Sojourn times of trapping rays and the behavior of the modified resolvent of the Laplacian

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1 Introduction

Let $\Omega \subset \mathbf{R}^n, n \geq 3, n$ odd, be an open and connected domain with C^∞ boundary $\partial\Omega$ and bounded complement

$$K = \overline{\mathbf{R}^n \setminus \Omega} \subset \{x \in \mathbf{R}^n : \mid x \mid \leq \rho_0\}.$$

Consider the problem

$$\begin{cases} (\partial_t^2 - \Delta_x)u = 0 \ \text{ in } \ \mathbf{R} \times \Omega, \\ u = 0 \ \text{ on } \ \mathbf{R} \times \partial\Omega, \\ u(0,x) = f_1(x), \partial_t u(0,x) = f_2(x). \end{cases} \tag{1}$$

We can associate to (1) a *scattering operator*

$$S(\lambda) : L^2(\mathbf{R}^{n-1}) \longrightarrow L^2(\mathbf{R}^{n-1}), \ \ \lambda \in \mathbf{R},$$

whose kernel $a(\lambda, \theta, \omega)$, called *scattering amplitude*, depends analytically on $\omega, \theta \in S^{n-1}$ (see [LP]). For fixed $(\theta, \omega) \in S^{n-1} \times S^{n-1}$, $a(\lambda, \theta, \omega)$ is a tempered distribution in λ and

$$a(\lambda, \theta, \omega) = (\frac{2\pi}{i\lambda})^{(n-1)/2} \mathcal{F}_{t \to \lambda} s(t, \theta, \omega).$$

Here $\mathcal{F}_{t\to\lambda}$ denotes the *Fourier transform* and the distribution $s(t, \theta, \omega)$ is called the *scattering kernel* (see [Ma], [P1]).

The operator $S(\lambda)$ and the distribution $a(\lambda, \theta, \omega)$ admit meromorphic continuation in $\mathbf{C}$ with poles λ_j, Im $\lambda_j < 0$, which are independent of θ and ω. Notice that in [LP] the scattering poles are related to $\overline{a(\lambda, \theta, \omega)}$.

One can characterize the poles λ_j using the modified resolvent of the Laplacian in Ω given by

$$R_{\varphi,\psi}(\lambda) = \varphi(x) R(\lambda) \psi(x).$$

Here the operator

$$R(\lambda) : C_0^\infty(\overline{\Omega}) \ni f \mapsto u(x,\lambda) \in C^\infty(\overline{\Omega}),\ \mathrm{Im}\lambda \geq 0,$$

is determined by the $(-i\lambda)$-outgoing solution of the Dirichlet problem for the reduced wave equation (see Section 2). The functions $\varphi(x), \psi(x) \in C_0^\infty(\mathbf{R}^n)$ are chosen to be equal to 1 in some neighbourhood of K. Then $R_{\varphi,\psi}(\lambda)$ admits a meromorphic continuation in $\mathbf{C}$ the poles of which and their multiplicities coincide with these of the λ_j's. Moreover, the poles λ_j do not depend on the choice of φ and ψ (see [LP], [V]).

It is well known that if K is non-trapping, there exist $\epsilon > 0$ and $d > 0$ such that $S(\lambda)$ has no poles in the domain

$$U_{\epsilon,d} = \{\lambda \in \mathbf{C} : d - \epsilon \mathrm{Log}(1+ \mid \lambda \mid) \leq \ \mathrm{Im}\ \lambda \leq 0\,\}. \tag{2}$$

On the other hand, Vainberg [V] proved that for non-trapping obstacles the estimate

$$\|R_{\varphi,\psi}(\lambda) f\|_{H^1(\Omega)} \leq C e^{\alpha|\mathrm{Im}\lambda|} \|f\|_{L^2(\Omega)} \tag{3}$$

holds for each $\lambda \in U_{\epsilon,d}$. Here the constants $C > 0, \alpha > 0$ depend on supp $\varphi \cup$ supp ψ.

The situation changes when the obstacle is trapping. In this case one expects to have an infinite number of scattering poles in $U_{\epsilon,d}$ for all $\epsilon > 0$ and all $d > 0$. Such a result is proved in [BGR] provided the obstacle is a union of two disjoint strictly convex bodies. In the same case more precise results concerning the distribution of poles are obtained by Ikawa [I1] and Gérard [G].

The case $K = \bigcup_{j=1}^N K_j, K_i \cap\ K_j = \emptyset,\ i \neq j,\ K_j$ strictly convex, has been examined by Ikawa [I2], [I3] under the condition

$$\text{(H)} \qquad \begin{cases} K_\nu \cap \text{ convex hull } (K_i \cup K_j) = \emptyset \text{ for each} \\ \text{triple } (i,j,\nu) \in \{1,\dots,N\}^3, i \neq j, i \neq \nu, j \neq \nu. \end{cases}$$

This condition implies the non-existence of periodic rays in Ω with segments tangent to $\partial\Omega$. The case when the obstacles K_j are in generic position has been studied in [PS1].

To describe the results in this paper we need several definitions. Given an obstacle K, fix an open ball B_0 of radius ρ_0 containing K. For each ξ we shall denote by Z_ξ the hyperplane tangent to B_0 and orthogonal to ξ such that the halfspace H_ξ, determined by Z_ξ and having ξ as an inner normal, contains K. Let $\omega \in S^{n-1}, \theta \in S^{n-1}$ and let γ be an (ω,θ)-ray in Ω, that is an infinite ray in Ω with incoming direction ω and outgoing direction θ (see [PS3] for the precise definition). The *sojourn time* T_γ of γ is defined by $T_\gamma = T'_\gamma - 2\rho_0$, where T'_γ is the length of this part of γ which is contained in $H_\omega \cap H_{-\theta}$. If γ has no gliding segments on $\partial\Omega$ and only finitely many reflection points, then γ is called a *reflecting* (ω,θ)*-ray* in Ω. If moreover γ has no segments tangent to ∂K, then it is called an *ordinary* reflecting (ω,θ)-ray.

Let γ be an ordinary reflecting (ω, θ)-ray in Ω with successive reflection points $x_1, \ldots, x_k$ on ∂K. Note that in this case

$$T_\gamma = \langle \omega, x_1 \rangle + \sum_{i=1}^{k-1} \|x_i - x_{i+1}\| - \langle \theta, x_k \rangle$$

where $< , >$ denotes the standard inner product in $\mathbf{R}^n$. (see Section 2.4 in [PS3]). Denote by u_γ the orthogonal projection of x_1 on $Z = Z_\omega$. Then there exists a neighbourhood $W = W_\gamma$ of u_γ in Z such that for every $u \in W$ there are unique $\theta(u) \in S^{n-1}$ and points $x_1(u), \ldots, x_k(u) \in \partial K$ which are the successive reflection points of a reflecting $(\omega, \theta(u))$-ray in Ω passing through u. Setting $J_\gamma(u) = \theta(u)$, one obtains a smooth map

$$J_\gamma : W_\gamma \longrightarrow S^{n-1}.$$

The ray γ is called *non-degenerate* if $\det dJ_\gamma(u_\gamma) \neq 0$.

The aim of this paper is to establish a link between the existence of a sequence $\{-T_m\}$ of singularities of $s(t, \theta_m, \omega_m)$ and the estimate

$$\|R_{\varphi,\psi}(\lambda) f\|_{H^1(\Omega)} \leq C e^{\alpha |\mathrm{Im}\lambda|} (1 + |\lambda|)^p \|f\|_{H^k(\Omega)}, \tag{4}$$

for $\lambda \in U_{\epsilon,d}$ and some constants $C > 0, \alpha \geq 0$, $p \in \mathbf{N}, k \in \mathbf{N}$. It is proved in Theorem 2..3 below that if there exist ordinary reflecting (ω_m, θ_m)-rays γ_m with sojourn times $T_m \to \infty$ such that

$$-T_m \in \text{sing supp}\, s(t, \theta_m, \omega_m), \tag{5}$$

then either there exist poles in $U_{\epsilon,d}$ for all $\epsilon > 0, d > 0$, or there exist $\epsilon > 0, d > 0$ for which the domain $U_{\epsilon,d}$ is free of poles but (4) fails for every choice of C, α, p and k.

If K is a trapping obstacle, the existence of a sequence $\{\gamma_m\}$ of reflecting (ω_m, θ_m)-rays with $T_{\gamma_m} \to \infty$ follows from the continuity of the generalized Hamiltonian flow related to the wave operator $\Box = \partial_t^2 - \Delta_x$ (see Section 5). The main difficulty is to prove that these rays produce singularities leading to (5). According to the results in [CPS] (see also Chapters 8 and 9 in [PS3]), to obtain (5) it is sufficient that for all m the pair (ω_m, θ_m) has the following three properties:

(a) if δ and γ are different ordinary reflecting (ω_m, θ_m)-rays, then $T_\delta \neq T_\gamma$;

(b) for every m, γ_m is non-degenerate;

(c) there are no (ω_m, θ_m)-rays in Ω containing gliding segments on $\partial\Omega$ or tangent segments to $\partial\Omega$.

In Section 4 we show that given an arbitrary obstacle K, for almost all $(\omega, \theta) \in S^{n-1} \times S^{n-1}$ (with respect to the Lebesgue measure in $S^{n-1} \times S^{n-1}$) every two different ordinary reflecting (ω, θ)-rays have distinct sojourn times. Using this, the property (a) can be arranged by an apropriate approximation. To obtain an analogue of this for (c), one must study all (ω, θ)-rays in Ω, i.e. all projections of generalized (ω, θ)-bicharacteristics of the wave operator. In general this is a difficult problem, especially when the set $G^\infty(K)$ of points $z \in \partial K$, where the curvature of K along some direction $\xi \in T_z(\partial K)$ vanishes of infinite order, is not empty. In Section 3 we prove that for each obstacle K for almost all $(\omega, \theta) \in S^{n-1} \times S^{n-1}$ all reflecting (ω, θ)-rays in Ω are ordinary. This result can be applied in the case when the obstacle has the form

$$K = \bigcup_{j=1}^{N} K_j \, , K_i \cap K_j = \emptyset \text{ for } i \neq j, \qquad K_j \text{ convex for all } j = 1, \dots, N, \tag{6}$$

to show that (c) is satisfied for almost all $(\omega, \theta) \in S^{n-1} \times S^{n-1}$ even if $G^\infty(K) \neq \emptyset$.

To arrange (b) for obstacles of the form (6) we use the representation of the (linear) Poincaré map (see [PS3], Proposition 2.4.2). On the other hand, using Sard's theorem, it is possible to arrange (b) for a dense set of directions $(\omega, \theta) \in S^{n-1} \times S^{n-1}$. The results in Sections 2-5 are established for obstacles with arbitrary geometry. For several convex obstacles we make two applications.

1.1. Theorem. *Let K have the form* (6). *Assume that there are no points $z \in \partial K$ such that the Gauss curvature $\mathcal{K}(u)$ of ∂K at u vanishes for each u in some neighbourhood U_z of z in ∂K. Then there exists a sequence of ordinary reflecting (ω_m, θ_m)-rays in Ω with sojourn times $T_m \to \infty$ such that for t near $-T_m$ we have*

$$s(t, \theta_m, \omega_m) = A_m \delta^{(n-1)/2}(t + T_m) + \text{ lower order singularities,}$$

with $A_m \neq 0$. Moreover, either the assertion (i) *or the assertion* (ii) *in Theorem 2.3 in Section 2 holds.*

1.2. Theorem. *Let K have the form* (6). *Then there exists a set $\mathcal{R} \subset S^{n-1} \times S^{n-1}$, the complement of which has zero Lebesgue measure in $S^{n-1} \times S^{n-1}$, such that for all $(\omega, \theta) \in \mathcal{R}$ we have*

$$\text{sing supp} s(t, \theta, \omega) = \{-T_\gamma : \gamma \in \mathcal{L}_{\omega,\theta}\},$$

where $\mathcal{L}_{\omega,\theta}$ is the set of all (ω, θ)-rays in Ω. Moreover, for t near $-T_\gamma$ we have

$$s(t, \theta, \omega) = c_\gamma \delta^{(n-1)/2}(t + T_\gamma) + \text{ lower order singularities}$$

with $c_\gamma \neq 0$.

For $n = 3$ the assumption of Theorem 1.1 means that there are no points $z \in \partial K$ such that the standard metric on ∂K is locally flat around z. For strictly convex obstacles the assertion of Theorem 1.2 was proved in [PS2].

In the works of Ikawa [I2], [I3] and Sjöstrand and Zworski [SjZ] the singularities of the distribution

$$u(t) = \begin{cases} \sum_j e^{i\lambda_j t}\ , & t > 0, \\ \sum_j e^{-i\lambda_j t}\ , & t < 0, \end{cases}$$

have been exploited. Here the sumation is over all poles λ_j including their multiplicities.

It is well-known that sing supp$u(t)$ is contained in the set of the periods (lengths) of all periodic rays in Ω (see [PS3], Chapter 5). Thus, to prove that a period T_γ belongs to sing supp $u(t)$, it is sufficient to establish some properties $(a')-(c')$ similar to $(a)-(c)$. The reader may consult [PS3] for results in this direction. Notice that in general, even for several strictly convex obstacles, there might be different periodic rays with rationally dependent periods or periodic reflecting rays with segments tangent to the boundary. Thus, the singularities produced by periodic rays could be canceled and it seems difficult to prove that there exists a sequence $d_j \to \infty$ of singularities of $u(t)$.The condition (H) in [I2], [I3] has been used to avoid the existence of periodic rays with tangent segments. Notice also that the results in [SjZ] can be applied for all logarithmic domains if a sequence d_j of singularities going to infinity exists.

In this paper we study trapping (ω, θ)-rays with suitable directions ω, θ instead of periodic rays. This makes it possible to examine the general case of several convex obstacles and to find a sequence $T_m \to \infty$ of singularities of the scattering kernel. One could expect that such sequence of singularities exists for every trapping obstacle K with $G^\infty(K) = \emptyset$ and the conclusion of Theorem 2.3 holds for trapping obstacles of this type.

It is an open problem if the existence of a sequence $T_m \to \infty$ of singularities always implies the assertion (i) of Theorem 2.3. From general point of view there might exist some very degenerate examples of trapping obstacles for which there are logarithmic domains free of poles and the assertion (ii) of Theorem 2.3 holds.

2 The behavior of the resolvent of the Laplacian

Consider the problem

$$\begin{cases} (\Delta + \lambda^2) u(x, \lambda) = f \ \text{ in } \ \Omega \\ u(x, \lambda) = 0 \ \text{ on } \ \partial\Omega \\ u(x, \lambda) \ \text{ is } \ (-i\lambda)\text{-outgoing.} \end{cases} \tag{7}$$

The last condition means that for $x = \mid x \mid \omega, \omega \in S^{n-1}$, and $\mid x \mid \to \infty$ we have

$$u(\mid x \mid \omega, \lambda) = \frac{e^{i\lambda |x|}}{\mid x \mid^{(n-1)/2}} (b(\omega, \lambda) + \mathrm{O}(\frac{1}{\mid x \mid})).$$

For Im $\lambda \geq 0$ the problem (7) has an unique solution for $f \in L^2(\Omega)$ and the operator

$$R(\lambda) : C_0^\infty(\overline{\Omega}) \ni f \mapsto u(x,\lambda) \in C^\infty(\overline{\Omega}) \ , \mathrm{Im}\lambda \geq 0,$$

admits a meromorphic extension in $\mathbf{C}$ with poles λ_j, Im $\lambda_j < 0$ (see [LP]).

Let $\hat{s}(\lambda,\theta,\omega) = \mathcal{F}_{t\to\lambda} s(t,\theta,\omega)$ be the *Fourier transform* of the scattering kernel $s(t,\theta,\omega)$. It is well known ([LP], [P1], [Ma]) that $\hat{s}(\lambda,\theta,\omega)$ has the representation

$$\hat{s}(\lambda,\theta,\omega) = c_n \lambda^{n-2} \int_{\partial\Omega} e^{-i\lambda\langle x,\theta\rangle} (\frac{\partial v}{\partial \nu} + i\lambda \langle \nu,\theta\rangle v)(x,\lambda) dS_x,$$

where $c_n =$ const, $\nu(x)$ is the *unit normal* to $\partial\Omega$ at $x \in \partial\Omega$ pointing into Ω, and $v(x,\lambda)$ is the solution of the problem

$$\begin{cases} (\Delta + \lambda^2) v(x,\lambda) = 0 \ \text{ in } \ \Omega \\ v + e^{i\lambda\langle x,\omega\rangle} = 0 \ \text{ on } \ \partial\Omega \\ v \ \text{ is } \ (-i\lambda)-\text{outgoing.} \end{cases}$$

We shall express $\hat{s}(\lambda,\theta,\omega)$ by using the operator $R(\lambda)$. Let $a > \rho_0$. Consider a function $\varphi_a \in C_0^\infty(\mathbf{R}^n)$ such that

$$\varphi_a(x) = 1 \ \text{ for } \ | x | \leq a.$$

Setting

$$F_a(\lambda) = [\Delta\varphi_a + 2i\lambda\langle\nabla\varphi_a,\omega\rangle] e^{i\lambda\langle x,\omega\rangle},$$

we get

$$v(x,\lambda) + \varphi_a(x) e^{i\lambda\langle x,\omega\rangle} = R(\lambda)[(\Delta+\lambda^2)(\varphi_a e^{i\lambda\langle x,\omega\rangle})] = R(\lambda) F_a(\lambda).$$

Next, choose a function $\chi_b(x) \in C_0^\infty(\mathbf{R}^n)$ such that $\chi_b(x) = 1$ on a neighbourhood of K and $\varphi_a(x) = 1$ on supp χ_b. Then the normal derivative becomes

$$\frac{\partial v}{\partial \nu}_{|\partial\Omega} = -i\lambda e^{i\lambda\langle x,\omega\rangle} \langle \nu,\omega\rangle_{|\partial\Omega} + \frac{\partial}{\partial\nu}(\chi_b R(\lambda) F_a(\lambda))_{|\partial\Omega}.$$

On the other hand, for $\theta \neq -\omega$ by Green's formula we have

$$\int_{\partial\Omega} e^{-i\lambda\langle x,\theta-\omega\rangle} \langle \nu,\theta+\omega\rangle dS_x = \int_K \frac{\partial}{\partial(\theta+\omega)} (e^{-i\lambda\langle x,\theta-\omega\rangle}) dx = 0.$$

Thus, using Green's formula once more, we obtain

$$\begin{aligned} \hat{s}(\lambda,\theta,\omega) &= -c_n\lambda^{n-2} \int_\Omega e^{-i\lambda\langle x,\theta\rangle} (\Delta+\lambda^2)(\chi_b R(\lambda) F_a) dx \\ &= -c_n\lambda^{n-2} \int_\Omega e^{-i\lambda\langle x,\theta\rangle} [(\Delta\chi_b) R(\lambda) F_a(\lambda) + 2\langle \nabla_x \chi_b, \nabla_x (R(\lambda)F_a(\lambda)\rangle] dx. \quad (8) \end{aligned}$$

Let $\psi_c(x) \in C_0^\infty(\mathbf{R}^n), c = a, b$, be cut-off functions such that

$$\psi_a(x) = 1 \text{ on supp}\varphi_a \ , \ \psi_b(x) = 1 \quad \text{on supp}\chi_b.$$

Then in the representation of $\hat{s}(\lambda, \theta, \omega)$ we can replace the resolvent $R(\lambda)$ by the *modified resolvent*

$$R_{a,b}(\lambda) = \psi_b(x) R(\lambda) \psi_a(x).$$

Below we assume that ψ_a and ψ_b are fixed.

Next, consider the domain $U_{\epsilon,d}$ defined in the introduction and assume that $R_{a,b}(\lambda)$ satisfies the condition

$$\text{(B)} \qquad \begin{cases} R_{a,b}(\lambda) \text{ is analytic in } U_{\epsilon,d} \text{ and there exist } C > 0, \alpha \geq 0, p \in \mathbf{N}, k \in \mathbf{N} \\ \text{such that } \|R_{a,b}(\lambda)\varphi\|_{H^1(\Omega)} \leq C(1+ \mid \lambda \mid)^p e^{\alpha|\mathrm{Im}\lambda|} \|\varphi\|_{H^k(\Omega)} \\ \text{for each } \varphi \in C_0^\infty(\Omega) \text{ and each } \lambda \in U_{\epsilon,d}. \end{cases}$$

Since the integration in (8) is over a compact set, we conclude that the condition (B) implies the estimate

$$\mid \hat{s}(\lambda, \theta, \omega) \mid \leq C_1(1+ \mid \lambda \mid)^m e^{\beta|\mathrm{Im}\lambda|} \ , \ \lambda \in U_{\epsilon,d}, \tag{9}$$

with $\beta \geq 0, m \in \mathbf{N}$, uniformly with respect to $(\theta, \omega) \in S^{n-1} \times S^{n-1}$.

2.1. Remark. The analysis in Chapter III of [LP] shows that the norm of the *scattering operator*

$$S(\lambda) : L^2(S^{n-1}) \longrightarrow L^2(S^{n-1}) \ , \ \lambda \in U_{\epsilon,d},$$

is less than the norm of the modified resolvent $\varphi(x)R(\lambda)\psi(x)$, where $\varphi(x), \psi(x) \in C_0^\infty(\overline{\Omega})$ and $\varphi(x) = 1$, $\psi(x) = 1$ in a neighbourhood of K. The inequality (9) is sharper since it gives an estimate for the kernel of the scattering operator $S(\lambda)$.

Now we shall deduce from (9) some regularity of $s(t, \theta, \omega)$ for $t \to -\infty$. To do this we need the following lemma, which is similar to Theorem 7.3.8 in [H1].

2.2. Lemma. *Let $u \in \mathcal{S}'(\mathbf{R})$ be a distribution with* supp $u \subset \{t : t \leq \tau\}$. *Assume that the Fourier transform $\hat{u}(\xi)$ admits an analytic continuation in the domain $U_{\epsilon,d}$ such that for all $\zeta \in U_{\epsilon,d}$ we have*

$$\mid \hat{u}(\zeta) \mid \leq C(1+ \mid \zeta \mid)^N e^{\alpha|\mathrm{Im}\zeta|} \ , \ \alpha \geq 0. \tag{10}$$

Then for each $q \in \mathbf{N}$ there exist $t_q < \tau$ and $v_q \in C^q(\mathbf{R})$ such that $u = v_q$ for $t \leq t_q$.

Proof. Choose a function $\varphi \in C_0^\infty(\mathbf{R})$ such that supp $\varphi \subset (-1, 1)$ and $\int_{\mathbf{R}} \varphi(t)dt = 1$. Set $\varphi_\delta(t) = \frac{1}{\delta}\varphi(\frac{t}{\delta})$, $0 < \delta \leq 1$, and consider $u * \varphi_\delta$. Consider the path

$$\Gamma_\epsilon : \mathbf{R} \setminus [-\gamma, \gamma] \ni \xi \mapsto \zeta = \xi + i(d - \epsilon \,\mathrm{Log}\,(1+ \mid \xi \mid) \in U_{\epsilon,d},$$

where $\gamma = \exp(\frac{d}{\epsilon}) - 1$ and d is given in the definition of $U_{\epsilon,d}$. Clearly, (10) implies

$$\mid \hat{u}(\zeta)\hat{\varphi}(\delta\zeta) \mid \leq C_M(1+ \mid \zeta \mid)^N (1+ \mid \delta\zeta \mid)^{-M} e^{(a+\delta)|\mathrm{Im}\zeta|} \ , \ \zeta \in U_{\epsilon,d}.$$

Using the analyticity of $\hat{u}(\zeta)$ in $U_{\epsilon,d}$ and the above estimate for fixed $\delta > 0$, we can write the integral

$$(u * \varphi_\delta)(t) = (2\pi)^{-1} \int_{\mathbf{R}} e^{it\xi} \hat{u}(\xi)\hat{\varphi}(\delta\xi) d\xi$$

as a sum of two integrals

$$(2\pi)^{-1} \int_{\Gamma_\epsilon} e^{it\zeta} \hat{u}(\zeta)\hat{\varphi}(\delta\zeta) d\zeta + (2\pi)^{-1} \int_{|\xi| \leq \gamma} e^{it\zeta} \hat{u}(\xi)\hat{\varphi}(\delta\xi) d\xi.$$

The second integral is over a compact interval, therefore passing to a limit as $\delta \to 0$, we obtain a C^∞ function.

Next, for $\zeta \in \Gamma_\epsilon, 0 < \delta \leq 1$, we have the estimate

$$\mid \hat{u}(\zeta)\hat{\varphi}(\delta\zeta) e^{it\zeta} \mid \leq C(1+ \mid \zeta \mid)^N e^{-(a+1+t)\mathrm{Im}\zeta} \leq C'(1+ \mid \mathrm{Re}\zeta \mid)^{N+\epsilon(a+1+t)},$$

provided $a + 1 + t < 0$. Given $q \in \mathbf{N}$, take

$$t_q = \frac{1}{\epsilon}(-N - n - 1 - q - \epsilon(a+1)).$$

Then $t \leq t_q$ implies $\epsilon(a+1+t) \leq -N-n-1-q$ and, since $d\zeta = F(\xi)d\xi \to d\xi$ as $\mid \xi \mid \to \infty$, for $t \leq t_q$, the integral on Γ_ϵ is uniformly convergent for $0 \leq \delta \leq 1$. The same is true if we take the derivatives with respect to t up to order q. Letting $\delta \to 0$ and using the fact that $\hat{\varphi}(\delta\zeta) \to 1$, by Lebesgue theorem we conclude that for $t \leq t_q$ we have

$$u * \varphi_\delta \longrightarrow_{\delta \to 0} f,$$

f being a C^q function. This completes the proof of the lemma.

Combining the estimate (9) and Lemma 2.2, we obtain the following.

2.3. Theorem. *Assume that there exists a sequence of ordinary reflecting (ω_m, θ_m)-rays in Ω with sojourn times T_m such that*

$$T_m \longrightarrow_{m \to \infty} \infty. \tag{11}$$

Let $\Phi \in C_0^\infty(\mathbf{R})$ be such that $\mathrm{supp}\, \Phi \subset (-1,1)$, *$\Phi(t) = 1$ for $\mid t \mid \leq \frac{1}{2}$. Assume that there exists a sequence $\gamma_m \to 0$ of non-zero real numbers and an integer k independent of m such that*

$$\mid \mathcal{F}_{t \to \lambda}[\Phi(\frac{t+T_m}{\gamma_m}) s(t, \theta_m, \omega_m)] \mid \geq (c_m - o_m(1)) \mid \lambda \mid^k \ , \ \mid \lambda \mid \to \infty, \tag{12}$$

where $c_m > 0$. Then there are two possibilities:

(i) *For each $\epsilon > 0$ and each $d > 0$, $R_{a,b}(\lambda)$ and $S(\lambda)$ have poles in the domain $U_{\epsilon,d}$;*

(ii) *For some $\epsilon > 0, d > 0$, $R_{a,b}(\lambda)$ is analytic in $U_{\epsilon,d}$ but for all $\alpha \geq 0, p \in \mathbf{N}, k \in \mathbf{N}$ we have*

$$\sup_{\lambda \in U_{\epsilon,d}, \|\varphi\|_{H^k(\Omega)}=1} (1+ \mid \lambda \mid)^{-p} e^{-\alpha|\mathrm{Im}\lambda|} \|R_{a,b}(\lambda)\varphi\|_{H^1(\Omega)} = +\infty.$$

2.4. Remark. Notice that if $R_{a,b}(\lambda)$ is analytic in $U_{\epsilon,d}$, the same is true for $\hat{s}(\lambda, \theta, \omega)$ for all $\omega, \theta \in S^{n-1}$, and this implies the analyticity of the

operator $S(\lambda)$ in $U_{\epsilon,d}$. This shows that the cases (i) and (ii) do not depend on the choice of $\psi_a(x)$ and $\psi_b(x)$.

Now we are going to find a link between (ii) and the behavior of the resolvent of the generator B_c of the Lax-Phillips semigroup $Z_c(t)$. Let $H_D(\Omega)$ be the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|\varphi\|_{H_D} = (\int_\Omega \mid \nabla\varphi \mid^2 dx)^{1/2}.$$

In the energy space $H = H_D(\Omega) \oplus L^2(\Omega)$ consider the unitary group $U(t) = e^{tG}$ with generator

$$G = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix},$$

whose domain is

$$D(G) = \{(f_1, f_2) \in H : \ \Delta f_1 \in L^2(\Omega), \nabla f_2 \in H_D(\Omega) \ \}.$$

Let $H_0 = H_D(\mathbf{R}^n) \oplus L^2(\mathbf{R}^n)$ be the energy space for Ω replaced by $\mathbf{R}^n$, and let $U_0(t) = e^{tG_0}$ be the unitary group related to the Cauchy problem for $\partial_t^2 - \Delta_x$ (see [LP]). Consider the spaces

$$D_\pm^c = \{f \in H_0 : U_0(t)f = 0 \ \text{for} \ \ \mid x \mid \leq c \pm t, \pm t \geq 0 \ \},$$

where $c \geq \rho_0$. Denote by $P_\pm^c$ the orthogonal projections on $(D_\pm^c)^\perp$ and by B^c the generator of the semigroup $Z_c(t) = P_+^c U(t) P_-^c$.

Let $\chi \in C_0^\infty(\mathbf{R}^n)$ be a function such that $\chi \equiv 1$ in a neighbourhood of K and $\chi(x) = 1$ on supp $\psi_a(x) \cup$ supp $\psi_b(x)$. Therefore

$$R_{a,b}(\lambda)f = \psi_b \chi R(\lambda) \chi \psi_a f,$$

and an estimate for $R_{a,b}(\lambda)$ can be deduced from that for $\|\chi R(\lambda)\chi\|_{H^1(\Omega)}$, which in turn can be estimated by $\|\chi(G + i\lambda)^{-1}\chi\|_H$. Moreover, if supp $\chi \subset \{x :\mid x \mid \leq c\}$, we have

$$(\chi(G + i\lambda)^{-1}\chi f, g)_H = ((B^c + i\lambda)^{-1}\chi f, \chi g)_H,$$

where (,) denotes the inner product in H. Thus an estimate

$$\|(B^c-\mu)^{-1}\|_H \leq Ce^{\alpha|\mathrm{Re}\mu|}(1+\mid\mu\mid)^p \tag{13}$$

in the domain $W_{\epsilon,d} = \{\mu \in \mathbf{C} : d - \epsilon \mathrm{Log}(1+ \mid \mu \mid) < \mathrm{Re}\mu \leq 0\}$ leads to an estimate for $R_{a,b}(\lambda)$ in $U_{\epsilon,d}$, which is a contradiction with (ii).

The existence of ordinary reflecting trapping rays implies that for some $t_0 > 0$ the spectral radius of the semigroup $Z^c(t_0)$ is 1 (see [R]). Therefore, there exists $\alpha \in \mathbf{R}$ such that

$$e^{it_0\alpha} \in \sigma(Z^c(t_0)),$$

$\sigma(Z^c(t_0))$ being the spectrum of $Z^c(t_0)$. Consequently, the result of I.Herbst [H] shows that we must have

$$\sup_{\mathrm{Re}\mu=0} \|(B^c-\mu)^{-1}\|_H = +\infty.$$

The estimate (13) concerns the case when $\mu \in \mathbf{C}$ runs over a logarithmic domain, and the assumptions in Theorem 2.3 imply that even a polynomial bound of $\sup_{\mu\in W_{\epsilon,d}} \|(B^c-\mu)^{-1}\|_H$ is impossible.

3 Directions of tangency

Let X be a compact smooth $(n-1)$-dimensional submanifold of $\mathbf{R}^n, n \geq 2$. In this section we consider pairs $(\omega, \theta) \in S^{n-1} \times S^{n-1}$ for which there exists at least one (ω, θ)-ray for X which is not ordinary, i.e. it is tangent to X at some of its points. It is shown that the set of these pairs has Lebesgue measure zero in $S^{n-1} \times S^{n-1}$.

Before proceeding with the statement of the main result in this section, we introduce a notion that is slightly different from the notion of an (ω, θ)-ray but it is rather convenient for our next considerations. It was also used in our paper [CPS].

A curve γ in $\mathbf{R}^n$ is called an *(ω, θ)-trajectory for X* if it has the form

$$\gamma = \bigcup_{i=0}^{s} l_i,$$

where $l_i = [x_i, x_{i+1}], i = 1, \ldots, s-1, x_i \in X$ for all $i = 1, \ldots, s$, while l_0 (resp. l_s) is the infinite ray starting at x_1 (resp. x_s) with direction $-\omega$ (resp. θ), and for every $i = 0, 1, \ldots, s-1$, l_i and l_{i+1} satisfy the law of reflection at x_i with respect to X.

Clearly, every reflecting (ω, θ)-ray is a (ω, θ)-trajectory, but the converse is not true in general, since an (ω, θ)-trajectory may intersect transversally X.

3.1. Theorem. *There exists $\mathcal{R} \subset S^{n-1} \times S^{n-1}$ the complement of which is a countable union of compact subsets of measure zero in $S^{n-1} \times S^{n-1}$ such that for every pair $(\omega, \theta) \in \mathcal{R}$*

all (ω,θ)-trajectories for X are ordinary.

Fix a hyperplane Z in $\mathbf{R}^n$ such that X is contained in one of the open halfspaces determined by Z.

As a simple corollary of our argument in this section we get also the following theorem, which is in fact a consequence of a result of Melrose and Sjöstrand [MS], see also Chapter 24 in [H2].

3.2. Theorem. *There exists $\mathcal{T} \subset Z \times S^{n-1}$, the complement of which is a countable union of compact subsets of measure zero in $Z \times S^{n-1}$, such that for every $(x,\omega) \in \mathcal{T}$ the trajectory of the generalized geodesic flow (in the exterior domain determined by X) starting at x in direction ω has no tangencies to X.*

We now turn to the proofs of the above statements.

Fix two integers k and s with $s \geq 1$ and $0 \leq k \leq s$. Denote by $M(s,k)$ the set of those

$$\tilde{\xi} = (\omega; x; y; \theta) \in M_s = S^{n-1} \times X^{(s)} \times X \times S^{n-1},$$

with $x = (x_1,\dots,x_s)$, such that there exists an (ω,θ)-trajectory for X with successive (transversal) reflection points $x_1,\dots,x_s$, the segment $[x_k,x_{k+1}]$ of which is tangent to X at the point $y \in (x_k,x_{k+1})$. Here by x_0 (resp. x_{s+1}) we denote the orthogonal projection of x_1 on Z_ω (resp. of x_s on $Z_{-\theta}$), and by definition

$$X^{(s)} = \{(x_1,\dots,x_s) \in X^s \,:\, x_i \neq x_j\,, i \neq j\,\}.$$

It is convenient to consider also the following subsets of $M(s,k)$:

$$M_r(s,k) = \{(\omega;x;y;\theta) \in M(s,k) : x_k^{(r)} \neq x_{k+1}^{(r)}\} \;\;,\; r = 1,\dots,n.$$

Clearly, these sets are open in $M(s,k)$ and

$$M(s,k) = \bigcup_{r=1}^{n} M_r(s,k).$$

3.3. Lemma. *Each $M_r(s,k)$ is a smooth submanifold of M_s of codimension $2n-3$.*

Proof. We consider only the case $r = n$, the other cases are the same.

Fix an arbitrary $\eta = (\tilde{\omega};\tilde{x};\tilde{y};\tilde{\theta}) \in M_n(s,k)$. Choose smooth charts

$$\varphi_i : U_i \longrightarrow X$$

of X around $\tilde{x}_i$ and

$$\psi : V \longrightarrow X$$

of X around $\tilde{y}$ such that $\varphi_i(U_i) \cap \varphi_{i+1}(U_{i+1}) = \emptyset$, $i = 1, \ldots, s-1$, $\varphi_k(U_k) \cap \psi(V) = \emptyset$, $\varphi_{k+1}(U_{k+1}) \cap \psi(V) = \emptyset$ (for $k = 0$ or s the corresponding condition is to be deleted). Assuming $\tilde{\omega}_n > 0$, we may parametrize S^{n-1} around $\tilde{\omega}$ by

$$D_0 \ni \omega' = (\omega_1, \ldots, \omega_{n-1}) \mapsto \omega = (\omega'; \omega_n) \in S^{n-1},$$

where $\omega_n = (1 - {\omega_1}^2 - \ldots - {\omega_{n-1}}^2)^{1/2}$ and D_0 is the unit open ball in $\mathbf{R}^{n-1}$. Similarly, we may assume that S^{n-1} is parametrized around $\tilde{\theta}$ by $\theta'' = (\theta_2, \ldots, \theta_n) \in D_0$. In this way we get a chart

$$\chi : U = D_0 \times U_1 \times \ldots \times U_s \times V \times D_0 \longrightarrow D \subset M_s,$$

defined by

$$\chi(\xi) = (\omega; \varphi_1(u_1), \ldots, \varphi_s(u_s); \psi(v); \theta)$$

for

$$\xi = (\omega'; u; v; \theta'') \in U.$$

Here $\omega = (\omega'; \omega_n), \theta = (\theta_1; \theta'')$, $u_i = (u_i^{(1)}, \ldots, u_i^{(n-1)}) \in U_i$.

Next, define

$$F : U \longrightarrow \mathbf{R}$$

by

$$F(\xi) = \sum_{i=1}^{s-1} \|\varphi_i(u_i) - \varphi_{i+1}(u_{i+1})\|.$$

case 1. $0 < k < s$. Let

$$\xi = (\omega'; u; v; \theta'') \in U$$

be such that $\chi(\xi) \in M_n(s, k)$. Then we have

$$\mathrm{grad}_{u_i} F(\xi) = 0 \ , \ i = 2, \ldots, s-1, \tag{14}$$

$$\langle \frac{\varphi_2(u_2) - \varphi_1(u_1)}{\|\varphi_2(u_2) - \varphi_1(u_1)\|} - \omega, \frac{\partial \varphi_1}{\partial u_1^{(j)}}(u_1) \rangle = 0 \ , \ j = 1, \ldots, n-1 \tag{15}$$

$$\langle \frac{\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})}{\|\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})\|} - \theta, \frac{\partial \varphi_s}{\partial u_s^{(j)}}(u_s) \rangle = 0 \ , \ j = 1, \ldots, n-1, \tag{16}$$

$$\frac{\psi(v) - \varphi_k(u_k)}{\|\psi(v) - \varphi_k(u_k)\|} + \frac{\psi(v) - \varphi_{k+1}(u_{k+1})}{\|\psi(v) - \varphi_{k+1}(u_{k+1})\|} = 0, \tag{17}$$

$$\langle \varphi_{k+1}(u_{k+1}) - \varphi_k(u_k), N(\xi) \rangle = 0, \tag{18}$$

where

$$N(\xi) = \det \begin{pmatrix} f_1 & f_2 & \dots & f_n \\ \frac{\partial \psi^{(1)}}{\partial v^{(1)}}(v) & \frac{\partial \psi^{(2)}}{\partial v^{(1)}}(v) & \dots & \frac{\partial \psi^{(n)}}{\partial v^{(1)}}(v) \\ \dots & \dots & \dots & \dots \\ \frac{\partial \psi^{(1)}}{\partial v^{(n-1)}}(v) & \frac{\partial \psi^{(2)}}{\partial v^{(n-1)}}(v) & \dots & \frac{\partial \psi^{(n)}}{\partial v^{(n-1)}} \end{pmatrix}$$

is a normal vector to X at $\psi(v)$. Here $f_1, \dots, f_n$ are the standard basis vectors in $\mathbf{R}^n$. In correspondence with these conditions we introduce the functions

$$K_i^{(j)}(\xi) = \frac{\partial F}{\partial u_i^{(j)}}(\xi) \quad, i = 2, \dots, s-1, j = 1, \dots, n-1,$$

$$L_j(\xi) = \langle \frac{\varphi_2(u_2) - \varphi_1(u_1)}{\|\varphi_2(u_2) - \varphi_1(u_1)\|} - \omega, \frac{\partial \varphi_1}{\partial u_1^{(j)}}(u_1) \rangle \ , j = 1, \dots, n-1,$$

$$M_j(\xi) = \langle \frac{\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})}{\|\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})\|} - \theta, \frac{\partial \varphi_s}{\partial u_s^{(j)}}(u_s) \rangle \ , j = 1, \dots, n-1,$$

$$P_j(\xi) = \frac{\psi^{(j)}(v) - \varphi_k^{(j)}(u_k)}{\|\psi(v) - \varphi_k(u_k)\|} + \frac{\psi^{(j)}(v) - \varphi_{k+1}^{(j)}(u_{k+1})}{\|\psi(v) - \varphi_{k+1}(u_{k+1})\|} \ , j = 1, \dots, n-1,$$

$$Q(\xi) = \langle \varphi_{k+1}(u_{k+1}) - \varphi_k(u_k), N(\xi) \rangle.$$

Finally, define the map

$$G : U \longrightarrow (\mathbf{R}^{n-1})^{s-2} \times \mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R}$$

by

$$G(\xi) = ((K_i^{(j)}(\xi))_{2 \le i \le s-1}^{1 \le j \le n-1}; (L_j(\xi))_{1 \le j \le n-1}; (M_j(\xi))_{1 \le j \le n-1}; (P_j(\xi))_{1 \le j \le n-1}; Q(\xi)).$$

Then G is smooth and, according to the above, we have

$$\chi^{-1}(D \cap M_n(s,k)) = G^{-1}(0).$$

Thus, to prove the lemma we have to establish that $G^{-1}(0)$ is a smooth submanifold of U of dimension $2n-3$. To do this it is sufficient to show that G is submersion at any point of $G^{-1}(0)$. Indeed, if this is true, then $G^{-1}(0)$ is a smooth submanifold with

$$\dim G^{-1}(0) = (s+3)(n-1) - [(s+1)(n-1) + 1] = 2n-3.$$

Fix $\xi \in G^{-1}(0)$ and assume that

$$\sum_{i=2}^{s-1} \sum_{j=1}^{n-1} A_i^{(j)} \mathrm{grad} K_i^{(j)}(\xi) + \sum_{j=1}^{n-1} B_j \mathrm{grad} L_j(\xi) + \sum_{j=1}^{n-1} C_j \mathrm{grad} M_j(\xi) + \sum_{j=1}^{n-1} p_j \mathrm{grad} P_j(\xi) + q \mathrm{grad} Q(\xi) = 0 \tag{19}$$

for some real coefficients $A_i^{(j)}, B_j, C_j, p_j, q$. We have to show that these constants are zero.

First, consider in (19) the derivatives with respect to $\omega_1, \ldots, \omega_{n-1}$. According to $\omega_n = (1 - \omega_1^2 - \ldots - \omega_{n-1}^2)^{1/2}$, we get

$$\sum_{j=1}^{n-1} B_j \big(-\frac{\partial \varphi_1^{(r)}}{\partial u_1^{(j)}}(u_1) + \frac{\omega_r}{\omega_n} \cdot \frac{\partial \varphi_1^{(n)}}{\partial u_1^{(j)}}(u_1)\big) = 0 \tag{20}$$

for $r = 1, \ldots, n-1$. Note that (20) holds also for $r = n$. Setting

$$c = \frac{1}{\omega_n} \sum_{j=1}^{n-1} B_j \frac{\partial \varphi_1^{(n)}}{\partial u_1^{(j)}}(u_1),$$

(20) implies

$$c\omega_r = \sum_{j=1}^{n-1} B_j \frac{\partial \varphi_1^{(r)}}{\partial u_1^{(j)}}(u_1) \quad , \ r = 1, \ldots, n,$$

that is

$$c\omega = \sum_{j=1}^{n-1} B_j \frac{\partial \varphi_1}{\partial u_1^{(j)}}(u_1). \tag{21}$$

Consequently, $c\omega$ is a tangent vector to X at $\varphi_1(u_1)$. However, $\xi = (\omega; u; v; \theta) \in G^{-1}(0)$ implies $\chi(\xi) \in M_n(s,k) \subset M(s,k)$, and so $\varphi_1(u_1)$ is the first (transversal) reflection point of an (ω, θ)-trajectory for X. In particular, ω is not tangent to X at $\varphi_1(u_1)$. Hence $c = 0$ and now (21) yields $B_1 = \ldots = B_{n-1} = 0$.

In a similar way, considering in (19) the derivatives with respect to $\theta_2, \ldots, \theta_n$, we find $C_1 = \ldots = C_{n-1} = 0$.

Next, we are going to show that $A_m^{(1)} = \ldots = A_m^{(n-1)} = 0$ for each $m = 2, \ldots, k$. Here we assume $k \geq 2$, for $k = 1$ there is nothing to be proved in this step. Since $k \geq 2$, the functions P_j, Q and $K_i^{(j)}$ for $i \geq 3$ do not depend on the variables $u_1^{(r)}$. On the other hand,

$$K_2^{(j)}(\xi) = \frac{\partial F}{\partial u_2^{(j)}}(\xi) = \langle \frac{\varphi_2(u_2) - \varphi_1(u_1)}{\|\varphi_2(u_2) - \varphi_1(u_1)\|} + \frac{\varphi_2(u_2) - \varphi_3(u_3)}{\|\varphi_2(u_2) - \varphi_3(u_3)\|}, \frac{\partial \varphi_2}{\partial u_2^{(j)}}(u_2) \rangle,$$

and therefore

$$\frac{\partial K_2^{(j)}}{\partial u_1^{(r)}}(\xi) = -a_1 [\langle \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1), \frac{\partial \varphi_2}{\partial u_2^{(j)}}(u_2) \rangle - \langle e_1, \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1) \rangle \langle e_1, \frac{\partial \varphi_2}{\partial u_2^{(j)}}(u_2) \rangle],$$

where

$$a_i = \frac{1}{\|\varphi_i(u_i) - \varphi_{i+1}(u_{i+1})\|} \quad , \quad e_i = \frac{\varphi_i(u_i) - \varphi_{i+1}(u_{i+1})}{\|\varphi_i(u_i) - \varphi_{i+1}(u_{i+1})\|}.$$

Now we consider in (19) the derivatives with respect to $u_1^{(r)}$ and find

$$\sum_{j=1}^{n-1} A_2^{j)} \frac{\partial K_2^{(j)}}{\partial u_1^{(r)}}(\xi) = 0. \tag{22}$$

Setting

$$w = \sum_{j=1}^{n-1} A_2^{(j)} \frac{\partial \varphi_2}{\partial u_2^{(j)}}(u_2) \in T_{\varphi_2(u_2)}X, \tag{23}$$

and using the expression for $\frac{\partial K_2^{(j)}}{\partial u_1^{(r)}}$ found above, we can rewrite (22) in the form

$$\langle \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1), w\rangle - \langle e_1, w\rangle\langle e_1, \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1)\rangle = 0.$$

That is,

$$\langle \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1), w - \langle e_1, w\rangle e_1\rangle = 0,$$

and this is true for all $r = 1, \ldots, n-1$. Consequently,

$$w - \langle e_1, w\rangle e_1 = \lambda N_1 \tag{24}$$

for some $\lambda \in \mathbf{R}$, N_1 being an unit normal vector to X at $\varphi_1(u_1)$. Note that $\langle e_1, N_1\rangle \neq 0$. Taking inner product of (24) with e_1 gives $0 = \lambda\langle N_1, e_1\rangle$, and so $\lambda = 0$. Thus, by (24), $w = \langle e_1, w\rangle e_1$. If $\langle e_1, w\rangle \neq 0$, the latter would imply $e_1 \in T_{\varphi_2(u_2)}X$, which would be a contradiction with $\xi \in G^{-1}(0)$. Hence $\langle e_1, w\rangle = 0$ and so $w = 0$. Now (23) yields $A_2^{(1)} = \ldots = A_2^{(n-1)} = 0$.

Using the above procedure, by induction, we get $A_m^{(j)} = 0$ for all $m = 2, \ldots, k$ and $j = 1, \ldots, n-1$.

Next, if $k < s-1$, we repeat the same argument, considering the derivatives with respect to $u_s^{(r)}$, to show that $A_{s-1}^{(j)} = 0$ for $j = 1, \ldots, n-1$. In a similar way, by induction, one gets $A_m^{(j)} = 0$ for all $m = s-1, s-2, \ldots, k+1$ and $j = 1, \ldots, n-1$. Therefore all coeffiecients $A_m^{(j)}$ in (19) are zero.

Now (19) has the form

$$\sum_{j=1}^{n-1} p_j \operatorname{grad} P_j(\xi) + q \operatorname{grad} Q(\xi) = 0. \tag{25}$$

Set for convenience

$$b_1 = \frac{1}{\|\psi(v) - \varphi_k(u_k)\|} \ , \ b_2 = \frac{1}{\|\psi(v) - \varphi_{k+1}(u_{k+1})\|},$$

and note that
$$\frac{\psi(v)-\varphi_k(u_k)}{\|\psi(v)-\varphi_k(u_k)\|}=-\frac{\psi(v)-\varphi_{k+1}(u_{k+1})}{\|\psi(v)-\varphi_{k+1}(u_{k+1})\|}=e_k.$$

We have
$$\frac{\partial P_j}{\partial u_k^{(r)}}(\xi)=-b_1[\frac{\partial \varphi_k^{(j)}}{\partial u_k^{(r)}}(u_k)-\langle e_k,\frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k)\rangle e_k^{(j)}]$$
and
$$\frac{\partial P_j}{\partial u_{k+1}^{(r)}}(\xi)=-b_2[\frac{\partial \varphi_{k+1}^{(j)}}{\partial u_{k+1}^{(r)}}(u_{k+1})-\langle e_k,\frac{\partial \varphi_{k+1}}{\partial u_{k+1}^{(r)}}(u_{k+1})\rangle e_{k+1}^{(j)}].$$

Further, set $p_n=0$ and $p=(p_1,\ldots,p_n)\in\mathbf{R}^n$, and consider in (25) the derivatives with respect to $u_k^{(r)}$. We get
$$-b_1\sum_{j=1}^{n-1}p_j[\frac{\partial \varphi_k^{(j)}}{\partial u_k^{(r)}}(u_k)-\langle e_k,\frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k)\rangle e_k^{(j)}]-q\langle\frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k),N(\xi)\rangle=0,$$
which is equivalent to
$$-b_1\langle p,\frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k)\rangle+b_1\langle e_k,p\rangle\langle e_k,\frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k)\rangle-q\langle N(\xi),\frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k)\rangle=0.$$
The latter can be rewritten as
$$\langle b_1p-\langle e_k,b_1p\rangle e_k+qN(\xi),\frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k)\rangle=0.$$
Since this is true for all $r=1,\ldots,n-1$, it implies
$$b_1p-\langle e_k,b_1p\rangle e_k+qN(\xi)=\mu N_k \tag{26}$$
for some $\mu\in\mathbf{R}$, N_k being a unit normal vector to X at $\varphi_k(u_k)$. Taking inner product of (26) with e_k, one finds
$$\mu\langle N_k,e_k\rangle=q\langle N(\xi),e_k\rangle=0,$$
since the segment $[\varphi_k(u_k),\varphi_{k+1}(u_{k+1})]$ is tangent to X at $\psi(v)$ and $N(\xi)$ is a normal vector to X at $\psi(v)$. On the other hand, $\langle N_k,e_k\rangle\neq 0$, and so $\mu=0$. Now (26) implies
$$b_1p-\langle e_k,b_1p\rangle e_k+qN(\xi)=0. \tag{27}$$
In a similar way, considering in (25) the derivatives with respect to $u_{k+1}^{(r)}$, one obtains
$$b_2p-\langle e_k,b_2p\rangle e_k-qN(\xi)=0. \tag{28}$$

Since $b = b_1 + b_2 > 0$, combining (27) and (28) gives

$$p = \langle e_k, p \rangle e_k. \tag{29}$$

First, note that the latter and (27) imply $q = 0$. Next, $\xi \in G^{-1}(0)$ yields $\chi(\xi) \in M_n(s,k)$ and so $\varphi_k^{(n)}(u_k) \neq \varphi_{k+1}^{(n)}(u_{k+1})$. Hence $e_k^{(n)} \neq 0$. On the other hand, $p_n = 0$ by definition, therefore (29) implies $0 = \langle e_k, p \rangle e_k^{(n)}$, i.e. $\langle e_k, p \rangle = 0$. Again by (29) we get $p = 0$. Thus, all coefficients in (19) are zero.

case 2. $k = s$. In this case for $\chi(\xi) \in M_n(s,k)$ the conditions (14), (15) and (16) remain the same, while (17) and (18) are replaced by

$$\frac{\psi(v) - \varphi_s(u_s)}{\|\psi(v) - \varphi_s(u_s)\|} = \theta$$

and

$$\langle \theta, N(\xi) \rangle = 0,$$

respectively. Correspondingly, we define the functions $K_i^{(j)}$, L_j and M_j in the same way and change the definitions of P_j and Q as follows:

$$P_j(\xi) = \frac{\psi^{(j)}(v) - \varphi_s^{(j)}(u_s)}{\|\psi(v) - \varphi_s(u_s)\|} - \theta_j \ , \ \ Q(\xi) = \langle \theta, N(\xi) \rangle.$$

Then the map G is defined in the same way.

Let $\xi \in G^{-1}(0)$ and assume (19) holds. Using the argument from case 1, we first show that $B_j = 0$ for all j. Then, by induction on $i = 2, \ldots, s-1$, considering the derivatives with respect to $u_{i-1}^{(r)}$, respectively, we show that $A_i^{(j)} = 0$ for all $i = 2, \ldots, s-1, j = 1, \ldots, n-1$. Here we use the fact that P_j, Q and M_j do not depend on u_i for $i \leq s-2$. Next, considering in (19) the derivatives with respect to $u_{s-1}^{(r)}$ and $u_s^{(r)}$, we find $C_j = 0$ and $p_j = 0$ for all $j = 1, \ldots, n-1$. Finally, $q = 0$ follows easily considering the derivatives with respect to θ_r. Hence in this case G is again submersion at any point of $G^{-1}(0)$, and so $M_n(s,k)$ is a smooth submanifold of M_s of dimension $2n-3$.

case 3. $k = 0$. The same as case 2.

This concludes the proof of the lemma.

Proof of Theorem 3.1. For given s and k consider the projection

$$\pi_s : M_s = S^{n-1} \times X^{(s)} \times X \times S^{n-1} \longrightarrow S^{n-1} \times S^{n-1},$$

defined by

$$\pi_s(\omega; x; y; \theta) = (\omega, \theta).$$

Since π_s is smooth and $M_r(s,k)$ is a smooth submanifold of dimension

$$2n-3 < \dim(S^{n-1}\times S^{n-1}),$$

the set

$$L_r(s,k) = \pi_s(M_r(s,k)) \subset S^{n-1}\times S^{n-1}$$

has measure zero. Let $M_r(s,k) = \cup_{j=1}^{\infty} K_j$ with K_j compact. Then $L_r(s,k) = \cup_{j=1}^{\infty}\pi_s(K_j)$ is a countable union of compact subsets of $S^{n-1}\times S^{n-1}$ with measure zero.

Finally, set

$$L = \bigcup_{0\le k\le s}\bigcup_{r=1}^{n} L_r(s,k)$$

and $\mathcal{R} = (S^{n-1}\times S^{n-1})\setminus L$. Then $\mathcal{R}$ has the desired properties.

Proof of Theorem 3.2. Consider the projections

$$p_s : M_s \longrightarrow S^{n-1}\times X,$$

defined by

$$p_s(\omega; x_1,\dots,x_s; y;\theta) = (\omega, x_1),$$

and set

$$\mathcal{T} = (S^{n-1}\times X)\setminus \bigcup_{0\le k\le s}\bigcup_{r=1}^{n} p_s(M_r(s,k)).$$

The same argument as that in the proof of Theorem 1 shows that $\mathcal{T}$ has the desired properties.

3.3. Remark. In the case when K has the form (6) with $G^{\infty}(K)=\emptyset$ it is possible to prove a slightly stronger result than Theorem 3.1. Namely, for each fixed $\omega\in S^{n-1}$ there exists a set $\mathcal{R}(\omega)\subset S^{n-1}$, the complement of which has Lebesgue measure zero, such that for each $\theta\in\mathcal{R}(\omega)$ all (ω,θ)-rays are ordinary reflecting ones (see [P2]).

4 Sojourn times

Let Ω be a domain in $\mathbf{R}^n, n\ge 2$, with bounded complement and smooth boundary $X=\partial\Omega$, and let ω be a fixed unit vector in $\mathbf{R}^n$. Our aim in this section is to prove the following.

4.1. Proposition. *There exists $\mathcal{S}(\omega)\subset S^{n-1}$ the complement of which is a countable union of compact subsets of measure zero of S^{n-1} such that if $\theta\in\mathcal{S}(\omega)$, then any two different ordinary reflecting (ω,θ)-rays in Ω have distinct sojourn times.*

The rest of the section is devoted to the proof of this proposition.

Let $B = B_R$ be the open ball in $\mathbf{R}^n$ with center 0 and radius R and fix R so big that B contains X. Let $Z = Z_\omega$ be the hyperplane tangent to B and orthogonal to ω and such that the halfspace determined by Z and having ω as an inner normal contains X.

For a given integer $k \geq 1$ denote by U_k the set of those $u \in Z$ such that the trajectory $\gamma(u)$ of the generalized deodesic flow in Ω is an ordinary reflecting ray with exactly k reflection points. Denote by $J_k(u) \in S^{n-1}$ the direction of $\gamma(u)$ after the last reflection. Clearly, U_k is open in Z and

$$J_k : U_k \longrightarrow S^{n-1}$$

is smooth.

Next, we fix two arbitrary integers $k \geq 1, s \geq 1$. For $u \in U_k$ denote by $f(u)$ the sojourn time of the scattering ray $\gamma(u)$ (more precisely, the scattering ray determined by $\gamma(u)$). Thus, we get a smooth function $f : U_k \longrightarrow \mathbf{R}$. For convenience the same function on U_s will be denoted by g, so $g : U_s \longrightarrow \mathbf{R}$.

For $u \in U_k$ let $x_1(u) \in X, \ldots, x_k(u) \in X$ be the successive reflection points of $\gamma(u)$. Then $x_i : U_k \longrightarrow X$ are smooth maps. Let for $y \in X$, $N(y)$ denotes the *unit normal to* X at y pointing into Ω. Then for $u \in U_k$ we have

$$J_k(u) = \frac{x_k(u) - x_{k-1}(u)}{\|x_k(u) - x_{k-1}(u)\|} - 2\langle \frac{x_k(u) - x_{k-1}(u)}{\|x_k(u) - x_{k-1}(u)\|}, N(x_k(u))\rangle N(x_k(u))$$

and

$$f(u) = \sum_{i=0}^{k-1} \|x_{i+1}(u) - x_i(u)\| + t - 2R,$$

where $x_0(u)$ (resp. $x_{k+1}(u)$) denotes the orthogonal projection of $x_1(u)$ (resp. $x_k(u)$) on Z (resp. $Z_{-\theta}$), $\theta = J_k(u)$, and $t = \|x_k(u) - x_{k+1}(u)\|$. It is easy to see that

$$\langle \theta, x_k + t\theta - R\theta \rangle = 0.$$

Therefore $t = R - \langle \theta, x_k \rangle$, and so

$$f(u) = \sum_{i=0}^{k-1} \|x_{i+1}(u) - x_i(u)\| - \langle x_k(u), J_k(u) \rangle - R.$$

For $v \in U_s$ the successive reflection points of $\gamma(v)$ will be denoted by $y_1(v), \ldots, y_s(v)$. We set $y_0(v) = v$ and define $y_{s+1}(v)$ in the same way as $x_{k+1}(u)$.

Further, denote by $W(k,s)$ the set of those $(u,v) \in U_k \times U_s$ such that $J_k(u) = J_s(v)$, $f(u) = g(v)$ and rank $\mathrm{d}J_k(u) =$ rank $\mathrm{d}J_s(v) = n-1$.

4.2. Lemma. *$W(k,s)$ is a smooth $(n-2)$-dimensional submanifold of $U_k \times U_s$.*

Proof. Consider an arbitrary point $w_0 = (u_0, v_0)$ in $W(k,s)$. Then rank $\mathrm{d}J_k(u_0) =$ rank $\mathrm{d}J_s(v_0) = n-1$. Clearly, there exists a neighbourhood U of w_0 in $U_k \times U_s$ such that for every $(u,v) \in W$ we have rank $\mathrm{d}J_k(u) =$ rank $\mathrm{d}J_s(v) = n-1$. Define the map

$$L : U \longrightarrow \mathbf{R}^n$$

by

$$L(u,v) = (\lambda(u,v); (\chi^{(j)}(u,v))_{1\leq j\leq n-1}),$$

where

$$\lambda(u,v) = f(u) - g(v) \ , \ \ \chi(u,v) = J_k(u) - J_s(v).$$

Clearly,

$$W(k,s) \cap U \subset L^{-1}(0),$$

and so it is sufficient to show that L is submersion at any point of $L^{-1}(0)$.

We shall show that L is submersion at w_0. For the other points of $L^{-1}(0)$ the argument is the same. Set $\theta = J_k(u_0)$. Without loss of generality we may assume that

$$\theta^{(n)} \neq 0. \tag{30}$$

Suppose that

$$\sum_{j=1}^{n-1} A_j \mathrm{grad}\, \chi^{(j)}(w_0) + C \mathrm{grad}\, \lambda(w_0) = 0 \tag{31}$$

for some constants A_j, C. Set $A_n = 0$ and $A = (A_1, \ldots, A_n) \in \mathbf{R}^n$.

Before going on, we have to compute several derivatives. Setting

$$e_i = \frac{x_{i+1}(u_0) - x_i(u_0)}{\|x_{i+1}(u_0) - x_i(u_0)\|},$$

for $p = 1, \ldots, n-1$ and $i = 1, \ldots, k-1$ we have

$$\frac{\partial}{\partial u_p}\|x_{i+1} - x_i\|(u_0) = \frac{1}{\|x_{i+1} - x_i\|}\langle x_{i+1} - x_i, \frac{\partial x_{i+1}}{\partial u_p}(u_0) - \frac{\partial x_i}{\partial u_p}(u_0)\rangle = \langle e_i, \frac{\partial x_{i+1}}{\partial u_p}(u_0) - \frac{\partial x_i}{\partial u_p}(u_0)\rangle.$$

Note that

$$\langle e_{i-1}, \frac{\partial x_i}{\partial u_p}(u_0)\rangle = \langle e_i, \frac{\partial x_i}{\partial u_p}(u_0)\rangle,$$

since $\frac{\partial x_i}{\partial u_p}(u_0)$ is tangent to X at $x_i(u_0)$. Consequently,

$$\frac{\partial f}{\partial u_p}(u_0) = \sum_{i=0}^{k-1}\langle e_i, \frac{\partial x_{i+1}}{\partial u_p}(u_0) - \frac{\partial x_i}{\partial u_p}(u_0)\rangle - \langle \frac{\partial x_k}{\partial u_p}(u_0), J_k(u_0)\rangle - \langle x_k, \frac{\partial J_k}{\partial u_p}(u_0)\rangle,$$

and using the fact that

$$\langle \frac{\partial x_k}{\partial u_p}(u_0), J_k(u_0)\rangle = \langle e_k, \frac{\partial x_k}{\partial u_p}(u_0)\rangle,$$

we find

$$\frac{\partial f}{\partial u_p}(u_0) = \sum_{i=1}^{k}\langle e_{i-1}, \frac{\partial x_i}{\partial u_p}(u_0)\rangle - \sum_{i=0}^{k-1}\langle e_i, \frac{\partial x_i}{\partial u_p}(u_0)\rangle - \langle e_k, \frac{\partial x_k}{\partial u_p}(u_0)\rangle - \langle x_k(u_0), \frac{\partial J_k}{\partial u_p}(u_0)\rangle.$$

Since $e_0 = \omega$, this yields

$$\frac{\partial f}{\partial u_p}(u_0) = -\langle e_0, \frac{\partial x_0}{\partial u_p}(u_0)\rangle - \langle x_k(u_0), \frac{\partial J_k}{\partial u_p}(u_0)\rangle = -\langle x_k(u_0), \frac{\partial J_k}{\partial u_p}(u_0)\rangle.$$

According to $\langle x_k(u_0), \theta\rangle = 0$, we deduce

$$\frac{\partial \lambda}{\partial u_p}(u_0) = \frac{\partial f}{\partial u_p}(u_0) = -\langle x_k(u_0), \frac{\partial J_k}{\partial u_p}(u_0)\rangle = 0.$$

Moreover, we have

$$\frac{\partial \chi^{j)}}{\partial u_p}(u_0) = \frac{\partial J_k^{(j)}}{\partial u_p}(u_0).$$

Hence, considering in (31) the derivatives with respect to u_p, we obtain

$$\begin{aligned} 0 &= \sum_{j=1}^{n-1} A_j \frac{\partial J_k^{(j)}}{\partial u_p}(u_0) - C\langle x_k(u_0), \frac{\partial J_k^{(j)}}{\partial u_p}(u_0)\rangle \\ &= \langle A, \frac{\partial J_k}{\partial u_p}(u_0)\rangle - C\langle x_k(u_0), \frac{\partial J_k^{(j)}}{\partial u_p}(u_0)\rangle = \langle A - Cx_k(u_0), \frac{\partial J_k^{(j)}}{\partial u_p}(u_0)\rangle \end{aligned}$$

for all $p = 1, \ldots, n-1$. Since rank $\mathrm{d}J_k(u_0) = n-1$, this yields

$$A - Cx_k(u_0) = a\theta \tag{32}$$

for some $a \in \mathbf{R}$. In a similar way, considering in (31) the derivatives with respect to v_p, one gets

$$A - Cy_s(v_0) = b\theta \tag{33}$$

for some $b \in \mathbf{R}$. Now combining (32) and (33) gives $C(x_k(u_0) - y_s(v_0)) = 0$. Since $x_k(u_0) \neq y_s(v_0)$ and the vector $x_k(u_0) - y_s(v_0)$ is orthogonal to θ, this implies $C = 0$. Using (32) again and taking into account that $A_n = 0$ by definition, we find $a\theta^{(n)} = 0$. Hence, by (30), $a = 0$. Applying again (32), we find $A = 0$, i.e. $A_1 = \ldots = A_{n-1} = 0$. Thus, L is submersion at w_0 which concludes the proof of the lemma.

Proof of Proposition 4.1. Define the map

$$\varphi : U_k \times U_s \longrightarrow S^{n-1} \times S^{n-1}$$

by $\varphi(u, v) = J_k(u)$. Then φ is smooth and since $\dim W(s,k) = n-2$, $\varphi(W(s,k))$ is a countable union of compact subsets of S^{n-1} of measure zero.

Set

$$F_k = \{u \in U_k : \operatorname{rank} \mathrm{d}J_k(u) \leq n-2\}.$$

Then F_k is closed in U_k and so it can be represented as a countable union of compact subsets

$$F_k = \cup_i F_{k,i}.$$

Then by Sard's theorem $J_k(F_{k,i})$ has measure zero in S^{n-1} for all k and i. Therefore

$$F = \cap_k \cap_i J_k(F_{k,i})$$

has measure zero in S^{n-1}. Finally, setting

$$\mathcal{S}(\omega) = S^{n-1} \setminus (F \cup \cup_k \cup_s J_k(W(k,s))),$$

we get a subset of S^{n-1} the complement of which is a countable union of compact subsets of S^{n-1} with measure zero.

To show that $\mathcal{S}(\omega)$ has the desired property, consider and arbitrary $\theta \in \mathcal{S}(\omega)$. Assume that there exists $u \neq v$ in Z which determine ordinary reflecting (ω, θ)-rays in Ω with coinciding sojourn times. Then we have $u \in U_k$ and $v \in U_s$ for some k and s. If $\operatorname{rank} \mathrm{d}J_k(u) \leq n-2$, then $u \in F_k$ and so $u \in F_{k,i}$ for some i. However, this implies $\theta = J_k(u) \in J_k(F_{k,i})$ which is a contradiction with $\theta \in \mathcal{S}(\omega)$. Hence $\operatorname{rank} \mathrm{d}J_k(u) = n-1$. In the same way we find $\operatorname{rank} \mathrm{d}J_s(v) = n-1$ and therefore $(u, v) \in W(k,s)$. This implies $\theta = J_k(u) = \varphi(u,v)$ which is again a contradiction with $\theta \in \mathcal{S}(\omega)$. In this way we have seen that any two different ordinary reflecting (ω,θ)-rays in Ω have distinct sojourn times which proves the proposition.

Introduce the set

$$\mathcal{S} = \{(\omega, \theta) \in S^{n-1} \times S^{n-1} : \theta \in \mathcal{S}(\omega)\}.$$

It follows by the properties of the sets $\mathcal{S}(\omega)$ that the complement of $\mathcal{S}$ in $S^{n-1} \times S^{n-1}$ has measure zero and for each $(\omega, \theta) \in \mathcal{S}$ any two different ordinary reflecting (ω,θ)-rays in Ω have distinct sojourn times.

5 Existence of scattering rays with sojourn times tending to infinity

Let Ω be a closed domain in $\mathbf{R}^n, n \geq 2$, with bounded complement and smooth boundary $\partial\Omega$. In this section we show that if the obstacle

$$K = \mathbf{R}^n \setminus \Omega^\circ$$

is trapping, then there exists a sequence of reflecting (ω_m, θ_m)-rays γ_m in Ω with sojourn times $T_{\gamma_m} \to \infty$.

Set $\mathbf{Q} = \mathbf{R} \times \Omega$ and denote by (τ, ξ) the variables dual to (t, x) in $T^*(\mathbf{Q})$. The *characteristic set* of the wave operator $\Box$ is defined by

$$\Sigma = \{(t, x, \tau, \xi) \in T^*(\overline{\mathbf{Q}}) \setminus \{0\} : \tau^2 =\mid \xi \mid^2\}.$$

For $z = (t, x) \in \mathbf{R} \times \partial\Omega$ consider the *compressed cotangent bundle*

$$\tilde{T}^*_z(\overline{\mathbf{Q}}) = T^*_z(\overline{\mathbf{Q}})/N_z(\partial\mathbf{Q}) \cong T^*_z(\partial\mathbf{Q}),$$

$N_z(\partial\mathbf{Q})$ being the fiber of one-forms vanishing on $T_z(\partial\Omega)$. Setting

$$\tilde{T}^*(\overline{\mathbf{Q}}) = T^*(\mathbf{Q}^\circ) \cup T^*(\partial\mathbf{Q}),$$

introduce the map

$$\sim: T^*(\overline{\mathbf{Q}}) \longrightarrow \tilde{T}^*(\overline{\mathbf{Q}})$$

which coincides with the identity on $\mathbf{Q}^\circ$, while for $(z, \zeta) \in T^*(\overline{\mathbf{Q}}), z \in \partial\mathbf{Q}$, we set

$$\sim (z, \zeta) = (z, \eta) \in T^*_z(\partial\mathbf{Q}) \ \ , \ \ \eta = \zeta \mid_{T_z(\partial\mathbf{Q})} .$$

The image $\Sigma_b = \tilde{\Sigma} = \sim(\Sigma)$ is called the *compressed characteristic set*, and if γ is a generalized bicharacteristic of $\Box$, its image $\tilde{\gamma} = \sim(\gamma)$ is called a *compressed generalized bicharacteristic (ray).*

Let $\rho_0 > 0$ be fixed so that

$$K \subset B_0 = \{x \in \mathbf{R}^n : \mid x \mid \leq \rho_0 \}.$$

Given a point $\nu = (0, x, 1, \xi) \in \Sigma_b, (x, \xi) \in T^*(\partial\Omega)$, consider the compressed generalized bicharacteristic

$$\gamma_\nu(t) = (t, x(t), 1, \xi(t)) \in \tilde{T}^*(\overline{\mathbf{Q}})$$

of $\Box$, parametrized by the time t and passing through ν for $t = 0$. Denote by $T(\nu) \in \mathbf{R}^+ \cup \infty$ the maximal $T > 0$ such that

$$x(t) \in B_0 \ \text{ for } \ 0 \leq t \leq T(\nu).$$

Denote by Σ_∞ the set of those $\nu = (0, x, 1, \xi) \in \Sigma_b, (x, \xi) \in T^*(\partial\Omega)$, such that $T(\nu) = \infty$. Using the continuity of the generalized Hamiltonian flow of $\Box$ (cf. [MS]), it is easy to see that Σ_∞ is closed in Σ_b. On the other hand, $\Sigma_\infty \neq \Sigma_b$. Indeed, take a hyperplane Π tangent to $\partial\Omega$ such that K is contained in a half-space determined by Π. Consider an arbitrary $\nu_0 = (0, x_0, 1, \xi_0) \in \Sigma_b$ with $(x_0, \xi_0) \in T^*(\partial\Omega)$, $x_0 \in \partial\Omega \cap \Pi$. Then we have $T(\nu_0) < \infty$, since $\gamma_{\nu_0}(t)$ leaves B_0 for $t > 2\rho_0$.

By definition the obstacle K is *trapping* if $\Sigma_\infty \neq \emptyset$. Therefore the boundary $\partial\Sigma_\infty$ in Σ_b is not empty. Take an arbitrary $\hat{\nu} \in \partial\Sigma_\infty$. Since $\Sigma_b \setminus \Sigma_\infty \neq \emptyset$, there exists a sequence of elements $\nu_m = (0, x_m, 1, \xi_m)$ of Σ_b with $(x_m, \xi_m) \in T^*(\partial\mathbf{Q})$ such that $\nu_m \notin \Sigma_\infty$ for all m and $\nu_m \to \hat{\nu}$. Consider the compressed generalized bicharacteristics

$$\gamma_{\nu_m}(t) = (t, x_m(t), 1, \xi_m(t))$$

passing through ν_m for $t = 0$ and such that $T(\nu_m) < \infty$. If the sequence $\{T(\nu_m)\}$ is bounded, this would imply $T(\hat{\nu}) < \infty$ in contradiction with $\hat{\nu} \in \Sigma_\infty$. Therefore $\{T(\nu_m)\}$ is unbounded and we may assume

$$T(\nu_m) \longrightarrow_{m\to\infty} +\infty.$$

Set

$$y_m = x_m(T(\nu_m)) \in \partial B_0 \ , \ \ \omega_m = \xi_m(T(\nu_m)) \in S^{n-1}.$$

Passing to a subsequence, we may assume that $y_m \to z \in \partial B_0$ and $\omega_m \to \omega \in S^{n-1}$. Consider the generalized bicharacteristic $\gamma_\mu(t) = (t, y(t), 1, \xi(t))$ of $\Box$ issued from $\mu = (0, z, 1, \omega)$. By continuity we have $T(\mu) = \infty$ and $y(t) \in B_0$ for $t \geq 0$.

Let Z_ω be the hyperplane passing through z and orthogonal to ω. Denote by Z_∞ the set of those points $y \in Z_\omega$ such that the generalized bicharacteristic γ_{μ_y} passing through $\mu_y = (0, y, 1, \omega)$ has the property $T(\mu_y) = \infty$. A simple argument shows that Z_∞ is closed in Z_ω. Clearly $Z_\infty \neq Z_\omega$. Consequently, there exists a sequence

$$z_m \to y_0 \ , \ \ z_m \in Z_\omega \setminus Z_\infty$$

such that

$$T(\mu_{z_m}) < \infty \ \ , \ \ T(\mu_{z_m}) \to \infty.$$

In general the bicharacteristic $\gamma_{\mu_{z_m}}$ could contain gliding or glancing segments. However, if $G^\infty(K) = \emptyset$, then γ_{μ_z} can be approximated by multiple reflecting rays (cf. [MS]). Thus, taking (z'_m, ω'_m) sufficiently close to (z_m, ω), we obtain the following result.

5.1. Proposition. *Let the obstacle K be such that $G^\infty(K) = \emptyset$ and $\Sigma_\infty \neq \emptyset$. Then there exists a sequence of ordinary reflecting (ω'_m, θ'_m)-rays γ_m such that*

$$T_{\gamma_m} \longrightarrow_{m\to\infty} \infty.$$

5.2. Corollary. *Let the obstacle K have the form* (6) *and let $\Sigma_\infty \neq \emptyset$. Then there exists a sequence of ordinary reflecting (ω'_m, θ'_m)-rays γ_m such that*

$$T_{\gamma_m} \longrightarrow_{m\to\infty} \infty.$$

Proof. In this case every generalized bicharacteristic γ of $\Box$ in Ω is uniquelly determined by each of its points. Moreover, every such bicharacteristic can be approximated by ordinary reflecting ones (see [MS]). Thus, the argument of the proof of Proposition 5.1 works even if $G^\infty(K) \neq \emptyset$.

Next, we consider a fixed ordinary reflecting (ω'_m, θ'_m)-ray γ_m which is non-degenerate (see Section 1). We wish to replace (ω'_m, θ'_m) by a pair (ω''_m, θ''_m) sufficiently close to (ω'_m, θ'_m) for which there exist ordinary reflecting (ω''_m, θ''_m)-rays δ_m such that $\{T_{\delta_m}\}$ is an infinite sequence of isolated points in sing supp $s(-t, \theta''_m, \omega''_m)$.

Let $\mathcal{S}$ be the subset of $S^{n-1} \times S^{n-1}$ introduced at the end of Section 4. Assume that K satisfies the following condition:

$$(\mathbf{G}) \qquad \begin{cases} \text{There exits a subset } \mathcal{G} \subset S^{n-1} \times S^{n-1} \text{ the complement of} \\ \text{which has Lebesgue measure zero such that for each } (\omega, \theta) \in \mathcal{G} \\ \text{all } (\omega, \theta) - \text{rays in } \Omega \text{ are ordinary reflecting ones.} \end{cases}$$

Setting $\Xi = \mathcal{G} \cap \mathcal{S}$, it suffices to take an approximation by $(\omega''_m, \theta''_m) \in \Xi$.

On the other hand, to guarantee the existence of such (ω''_m, θ''_m)-rays, we shall use a corollary of the inverse mapping theorem (cf. [H1], Theorem 1.1.7). Let U and V be open subsets of $\mathbf{R}^m$ and let

$$F : U \ni x \mapsto f(x) \in V$$

be a C^∞ map. Suppose that $x_0 \in U$ is such that $\det df(x_0) \neq 0$. Then

$$\alpha = 1/\|df(x_0)^{-1}\| > 0,$$

$\|.\|$ being the standard norm in $\mathcal{L}(\mathbf{R}^m, \mathbf{R}^m)$ – the space of linear maps. Set $y_0 = f(x_0)$ and choose $\delta > 0$ so small that

$$U_\delta = \{x \in \mathbf{R}^m : \|x - x_0\| < \delta\} \subset U,$$

$$\|df(x) - df(x_0)\| \leq \frac{\alpha}{2} \text{ for } x \in U_\delta,$$

$$V_\delta = \{y \in \mathbf{R}^m : \|y - y_0\| < \frac{\delta\alpha}{2}\} \subset V.$$

Then it follows by the inverse mapping theorem that the map f is injective on U_δ and surjective on V_δ.

In what follows we are going to construct suitable approximations for $x_0 = z'_m$ and $\omega_0 = \omega'_m$. Consider the hyperplane $Z = Z_{\omega_0}$. For ω sufficiently close to ω_0, the (ω, θ)-rays issued from $y \in Z_\omega$ in direction ω can be considered as suitable (ω, θ)-rays issued from a point $x \in Z$, provided y is close enough to x_0. Thus we obtain a C^∞ map

$$U = \mathcal{O} \times \Gamma \ni (x, \omega) \mapsto f(x, \omega) \in S^{n-1}.$$

Here $\mathcal{O} \subset Z$ is a small neighbourhood of x_0, $\Gamma \subset S^{n-1}$ is a small neighbourhood of ω_0, and $f(x, \omega)$ is the outgoing direction of the ray issued from x in direction ω. Since γ_m is non-degenerate by assumption, we have

$$\det f'_x(x_0, \omega_0) \neq 0.$$

We may assume that U is chosen so small that $\det f'_x(x, \omega) \neq 0$ holds for all $(x, \omega) \in \overline{U}$. Set

$$\max_{(x,\omega)\in U} \|(f'_x(x, \omega))^{-1}\| = \frac{1}{\alpha}.$$

Then there exists $\delta > 0$ such that for $(x, \omega) \in U$ with $\|x - x_0\| < \delta, \|\omega - \omega_0\| < \delta$ we have

$$\|f'_x(x, \omega) - f'_x(x_0, \omega_0)\| \leq \frac{1}{4}\alpha.$$

We may assume that δ is so small that

$$\mathcal{O}_\delta = \{x \in Z : \|x - x_0\| < \delta\} \subset \mathcal{O} \ , \ \Gamma_\delta = \{\omega \in S^{n-1} : \|\omega - \omega_0\| < \delta\} \subset \Gamma.$$

Clearly, for $\omega \in \Gamma_\delta$ fixed the map

$$\mathcal{O}_\delta \ni x \mapsto f(x, \omega) \in S^{n-1}$$

is injective. Denote $\theta_0 = f(x_0, \omega_0)$ and consider the set

$$W_\delta = \{\theta \in S^{n-1} : \|\theta - \theta_0\| < \frac{\delta\alpha}{4}\}.$$

Choose $\delta' \in (0, \delta)$ so small that

$$\|f(x_0, \omega) - \theta_0\| < \frac{\delta\alpha}{4} \ \text{ for } \ \omega \in \Gamma_{\delta'}.$$

Then for $\omega \in \Gamma_{\delta'}$ and $\theta \in W_\delta$ we deduce

$$\|\theta - f(x_0, \omega)\| < \frac{\delta\alpha}{2},$$

therefore, according to the above remark, for each fixed $\omega \in \Gamma_{\delta'}$ and each fixed $\theta \in W_\delta$ we can find $x_{(\omega,\theta)} \in \mathcal{O}_\delta$ with

$$f(x_{(\omega,\theta)}, \omega) = \theta.$$

This shows that, exploiting the density of Ξ in $S^{n-1} \times S^{n-1}$, we can approximate $(\omega_0, \theta_0) = (\omega'_m, \theta'_m)$ by pairs $(\omega_j, \theta_j) \in \Xi$ for which there exists (ω_j, θ_j)- rays ν_j with sojourn times T_j

converging to T_{γ_m} as $j \to \infty$. Applying the results of [CPS], [PS3], we obtain a sequence of ordinary reflecting (ω_m'', θ_m'')-rays δ_m such that

$$T_{\delta_m} \longrightarrow_{m\to\infty} \infty,$$

and

$$-T_{\delta_m} \in \text{sing supp}\, s(t, \theta_m'', \omega_m'') \ \ , \ \ m = 1, 2, \ldots .$$

Moreover, near $-T_{\delta_m}$ the scattering kernel $s(t, \theta_m'', \omega_m'')$ has a singularity with leading term $A_m \delta^{(n-1)}(t + T_{\delta_m})$ with $A_m \neq 0$. Thus, if the reflecting rays γ_m in Proposition 5.1 are non-degenerate, we conclude that the assertion of Theorem 2.3 is true.

Finally, notice that by Theorem 3.1 the condition **(G)** holds for obstacles K having the form (6). It seems natural to make the following

Conjecture. *For each obstacle K the condition* **(G)** *is satisfied.*

6 Proofs of Theorems 1.1 and 1.2

Throughout this section we assume that K has the form (6). For $z \in \partial K$ we denote by $\mathcal{K}(z)$ the *Gauss curvature* of ∂K at z. Following the argument at the end of the previous section, we need to construct a sequence of ordinary reflecting non-degenerate (ω_m, θ_m)-rays γ_m in Ω with $T_m = T_{\gamma_m} \to \infty$. To do this we use the following.

6.1. Proposition. *Let K have the form* (6) *and let γ be an ordinary reflecting (ω, θ)-ray with reflection points $x_1, \ldots, x_k$. Assume that there exists j such that $\mathcal{K}(x_j) > 0$. Then γ is non-degenerate.*

Proof. Consider the map J_γ defined in Section 1. Then the map $\mathrm{d}\,J_\gamma(u_\gamma)$ has the following representation (see [PS3], Proposition 2.4.2)

$$\mathrm{d}J_\gamma(u_\gamma)u = M_k\sigma_k(I + \lambda_k M_{k-1})\sigma_{k-1}(I + \lambda_{k-1}M_{k-2}) \cdots \sigma_2(I + \lambda_2 M_1)\sigma_1 u.$$

Here

$$\lambda_i = \|x_{i-1} - x_i\| \ , \ i = 1, \ldots, k, \ x_0 = u_\gamma,$$

σ_i is a linear map related to the symmetry with respect to the tangent plane to ∂K at x_i, and M_i are symmetric linear maps defined by

$$M_1 = \tilde{\psi}_1 \ , \ M_i = \sigma_i M_{i-1}(I + \lambda_i M_{i-1})^{-1}\sigma_i + \tilde{\psi}_i \ \ , \ i = 2, \ldots, k,$$

$\tilde{\psi}_i$ being linear symmetric maps depending on the second fundamental form of ∂K at x_i. Since K_j are convex, we have $\tilde{\psi}_i \geq 0$ for all $i = 1, \ldots, k$. Hence $M_i \geq 0$ for each i. By assumption, there exists j with $\mathcal{K}(x_j) > 0$. Then $\tilde{\psi}_j > 0$, and so $M_i > 0$ for $i = j, j+1, \ldots, k$.

Consequently, $\mathrm{d}J_\gamma(u_\gamma)u = 0$ implies $u = 0$ which proves the assertion.

Proof of Theorem 1.1. By Corollary 5.2, there exists a sequence of ordinary reflecting (ω_m', θ_m')-rays δ_m with sojourn times $T_{\delta_m} \to \infty$. Each point $z \in \partial K$ can be approximated by points $z' \in \partial K$ such that $\mathcal{K}(z') > 0$. Using this and Proposition 6.1, we find a sequence of ordinary reflecting non-degenerate (ω_m, θ_m)-rays γ_m with sojourn times $T_{\gamma_m} \to \infty$. To complete the proof we use the argument at the end of Section 5.

Proof of Theorem 1.2. Let $\mathcal{R}$ and $\mathcal{S}$ be the sets from Theorem 3.1 and the end of Section 4, respectively, and let

$$\Xi = \{(\omega, \theta) \in \mathcal{R} \cap \mathcal{S} : \omega \neq \theta\} \subset S^{n-1} \times S^{n-1}.$$

Given $(\omega, \theta) \in \Xi$, there are two possibilities:

1) there are no (ω, θ)-rays in Ω;

2) each (ω, θ)-ray is ordinary reflecting and different (ω, θ)-rays have different sojourn times.

Let $\mathcal{O}$ be a sphere in $\mathbf{R}^n$ which contains the obstacle K in its interior. Consider the set Γ of those $(x, \omega, y, \theta) \in \mathcal{O} \times S^{n-1} \times \mathcal{O} \times S^{n-1}$ such that $(\omega, \theta) \in \Xi$ and there exists an ordinary reflecting (ω, θ)-ray passing through x and y. Clearly, Γ is an open submanifold of $\mathcal{O} \times S^{n-1} \times \mathcal{O} \times S^{n-1}$. Since the projection

$$\pi : \Gamma \ni (x, \omega, y, \theta) \mapsto (\omega, \theta) \in S^{n-1} \times S^{n-1}$$

is smooth, it follows by Sard's theorem that there exists a set $\Sigma(\Gamma) \subset S^{n-1} \times S^{n-1}$ of measure zero such that $\mathrm{d}\pi(x, \omega, y, \theta) \neq 0$ whenever $(\omega, \theta) \notin \Sigma(\Gamma)$. As one can easily check, the last condition means that all (ω, θ)-rays with $(\omega, \theta) \notin \Sigma(\Gamma)$ are non-degenerate. Consequently, for $(\omega, \theta) \in \Xi \setminus \Sigma(\Gamma)$, all (ω, θ)-rays have the properties (a)-(c). Applying the results of Chapter 9 in [PS3], we complete the proof.

References

[BGR] C.Bardos, J.C.Guillot, J.Ralston: *Le relation de Poisson pour l'equation des ondes dans un ouvert non-borné.* Comm. PDE **7** (1982), 905-958.

[CPS] F.Cardoso, V.Petkov, L.Stoyanov: *Singularities of the scattering kernel for generic obstacles.* Ann. Inst. Henri Poincaré (Physique théorique) **53** (1990), 445-466.

[G] C.Gérard: *Asymptotique des pôles de la matrice de scattering pour deux obstacles strictement convexes.* Bull. de S.M.F., Mémoire n° 31, 116 (1988).

[H] I.Herbst: *Contraction semigroup and the spectrum of $A_1 \otimes I + I \otimes A_2$*. J. Operator Theory **7** (1982), 61-78.

[H1] L.Hörmander: *The Analysis of Linear Partial Differential Operators*, vol I. Springer-Verlag, Berlin, 1983.

[H2] L.Hörmander: *The Analysis of Linear Partial Differential Operators*, vol. III. Springer-Verlag, Berlin, 1985.

[I1] M.Ikawa: *Precise information on the poles of the scattering matrix for two strictly convex obstacles*. J. Math. Kyoto Univ. **27** (1987), 69-102.

[I2] M.Ikawa: *Decay of solutions of the wave equation in the exterior of several strictly convex bodies*. Ann. Inst. Fourier **38** (1988), 113-146.

[I3] M.Ikawa: *On the existence of the poles of the scattering matrix for several convex bodies*. Proc. Japan Acad., Ser. A **64** (1988), 91-93.

[LP] P.Lax, R.Phillips: *Scattering Theory*. Academic Press, New York, 1967.

[Ma] A.Majda: *A representation formula for the scattering operator and the inverse problem for arbitrary bodies*. Comm. Pure Appl. Math. **30** (1977), 165-194.

[MS] R.Melrose, J.Sjöstrand: *Singularities in boundary value problems*, I, II. Comm. Pure Appl. Math. **31** (1978), 593-617 and **35** (1982), 129-168.

[P1] V.Petkov: *Scattering Theory for Hyperbolic Operators*. North-Holland, Amsterdam, 1989.

[P2] V.Petkov: *Le comportement de la resolvante modifiée du Laplacien pour des obstacles captifs*. Séminaire EDP, Exposé XVIII, Ecole Polytechnique, 1991-1992.

[PS1] V.Petkov, L.Stoyanov: *Periods of multiple reflecting geodesics and inverse spectral results*. Amer. J. Math. **109** (1987), 619-668.

[PS2] V.Petkov, L.Stoyanov: *Singularities of the scattering kernel and scattering invariants for several strictly convex obstacles*. Trans. Amer. Math. Soc. **312** (1989), 203-235.

[PS3] V.Petkov, L.Stoyanov: *Geometry of Reflecting Rays and Inverse Spectral Problems*. John Wiley and Sons, Chishester, 1992.

[R] J.Ralston: *Solutions of the wave equation with localized energy*. Comm. Pure Appl. Math. **22** (1969), 807-823.

[SjZ] J.Sjöstrand, M.Zworski: *Lower bounds on the number of scattering poles.* Preprint, 1992.

[V] B.Vainberg: *Asymptotic Methods in Equations of Mathematical Physics.* Gordon and Breach Sci. Publ. Ltd., 1988.

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