ξ) to (y,η). Moreover, any point in $\partial\Sigma$ is a fixed point of B.

Defined in this way the billiard ball map is exact symplectic in the interior of Σ, indeed

$$B^*\sigma\ - \sigma\ =\ dT \tag{2.1}$$

where σ is the canonical one-form in $T^*\partial\Omega$ and $T(x,y) =|\ x - y\ |$ is the distance between x and y in $\mathbf{R}^2$, y being the first component of $B(x,\xi) = (y,\eta)$ (see Proposition 2.3 in [9]).

Near the boundary of Σ the billiard ball map B is a small perturbation of a completely integrable map for which the KAM theorem can be applied [17], [14]. In particular, there exists a large family of invariant circles $\Lambda(\omega)$ of B enumerated by their rotation numbers $\omega \in \Theta$, where Θ is a Cantor subset of the interval (0,1/2] with a positive measure (see Section 3). For each $\omega \in \Theta$, denote by $C(\omega)$ the corresponding caustic in Ω, i.e. the envelope of the rays $\{x + \tilde{e}(x,\xi) : t > 0\}, (x,\xi) \in \Lambda(\omega)$, issuing from $\Lambda(\omega)$. Then $C(\omega)$ is a smooth and strictly convex curve in Ω and the boundary $\partial\Omega$ is an evolute of $C(\omega)$ [14]. In other words, if we loop a string with a suitable length $T(\omega)$ around $C(\omega)$, lean a pen against it and draw, we get $\partial\Omega$. The Lazutkin parameter of $C(\omega)$ is defined by

$$t(\omega) = T(\omega) - \ell(\omega)$$

where $\ell(\omega)$ is the length of $C(\omega)$.

We consider a continuous deformation

$$[0,1] \ni s\ \longrightarrow\ \Omega_s \subset \mathbf{R}^2 \tag{2.2}$$

of bounded domains with smooth boundaries $\partial\Omega_s = \{x^s(t) : t \in \mathbf{T}\}$, such that the mapping $[0,1] \ni s \longrightarrow x^s(\cdot) \in C^\infty(\mathbf{T},\mathbf{R}^2)$ is continuous. For any strictly convex domain Ω_s we denote by B_s, $\Lambda_s(\omega)$ and $C_s(\omega)$ the corresponding billiard ball map, invariant circle and caustic with a rotation number ω. Consider the Cantor set Θ defined by (3.8). According to (3.18), the union Λ_s of the invariant circles $\Lambda_s(\omega)$, $\omega \in \Theta$, is a set of positive Lebesgue measure in $T^*\partial\Omega_s$. The main result in this paper is:

Theorem 2.1 *Let $[0,1] \in s \longrightarrow \Omega_s$ be a continuous deformation of bounded domains in $\mathbf{R}^2$ with smooth boundaries. Suppose that Ω_0 is strictly convex and*

$$\mathcal{L}(\Omega_s) = \mathcal{L}(\Omega_0)\ ,\ \ s \in [0,1]. \tag{2.3}$$

Then:

(i) Ω_s is strictly convex for any $s \in [0,1]$,

(ii) there exists a continuous family of smooth exact symplectic mappings

$$\chi_s : T^*\partial\Omega_0 \longrightarrow T^*\partial\Omega_s$$

such that

$$\chi_s(\Lambda_0(\omega)) \;=\; \Lambda_s(\omega),\ \forall\, \omega \in \Theta \tag{2.4}$$

$$\chi_s \circ B_0 = B_s \circ \chi_s \text{ on } \Lambda_0 \tag{2.5}$$

for any $s \in [0,1]$,

(iii) for any $\omega \in \Theta$ there exists a continuous family of caustics $[0,1] \longrightarrow C_s(\omega)$ in Ω_s and

$$\ell_s(\omega) = \ell_0(\omega),\ t_s(\omega) = t_0(\omega),\ s \in [0,1]. \tag{2.6}$$

3 KAM theorem

In this section we formulate a symplectic version of the KAM theorem for a family of exact symplectic mappings depending continuously on a parameter which will be the basic technical tool in the proof of the main results. This theorem is close to Theorem 1.1 in [21]. First we consider a continuous deformation of bounded strictly convex domains $\Omega_s,\ s \in [0,b],\ b > 0$, in $\mathbf{R}^2$, with smooth boundaries $\partial\Omega_s$ of length ℓ_s, and introduce action-angle coordinates for the approximated interpolating Hamiltonians of the corresponding billiard ball maps B_s.

Performing a suitable change of the variables in $\mathbf{R}^2$, we consider $\Omega_s,\ s \in [0,b]$, as a Riemannian manifold with a base $\Omega = \Omega_0$ and metric g_s depending continuously on s. The boundary $\partial\Omega_s$ is given by $\partial\Omega$ equipped with the induced metric g_s^0. The corresponding billiard ball map B_s is defined in the same manner as in Section 2. Its phase space coincides with the coball bundle

$$\Sigma_s = \{(x,\xi) \in T^*\partial\Omega :\ g_s^0(x,\xi) \le 1\}.$$

Let us denote by $\partial\Sigma_s^+$ one of the two components of the boundary of Σ_s. Since $\partial\Omega_s$ is strictly geodesically convex, B_s can be written as a small perturbation of a completely integrable map as follows (see [16]): there exists a smooth function ζ_s called approximated interpolating Hamiltonian which defines $\partial\Sigma_s^+$ ($\zeta_s = 0$ and $\nabla\zeta_s \neq 0$ on $\partial\Sigma_s$), $\zeta_s \ge 0$ on Σ_s, and such that in any local coordinates $\varrho = (x,\xi)$ in a local chart U in $T^*\partial\Omega$ we have

$$\begin{aligned} B_s(\varrho) \;&=\; \exp(-2\zeta_s(\varrho)^{1/2} H_{\zeta_s})(\varrho) + R_s(\varrho),\ \varrho \in \Sigma_s \cap U, && (3.1)\\ R_s(\varrho) \;&=\; O(\zeta_s^\infty(\varrho)) \text{ at } \partial\Sigma_s^+ \cap U. && (3.2)\end{aligned}$$

Here $t \longrightarrow \exp(tH_{\zeta_s})(\varrho)$ stands for the integral curve of the Hamiltonian vector field H_{ζ_s} starting at $\varrho \in \Sigma_s$, R_s is a continuous family of smooth functions in U, and (3.2) means that

$$|\ \partial_x^\alpha \partial_\xi^\beta R_s(x,\xi)\ | \le\ C_{N\alpha\beta}\, \zeta_s(x,\xi)^N, (x,\xi) \in U, \tag{3.3}$$

for any indices α, β, N. Moreover, the mapping $[0,b] \ni\ s \longrightarrow\ \zeta_s(\cdot)\ \in\ C^\infty(T^*\partial\Omega)$ is continuous.

We are going to describe action-angle coordinates for the Hamiltonian ζ_s. To simplify the notations we drop the index s. Denote by M_r the closed curve $\{\zeta = r\}$ in $T^*\partial\Omega$

where r varies in a small neighborhood of the origin. For any $\varrho \in M_r$ consider the map $\mathbf{R} \ni t \longrightarrow \exp(tH_{\zeta_s})(\varrho) \in M_r$ and denote by $2\pi\Pi(r)$ its period. Let S be a section transversal to M_0 in Σ. It is equipped with local coordinates $S \ni \varrho \to \zeta(\varrho)$. Denote by $\mathcal{O}$ the discrete group in $\mathbf{R} \times S$ generated by

$$\mathbf{R} \times S \ \ni \ (t, \zeta(\varrho)) \longrightarrow (t + 2\pi\Pi(\zeta(\varrho)), \zeta(\varrho)).$$

Let $(\mathbf{R} \times S)/\mathcal{O}$ be the corresponding factor space. It is a symplectic manifold, $d\zeta \wedge dt$ is a symplectic two-form on it, and the mapping

$$\mathbf{R} \times S \ \ni \ (t, \varrho) \longrightarrow \exp(tH_\zeta)(\varrho) \in \ T^*\partial\Omega$$

lifts to a symplectic diffeomorphism from $(\mathbf{R} \times S)/\mathcal{O}$ to a neighborhood of M_0 . Making suitable symplectic change of the variables

$$\theta \ = t/\Pi(\zeta), \ \ r = g(\zeta),$$

in $\mathbf{R}\times S$ we can suppose that $\mathcal{O}$ is generated by $(\theta, r) \longrightarrow (\theta + 2\pi, r)$ while the symplectic two-form becomes $d\theta \wedge dr$. It is easy to see that the first derivative $g'(\zeta) = -\Pi(\zeta)$ which yields

$$r(\zeta) = -\int_0^\zeta \Pi(t)\, dt + l, \ \ l = l(0) = \ell/2\pi. \tag{3.4}$$

Denote by $\zeta(r)$ the function inverse to $r(\zeta)$. We have obtained symplectic coordinates $(\theta_s(x,\xi), r_s(x,\xi))$ in a neighborhood of $\partial\Sigma_s^+$ in $T^*\partial\Omega$ with values in $\mathbf{T}\times\mathbf{R}$ such that $\partial\Sigma_s^+ = \{r_s = l_s\}$, $l_s = \ell_s/2\pi$ and $\Sigma_s \subset \{r_s \geq \ l_s\}$. Fix $\varepsilon > 0$ and set $\Gamma_s = (l_s - \varepsilon, l_s)$, $\mathbf{A_s} = \mathbf{T}\times\Gamma$.

The exact symplectic map B_s is generated in this coordinates by the function

$$G_s(\theta, r) = -\frac{4}{3}\,\zeta_s(r)^{3/2} + Q_s(\theta, r), \ \ (\theta, r) \in \ \mathbf{A_s}, \tag{3.5}$$

where

$$\zeta_s(l_s) = 0, \ \ \zeta_s'(l_s) = -1/\Pi_s(0) < 0, \ \ \partial_\theta^\alpha \partial_r^\beta Q_s(\theta, l_s) \ = 0, \ \forall\ \theta \in \mathbf{T}, \tag{3.6}$$

for any indices $\alpha \geq 0, \beta \geq 0$, and $s \in [0, b]$. Hereafter we say that G_s generates the exact symplectic map B_s in $\mathbf{A_s}$ if $\operatorname{graph}(B_s) = \{(B_s(x,\xi), (x,\xi) : \ (x,\xi) \in \mathbf{A}_s\}$ is parameterized by

$$\operatorname{graph}(B_s) = \{(\theta, r - \frac{\partial G_s}{\partial \theta}(\theta, r); \theta - \frac{\partial G_s}{\partial r}(\theta, r), r) : \ (\theta, r) \in \ \mathbf{A_s}\},$$

where

$$\left| \ \frac{\partial^2 G_s}{\partial\theta\partial r}(\theta, r) \ \right| < \ 1, \ \forall\ (\theta, r) \in \ \mathbf{A_s}.$$

Multiplying G_s by a cut-off function we can suppose that it is equal to zero for $r \leq l_s - 2\varepsilon/3$. From now on we denote by B_s the corresponding modified exact symplectic mappings. Note that B_s, Q_s, as well as the exact symplectic mappings ψ_s^0 defined by

$$(\psi_s^0)^{-1}(x,\xi) = (\theta_s(x,\xi), r_s(x,\xi))$$

depend continuously on s in the corresponding C^∞ semi-norms.

The billiard ball map B_s is a small perturbation of the completely integrable mapping generated in $\mathbf{A_s}$ by $\tau_s^0(r) = -\frac{4}{3}\zeta_s(r)^{3/2}$. In what follows we apply a symplectic version of the KAM theorem to B_s which is close to Theorem 1.1 in [21]. In contrast to [21], the generating function $\tau_s^0(r)$ has singularity at $r = l_s$.

As a consequence of (2.3) and Lemma 5.2 we easily obtain $\ell_s = \ell_0$ in $[0,1]$. Indeed, the continuous function $[0,1] \ni s \to \ell_s$ takes values in the set $\mathcal{L}(\Omega_s) = \mathcal{L}(\Omega_0)$ which does not contain intervals, according to Lemma 5.2. Hence, ℓ_s does not depend on s. To simplify the notations we set $\Gamma = \Gamma_0$ and $\mathbf{A} = \mathbf{A}_s$.

We are going to define the Cantor set Θ. Fix $\sigma > 1$, $\mu > 0$, and for any $a > 0$ and $N \in \mathbf{Z}_+$ define the Cantor set $\Theta(a, \mu, N)$ by the small denominator condition as follows:

$$\Theta(a,\mu,N) = \{\omega \in \mathbf{R} : |\, \omega k_1 - k_2 \,| \geq \mu a^N \,|\, k \,|^{-\sigma} \text{ for any } k = (k_1,k_2) \in \mathbf{Z}^2\backslash\{0\}\}.$$

Fix $0 < C < 1$ and $\omega_0 > 0$, and denote

$$\Theta^*(a,\mu,N) = \Theta(a,\mu,N) \cap [Ca, C^{-1}a],\ 0 < a < \omega_0. \tag{3.7}$$

Consider the Cantor subset of $[0,\infty)$ defined by

$$\Theta = \cup\{\Theta^*(a,\mu,N) : \quad 0 < a \leq \varepsilon(\mu,N),\ N \in \mathbf{Z}_+\} \cup \{0\} \tag{3.8}$$

where

$$\varepsilon(\mu,N) = \varepsilon_N \mu^{2/M},\ M = N^2 + 2N,$$

while the positive constants ε_N will be specified later.

This set is of a positive Lebesgue measure in $\mathbf{R}$ and even

$$\varepsilon - \mathrm{meas}(\Theta \cap [0,\varepsilon)) \leq C_p \varepsilon^p,\ 0 < \varepsilon < \varepsilon_0, \tag{3.9}$$

for any $p \geq 1$. The following theorem provides a symplectic normal form for the family of symplectic mappings B_s in a neighborhood of $\partial\Sigma_s^+$.As above we assume that B_s is generated by a function G_s in $\mathbf{A}$ and that B_s coincides with the identity mapping in $\mathbf{T} \times [l_0 - \varepsilon, l_0 - 2\varepsilon/3]$.

Theorem 3.1 *Let $[0,b] \ni s \longrightarrow B_s \in C^\infty(\mathbf{A},\mathbf{A})$ be a continuous deformation of exact symplectic mappings. Suppose that the corresponding generating functions G_s satisfy (3.5) and (3.6), and $\ell_s = \ell_0$ for any s. Then there is a Cantor set Θ defined by (3.8) with suitable $\varepsilon_N > 0$ and there exist continuous in $[0,b]$ families of exact symplectic mappings $\psi_s \in C^\infty(\mathbf{T}\times\mathbf{R}, \mathbf{T}\times\mathbf{R})$ and functions $K_s \in C^\infty(\mathbf{R}), Q_s^0 \in C^\infty(\mathbf{T}\times\mathbf{R})$ such that:*

(i) $K_s(l_0) = 0$, $K_s'(l_0) < 0$, $K_s(t) > 0$ in Γ, and the exact symplectic map $B_s^0 = \psi_s^{-1} \circ B_s \circ \psi_s$ is generated in $\mathbf{A}$ by

$$G_s^0(\varphi, I) = \tau_s(I) + Q_s^0(\varphi, I),\ \tau_s(I) = -\frac{4}{3}K_s(I)^{3/2},\ (\varphi, I) \in \mathbf{A}, \tag{3.10}$$

where

$$Q_s^0(\varphi, I) = 0 \text{ on } \mathbf{T} \times E_s \tag{3.11}$$

and $E_s = \{I \in \Gamma : \ \tau_s^{'}(I)/2\pi \in \Theta\}$

(ii) K_s, Q_s^0, *and the generating function* $S_s(\theta, I)$ *of* ψ_s *satisfy the estimates*

$$\begin{aligned} | \, D_I^\alpha (K_s(I) - \zeta_s(I)) \, | \; + \; | \, D_I^\alpha D_\varphi^\beta Q_s^0(\varphi, I) \, | \quad &\leq \quad C_{\alpha\beta p} \, | \, t_0 - I \, |^p & (3.12) \\ | \, D_I^\alpha D_\theta^\beta S_s^0(\theta, I) \, | \quad &\leq \quad C_{\alpha\beta p} \, | \, t_0 - I \, |^p & (3.13) \end{aligned}$$

in $\mathbf{T} \times [l_0 - \varepsilon_0, l_0 + \varepsilon_0]$, $\varepsilon_0 > 0$, *for any* $s \in [0, b]$ *and any indices* $\alpha \geq 0$, $\beta \geq 0$ *and* $p > 0$.

The proof of Theorem 3.1 is given in the Appendix. First we construct exact symplectic mappings conjugating the billiard ball maps B_s in suitable domains away from the singularity set $\{r = l_0\}$ of B_s and then we patch them together using the uniqueness of the KAM circle with a given rotation number. An important role here plays Proposition A.1.

In view of (3.6), (3.12), and the equality $\ell_s = \ell_0$, the frequency map

$$\Gamma \ni I \;\longrightarrow\; \omega = \tau_s'(I)/2\pi \;\in\; (0, \omega_0), \; \omega_0 > 0, \tag{3.14}$$

is invertible if Γ is sufficiently small. Denote by $\mathcal{J}_s(\omega)$ the inverse map to (3.14) in $(0, \omega_0)$. Then $E_s = \mathcal{J}_s(\Theta)$ and we have

$$\mathcal{J}_s(\omega) \;=\; l_0 - c_0\omega^2 \;+\; O(\omega^4), \; l_0 = \ell_0/2\pi, \; c_0 = \frac{1}{4}\Pi_0(0)^{-1} > 0. \tag{3.15}$$

Moreover, $\mathcal{J}_s(\omega)$ can be extended to a smooth even function in $\mathbf{R}$. Set $\chi_s = \psi_s^0 \circ \psi_s$. Since E_s has no isolated points, (3.11) means that Q_s^0 has a zero of infinite order at $\mathbf{T} \times E_s$. In particular,

$$B_s^0(\varphi, I) \;=\; (\varphi \;+\; \tau_s'(I), I), \; (\varphi, I) \in \mathbf{T} \times E_s, \tag{3.16}$$

and

$$[0, b] \ni s \;\longrightarrow\; \Lambda_s(\omega) = \chi_s(\mathbf{T} \times \{\mathcal{J}_s(\omega)\}) \tag{3.17}$$

is a continuous family of invariant circles of B_s with a rotation number $\omega \in \Theta$ which accumulate at $\partial\Sigma_s^+ = \chi_s(\mathbf{T} \times \{l_0\})$ since $l_0 = \mathcal{J}_s(0)$. Denote by Λ_s the union of the invariant circles $\Lambda_s(\omega)$, $\omega \in \Theta$, and consider the function

$$h_s(x, \xi) = K_s(\chi_s^{-1}(x, \xi))$$

where K_s is introduced by Theorem 3.1. Since Q_s^0 has a zero of infinite order at E_s and $l_0 \in E_s$, h_s is an approximated interpolating hamiltonian of B_s. Thus we obtain

Corollary 3.1 *We have*

$$B_s(\varrho) = \exp(-2h_s(\varrho)^{1/2} H_{h_s})(\varrho) + R_s(\varrho), \;\; \varrho \in \Sigma_s \cap U,$$

in any local chart U *in* $T^*\partial\Omega$ *where* $R_s(\varrho) \in C^\infty(U)$ *is continuous with respect to* $s \in [0, b]$ *and* R_s *has a zero of infinite order at* $\Lambda_s \cap U$.

We are going to show in Proposition 4.1 that $h_s(\varrho)$ coincides with $(3\ell_s(\omega)/4)^{2/3}$ for any $\varrho \in \Lambda_s(\omega)$ and $\omega \in \Theta$ where $\ell_s(\omega)$ is the Lazutkin parameter of the invariant circle $\Lambda_s(\omega)$. Note that, according to (3.9)

$$1 - \mathrm{meas}(U \cap \Lambda_s) \leq \ C_N(\mathrm{meas}(U))^N, \ \ s \in [0, b], \tag{3.18}$$

for any sufficiently small neighborhood U of $\partial\Sigma_s^+$.

4 Caustics and Lazutkin's parameter

This section is devoted to the geometry of the caustics of a strictly convex domain. Our aim is to give a simple geometric interpretation of the function

$$\Theta \ni \omega \ \longrightarrow \ (\mathcal{J}_s(\omega), K_s(\mathcal{J}_s(\omega))$$

where K_s was introduced by Theorem 2.1 and $\mathcal{J}_s$ is the function inverse to the frequency map defined above. To simplify the notations we drop the index s.

Fix $\omega \in \Theta$ and consider the invariant circle $\Lambda(\omega)$ of B and the corresponding caustic $C(\omega)$ with a rotation number ω. Take $\varrho = (x, \xi)$ arbitrarily in $\Lambda(\omega)$. The projections $t \longrightarrow x + t\tilde{e}^{\pm}(\varrho)$ of the bicharacteristics of $S^*\mathbf{R}^2 = \{(y, \eta) \in T^*\mathbf{R}^2) : \mid \eta \mid = 1\}$ passing through $\pi^{\pm}(\varrho)$ are tangent to the caustic $C(\omega)$ which is smooth and strictly convex (see [14]). Let $y^{\pm} = y^{\pm}(\varrho)$ be the corresponding points of tangency. Denote by $\mid xy^{\pm} \mid$ the distance between x and $y^{\pm}$ and by $\mid y^- \frown y^+ \mid$ the length of the shortest arc in $C(\omega)$ connecting y^- with y^+. Lazutkin's parameter of the caustic $C(\omega)$ is defined by

$$t(\omega) \ = \mid xy^- \mid \ + \ \mid xy^+ \mid \ - \ \mid y^- \frown y^+ \mid$$

and it does not depend on the choice of $x \in \partial\Omega$ (see [14], [1]). As above denote by $\ell(\omega)$ the length of the caustic $C(\omega)$. The main result in this section is:

Proposition 4.1 *For any $\omega \in \Theta$ we have*

$$\mathcal{J}(\omega) \ = \ \ell(\omega)/2\pi, \ \ \tau(\mathcal{J}(\omega)) \ = \ -t(\omega). \tag{4.1}$$

Proof. Consider the flow-out

$$\mathcal{M}(\omega) \ = \ \{\exp(tH_g)(\pi^+(\varrho)) : \ \varrho \in \Lambda(\omega), \ 0 \leq t \leq q(\varrho)\}$$

of $\Lambda(\omega)$ with respect to the Hamiltonian $g(y, \eta) = \mid \eta \mid -1, \ (y, \eta) \in T^*\mathbf{R}^2$, where $q(\varrho)$ is the time t for which a point starting at x and travelling with unit speed along the ray $t \longrightarrow x + t\tilde{e}^+(\varrho)), \ t \geq 0$, reaches $y^+ \in C(\omega)$. Then $\mathcal{M}$ is a Lagrangian submanifold of $T^*\mathbf{R}^2$ whose boundary consists of two components, namely $\{\pi^+(x, \xi) : \ (x, \xi) \in \Lambda(\omega)\}$ and the cosphere bundle $S^*C(\omega) \ = \ \{(y, \eta) \in T^*\mathbf{R}^2 : \ y \in C(\omega), \ \mid \eta \mid = 1\}$. By Stoks' theorem

$$\ell(\omega) \ = \ \int_{S^*C(\omega)} \eta dy \ = \ \int_{\Lambda(\omega)} \xi dx \ = \ 2\pi\mathcal{J}(\omega)$$

since the map χ conjugating B to its symplectic normal form given by (3.10) and (3.11) is exact symplectic. This proves the first part of the claim.

To prove the second equality in (4.1) we use a symplectic trick which is due to V. Guillemin and R. Melrose [9] and Y. Colin de Verdiére [4]. Denote by $\sigma_0 = Id\varphi$ the symplectic one-form in $T^*\mathbf{T}$. Since $B^0 = \chi^{-1} \circ B \circ \chi$ is an exact symplectic map with a generating function $G^0(\varphi, I) = \tau(I) + Q^0(\varphi, I)$ given by Theorem 3.1, it is easy to see that

$$(B^0)^*\sigma_0 \; - \; \sigma_0 \; = \; df \tag{4.2}$$

where

$$f(\varphi, I) \; = \; I\tau'(I) \; - \; \tau(I) \; + \; F(\varphi, I), \; (\varphi, I) \in \mathbf{T} \times \Gamma, \tag{4.3}$$

and the function

$$F(\varphi, I) \; = \; I\frac{\partial Q^0}{\partial I}(\varphi, I) \; - \; Q^0(\varphi, I)$$

has a zero of infinite order on the Cantor set $\mathbf{T} \times E$ in view of (3.11). On the other hand, (2.1) implies

$$B^*\sigma \; - \; \sigma \; = \; dT^0, \; \sigma \; = \; \xi dx, \tag{4.4}$$

on Σ, where $T^0(x, \xi) = T(x, y(x, \xi)) =\mid x - y(x, \xi) \mid$ and $y(x, \xi)$ is the first component of $B(x, \xi)$. Since χ is exact symplectic

$$\chi^*\sigma \; - \; \sigma^0 \; = \; d\Phi, \; \Phi \in C^\infty(T^*\mathbf{T}). \tag{4.5}$$

Now, (4.2), (4.4) and (4.5) yield together the following useful equality :

$$f(\varphi, I) \; = \; T^0(\chi(\varphi, I)) \; + \; \Phi(\varphi, I) \; - \; \Phi(B^0(\varphi, I)) \; + \; C, \; (\varphi, I) \in \mathbf{A} \tag{4.6}$$

where C is a constant. Taking $I = l = \ell/2\pi$ we get $C = 0$.

We are ready to prove the second equality in Proposition 4.1. Take $g^0 \in \Lambda(\omega)$ and consider the orbit g of B defined by $g^j = B^j g^0 = (x_j, \xi_j), \; j = 0, 1, ...$ Denote by $\tilde{g}^j = (\varphi^j, \mathcal{J}(\omega)) = \chi^{-1}(g^j), \; j = 0, 1, ...,$ the corresponding orbit of B^0. For any $k \in \mathbf{N}$ denote by m_k the number of rotations that a point makes moving around $\partial\Omega$ in a counterclockwise direction from x_0 to x_k and passing successively through all $x_j, \; j \geq k$. Then

$$t(\omega) \; = \; \lim_{k\to\infty}(\frac{1}{k}\sum_{j=0}^{k} T^0(g^j) \; - \; \frac{m_k}{k}\ell(\omega)). \tag{4.7}$$

On the other hand, (3.11), (4.3) and (4.6) imply

$$T^0(g^j) \; = \; 2\pi\omega\mathcal{J}(\omega) - \tau(\mathcal{J}(\omega)) + \Phi(\varphi + 2\pi j\omega, \mathcal{J}(\omega)) - \Phi(\varphi + 2\pi(j+1)\omega, \mathcal{J}(\omega)).$$

Hence, the average action on g $= (g_0, g_1, ...)$ is equal to

$$\lim_{k\to\infty}(\frac{1}{k}\sum_{j=0}^{k} T^0(g^j)) \; = \; 2\pi\omega\mathcal{J}(\omega) \; - \; \tau(\mathcal{J}(\omega)), \tag{4.8}$$

the right hand side being just the Legendre transform of $\tau(I)/2\pi$ times 2π. Moreover,

$$\lim_{k\to\infty}\left(\frac{m_k}{k}\right) \ = \ \omega$$

while $\ell(\omega) = 2\pi\mathcal{J}(\omega)$. Now, (4.7) and (4.8) yield together the second equality in (4.1). This completes the proof of the proposition. □

Consider the approximated interpolating Hamiltonian $h(x,\xi) = K(\chi^{-1}(x,\xi))$ of B introduced by Corollary 3.2. For $r > 0$ small enough we set as in Section 3

$$M_r \ = \ \{(x,\xi) \in \Sigma : \ h(x,\xi) = r\}$$

and denote

$$\nu(r) \ = \ \int_{M_r} dt \tag{4.9}$$

where the Poisson bracket

$$\{h,t\} \ = \ H_h t \ = \ 1. \tag{4.10}$$

It is easy to show that the set of Taylor coefficients of $\nu(r)$ at $r = 0$ is algebraically equivalent to the set of Taylor's coefficients of $K(I)$ at $I = l$. Indeed, performing a symplectic change of the variables $(x,\xi) = \chi(\varphi,I)$, $(\varphi,I) \in \mathbf{T}\times\Gamma$, and using (4.10) we easily get

$$K'(I)\nu(K(I)) \ = \ 2\pi, \ I \in E. \tag{4.11}$$

Denote by $\mathcal{R}$ the function inverse to $I \to K(I)$ and set $\tilde{E} = \{K(I) : \ I \in E\}$. Then $0 \in \tilde{E}$ and (4.11) implies

$$\nu(r) \ = \ 2\pi\mathcal{R}'(r), \ r \in \tilde{E}. \tag{4.12}$$

The Taylor coefficients of $\nu(t)$ at $t = 0$, also called integral invariants, have been investigated by Sh. Marvizi and R. Melrose [16]. They are given by integrals on $\partial\Omega$ of certain polynomials of the curvature $\kappa(x)$ of $\partial\Omega$ and its derivatives. In particular, (4.6) in [16] and (4.12) yield together

$$\mathcal{R}'(0) \ = \ -\frac{1}{\pi}\int_0^\ell \kappa(x)^{2/3}dx \tag{4.13}$$

$$\mathcal{R}''(0) \ = \ \frac{1}{2160\pi}\int_0^\ell (9\kappa(x)^{4/3} \ + \ 8\kappa(x)^{-8/3}\kappa'(x)^2)dx. \tag{4.14}$$

5 Marked length spectrum and asymptitics of the average action

Fix $b > 0$ such that $\partial\Omega_s$ is strictly convex for any $0 \le s \le b$. Consider the set $\mathcal{G}(\Omega_s,m,n)$ of periodic broken geodesics γ of Ω_s with n vertices and winding number $m \le n/2$. The marked length spectrum $\mathcal{L}(\Omega_s,m,n)$ is defined as the set of lengths of all γ in $\mathcal{G}(\Omega_s,m,n)$. The set $\mathcal{L}(\Omega_s,m,n)$ is compact and we denote

$$T_{mn}(s) \ = \ \max\{t : \ t \in \mathcal{L}(\Omega_s,m,n)\}.$$

Following an idea due to G. Birkhoff we easily prove

Lemma 5.1 *The set $\mathcal{L}(\Omega_s, m, n)$ is not empty for any $s \in [0, b]$ and any integers $n \geq 2,\ m \leq n/2$. The function*

$$[0, b] \ni s \longrightarrow T_{mn}(s)$$

is continuous.

Proof. As above consider Ω_s as $\Omega = \Omega_0$ equipped with a suitable Riemannian metric $\| \cdot \|_s^2$ which depends continuously on s. Denote by $\theta : \mathbf{R} \to \partial\Omega$ a smooth covering of $\partial\Omega,\ \theta(x+1) = \theta(x),\ x \in \mathbf{R}$. Consider the function

$$S_s(x_1, \dots, x_n) = \|\theta(x_1) - \theta(x_2)\|_s^2 + \dots + \|\theta(x_n) - \theta(x_{n+1})\|_s^2,\ x_{n+1} = x_1 + m,$$

and set

$$M = \{(x_1, \dots, x_n) \in \mathbf{R}^n : \ x_1 \leq x_2 \leq \dots \leq x_n \leq x_{n+1} = x_1 + m,\ x_{j+1} - x_j \leq 1\}.$$

Obviously S_s is a continuous and periodic function in M with a period $e = (1, \dots, 1)$. Moreover, the factor space $M/\mathbf{Z}e$ is compact. The triangle inequality shows that the set

$$M_s^{max} = \{p \in M : \ S_s(p) = \max_M S_s\}$$

consists only of points internal for M. Hence, S_s is smooth on M_s^{max}. Moreover, $\nabla S_s(p) = 0$ at $p = (x_1, \dots, x_n) \in M_s$ if and only if $\theta(x_1), \dots, \theta(x_n)$ are successive vertices of a closed broken geodesic of Ω_s which belongs to $\mathcal{G}(\Omega_s, m, n)$. Therefore, the set $\mathcal{G}(\Omega_s, m, n)$ is not empty for any $s \in [a, b]$. Moreover,

$$T_{mn}(s) = \sup \{S_s(p) : \ p \in M\}$$

is continuous in $s \in [0, b]$. $\square$

Lemma 5.2 *The Lebesgue measure of $\mathcal{L}(\Omega_s)$ is zero.*

Proof. Let γ be a periodic broken geodesic in $\mathcal{G}(\Omega_s, m, n)$. Then length$(\gamma) = S_s(p)$ for some $p \in M$ such that $\nabla S_s(p) = 0$. Applying Sard's theorem we obtain that $\mathcal{L}(\Omega_s, m, n)$ has Lebesgue measure zero which proves the claim. $\square$

Using (2.3), Lemma 5.1 and Lemma 5.2 we easily obtain

$$T_{mn}(s) = T_{mn}(0),\ s \in [0, b]. \tag{5.1}$$

Indeed, according to (2.3) the continuous function $T_{mn}(s)$ takes values in $\mathcal{L}(\Omega_0)$ which does not contain intervals in view of Lemma 5.2. Hence, $T_{mn}(s)$ should be constant in $[0, b]$.

In what follows we shall evaluate the average action on the periodic orbits of the billiard ball map. To simplify the notations we drop the index s. Consider the set $\Gamma(m, n)$ of periodic orbits $g = (g_1, \dots, g_n)$ of B of period n and winding number m. Any such orbit gives rise to a periodic broken geodesic in $\mathcal{G}(\Omega, m, n)$. Denote by $L(g)$ the length of the periodic broken geodesic of Ω associated with the periodic orbit g of B.

According to (4.8), the average action of any orbit $\{g_0, g_1, \ldots\}$, $g_j = B^j(g_0)$, $g_0 \in \Lambda(\omega)$, of B on the invariant circle $\Lambda(\omega)$, $\omega \in \Theta$, is given by the Legendre transform

$$\mathcal{I}(\omega) \;=\; \omega\mathcal{J}(\omega) \;-\; \tau(\mathcal{J}(\omega))/2\pi \tag{5.2}$$

of $\tau(I)/2\pi$. Note that $\mathcal{I}(\omega)$ can be extended to a smooth odd function in $\mathbf{R}$ since $\mathcal{J}(\omega)$ is even, $\tau(I) = -\frac{4}{3}K(I)^{3/2}$, $K(l_0) = 0$, $K'(l_0) < 0$ and (3.15) holds. Moreover,

$$2\pi\mathcal{I}(\omega) \;=\; \omega\ell(\omega) \;-\; t(\omega),\ \forall\omega \in \Theta, \tag{5.3}$$

in view of Proposition 4.1.

Next we impose the following condition on the pair $(m, n) \in \mathbf{N}^2$

$$\mathrm{dist}(m/n, \Theta) \;\leq\; n^{-1/2} \tag{5.4}$$

where $\mathrm{dist}(z, \Theta)$ is the distance between z and Θ. Here is the main result in this section:

Theorem 5.1 *For any $(m, n) \in \mathbf{N}^2$ satisfying (5.4) and any $g \in \Gamma(m, n)$ we have*

$$\mid L(g)/n \;-\; \mathcal{I}(m/n) \mid \;\leq\; C_N n^{-N},\ \forall\ N > 0, \tag{5.5}$$

where C_N depends only on N and Ω.

Proof of Theorem 2.1. Fix $\omega \in \Theta$ and choose a sequence $(m_j, n_j) \in \mathbf{N}^2, j \in \mathbf{N}$, such that

$$\mid \omega - m_j/n_j \mid \;\leq\; n_j^{-1/2},\ j = 1, 2, \ldots\ .$$

For any $j \in \mathbf{N}$ pick a periodic orbit $g_s^j \in \Gamma_s(m_j, n_j)$ of B_s such that

$$L_s(g_s^j) \;=\; \max\{t :\ t \in \mathcal{L}(\Omega_s, m_j, n_j)\} \;=\; T_{m_j n_j}(s),\ s \in [0, b].$$

Theorem 5.1 yields

$$\mathcal{I}_s(\omega) \;=\; \lim_{j\to\infty}(T_{m_j n_j}(s)/n_j),\ s \in [0, b],$$

and taking into account (5.1) we obtain

$$\mathcal{I}_s(\omega) \;=\; \mathcal{I}_0(\omega),\ \forall\omega \in \Theta,\ \forall s \in [0, b].$$

Since Θ has no isolated points, differentiating the last equality with respect to ω we obtain

$$\mathcal{J}_s(\omega) \;=\; \mathcal{J}_0(\omega),\ K_s(\mathcal{J}_0(\omega)) \;=\; K_0(\mathcal{J}_0(\omega)),\ \forall\omega \in \Theta,\ \forall s \in [0, b],$$

which proves (2.6). In particular,

$$E_s = E_0,\ K_s(I) = K_0(I),\ \forall\ I \in E_0, \tag{5.6}$$

and using Theorem 3.1 we prove (ii), Theorem 2.1 for $s \in [0, b]$.

It remains to show that Ω_s is strictly convex for any $s \in [0,1]$. Suppose that Ω_s is strictly convex for $s < b_0$ but only convex for $s = b_0$. Consider the function $\mathcal{R}_s(r)$ inverse to $r = K_s(I)$ and set $\tilde{E}_s = \{K_s(I): \ I \in E\}$. Then (5.6) yields

$$\tilde{E}_s = \tilde{E}_0, \ \mathcal{R}_s(r) = \mathcal{R}_0(r), \ \forall \ r \in \tilde{E}_0,$$

and since $0 \in \tilde{E}_0$ we obtain

$$\mathcal{R}_s'(0) \ = \ \mathcal{R}_0'(0), \ \mathcal{R}_s''(0) \ = \ \mathcal{R}_0''(0), \ s \in [0, b_0). \tag{5.7}$$

Denote by $\kappa_s(x), \ x \in \partial\Omega_s$, the curvature of $\partial\Omega_s$, set $f_s(x) = \kappa_s(x)^{-1/3}$ for $s < b_0$, and define $f_{b_0}(x) = \kappa_{b_0}(x)^{-1/3}$ if $\kappa_{b_0}(x) \neq 0, \ f_{b_0}(x) = 0$ if $\kappa_{b_0}(x) = 0$. The second equality of (5.7) and (4.14) yield together

$$\int_{\partial\Omega_s} | \ f_s'(x) \ |^2 \ dx \ \leq \ C, \ s \in [0, b_0). \tag{5.8}$$

On the other hand, the first equality of (5.7) and (4.14) imply

$$\kappa_s(x_s) > C_1 > 0, \ s \in [0, b_0),$$

for suitable $x_s \in \partial\Omega_s$. Then $f_s(x_s) \leq C_2, \ s \in [0, b_0)$, and using Taylor's formula and (5.8) we obtain the estimate

$$\int_{\partial\Omega_s} (| \ f_s(x) \ |^2 \ + \ | \ f_s'(x) \ |^2) dx \ \leq \ C_3, \ s \in [0, b_0),$$

which means that $\{f_s : \ s \in [0, b_0)\}$ is a compact subset of $L^2(\partial\Omega)$ (we regard $\partial\Omega_s$ as $\partial\Omega$ equipped with a suitable Riemannian metric). In particular, $f_{b_0} \in L^2(\partial\Omega)$. On the other hand, the curvature $k_{b_0}(x) \geq 0$ and it has a zero of at least second order at a point $x_0 \in \partial\Omega$. Hence,

$$| \ f_{b_0}(x) \ | \geq \ C \ | \ x - x_0 \ |^{-2/3}$$

in any local coordinates in a neighborhood of x_0 in $\partial\Omega$ which implies $f_{b_0} \not\in L^2(\partial\Omega)$. Hence, $\partial\Omega_s$ is strictly convex for any $s \in [0,1]$. The proof of Theorem 2.1 is complete. □

Proof of Theorem 5.1. The proof is based on a suitable approximation of $(B^0)^j(\varphi, I), \ j \leq n$, where B^0 is introduced by Theorem 3.1. Fix $\varepsilon_0 > 0$ and consider a neighborhood $V = \{\varrho \in \Sigma : \ 0 \leq h(\varrho) \leq \varepsilon_0\}$ of $\partial\Sigma_s^+$ in Σ, where h is a continuous family of approximated interpolating Hamiltonians of B introduced by Corollary 3.2. There exists $\varepsilon > 0$ such that if $m/n < \varepsilon$ and $(g_1, \ldots, g_n) \in \Gamma(m,n)$, then $g_j \in V$ for each $j \leq n$. Indeed, denote by t the maximal length of the segments with end points x_j and $x_{j+1}, \ j = 1, \ldots, n$, where $g_j = (x_j, \xi_j), \ x_{n+1} = x_1$. Then $t < m\ell_0/n < \varepsilon$ which implies $g_j \in V$ if ε is sufficiently small. Let $\omega \in \Theta \subset [0, \omega_0], \ \omega_0 < \varepsilon_0$ and $m/n \leq \varepsilon_0$. We have $g_j \in V, \ j \leq n$, for any periodic orbit $g = (g_1, \ldots, g_n) \in \Gamma(m,n)$. Let $\tilde{g} = (\tilde{g}_1, \ldots, \tilde{g}_n), \ \tilde{g} = (\varphi_j, I_j) = \chi^{-1}(g_j)$, where χ has been introduced in Section 3. Then $\tilde{g}$ is a periodic orbit of B^0 of period n and winding number m, and $\tilde{g}_j \in \mathbf{T} \times \Gamma, \ \Gamma = (l_0 - \varepsilon, l_0)$. According to Theorem 3.1, the map B^0 has the form

$$B^0(\varphi, I) = (\varphi + \tau'(I) + Q_1^0(\varphi, I), I + Q_2^0(\varphi, I)) \tag{5.9}$$

where Q_1^0 and Q_2^0 have a zero of infinite order at $\mathbf{T} \times E$ and $E = \mathcal{J}(\Theta) \subset [l_0 - \varepsilon, l_0]$. Consider the sets

$$V_n = \{\xi \in (\frac{1}{2n}, \omega_0) : \ \mathrm{dist}(\xi, \Theta) \leq 2n^{-1/2}\}, \ J_n = \mathcal{J}(V_n).$$

There exists $C_0 > 0$ such that

$$\mathrm{dist}(I, E) \ \leq \ C_0 n^{-1/2} \text{ and } l_0 - I \geq C_0^{-1} n^{-2}, \ \forall I \in V_n, \ \forall n \in \mathbf{Z}_+, \tag{5.10}$$

where $E = \mathcal{J}(\Theta)$.

Proposition 5.1 *For any integers $\alpha \geq 0$, $\beta \geq 0$, and $N \geq 1$, there exists a constant $C_{N\alpha\beta}$ such that*

$$| \ \partial_\varphi^\alpha \partial_I^\beta ((B^0)^j(\varphi, I) - (\varphi + j\tau'(I), I)) \ | \leq \ C_{N\alpha\beta} \, n^{-N}, \ 1 \leq j \leq n, \ \forall n \in \mathbf{N}, \tag{5.11}$$

in $(\varphi, I) \in \mathbf{T} \times J_n$.

Proof. Set $U(\varphi, \xi) = (\varphi, \mathcal{J}(\xi))$, $(\varphi, \xi) \in \mathbf{T} \times \mathbf{R}$, and consider the map $B = U^{-1} \circ B^0 \circ U$ in $\mathbf{T} \times (0, \omega_0)$. We have

$$B(\varphi, \xi) = (\varphi + 2\pi\xi, \xi) + R(\varphi, \xi)$$

where $R = (R_1, R_2)$ can be extended as a smooth mapping in $\mathbf{T} \times \mathbf{R}$ through $\xi = 0$. Indeed, we have

$$R_1(\varphi, \xi) = Q_1^0(\varphi, \mathcal{J}(\xi)), \ R_2(\varphi, \xi) = \frac{1}{2\pi}\tau'(\mathcal{J}(\xi) + Q_2^0(\varphi, \mathcal{J}(\xi)) - \xi$$

where Q_j^0, $j = 1, 2$, are given by (5.9), they are smooth in $\mathbf{T} \times \mathbf{R}$, and have a zero of infinite order at $\mathbf{T} \times E$. On the other hand, the singularity of τ' at $I = l$ is described by

$$\tau'(I) = -2K(I)^{1/2} K'(I), \ K(l) = 0, \ K'(l) < 0,$$

and we prove easily that R_j are smooth at $\xi = 0$ as $l \in E$. Moreover,

$$| \ \partial_\varphi^\alpha \partial_\xi^\beta R_j(\varphi, \xi) \ | \leq \ C_{N\alpha\beta} \, n^{-3N}, \ j = 1, 2, \tag{5.12}$$

for any $(\varphi, \xi) \in \mathbf{T} \times \mathbf{R}$ such that $\mathrm{dist}(\Theta, \xi) \leq 3n^{-1/2}$. Set

$$B^j(\varphi, \xi) = (\Phi_j(\varphi, \xi), \Xi_j(\varphi, \xi)), \ (\varphi, \xi) \in V_n, \ j \leq n.$$

First we prove by induction with respect to $j \leq n$ the inequalities

$$\begin{aligned} &| \ \Xi_j - \xi \ | \leq jn^{-4}, \ | \ \Phi_j - \varphi - 2\pi j\xi \ | \leq \pi j^2 n^{-4}, \\ &\mathrm{dist}(\Xi_j, \Theta) \leq 2n^{-1/2} + jn^{-4} < 3n^{-1/2}, \ j \leq n, \end{aligned} \tag{5.13}$$

for any $(\varphi,\xi)\in \mathbf{T}\times V_n$ and $n\geq n_1$ where n_1 is sufficiently large. In the same way, making use the third inequality in (5.13) as well as (5.12) we obtain

$$| \, \partial_\varphi^\alpha \partial_\xi^\beta (\Xi_j(\varphi,\xi)-\xi) \, | \ + \ | \, \partial_\varphi^\alpha \partial_\xi^\beta (\Phi_j(\varphi,\xi)-\varphi-2\pi j\xi) \, | \leq \ C_{N\alpha\beta}\, n^{-3N}, \ j\leq n, \ n\geq 1,$$

in $(\varphi,\xi)\in \mathbf{T}\times V_n$ for any nonnegative integers α, β, and N. Conjugating B with U and using the estimate

$$| \, \partial_I^\alpha \tau(I) \, | \leq \ C_\alpha n^{2\alpha}, \ I\in J_n,$$

which follows from (5.10) we complete the proof of Proposition 5.1. □

Consider the set W_n of all $(\varphi',\varphi)\in \mathbf{R}^2$ such that

$$(\varphi'-\varphi)/2\pi n \ \in \ V_n.$$

Set

$$P^0(\varphi,I)=(B^0)^n(\varphi,I)=(\varphi+n\tau'(I),I)+Q(\varphi,I)$$

where $Q=(Q_1,Q_2)$ is a smooth function in $\mathbf{T}\times\mathbf{R}$. Using Proposition 5.1 we solve the equation

$$\varphi'=\varphi+n\tau'(I)+Q_1(\varphi,I)$$

with respect to $I\in J_n$ when $(\varphi',\varphi)\in W_n$ and n is sufficiently large. This equation is equivalent to

$$I=\mathcal{J}((\varphi'-\varphi)/2\pi n)+Q_3(\varphi',\varphi,I), \ I\in J_n, \ (\varphi',\varphi)\in W_n \tag{5.14}$$

where

$$| \, \partial_{\varphi'}^\alpha \partial_\varphi^\beta \partial_I^p Q_3(\varphi',\varphi,I) \, | \leq \ C_{N\alpha\beta p}\, n^{-N}, \ (\alpha,\beta,p)\in \mathbf{Z}_+^3,$$

in $W_n\times J_n$. Hence, (5.14) can be solved by successive iterates for $n\geq n_0$ and n_0 sufficiently large. Denote by $I(\varphi',\varphi), \ (\varphi',\varphi)\in W_n, \ n\geq n_0$, the solution of (5.14). Then

$$| \, I(\varphi',\varphi)-\mathcal{J}((\varphi'-\varphi)/2\pi n) \, | \leq \ C_N\, n^{-N}, \ (\varphi',\varphi)\in W_n. \tag{5.15}$$

In particular, graph(P^0) can be parameterized over $\mathbf{T}\times J_n$ by $(\varphi',\varphi)\in W_n$ as follows:

$$\mathrm{graph}(P^0) \ = \ \{(\varphi',\frac{\partial H_n}{\partial\varphi'}(\varphi',\varphi),\varphi,-\frac{\partial H_n}{\partial\varphi}(\varphi',\varphi)): \ (\varphi',\varphi)\in W_n\} \tag{5.16}$$

where H_n is a smooth and 2π - periodic function function on $\mathbf{R}^2$ satisfying the equality

$$\frac{\partial H_n}{\partial\varphi}(\varphi',\varphi)=I(\varphi',\varphi).$$

Choose $m\in\mathbf{N}$ and suppose that (5.4) holds for the pair $(m,n)\in\mathbf{N}^2, \ n\geq n_0$. Then $(\varphi+2\pi m,\varphi)\in W_n$ for any $\varphi\in\mathbf{R}$. Set

$$h(\varphi) \ = \ h_{mn}(\varphi) \ = \ H_n(\varphi+2\pi m,\varphi), \ \varphi\in\mathbf{R}. \tag{5.17}$$

The function $h(\varphi)$ is smooth and 2π-periodic in $\mathbf{R}$. According to (5.16) there is one - one correspondence between the critical points of h_{mn} in $\mathbf{T}$ and the fixed points of P^0 in $\mathbf{T}\times\Gamma$ which is given by

$$\mathrm{Crit}(h_{mn}) \ni \varphi \longrightarrow g_1(\varphi) = (\varphi, I(\varphi + 2\pi m, \varphi)) \in \mathrm{Fix}(P^0). \tag{5.18}$$

Then

$$\begin{gathered}\mathrm{Crit}(h_{mn}) \ni \varphi \longrightarrow g(\varphi) = (g_1(\varphi), \dots, g_n(\varphi)) \in \Gamma(m,n), \\ g_j(\varphi) = B^{j-1} g_1(\varphi),\ j = 1, \dots, n,\end{gathered}$$

is a one - one correspondence between the critical points of h_{mn} and the periodic orbits of B in $\Gamma(m,n)$, and (5.15) implies

$$| I - \mathcal{J}(m/n) | \le C_N n^{-N}, \tag{5.19}$$

at any periodic point (φ, I) of B^0 of period n and winding number m.

Take $g = (g_1, \dots, g_n) \in \Gamma(m,n)$ and denote as before $(\varphi_j, I_j) = \chi^{-1}(g_j)$. Using (4.6) we obtain

$$| L(g)/n - \mathcal{J}(m/n) | \le \frac{1}{n} \sum_{j=1}^{n} | f(\varphi_j, I_j) - \mathcal{I}(m/n) | .$$

On the other hand,

$$| f(\varphi_j, I_j) - \mathcal{I}(m/n) | \le | \tau(I_j) - \tau(\mathcal{J}(m/n)) | + | I_j \tau'(I_j) - \mathcal{J}(m/n)\tau'(\mathcal{J}(m/n)) | + C_N n^{-N}.$$

We evaluate the right hand side of the inequality above using Taylor's formula. We have $| \tau'(I) | \le C$ in Γ. Moreover,

$$\tau'(I_j) - \tau'(\mathcal{J}(m/n)) = (I_j - \mathcal{J}(m/n))\tau''(\tilde{I}_j)$$

where

$$| \tilde{I}_j - \mathcal{J}(m/n) | \le | I_j - \mathcal{J}(m/n) | \le C_N n^{-N}$$

according to (5.19). Then

$$| \tau''(\tilde{I}_j) | \le | K(\tilde{I}_j)^{-1/2} K'(\tilde{I}_j) | + C \le C n^2.$$

Hence, $| f(\varphi_j, I_j) - \mathcal{I}(m/n) | \le C_N n^{-N}$ which completes the proof of Theorem 5.1. $\square$

The function h introduced by (5.17) is going to play an important role in Section 6. Note that it is uniquely determined by (5.16) and (5.17) up to a constant and we normalize it by taking $H_n(\varphi_0, \varphi_0 + 2\pi m) = L(g(\varphi_0))$ where φ_0 is a point in $\mathrm{Crit}(h_{mn})$. Then we obtain

Lemma 5.3 *We have*

$$h_{mn}(\varphi) = L(g(\varphi)),\ \forall \varphi \in \mathrm{Crit}(h_{mn}).$$

Proof. Taking into account (4.4) we get

$$(B^n)^*(\sigma) - \sigma \;=\; dT_n,\; T_n = \sum_{j=0}^{n} (B^j)^* T^0.$$

On the other hand,

$$(P^0)^*(Id\varphi) - Id\varphi \;=\; dH_n,\; P^0 = (\chi^{-1} \circ B \circ \chi)^n,$$

in $\mathbf{T} \times J_n$, and

$$\chi^*(\sigma) - Id\varphi \;=\; d\Phi,\; \Phi \in C^\infty(\mathbf{T} \times \mathbf{R}),$$

since χ is exact symplectic. Set $T_n^0(\varphi, I) = T_n(\chi(\varphi, I))$. Taking into account the equalities above as well as the normalization of H_n we easily obtain

$$H_n(\varphi', \varphi) = T_n^0(\varphi, I(\varphi', \varphi)) - \Phi(P(\varphi, I(\varphi', \varphi))) + \Phi(\varphi, I(\varphi', \varphi)),\; (\varphi', \varphi) \in W_n. \tag{5.20}$$

Using (5.17) and (5.18) we complete the proof of the lemma. □

The following result is a generalization of Theorem 5.15 in [16]

Corollary 5.1 *Fix $m \in \mathbf{Z}_+$ and pick arbitrarily $g_{mn} \in \Gamma(m, n)$. Then*

$$\mid L(g_{mn}) \;-\; \sum_{k=0}^{N} c_{mk} n^{-2k} \mid \;\le\; C_{Nm} n^{-2N-2}$$

where

$$c_{mk} \;=\; \frac{m^{2k+1}}{(2k+1)!} \mathcal{I}^{(2k+1)}(0),\; k = 0, 1, ...$$

Since $0 \in \Theta$, the pair (m, n) satisfies (5.4) if m is fixed and $n > n_1(m)$. Let us expand $\mathcal{I}(t)$ in Taylor series at $t = 0$. The derivatives $\mathcal{I}^{(2k)}(0) = 0$ since the function $\mathcal{I}(t)$ is odd. Applying Theorem 5.1 we prove the assertion.

Remark 5.1. Since $0 \in \Theta$, we can write the coefficients c_{mk} explicitly in terms of the Taylor series of $t = 0$ of the restriction of the function $\mathcal{I}(\omega)$ on Θ, which is given by (5.3). Moreover, the relation between c_{mk} and the integral invariants of Marvizi and Melrose is explicitly given by (4.11) and (5.2).

Consider two strictly convex domains Ω_1 and Ω_2. Let $B_j,\; j = 1, 2$ be the corresponding billiard ball maps. Choosing the constant ω_0 in (3.6) sufficiently small we obtain for any $\omega \in \Theta$ an invariant curve $\Lambda_j(\omega)$ of $B_j,\; j = 1, 2$. The following statement is a discrete version of Theorem 2.1.

Theorem 5.2 *Let $\Omega_j,\; j = 1, 2$ be strictly convex domains in $\mathbf{R}^2$ and*

$$\mathcal{L}(\Omega_1, m, n) \;=\; \mathcal{L}(\Omega_2, m, n),\; \forall (m, n),\; m \le n/2.$$

Then

$$\ell_1(\omega) \;=\; \ell_2(\omega),\; t_1(\omega) \;=\; t_2(\omega), \forall\, \omega \in \Theta,$$

and there exists an exact symplectic mapping $\chi : \Sigma_1 \to \Sigma_2$ such that

$$\chi(\Lambda_1(\omega)) \;=\; \Lambda_2(\omega) \text{ and } \chi \circ B_1 \;=\; B_2 \circ \chi \text{ on } \Lambda_1(\omega),\; \forall\, \omega \in \Theta.$$

Theorem 5.2 follows from Theorem 5.1 and the arguments at the end of the proof of Theorem 2.1.

6 Spectral invariants

Let Ω be a bounded domain in R^2 with a smooth boundary $\partial\Omega$. Consider the Laplacian Δ in Ω with Dirichlet (Neumann) boundary conditions, and the related distribution $Z(t)$ defined by (1.7).

Denote by $\mathcal{G}(\Omega)$ the set of all periodic generalized geodesics of Ω and by $\mathcal{G}(\Omega; m, n)$ the set of the periodic broken geodesics in Ω corresponding to the periodic orbits of B in $\Gamma(m, n)$. Let $\mathcal{L}_{mn}(\Omega)$ be the set of lengths of all periodic broken geodesics in $\mathcal{G}(\Omega)\backslash\mathcal{G}(\Omega, m, n)$. In order to prevent cancellation of the singularity of $Z(t)$ created by the geodesics of maximal length T_{mn} in $\mathcal{G}(\Omega)$, we impose the following condition

$$T_{mn} \notin \mathcal{L}_{mn}(\Omega). \tag{6.1}$$

This condition is satisfied for generic domains Ω (see [18]).

Theorem 6.1 *Let Ω be a strictly convex bounded domain in $\mathbf{R}^2$ with a smooth boundary. Choose $(m, n) \in \mathbf{N}^2$ such that $\operatorname{dist}(m/n, \Theta) \le n^{-1/2}$ and suppose (6.1) to be fulfilled. Then*

$$T_{mn} \in \text{sing.supp.}(Z) \tag{6.2}$$

holds if $n \ge n_0$ and $n_0 = n_0(\Omega)$ is sufficiently large.

This statement generalizes Theorem 6.4 in [16] where it has been proved for $m = 1$ and n sufficiently large. The main idea in [16] is to find suitable representation of $Z(t)$ in a neighborhood of T_{mn} as a Fourier integral on $\partial\Omega$ (see Proposition 6.11 in [16]) and then to apply a result of Soga [22]. When m is fixed and $n \ge n_1(m)$ is sufficiently large Proposition 6.11 in [16] still holds. In the general case when (m, n) satisfies (5.4) and $n \ge n_0(\Omega)$, we use another representation of $Z(t)$ in a neighborhood of T_{mn} which is close to that obtained in Proposition 5.4, [21]. Consider the function $h(\varphi) = h_{mn}(\varphi), \varphi \in \mathbf{T}$, defined by (5.17). For any $z \in \mathbf{C}$ denote by $\Re z$ its real part.

Proposition 6.1 *Suppose that (m, n) satisfies (5.4) and $n \ge n_0$. If $n_0 = n_0(\Omega)$ is sufficiently large, we have*

$$Z(t) \;=\; \exp(i\pi\mu_n)\int_0^\infty \int_T \Re(\exp(i\tau(t - h(\varphi)))a(\varphi, \tau))d\varphi d\tau \;+\; \tilde{Z}(t)$$

where $\tilde{Z}(t)$ is smooth at $t = T_{mn}$, μ_n is a Maslov's index, and $a(\varphi, \tau)$ is a classical symbol of order one with respect to τ. Moreover, $a(\varphi, \tau) = 0$ for $\tau \le 1$ and the principal part of a is equal to $a_1(\varphi)\tau$ for $\tau \ge 2$ where $a_1 > 0$ on $\mathbf{T}$.

Applying Lemma 5.5 in [21] to the oscillatory integral given by Proposition 6.1 we prove Theorem 6.1.

Proof of Proposition 6.1. Consider the fundamental solution $E(t, x, y)$ of the mixed problem

$$(D_t^2 \;-\; D_x^2)E \;=\; 0,\; E_{|x\in\partial\Omega} \;=\; 0,\; E(0, x, y) \;=\; \delta(x - y),\; (D_tE)(0, x, y) \;=\; 0,$$

where $D_t = -i\partial/\partial t,\ D_x^2 = D_{x_1}^2 + D_{x_2}^2,\ D_{x_j} = -i\partial/\partial x_j,\ j = 1, 2$. The distribution E is just the kernel of the operator $\cos(t\sqrt{\Delta})$. Denote by $E^{\pm}$ the Schwartz kernel of the operator $\exp(\mp it\sqrt{\Delta})$ and consider

$$Z^{\pm}(t) \ = \ \int_{\Omega} E^{\pm}(t, x, x)dx.$$

Then, $Z(t) = Z^+(t) + Z^-(t)$ and $Z^- = \overline{Z^+}$ in a distribution sense where $\overline{z}$ is the complex conjugated number to $z \in \mathbf{C}$. According to (6.14), [16], we have

$$Z^+(t) \ = \ \int_{\partial\Omega} K^+(t, x, x)dx \ + \ Z_1^+(t)$$

where Z_1^+ is smooth at T_{mn} and K^+ belongs to the Hörmander's class $I^{-1/4}(\mathbf{R} \times \partial\Omega \times \partial\Omega; C'_+)$ of Lagrangian distributions associated with the Lagrangian manifold

$$C'_+ = \{(t, x, y; \tau, \xi, \eta) \in T^*(\mathbf{R}\times\partial\Omega\times\partial\Omega) \, : \, t = T_n(y, -\eta/\tau), \ (x, \xi/\tau) = B^n(y, -\eta/\tau), \ \tau > 0\}$$

where $T_n(y, \eta)$ is introduced in Lemma 5.3. Choose neighborhoods $V_1 \subset V_2 \subset C'_+$ of the set

$$C_{mn} = \{(t, x, y; \tau, \xi, \eta) \in C'_+; \ (y, -\eta/\tau) \in \Gamma(m, n)\}.$$

Without loss of generality we can suppose that the complete symbol of K_+ vanishes outside V_2 while its principal symbol is a positive function in V_1 modulo a Maslov's factor. Denote by C'_0 a Lagrangian submanifold of $T^*(\partial\Omega \times \partial\Omega)$ associated with the graph of B^n

$$C'_0 = \{(x, y; \xi, \eta) \in T^*(\partial\Omega \times \partial\Omega); \ (x, \xi) = B^n(y, -\eta), \ (y, -\eta) \in \Sigma\},$$

and let $V_1^0 \subset V_2^0 \subset C'_0$ be neighborhoods of the set

$$\{(x, y; \xi, \eta) \in C'_0; \ (y, -\eta) \in \Gamma(m, n)\}.$$

As in Lemma 5.7, [21], we find a Fourier oscillatory integral $R(x, y, \tau)$ of the class $I^0(\partial\Omega \times \partial\Omega; C'_0, \tau)$ such that

$$K_+(t, x, y) \ = \int e^{-it\tau} R(x, y, \tau)d\tau.$$

Since B^n is an exact symplectic mapping, the Liouville class of C'_0 in $H^1(C'_0; \mathbf{R})$ given by the restriction of the canonical symplectic one form of $T^*(\partial\Omega\times\partial\Omega)$ on C'_0 is trivial. In this case there is a complete analogy between the theory of the Lagrangian distributions and the Fourier oscillatory integrals (see [7]). The only difference is that the principal symbol of a Fourier oscillatory integral associated with C'_0 has an additional Liouville factor $\exp(i\tau f(\varrho))$, $\varrho \in C'_0$, where f is given by the restrictions on C'_0 of suitable phase functions generating C'_0. In our case, the principal symbol of R is equal to $\exp(i\tau T_n(y, -\eta) + i\pi\mu_n)$ times a positive function in V_1^0, the complete symbol of R vanishes outside V_2^0 and $R(x, y, \tau) = O_N(\tau^{-N})$ as $\tau \to -\infty$ for any $N > 0$.

Denote by $R(\tau)$ the Fourier integral operator with a large parameter τ whose Schwartz kernel is $R(x, y, \tau)$. Using Theorem 3.1 (here s is fixed) we shall conjugate $R(\tau)$ to a

Fourier integral operator on $\mathbf{T}$ with a large parameter τ and phase function $h_{mn}(\varphi)$. Denote by C_1' the Lagrangian manifold

$$C_1' = \{(x, \varphi; \xi, I) \in T^*(\partial\Omega \times \mathbf{T};\ (x, \xi) = \chi(\varphi, -I)\}.$$

and observe that the Maslov bundle of C_1' is trivial over $\mathbf{Z}_4$ since the projection

$$C_1' \ni (x, \varphi; \xi, I) \longrightarrow (x, \varphi) \in \partial\Omega \times \mathbf{T}$$

is a diffeomorphism. Indeed, we can replace χ by $\chi\circ\psi$ where $\psi(\varphi, I) = (\varphi + CI, I),\ C > 0$. Then

$$(\chi \circ \psi)^{-1}(x, \xi) \ = \ (\Phi(x, \xi) + C\Xi(x, \xi), \Xi(x, \xi)),\ \partial\Xi/\partial\xi \neq 0 \text{ on } \mathbf{T} \times \{l\},$$

and we can solve the equation $\varphi = \Phi(x, \xi) + C\Xi(x, \xi)$ with respect to ξ if C is sufficiently large.

As in [21], Lemma 5.8, (see also [5]), we easily obtain

Lemma 6.1 *Let $\Psi(\tau)$ be a pseudodifferential operator with a large parameter τ with symbol equal to one in $\mathbf{T} \times V_n$ and equal to zero outside a neighborhood of this set in $\mathbf{T} \times \Gamma$. There exists a Fourier integral operator $A(\tau)$ of the class $I^0(\partial\Omega \times \mathbf{T}, C_1', \tau)$ such that*

$$A^*(\tau)A(\tau) \ = \ \Psi(\tau),$$

the principal symbol of $A(\tau)$ is equal to one on the lifting of $\mathbf{T}\times V_n$ in C_1' and the complete symbol of $A(\tau)$ vanishes outside a small neighborhood of it.

The operator

$$R_1(\tau) \ = \ A^*(\tau)R(\tau)A(\tau) : L^2(\mathbf{T}) \ \longrightarrow \ L^2(\mathbf{T})$$

has a distribution kernel $R_1(\varphi', \varphi, \tau)$ in $I^0(\mathbf{T} \times \mathbf{T}, C_2', \tau)$ where

$$C_2' = \{(\varphi', \varphi; I', I) \in T^*\,\mathbf{T}^2;\ (\varphi', I') = P^0(\varphi, -I)\},\ P^0 = (B^0)^n.$$

According to (5.16), C_2' is generated by $H_n(\varphi', \varphi) + C,\ (\varphi', \varphi) \in W_n$ where C is constant. Hence we get

$$R_1(\varphi', \varphi, \tau) \ = \ \exp(i\tau(H_n(\varphi', \varphi) + C) + i\pi\mu_n)\, b(\varphi', \varphi, \tau)$$

where $b = 0$ for $\tau < 1$ and $b(\varphi', \varphi, \tau) = b_1(\varphi', \varphi)\tau + b_0(\varphi', \varphi) + \ldots$ is a classical symbol of order one as $\tau \to +\infty$, $b = 0$ outside a neighborhood of W_n, and b_0 is a positive function in W_n. On the other hand, comparing the Liouville factors of $R(x, y, \tau)$ and $R_1(\varphi', \varphi, \tau)$ on C_{mn}, we obtain as in Section 5.4 in [21] and Section 3.2 in [3] that $C = 0$.

The L^2-trace of $R(\tau)$ is equal to

$$\operatorname{trace} R(\tau) \ = \ \operatorname{trace} R_1(\tau) \ = \ \int_{\mathbf{T}} \exp(i\tau h(\varphi) + i\pi\mu_n)\, b(\varphi + 2\pi m, \varphi, \tau)\, d\varphi\,,$$

which completes the proof of Proposition 6.1. □

Using Theorem 6.1 and certain arguments form Section 5 we prove that the invariant circles $\Lambda_s(\omega)$ and the restriction of the billiard ball map on them are spectral invariants of the Laplacian for suitable continuous deformations of a strictly convex domain. Define $\mathcal{T}_{mn}(s)$ as $T_{mn}(s)$ when $\Gamma_s(m,n)$ is not empty and set $\mathcal{T}_{mn}(s) = 0$ otherwise. Fix $n_1 > 0$. We say that the deformation (2.2) satisfies the condition $(\mathcal{R})$, if for any $(m,n) \in \mathbf{N}^2$ fixed so that

$$\operatorname{dist}(m/n, \Theta) \;\leq\; n^{-1/2},\ n \geq n_0,$$

the relation

$$\mathcal{T}_{mn}(s) \;\not\in\; \mathcal{L}_{mn}(\Omega_s)$$

is fulfilled for s in a dense subset of $[0,1]$. Note that by definition $\mathcal{T}_{mn}(s)$ does not belong to the length spectrum of Ω_s if the set $\Gamma_s(m,n)$ is empty. Moreover, using arguments from [18] it could be proved that $(\mathcal{R})$ is generic for continuous deformations of the domain.

Theorem 6.2 *Let Ω_0 be a strictly convex bounded domain with a smooth boundary. Suppose that (2.2) is a continuous deformation of Ω_0 satisfying $(\mathcal{R})$ and that*

$$\operatorname{Spec}(\Delta_s) \;=\; \operatorname{Spec}(\Delta_0),\ 0 \leq s \leq 1. \tag{6.3}$$

Then:

(i) Ω_s is strictly convex for any $s \in [0,1]$,

(ii) there exists a continuous family of smooth exact symplectic mappings

$$\chi_s : T^*\partial\Omega_0 \longrightarrow T^*\partial\Omega_s$$

such that

$$\chi_s(\Lambda_0(\omega)) \;=\; \Lambda_s(\omega),\ \text{and}\ \chi_s \circ B_0 = B_s \circ \chi_s\ \text{on}\ \Lambda_0(\omega)$$

for any $\omega \in \Theta_0$ and any $s \in [0,1]$,

(iii) for any $\omega \in \Theta_0$ there exists a continuous family of caustics $[0,1] \longrightarrow C_s(\omega)$ in Ω_s and

$$\ell_s(\omega) = \ell_0(\omega),\ t_s(\omega) = t_s(\omega),\ s \in [0,1].$$

Proof. Take $b > 0$ such that Ω_s are strictly convex for any $s \in [0,b]$. Then $\mathcal{T}_{mn}(s) = T_{mn}(s)$ is continuous in $[0,b]$. Fix $\omega \in \Theta$ and suppose that the pair $(m,n) \in \mathbf{N}$ satisfies (5.5). We are going to prove that

$$T_{mn}(s) \;\in\; \text{sing.supp.} Z_0,\ \forall\ s \in [0,b]. \tag{6.4}$$

Take $s_0 \in [0,b]$ and choose a sequence s_j tending to s_0 such that $T_{mn}(s_j) \not\in \mathcal{L}_{mn}(\Omega_{s_j})$. Theorem 6.1 implies

$$T_{mn}(s_j) \in \text{sing.supp.} Z_{s_j}.$$

On the other hand, $Z_s(t) \equiv Z_0(t)$ for any s in view of (6.3) and we get (6.4) since sing.supp.Z_0 is a closed set and $T_{mn}(s)$ is continuous. Using the Poisson relation (1.8) we obtain

$$T_{mn}(s) \in \mathcal{L}(\Omega_0),\ \forall\ s \in [0,b].$$

Hence,

$$T_{mn}(s) \;=\; T_{mn}(0),\ \forall\ s \in [0,b],$$

and as in Theorem 2.1 we complete the proof of Theorem 6.2. □

Appendix

We are going to prove Theorem 3.1. As in Section 3 we fix $N \in \mathbf{Z}_+$ and $\mu > 0$, and denote

$$\varepsilon(\mu, N) = \varepsilon_N \mu^{2/M},\ \ M = N^2 + 2N,$$

where the positive constants ε_N will be specified later. Fix $0 < C_0 < C < 1,\ C_1 >> 1$ - sufficiently large, set $\Gamma_a = [l_0 - C_1 a^2, l_0 - C_1^{-1} a^2]$, and denote

$$\Theta^0(a,\mu,N) = \Theta(a,\mu,N) \cap (C_0 a, C_0^{-1} a),\ \ 0 < a < \omega_0,$$

where C is fixed in (3.7). The following result is a counterpart of Theorem 1.1, [21]:

Theorem A .1 *Let $[0,b] \ni s \longrightarrow\ B_s \in C^\infty(\mathbf{A},\mathbf{A})$ be a continuous deformation of exact symplectic mappings satisfying the assumptions of Theorem 3.1 for any s in $[0,b]$. Then for any $N \in \mathbf{Z}_+$ there is $\varepsilon_N > 0$, and for any $a \in (0, \varepsilon(\mu,N))$ there exist continuous in $s \in [0,b]$ families of exact symplectic mappings $\chi_{sa} \in C^\infty(\mathbf{T}\times\mathbf{R}, \mathbf{T}\times\mathbf{R})$ and functions $K_{sa} \in C^\infty(\mathbf{R}), Q_{sa} \in C^\infty(\mathbf{T}\times\mathbf{R})$ such that $K_{sa}(I) > 0$ in Γ and:*

(i) the exact symplectic mappings $B^0_{sa} = \chi_{sa}^{-1} \circ B_s \circ \chi_{sa}$ is generated in $\mathbf{A}$ by

$$G^0_{sa}(\varphi, I) = \tau_{sa}(I) + Q^0_{sa}(\varphi, I),\ \ \tau_{sa}(I) = -\frac{4}{3} K_{sa}(I)^{3/2},\ \ (\varphi, I) \in \mathbf{A}, \tag{A.1}$$

and

$$Q^0_{sa}(\varphi, I) = 0 \text{ on } \mathbf{T} \times E_{sa} \tag{A.2}$$

where $E_{sa} = \{I \in \Gamma :\ \tau'_{sa}(I)/2\pi \in \Theta^0(a,\mu,N)\} \subset \Gamma_a$,

(ii) $\chi_{sa} = 0$ outside $\mathbf{T}\times\Gamma_a$, and $K_{sa},\ Q^0_{sa}$, and the generating function $S_{sa}(\theta, I)$ of χ_{sa} satisfy the estimates

$$\begin{aligned} |\ D_I^\alpha(K_{sa}(I) - \zeta_s(I))\ | + |\ D_I^\alpha D_\varphi^\beta Q^0_{sa}(\varphi, I)\ | \ &\le\ C_{\beta N} \mid t_0 - I \mid^{N-\alpha} && \text{(A.3)}\\ |\ D_I^\alpha D_\theta^\beta S^0_{sa}(\theta, I)\ | \ &\le\ C_{\beta N} \mid t_0 - I \mid^{N-\alpha} && \text{(A.4)} \end{aligned}$$

in $\mathbf{T}\times[l_0 - \varepsilon_0, l_0 + \varepsilon_0]$, $\varepsilon_0 > 0$, for $s \in [0,b]$ and any indices $0 \le \alpha \le N,\ \beta \ge 0$, where $C_{\beta N}$ depend neither on a nor on s.

We point out that $K_{sa},\ Q_{sa},\ \chi_{sa}$, as well as the Cantor set E_{sa} in Theorem A.1 depend on the parameters a and s but the constants $C_{\beta N}$ in (A.3) and (A.4) are independent of a and s.

The proof of Theorem A.1 is close to that of Theorem 1.1 in [21] and we are going only to sketch it. It is based on Theorem A, [19] and on an idea of R. Douady [6] (see also Appendix, [21]) to transform the initial problem for symplectic mappings to a similar problem for Hamiltonian systems.

First we write the generating function $G_s(\theta, r)$ of B_s in the form

$$G_s(\theta, r) = -\frac{4}{3}\,\zeta_s^0(r)^{3/2} + R_s(\theta, r), \ (\theta, r) \in \ \mathbf{A}, \tag{A.5}$$

where $\zeta_s^0(r)$ is the Taylor polynomial of $\zeta_s(r)$ at $r = l_0$ up to order $M_1 = N^2 + 4N + 4$ while R_s satisfies

$$\mid \partial_\theta^\alpha \partial_r^\beta R_s(\theta, r) \mid \leq \ C \mid r - l_0 \mid^{M_1 - \beta}, \ (\theta, r) \in \mathbf{A}, \tag{A.6}$$

for any indices $\alpha \geq 0, \beta \geq 0$, and $s \in [0, b]$, and R_s depends continuously on s in $C^\infty(\mathbf{A})$. As in Section 3, we suppose that $R_s(\theta, r) = 0$ for $r \in [l_0 - \varepsilon, l_0 - 2\varepsilon/3]$. Fix $d = a^2$ and set

$$\mathbf{D} = \Gamma \times (-\varepsilon_0, \varepsilon_0), \ \mathbf{D}_d = (l_0 - C_2 d, l_0 - C_2^{-1} d) \times (-C_2 d, C_2 d)$$

where $C_2 > C_1 > 1$. Define $\tilde{\mathbf{D}}_d$ the same way as $\mathbf{D}_d$ with a constant $C_3 > C_2$ and set

$$\tilde{\mathbf{A}} = \mathbf{T}^2 \times \mathbf{D}, \ \tilde{\mathbf{A}}_d = \mathbf{T}^2 \times \tilde{\mathbf{D}}_d.$$

Denote $y = (y_1, y_2) \in \mathbf{T}^2, \ \eta = (\eta_1, \ \eta_2) \in \mathbf{D}, \ y_1 = \theta, \ \eta_1 = r$ and $\tau_s^0(\eta) = -\frac{4}{3}\,\zeta_s^0(\eta_1)^{3/2}$. As in [6], [21], we first construct a Hamiltonian $\tilde{H}_s(y, \eta)$ in $\tilde{\mathbf{A}}$ close to $\tilde{H}_s^0(\eta) = 2\pi\eta_2 + \tau_s^0(\eta_1)$ and such that the Poincaré map corresponding to the Hamiltonian flow $F_s^t(\varrho)$ of $\tilde{H}_s$ on the level surface $\{\tilde{H}_s = 0\}$ coincides with B_s. Set

$$\mathbf{A}' \ = \ \{(y, \eta) \in \tilde{\mathbf{A}} : \ \tilde{H}_s(\eta) = 0, \ y_2 = 0\}$$

and denote by $\imath_s : \mathbf{A} \longrightarrow \mathbf{A}'$ the inclusion map

$$\imath_s(y_1, \eta_1) \ = \ (y_1, 0, \eta_1, -\tau_s^0(\eta_1)/2\pi).$$

Taking into account (A.5) and (A.6) we prove as in [13] and [21]

Lemma A .1 *There exists a continuous in $[0, b]$ family of Hamiltonians $\tilde{H}_s \in C^\infty(\tilde{\mathbf{A}})$ such that*

$$\mid D_y^\alpha D_\eta^\beta (\tilde{H}_s(y, \eta) \ - \ \tilde{H}_s^0(\eta)) \mid \leq \ C_{\alpha\beta} \mid \eta_1 \mid^{M_1 - |\beta|},$$

$$\tilde{H}_s(y, \eta) \ = \ H_s^0(\eta) \textit{ in a neighborhood of } \mathbf{A}' \textit{as well as outside } \tilde{\mathbf{A}}_d,$$

$$B_s \ = \ \imath_s^{-1} \circ F_s^1 \circ \imath_s.$$

Set $H_s^0 = \tilde{H}_s^0 + (\tilde{H}_s^0)^2, \ H_s' = \tilde{H}_s + (\tilde{H}_s)^2$. Next we apply a KAM theorem to the pair $H_s^0, \ H_s'$, which is a variant of Theorem A, [19], proved by J. Pöschel (see also Theorem 5.4 in [13]). Fix $p > M_1$. As in [19] denote by $\sigma_\gamma(y, \eta)$ the map $(y, \eta) \longrightarrow (y, \gamma\eta)$ and by $\| \cdot \|_{p, \tilde{A}_d}$ the respective Hölder norms of the functions in $\tilde{A}_d$ as well as

$$\| \ f \ \|_{p, \tilde{A}_d; \gamma} \ = \ \| \ f \circ \sigma_\gamma \ \|_{p, \sigma_\gamma^{-1}(\tilde{A}_d)}$$

Denote by $\tilde{\Gamma}$ the intersection of a neighborhood of the point $\{(l_0, 0)\}$ in $\mathbf{C}$ with $\{\eta \in \mathbf{C} : \Re\eta_1 < l_0\}$, and set $d = a^2$ and

$$\mathbf{D}_d + \varrho \;=\; \{z \in \tilde{\Gamma} : \mid z - \eta \mid \leq \varrho \text{ for some } \eta \in \tilde{\mathbf{D}}_d\}$$

Next fix $\sigma > 1,\ \mu > 0$, as in Section 3, set $\gamma_d = \mu d^{N/2} = \mu a^N$, and consider the Cantor set

$$\Theta^d = \{\omega \in \mathbf{R}^2 : |< \omega, k >| \geq \ \gamma_d \mid k \mid^{-\sigma} \ \text{ for any } k = (k_1, k_2) \in \ \mathbf{Z}^2 \backslash \{0\}\}.$$

Fix $s_0 \in [0, b]$. The following KAM theorem is a variant of Theorem A in [19].

Theorem A .2 *Let $[0, b] \ni s \longrightarrow H_s^0(\eta)$ be a continuous family of analytic functions in $\tilde{\Gamma}$ such that*

$$\| \partial H_s^0/\partial\eta^2 \|_{\mathbf{D}_{d+\varrho}} \ ,\ \| ((\partial H_s^0/\partial\eta^2)^{-1} \|_{\mathbf{D}_{d+\varrho}} \ \leq \ C d^{-1/2},\ d \in (0, d_0), \tag{A.7}$$

where $\varrho = cd$, $0 < c < 1$, $C > 0$, and assume the map $\partial H_s'/\partial\eta : \ \tilde{\Gamma} \to \mathbf{C}^2$ to be invertible.

For any fixed $\lambda > \sigma + 1 > 2$, and $\alpha > 1,\ \alpha \not\in \Lambda = \{i/\lambda + j : \ i, j \geq 0 \text{ integer}\}$, there is a positive ε independent of d and s such that if $H_s' \in C^\infty(\tilde{\mathbf{A}})$ is continuous in $[0, b]$, and

$$\| H_s' - H_s^0 \|_{p, \tilde{\mathbf{A}}_d; \gamma_d} \ \leq \ \gamma_d^2 d^2 \varepsilon,\ p = \alpha\lambda + \lambda + \sigma, \tag{A.8}$$

then:

(i) for any $d \in (0, d_0)$ there is a neighborhood $U(s_0)$ of s_0 and continuous with respect to $s \in U(s_0)$ families of functions $S_{sd} \in C^\infty(\tilde{\mathbf{A}}),\ H_{sd} \in C^\infty(\mathbf{R}^2)$ such that $\tilde{S}_{sd} = 0$ outside $\tilde{\mathbf{A}}_d$ and

$$\partial_\xi^q (H_s'(y, \xi - \nabla_y \tilde{S}_{sd}(y, \xi)) \ - \ H_{sd}(\xi)) \ = \ 0\ ,\ 0 \leq q \leq 1,$$

on $\mathbf{T}^2 \times \tilde{E}_{sd}$ where

$$\tilde{E}_{sd} \ = \ \{\xi \in \mathbf{D}_d : \ \nabla_\xi H_{sd}(\xi)/2\pi \in \Theta^d\},$$

(ii) for any $\beta \geq \alpha$,

$$\| \tilde{S}_{sd} \|_{\tilde{\beta}, \tilde{\mathbf{A}}_d; \gamma_d} \ \leq \ C_\beta \gamma_d^{-1} d^{-(\beta+1)/2} \| H_s' - H_s^0 \|_{\tilde{p}}$$

where $\tilde{p} = \beta\lambda + \lambda + \sigma$ and $\tilde{\beta} = \beta - (\lambda - \sigma)/\lambda$ is not in Λ.

The proof of Theorem A.2 is similar to that of Theorem A in [19] (see also Theorem 5.4 and the Appendix in [13]) and we omit it. The continuity of $\tilde{S}_{sd}$ and H_{sd} with respect to $s \in U(s_0)$ in the corresponding C^∞ spaces follows from the arguments in A.2, Appendix, in [21].

Consider the pair $H_s^0,\ H_s'$ defined above. Obviously, H_s^0 satisfies (A.7) if c is sufficiently small. Fix $\lambda > \sigma + 1$ and $\beta > 0$ such that $\tilde{\beta} = \beta - (\lambda - \sigma)/\lambda$ is not in Λ and $2N - 1 < \tilde{\beta} \leq 2N$. Set $p = \beta\lambda + \lambda + \sigma$. Lemma A.2 implies

$$\| H_s' - H_s^0 \|_{p, \tilde{\mathbf{A}}_d; \gamma_d} \ \leq \ C_N a^{M_1} \ \leq \ \gamma_d^2 d^2 \varepsilon,\ M_1 = N^2 + 4N + 4,$$

if $d = a^2,\ a \leq \varepsilon_N \mu^{2/M},\ M = N^2 + 2N$, and $\varepsilon_N = (\varepsilon/C_N)^{1/M}$ is sufficiently small. Hence, we can apply Theorem A.2. The corresponding function $\tilde{S}_{sd}$ satisfies the estimate

$$\| \tilde{S}_{sd} \|_{\tilde{\beta},\tilde{\mathbf{A}}_d;\gamma_d} \ \leq \ C\gamma_d^N$$

which implies

$$| \partial_\theta^p \partial_\xi^q \tilde{S}_{sd}(\theta,\xi) | \leq \ C_N a^N \ , \ | q | \leq N. \tag{A.9}$$

Moreover, $\tilde{S}_{sd} = 0$ outside a neighborhood of $\mathbf{T} \times \mathbf{D}_d$ in $\tilde{\mathbf{A}}_d$ and it generates an exact symplectic transformation if d is sufficiently small. As in Appendix, [21] we complete the proof of Theorem A.1

We are going to patch together the exact symplectic mappings χ_{sa}. Fix $N_j \in \mathbf{Z}_+$ and $a_j \in (0, \varepsilon(\mu, M_j)),\ M_j = N_j^2 + 2N_j,\ j = 1,2$, and consider the corresponding functions K_{sj} and S_{sj} given by Theorem A.1. Let χ_{sj} be the exact symplectic mapping with a generating function S_{sj}. Denote by $\mathcal{J}_{sj}(\omega)$ the inverse to the frequency mapping $\Gamma \ni I \to \tau'_{sj}(I)/2\pi$.

Proposition A .1 *We have*

$$\mathcal{J}_{s1}(\omega) \ = \ \mathcal{J}_{s2}(\omega),\ K_{s1}(\mathcal{J}_{s1}(\omega)) \ = \ K_{s2}(\mathcal{J}_{s1}(\omega)) \tag{A.10}$$

$$\chi_{s1}(\theta, \mathcal{J}_{s1}(\omega)) \ = \ \chi_{s2}(\theta + r(\mathcal{J}_{s1}(\omega)),\, \mathcal{J}_{s1}(\omega)), \tag{A.11}$$

for any $\omega \in \Theta^{1,2} := \Theta^0(a_1, N_1) \cap \Theta^0(a_2, N_2)$ *where* $r_{sj}(I) = S_{s1}(0,I) - S_{s2}(0,I)$.

Proof. The proof is close to that of Proposition A.5 in [21]. To simplify the notations we drop the index s. Suppose that $\Theta^{1,2}$ is not empty. Set $B_j = \chi_j^{-1} \circ B \circ \chi_j$ and consider $\psi = \chi_1^{-1} \circ \chi_2$. Then $B_2 = \psi^{-1} \circ B_1 \circ \psi$ and (4.2) implies

$$B_j^* \sigma_0 \ - \ \sigma_0 \ = \ df_j,\ \sigma_0 = I d\varphi,\ j = 1,2, \tag{A.12}$$

where

$$f_j(\varphi, \mathcal{J}_j(\omega)) \ = \ 2\pi \mathcal{I}_j(\omega),\ \omega \in \Theta^0(a_j, N_j),\ j = 1,2, \tag{A.13}$$

according to (4.3). On the other hand,

$$\psi^* \sigma_0 \ - \ \sigma_0 \ = \ d\phi \tag{A.14}$$

where ϕ is a smooth function in $\mathbf{A}$. As in the proof of (4.6) we deduce from (A.12) and (A.14) that

$$f_2(\varphi, I) \ = \ f_1(\psi(\varphi, I)) + \phi(\varphi, I) - \phi(B_2(\varphi, I)) + C,\ (\varphi, I) \in \mathbf{A}, \tag{A.15}$$

where C is a constant. As the invariant circles $\Lambda(\omega)$ are uniquely determined by their rotation numbers $\omega \in \Theta$, we have

$$\psi(\mathbf{T} \times \mathcal{J}_2(\omega)) \ = \ \mathbf{T} \times \mathcal{J}_1(\omega),\ \omega \in \Theta^{1,2}. \tag{A.16}$$

Using (A.13) and (A.16) we obtain

$$2\pi \mathcal{J}_2(\omega) \ = \ 2\pi \mathcal{J}_1(\omega) + \phi(\varphi, \mathcal{J}_2(\omega)) - \phi(\varphi + \omega, \mathcal{J}_2(\omega)) + C,\ (\varphi, \omega) \in \mathbf{T} \times \Theta^{1,2},$$

which implies

$$2\pi \mathcal{J}_2(\omega) \;=\; 2\pi \mathcal{J}_1(\omega) + C,\;\; \omega \in \Theta^{1,2}.$$

Differentiating the last equality with respect to ω we prove (A.10).

According to (A.10) and (A.16) we have

$$\psi(\mathbf{T} \times \mathcal{J}_2(\omega)) \;=\; \mathbf{T} \times \mathcal{J}_2(\omega),\;\; \omega \in \Theta^{1,2},$$

and we obtain

$$\nabla_\theta S_1(\theta, \mathcal{J}_2(\omega)) \;=\; \nabla_\theta S_2(\theta, \mathcal{J}_2(\omega)),\;\; (\theta, \omega) \in \mathbf{T} \times \Theta^{1,2}.$$

This proves (A.11). □

Proof of Theorem 3.1. First we fix N in $\mathbf{Z}_+$. Take $a_j = C_0^j,\; \gamma_j = \mu_0 a_j^M,\; M = N^2 + 2N,\; \mu_0 < \mu$, and denote by $K_{sj}(I)$ and $S_{sj}(\theta, I)$ the corresponding functions given by Theorem A.1 for $a = a_j,\; j = 1, 2, \ldots$. Replacing $S_{sj}(\theta, I)$ by $S_{sj}(\theta, I) - S_{sj}(0, I)$ we can suppose that $S_{sj}(0, I) = 0$. Using Proposition A.1, (A.3) and (A.4), and applying a suitable Whitney extension theorem we find smooth functions $\mathcal{J}_s^N(\omega),\; K_s^N(I)$ and $S_s^N(\theta, I)$ in Γ and $\mathbf{T} \times \Gamma$ respectively such that

$$\mathcal{J}_s^N(\omega) \;=\; \mathcal{J}_{sj}(\omega),\;\; K_s^N(\mathcal{J}_s^N(\omega)) \;=\; K_{sj}(\mathcal{J}_s^N(\omega))$$

$$S_s^N(\theta, \mathcal{J}_s^N(\omega)) \;=\; S_{sj}(\theta,\, \mathcal{J}_s^N(\omega)),$$

for any $(\theta, \omega) \in \mathbf{A}^j \equiv \mathbf{T} \times (\Theta^0(a_j, \mu_0, N) \cap [a_{j+1}, a_j)]$. Denote by χ_s^N the exact symplectic mapping generated by S_s^N and by (φ, I) the corresponding symplectic coordinates in $\mathbf{T}\times\mathbf{R}$. Then $B_s^0 = (\chi_s^N)^{-1} \circ B_s \circ \chi_s^N$ is generated by $\tau_s^N(I) + Q_s^N(\varphi, I),\; \tau_s^N(I) = -\frac{4}{3} K_s^N(I)^{3/2}$ where $Q_s^N(\varphi, I)$ is a smooth function and $Q_s^N(\varphi, \mathcal{J}_s^N(\omega))$ has a zero of infinite order on each $\mathbf{A}^j,\; j = 1, 2, \ldots$. Fix $\mu_0 = \mu C_0^N$ and take $a \in [a_{j+1}, a_j],\; d = a^2$. Then $\gamma_j < \gamma_{j-1} < \gamma_d$ and the Cantor set $\Theta^*(a, \mu, N)$ is contained in the union of the sets $\mathbf{A}^p,\; p = j-1, j$. Hence, $Q^0(\varphi, \mathcal{J}_s^N(\omega))$ has a zero of infinite order on

$$\Theta_N = \cup\{\Theta^*(a, \mu, N) :\; 0 < a \leq \varepsilon(\mu, N)\}.$$

Moreover, $K_s^N,\; Q_s^N$ and S_s^N satisfy (A.3) and (A.4). Take integers $N_2 > N_1 \geq 1$ pick a in $(0, \varepsilon(\mu, N_2)]$ and set $\gamma_j = \mu a^{N_j}$. Then $\gamma_2 < \gamma_1$ and we obtain

$$\Theta_{N_1} \cap (0, \varepsilon(\mu, N_2)] \subset \Theta_{N_2}.$$

Using Proposition A.1 as well as (A.3) and (A.4) we obtain smooth functions (in the sense of Whitney) $\mathcal{J}_s(\omega),\; K_s(\mathcal{J}_s(\omega))$ and $S_s(\theta, \mathcal{J}_s(\omega))$ in Θ and $\mathbf{T} \times \Theta$ respectively such that

$$\mathcal{J}_s(\omega) \;=\; \mathcal{J}_s^N(\omega),\;\; K_s(\mathcal{J}_s(\omega)) \;=\; K_s^N(\mathcal{J}_s(\omega))$$

$$S_s(\theta, \mathcal{J}_s(\omega)) \;=\; S_s^N(\theta,\, \mathcal{J}_s(\omega)),$$

for any $(\theta, \omega) \in \mathbf{T}\times\Theta$. Denote by $\tilde{S}_s(\theta, \omega)$ and $\tilde{\mathcal{J}}_s(\omega)$ suitable smooth Whitney extensions of $\tilde{S}_s(\theta, \mathcal{J}_s(\omega)),\; (\theta, \omega) \in \mathbf{T} \times \Theta$ and $\mathcal{J}_s(\omega),\; \omega \in \Theta$, which depend continuously on s. Let

$\tilde{K}_s'(I)$ be the function inverse to $\omega \to \mathcal{J}_s(\omega)$. Taking χ_s to be the exact symplectic mapping generated by $S_s(\theta, I) = \tilde{S}_s(\theta, \tilde{K}_s'(I))$ and using (A.3) and (A.4) we complete the proof of Theorem 3.1. $\Box$

Acknowledgments. This paper has been written under the support of the Alexander von Humboldt foundation. The main part of it was prepared during my stay in TH Darmstadt and I would like to thank Hans-Dieter Alber for the hospitality and the stimulating discussions. I would like to thank Jürgen Pöschel for the helpful discussions about KAM theory.

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