Topological Quantum Groups, Star Products and their relations

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Dedicated to our friend Ludwig Faddeev on his 60th birthday.

Abstract.

This short summary of recent developments in quantum compact groups and star products is divided into 2 parts. In the first one we recast star products in a more abstract form as deformations and review its recent developments. The second part starts with a rapid presentation of standard quantum group theory and its problems, then moves to their completion by introduction of suitable Montel topologies well adapted to duality. Preferred deformations (by star products and unchanged coproducts) of Hopf algebras of functions on compact groups and their duals, are of special interest. Connection with the usual models of quantum groups and the quantum double is then presented.

0 - INTRODUCTION.

The idea that quantum theories are deformations of classical theories was presumably in the back of the mind of many scientists, even before the mathematical notion of deformation was formalized by Gerstenhaber [G] for algebraic structures. We were even told by witnesses (many of whom contribute to this volume) that Ludwig Faddeev mentioned that idea in his lectures on quantum mechanics in Leningrad in the early 70's, around the time when the so-called geometric quantization was developed.

However in all these approaches people were always considering that in the end quantum theories have to be formulated in operator language, while an essential point in our approach ([FS1], [B*ea*]) is that quantum theories can be developed in an autonomous manner on the algebras of classical observables by deforming the algebraic structures. The connection with operatorial formulation, whenever possible, comes only afterwards and is optional. This applies both to quantum mechanics and quantum field theories. Our approach is often referred to as star-products, or deformation quantization.

Around the beginning of the 80's, when it became rather clear that constructive quantum field theory (at least in 4 dimensions) was facing tremendous analytical problems, the school of Faddeev tried a new approach to quantization of field theories, first with 2-dimensional integrable models. Doing so they discovered [KR] the beginning of what turned to be [FRT] a mathematical gold mine, to which both mathematicians and theoretical physicists rushed (and the rush is still in full speed): quantum groups.

In this short Note we shall present both theories in a context that makes the relations between both quite natural. This presentation (especially its second part) relies on a paper [BFGP]) now being published, where necessary details can be found.

1 - DEFORMATIONS, QUANTIZATIONS AND STAR PRODUCTS.

1.1. The framework. Let A be an algebra. In the following it can be an associative algebra (vector space with product and unit), a Lie algebra, a bialgebra (associative algebra with coproduct), a Hopf algebra (bialgebra with counit and antipode), etc., with the usual compatibility relations between algebraic laws. For simplicity of notations we shall take the base field to be $\mathbb{C}$ (the complex numbers). It can also be a topological algebra, i.e. any of the above when the vector space is endowed with a topology such that all algebraic laws are continuous mappings. We shall specify the kind of algebra considered whenever needed.

An example of such an algebra is given by the Hopf algebra $\mathbb{C}[t]$ of complex polynomials in one variable t, with product $t^n \times t^p = \begin{pmatrix} n+p \\ p \end{pmatrix} t^{n+p}$, coproduct $\delta(t^n) = \sum_{i=0}^{n} t^i \otimes t^{n-i}$, counit $\varepsilon(t^n) = \delta_{n0}$ (Kronecker δ) and antipode $S(t^n) = (-1)^n \, t^n$.

Its dual (in a sense we shall make precise in the following) is the bialgebra of formal series $\mathbb{C}[[t]]$, with usual product and coproduct given by $\Delta f(t,t') = f(t+t') \in \mathbb{C}[[t,t']]$ for $f \in \mathbb{C}[[t]]$.

Now if we extend the base field to the ring $\mathbb{C}[[t]]$, we get from A the module $\tilde{A} = A[[t]]$ of formal series in t with coefficients in A, on which we can consider algebra structures.

1.2. Definition. *A deformation of an algebra A is a (topologically free in the case of* *topological algebras)* $\mathbb{C}[[t]]$ *algebra* $\tilde{A}$ *such that the quotient of* $\tilde{A}$ *by the ideal* $t\tilde{A}$ *generated by* t *is isomorphic to* A.

For an associative algebra this means that on $\tilde{A}$ there is a new product, denoted by $*$, such that for $a, b \in A$:

$$a * b = \sum_{r=0}^{\infty} t^r \; C_r(a,b) \tag{1}$$

where $C_0(a,b) = ab$ (the product of A), and the cochains $C_r \in \mathcal{L}(A\hat{\otimes}A, A)$, the space of linear (continuous) maps from the (completed, for some adequate topology, in the topological case) tensor product $A \otimes A$ into A. The associativity condition for $*$ gives as usual [G] conditions on the cochains C_r (e.g. C_1 is a cocycle for the Hochschild cohomology).

For a Lie algebra one has similar relations (with Chevalley cohomology), and for bialgebras an adequate cohomology can be introduced [B1].

For a bialgebra, denoting by $\otimes_t$ the tensor product of $\mathbb{C}[[t]]$ modules, one can identify $\tilde{A}\hat{\otimes}_t\tilde{A}$ with $(A\hat{\otimes}A)[[t]]$ and therefore the deformed coproduct is defined by

$$\tilde{\Delta}(a) = \sum_{r=0}^{\infty} t^r D_r(a), \;\; a \in A \tag{2}$$

where $D_i \in \mathcal{L}(A, A\hat{\otimes}A)$ and D_0 is the coproduct Δ of A.

For a Hopf algebra, the deformed (Hopf) algebra has same unit and counit, but in general not the same antipode.

As in the algebraic theory [G], two deformations are said *equivalent* if they are isomorphic as$\mathbb{C}[[t]]$ (topological) algebras, the isomorphism being the identity in degree 0 (in t). And a deformation $\tilde{A}$ is said *trivial* if it is equivalent to the deformation obtained by base field extensions from the algebra A.

1.3. Example. Star products. We take $A = C^\infty(W)$, with W a symplectic (or Poisson) manifold with 2-form ω. On A we have a Poisson bracket $(a,b) \mapsto P(a,b)$, which is a bidifferential operator of order (1,1). We say that (1) defines a *star product* on the associative algebra A (with pointwise multiplication) if in addition:

$$C_1(a,b) - C_1(b,a) = 2P(a,b) \quad a, b \in A. \tag{3}$$

We do not assume here that the C_r are bidifferential operators, nor n.c. (null on constant functions, which implies that the function 1 is a unit for the deformed algebra as well). If we do, then [B*ea*] it is coherent to restrict oneself to the corresponding Hochschild cohomologies. But in star representations (see below) one encounters often bipseudodifferential cochains C_r.

¿From (3) follows that the star product defines a deformation of the Lie algebra (A, P) by :

$$[a,b]_* \equiv \frac{1}{2t}(a*b - b*a) = P(a,b) + \sum_{r=2}^{\infty} t^{r-1}(C_r(a,b) - C_r(b,a)). \tag{4}$$

This allows (in the differentiable case) to use instead of the infinite-dimensional Hochschild cohomologies, the finite-dimensional Chevalley cohomology spaces. E.g. the dimension of

Chevalley 2-cohomology is (in the n.c. case) $1 + b_2(W)$ where $b_2(W)$ is the second Betti number of W which permits (as in [B*ea*]) to show that at each level there are only $1+b_2(W)$ choices.

1.4. Typical example : Moyal on $\mathbb{R}^{2n}$. In 1927, H. Weyl [W] gave a rule for passing from a classical observable $a \in A = C^\infty(\mathbb{R}^{2l})$ to an operator in $L^2(\mathbb{R}^l)$ which represents a quantization of this observable. It can be written

$$A \ni a \mapsto \Omega_W(a) = \int \tilde{a}(\xi, \eta)\ exp(i(P\xi + Q\eta)/\hbar)\ w(\xi, \eta)\ d\xi\ d^l\eta \tag{5}$$

where $\tilde{a}$ is the inverse Fourier transform of a, P and Q satisfy the canonical commutation relations $[P_\alpha, Q_\beta] = i\hbar\ \delta_{\alpha\beta}\ (\alpha, \beta = 1, ..., l)$, w is a weight function ($= 1$ in the case of Weyl) and the integral is taken in the weak operator topology. An inverse formula was given a few years later by E. Wigner [Wi], and numerous variants exist. Whenever either side is defined, the trace can be given by:

$$Tr(\Omega_1(a)) = (2\pi\hbar)^{-l} \int_{\mathbb{R}^{2l}} a\ \omega^l \tag{6}$$

In the end of the 40's, starting from a point of view different from ours, Moyal [M] and Groenewold [Gr] found that the commutator and product (resp.) of quantum observables correspond, in the Weyl rule, to sine and exponential of the Poisson bracket (resp.), with the parameter $t = \frac{1}{2}\, i\hbar$. Thus $\Omega_1(a)\ \Omega_1(b) = \Omega_1(a *_M b)$ where $*_M$ is given by (1) with (for $r \geq 1$) $r! C_r(a, b) = P^r(a, b)$, the r^{th} power of the bidifferential operator P.

1.5. Quantizations. In 1975, inspired by our earlier works [FLS] on 1-differentiable deformations of the Lie algebras (A, P), J. Vey [V] obtained what turned to be the Moyal bracket as an example of differentiable deformation, and showed its existence on any symplectic W with $b_3(W) = 0$. We then not only made the connection with quantization but also showed, with examples, that quantization should in fact be considered as a deformation of a classical theory, with the same algebra of observables and a star-product [B*ea*]. Around the same time and independently, Berezin [B] had shown that the normal ordering of physicists (weight $w(\xi, \eta) = exp(-\frac{1}{4}(\xi^2 + \eta^2))$ in (5)) can be defined for more general manifolds than $\mathbb{R}^{2l}$. That ordering is the analogue (for complex coordinates $\xi \pm i\eta$) of the standard ordering (weight $w(\xi, \eta) = exp(-\frac{1}{2} i\xi\eta)$) which mathematicians are using in pseudodifferential operator theory, and is preferred for field theory quantization.

In our approach, we have an autonomous definition of the spectrum of an observable. To that effect we consider the star exponential (the analogue of the evolution operator)

$$Exp(sa) = \sum_{n=0}^{\infty} \frac{1}{n!}\ s^n (i\hbar)^{-n}\ (a*)^n \tag{7}$$

(the sums involved being taken in the distribution sense) and define the spectrum of the observable a to be that (in the sense of L. Schwartz) of the star exponential distribution, i.e. the support of its Fourier-Stieltjes transform (in s). For the harmonic oscillator for instance, one gets $(n + \frac{1}{2}l)\hbar$ with Moyal ordering ($n \in \mathbb{N}$) and $n\hbar$ with normal ordering (which explains why it is favored when $l \to \infty$). But many other examples can be treated, e.g. the hydrogen atom with $W = T^{\forall} S^3$ for manifold.

Star products can also be defined when dim $W = \infty$, and there one can e.g. find some cohomological cancellations of infinities [Di] by taking orderings "in the neighbourhood of normal ordering" : this amounts to substracting an infinite coboundary from an infinite cocycle to get a finite ("renormalized") cocycle.

1.6. Closed star products. Whenever there is a (generalized) Weyl mapping between $A = C^\infty(W)$ (plus possibly some distributions, or part of it only) and operators on a Hilbert space (typically a space of square integrable functions in "half" of the variables, via some polarization), some of these operators will have a trace. Therefore it is natural to ask whether a functional with the properties of a trace can be defined on the algebra $(A, *)$.

For Moyal ordering one has (6). For other orderings on $\mathbb{R}^{2l}$ that formula is valid modulo higher powers of $\hbar$. Therefore [CFS] a natural requirement is to look at the coefficient of $\hbar^l$ in $a * b$, where $a, b \in A[[\hbar]]$, and require that its integral over W is the same as that of $b * a$. Or equivalently :

$$\int_W C_r(a, b)\, \omega^l = \int_W C_r(b, a)\, \omega^l \tag{8}$$

whenever defined for $a, b \in A$ and $1 \leq r \leq l$. A star-product (1) satisfying (8) is called *closed*. If (8) is true for all r we call it *strongly closed*. Note that, in view of (3), (8) is always true for $r = 1$ - so that all star products on 2-dimensional manifolds are closed. It has been pointed out to us by Pierre Lecomte that, in view of (4) and Prop. 2.1. (iii) of [DW], all differentiable n.c. star products are closed. (There exist however non closed star products, that are not e.g. null on constants).

An interesting feature of closed star products [CFS] is that they are classified by *cyclic* cohomology [C], instead of only Hochschild cohomology. This suggests to define, in parallel to the similar notion for operator algebras [C], the *character* of a closed star-product as a cocycle φ in the cyclic cohomology bicomplex with components (non zero only for $l \leq 2k \leq 2l$) :

$$\varphi_{2k}(a_0, a_1, ..., a_{2k}) = \int_W a_0 * \tau(a_1, a_2) * ... * \tau(a_{2k-1}, a_{2k})\, \omega^l \tag{9}$$

where $\tau(a, b) = a * b - ab$ measures the noncommutativity of the star product. It can be shown [CFS] that for $W = T^*M, M$ compact Riemannian manifold, and for the star product of standard ordering (composition of symbols of pseudodifferential operators), the character coincides with that given by the trace on pseudodifferential operators. Therefore, using the algebraic index theorem of [CM], it is given by the Todd class $\mathrm{Td}(T^*M)$ as a current over T^*M.

1.7. Existence. Jacques Vey [V] had obtained the existence of star brackets for all symplectic manifolds with $b_3 = 0$, and this was extended ([NV], [L]) to star products (under the same hypothesis). The underlying idea is to "glue" Moyal products on Darboux charts, and the condition $b_3 = 0$ is needed to control multiple intersections of charts. But we knew from the beginning [Bea] that this condition is not necessary. Then M. Cahen and S. Gutt showed existence for $W = T^*M$, M parallelisable, and soon afterwards [LDW1] existence was shown for any W symplectic (or regular Poisson) manifold.

In 1985-86 (in obscure form, made more clear only recently) B. Fedosov [F] gave a geometrical and algorithmic construction of star products on any W by viewing $A[[t]]$ as a space of flat sections in the bundle of (formal) Weyl algebras on W (and pulling back the multiplication of sections; a flat connection on that bundle is algorithmically constructed starting with any symplectic connection on W). The geometric background of Fedosov's construction has been recently explicited further by several authors ([Gu], [EW]).

Using also Weyl algebras, but here essentially [LDW2] to build compatible local equivalences that allow to "glue together" Moyal products a Darboux charts, it has been possible [OMY] to give another and more concrete proof of existence of star products on any W, and even to do it in a way that proves directly also existence of closed star products.

1.8. Star representations. When a is a generator of a Lie algebra $\mathcal{G}$ of functions (e.g. on a coadjoint orbit of a Lie group G), the star exponential (7) gives the corresponding one-parameter group. And if the star commutator (4) coincides, for $a, b \in \mathcal{G}$, with $P(a, b)$, the Poisson bracket (which is the Lie bracket in this case), one can (by the Campbell-Hausdorff-Dynkin formula) generate a realization of $\tilde{G}$ (the connected and simply connected Lie group with Lie algebra $\mathcal{G}$) by the star exponentials (7) and their star products. Such a star product is said *covariant.*

It is said *invariant* if $[a, b] = P(a, b)\ \forall a \in \mathcal{G}$ and $b \in A$ (this is the geometric invariance of the star product under the action of G). There do not always exist invariant star products (e.g. for nilpotent groups of length > 2), but covariant ones always exist. For covariant star products, the geometric action of G is modified by a t-dependent multiplier.

We call *star representation* the distribution on G defined by the star exponential associated with a covariant star product. Such representations have been built for all compact and all solvable Lie groups, some series of representations of semi-simple groups (including some of those with unipotent orbits), and other examples. The cochains C_r obtained here are in general pseudodifferential.

2 - TOPOLOGICAL QUANTUM GROUPS.

2.1. The setting. Let G be a Poisson-Lie group, i.e. a Lie group with Poisson structure, such that for the usual coproduct Δ on the Hopf algebra $H = C^\infty(G)$, (i.e. $\Delta a(g, g') = a(gg')$; $g, g' \in G$), the Poisson bracket P (on G or $G \times G$) satisfies

$$\Delta\ P(a, b) = P(\Delta a,\ \Delta b) \quad a, b \in H \tag{10}$$

Equivalently we can consider the Lie bialgebra $\mathcal{G}$; the dual $\mathcal{G}^*$ has a bracket $\varphi^* : \mathcal{G}^* \wedge \mathcal{G}^* \to \mathcal{G}^*$ such that its dual φ is a 1-cocycle for the adjoint action. When φ is the coboundary of some $r \in \mathcal{G} \wedge \mathcal{G}$ (solution of the classical Yang-Baxter equation) it is said that the Poisson-Lie group is triangular. In that case there exists a G-invariant differentiable star product on H, and the associativity condition for that star product gives a solution to the quantum Yang-Baxter equation : the deformed algebra H is the realization of a quantum groups [D]. Furthermore [T] there exists a (non-invariant) equivalent star product $*'$ on H such that (for the same Δ as above)

$$\Delta(a *' b) = \Delta a *' \Delta b \tag{11}$$

and the same for the commutator, which is clearly a quantization of (10).

In the "dual" approach of Jimbo [J], one deforms Δ to some Δ_t on some completion $\mathcal{U}_t(\mathcal{G})$ of the enveloping algebra $\mathcal{U}(\mathcal{G})$. It is this deformation that was first discovered [KR], for $\mathcal{G} = sl(2)$: the commutation relations which define $\mathcal{U}_t$ have a deformed form (one of them becoming a sine instead of a linear function).

In line with our philosophy, it is thus natural to ask whether the deformed algebra $\mathcal{U}_t$ can be realized (instead of an operatorial realization) by classical functions and some star product giving the deformed commutators. It turns out that this is possible [FS], with a star-product using a new parameter $\hbar$ unrelated to t. In fact, since there is some duality between H and $\mathcal{U}_t$ (we shall make this more precise later), the two parameters t and $\hbar$ are in a way dual one to the other : the deformed algebra $H[[t]]$ (with star product) gives a deformed coproduct on $\mathcal{U}_t$ that induces deformed commutation relations expressible with another star product (with a new parameter $\hbar$). Moreover the latter expression is essentially unique [FS2] due to a strong invariance property that essentially characterizes the star-sine for the Moyal star product. These star realizations (with $\hbar$) can be given ([Lu], [FLuS]) for various series of classical Lie algebras.

We have just seen that duality plays an important rôle in the Hopf algebraic formulation of quantum groups. But there is a fundamental difficulty, that until recently was quietly avoided : the algebraic dual of an infinite-dimensional Hopf algebra A is not Hopf and the bidual is strictly larger than A. So (unless G is a finite group !) one has to be extremely careful in dualizing - or topologize in a suitable fashion.

2.2. Topological quantum groups : the classical case [BFGP].

a. Definition. A topological algebra (resp. bialgebra, Hopf algebra) A is said *well behaved* if the underlying (complete) topological vector space is nuclear and either Fréchet (F) or dual of Fréchet (DF) [Tr].

The topological dual A^* is then also well-behaved, and the bidual $A^{**} = A$. This is the case when A has countable dimension, with the strict inductive limit of finite-dimensional subspaces as topology. For example, $A = \mathbb{C}[t]$ (the polynomials) is well-behaved, and so is $A^* = \mathbb{C}[[t]]$.

b. The models. Let G be a compact connected Lie group. Then $H(G) = C^\infty(G)$ and its dual $A(G) = \mathcal{D}'(G)$ (the distributions) are well-behaved topological Hopf algebras.

Now G can be imbedded in $\mathcal{D}'(G)$ as Dirac distributions at points of G, and its linear span is dense in $\mathcal{D}'(G)$. The product on $\mathcal{D}'(G)$ is the convolution of (compactly supported) distributions, and the coproduct is defined by $\Delta(x) = x \otimes x$ for $x \in G$ (considered as a Dirac distribution).

We know that the enveloping algebra $\mathcal{U}(\mathcal{G})$ can be identified with differential operators on G, i.e. all distributions with support at the identity. Its "completion" $\mathcal{U}_t$ will involve some entire functions of Lie algebra generators, i.e. an infinite sum Dirac δ's and derivatives, and thus take us outside $\mathcal{D}'$. In order to include this model as well one will therefore have to restrict oneself to a subalgebra of H. The natural choice is the space $\mathcal{H}(G)$ of G-finite vectors of the regular representation, which is generated by the coefficients (matrix elements) of the irreducible (unitary) representations. Thus $\mathcal{H}(G) = \sum_{\rho\in\hat{G}} \mathcal{L}(V_\rho)$, where V_ρ is the space on which the representation $\rho \in \hat{G}$ is realized. Its dual is then

$$\mathcal{H}^*(G) = \mathcal{A}(G) = \prod_{\rho\in\hat{G}} \mathcal{L}(V_\rho) \supset \mathcal{D}'(G). \tag{11}$$

The imbedding $\mathcal{U}(\mathcal{G}) \ni u \mapsto i(u) = (\rho(u)) \in \mathcal{A}(G)$ has a dense image for the topology of $\mathcal{A}$ (the image is of course in $\mathcal{D}'(G)$, but is *not* dense for the $\mathcal{D}'$ topology).

2.3. Topological quantum groups : the deformations.

We shall restrict ourselves here to a summary of the main notions and results of the theory in the framework explained before, referring to [BFGP] and references quoted therein for more details.

Duality and deformations work very well together in our setting. More precisely :

Proposition 1. *Let $\tilde{A}$ be a bialgebra (resp. Hopf) deformation of a well-behaved topological bialgebra (resp. Hopf algebra) A. Then the $\mathbb{C}[[t]]$ dual $\tilde{A}^*_t$ is a deformation of the topological Hopf algebra A^*. Two deformations $\tilde{A}$ and $\tilde{A}'$ of A are equivalent iff $\tilde{A}^*_t$ and $\tilde{A}'^*_t$ are equivalent deformations of A^*.*

The known models of quantum groups lead us to select a special type of deformations:

Definition (see also [GS]). *A deformation of the bialgebra $\mathcal{H}(G)$ (resp. $C^\infty(G)$) with unchanged coproduct is called a preferred deformation.*

This definition is motivated by the following :

Proposition 2. *Let $(\mathcal{H}[[t]], *, \tilde{\delta})$ be a coassociative deformation of the bialgebra $\mathcal{H}$. Then, up to equivalence, one can assume that $\tilde{\delta} = \delta$ (the coproduct in $\mathcal{H}$) ; the product is quasi-commutative and quasi-associative, the counit unchanged, and if the product is associative then $\mathcal{H}[[t]]$ is a $\mathbb{C}[[t]]$ Hopf algebra with same unit and counit as $\mathcal{H}$. The same holds for H.*

(By quasi-associativity, etc., we means as usual that the associativity, etc., condition is satisfied up to a factor). That result is proved by using duality from the following results for the duals $\mathcal{A}(G) = \mathcal{H}(G)^*$ and $A(G) = H(G)^* = \mathcal{D}'(G)$:

Theorem 1. *Let A be either $\mathcal{A}(G)$ or $A(G)$. Then any associative algebra deformation of A is trivial, and A is rigid in the category of bialgebras; any associative bialgebra deformation of A is quasi-cocommutative and quasi-coassociative.*

More specifically $H^n(A, A) = 0$ $\forall n \geq 1$ and $H^1(A, A\hat{\otimes}A) = 0$ (for algebraic and continuous Hochschild cohomologies), which shows the rigidity of A as bialgebra in the sense of [B1]. Moreover, if $(A[[t]], \tilde{\Delta})$ is an associative bialgebra deformation of A with unchanged product, then there exists $\tilde{P} \in (A\hat{\otimes}A)[[t]]$ such that $\tilde{\Delta} = \tilde{P}\, \Delta_0\, P^{-1}$ (where Δ_0 is the coproduct in A), the counit is unchanged, and there exists an antipode $\tilde{S}$ for $A[[t]]$ that is given by $\tilde{S} = \tilde{a}\, S_0\, \tilde{a}^{-1}$ where S_0 is the antipode of A and $\tilde{a}$ is some element in $A[[t]]$. Our topological notion of duality also gives us, automatically, that the deformed product $*$ on the topological dual H (either $\mathcal{H}(G)$ or $C^\infty(G)$) of A is a *star product* (starting with the Poisson bracket) in the sense of part 1, for all G compact.

In addition, the restriction of a Hopf deformation of $H(G)$ defines a Hopf deformation of $\mathcal{H}(G)$. If Γ is a normal subgroup of G, any preferred deformation of $\mathcal{H}(G)$ gives a preferred deformation of $\mathcal{H}(G/\Gamma)$ (and the same with $H(G)$): we can define *quotient deformations*, a useful notion e.g. to pass from $SU(2)$ to $SO(3)$, etc.

2.4. Topological quantum groups : the models.

We shall now explain how the known models of quantum groups relate to the general framework presented in the previous section.

a. Generators of $\mathcal{H}(G)$. The algebra $\mathcal{H}(G)$, G compact, is a finitely generated domain. We say that a set $\{\pi_1, ..., \pi_2\} \subset \hat{G}$ of irreducible representations (irrep.) is *complete* if its coefficients generate $\mathcal{H}(G)$. For $SU(n), SO(n)$ and $Sp(n)$, the standard representation is in itself a complete set. For $Spin(n)$, we take the irreducible spin representation(s) (one for n odd, 2 for n even). For E_6 (resp. E_7) there exist(s) two (resp. 1) irrep. that form a complete set. For all other exceptional (simply connected compact) groups, any irrep. is a complete set.

Define $\pi_0 = \oplus_{i=1}^{r} \pi_i$, and call $\{C_{ij}\}$ the coefficients of π_0 in a given fixed basis : they form a topological generator system for the preferred Hopf deformation $(\mathcal{H}[[t]], *)$ of $\mathcal{H}$. The quasi-commutativity of that deformation can then be expressed as follows: if T is the matrix $[C_{ij}], T_1 = T \otimes Id,\ T_2 = Id \otimes T$, there exists an invertible R in $\mathcal{L}\ (V_{\pi_0} \otimes V_{\pi_0})[[t]]$ such that $R\ (T_1\ * T_2) = (T_1\ *\ T_2)\ R$.

b. The Drinfeld models [D1]. Let $\mathcal{U} = \mathcal{U}(\mathcal{G})$ be the enveloping algebra. Drinfeld has shown [D2] that it is rigid (as algebra), and there exists a Hopf deformation $\mathcal{U}_t$ of $\mathcal{U}$ (endowed with its natural topology) that is a topologically free complete $\mathbb{C}[[t]]$-module: there is an isomorphism $\tilde{\varphi} : \mathcal{U}_t \simeq \mathcal{U}[[t]]$ as $\mathbb{C}[[t]]$-modules, and also as algebras; we call such a $\tilde{\varphi}$ a *Drinfeld isomorphism.* The coproduct $\tilde{\Delta}$ of $\mathcal{U}_t$ is obtained from the original coproduct by a twist : $\tilde{\Delta} = \tilde{P}\ \Delta_0\ P^{-1}$ for some $\tilde{P} \in \mathcal{U}_t \hat{\otimes}_t\, \mathcal{U}_t$.

Using the fact that $\mathcal{U}(\mathcal{G}) \subset A(G) \subset \mathcal{A}(G)$ we can extend the Hopf deformation $\mathcal{U}_t$ to a Hopf deformation of $A(G)$ or $\mathcal{A}(G)$ with unchanged product, unit and counit. By $\mathbb{C}[[t]]$ duality this gives a preferred deformation of $H(G)$ or $\mathcal{H}(G)$ (resp.).

All this construction depends on the choice of a Drinfeld isomorphism $\tilde{\varphi}$, but in an inessential way : two Drinfeld isomorphisms $\tilde{\varphi}$ and $\tilde{\psi}$ give equivalent preferred deformations of $\mathcal{H}(G)$. Note that the above R-matrix can be specified to be a solution of the Yang-Baxter equation.

c. The Faddeev-Reshetikhin-Takhtajan models. These [FRT] models are recovered by a good choice of the Drinfeld isomorphism : if $\tilde{\rho}$ is a representation of $\mathcal{U}_t$ and $\pi = \rho_0 \in \hat{G}$ is its classical limit, then there is a Drinfeld isomorphism $\tilde{\varphi}$ such that $\tilde{\rho} = \pi \circ \tilde{\varphi}$.

When we apply this to $G = SU(n), SO(n)$ or $Sp(n)$ we recover the [FRT] quantizations of these groups as preferred Hopf deformations of $\mathcal{H}(G)$ that extend to preferred Hopf deformations of $C^\infty(G)$.

d. The Jimbo models [J]. These models are somewhat special, because we get here nontrivial deformations. We shall explain it here for the case $\mathcal{G} = sl(2)$. The general case is similar, the main difference being that there $\mathcal{U}(\mathcal{G})$ is extended by Rank$(\mathcal{G})$ parities.

Consider the quantum algebra A_t generated by 4 generators $\{F, F', S, C\}$ with relations :

$$[F, F'] = 2SC,\ FS = (S\ \mathrm{cos}t - C)\ F,\ FC = (C\ \mathrm{cos}t + S\ \sin^2 t)\ F \qquad (12a)$$

$$F'S = (S\cos t + C)\ F',\ F'C = (\cos t - S\ \sin^2 t)F',\ C^2 + S^2\sin^2 t = 1,\ [S,C] = 0. \quad (12b)$$

The more familiar form is obtained by setting $q = e^{it}$ ($t \notin 2\pi\mathbf{Q}$) and $S = \frac{K-K^{-1}}{q-q^{-1}}$, $C = \frac{1}{2}(K + K^{-1})$ for some new generators K and K^{-1}. But we prefer (12) because it is not singular at $t = 0$, and we can thus define $\tilde{A}_t$ as the $\mathbb{C}[[t]]$ algebra A_t when t is a formal parameter. The usual commutation rules of $sl(2)$ are obtained with SC, FC and $F'C$; therefore $A_0 \simeq \mathcal{U}(sl(2)) \otimes P$ where $P \simeq \mathbb{C}[x]/(x^2-1)$ is generated by a parity C ($C^2 = 1$ when $t = 0$).

The formal algebra $\tilde{A}_t$ is thus a deformation of A_0. But it is a domain, while A_0 is not and therefore the $\mathbb{C}[[t]]$ algebras $\tilde{A}_t$ and $A_0[[t]]$ cannot be isomorphic: the deformation is *nontrivial*.

Similarly A_t and A_0 cannot be isomorphic for $t \notin 2\pi\mathbf{Q}$. Furthermore, $\tilde{A}_{t_0+t}$ is a non trivial deformation of A_{t_0} because the Casimir element $Q_t = F'F + SC + S^2\cos t$ takes different values in A_{t_0} and A_{t_0+t} : in the $(2k+1)$-dimensional representation its value is $\sin(kt)\sin\,(k+1)t/\sin^2 t$. Therefore, in contradistinction with the other models, the Jimbo models are not rigid.

e. Topological quantum double. Now that we have good models with a nice duality between them, it is possible to have a good formulation of the quantum double. To this effect we shall consider $\mathcal{H}_t(G)\bar{\otimes}\mathcal{A}_t(G)$ (with inductive tensor product topology) ; its dual is $\mathcal{A}_t(G)\hat{\otimes}\mathcal{H}_t(G)$ (with the projective tensor product topology). Similarly we can consider $C_t^\infty(G)\bar{\otimes}\mathcal{D}'_t(G)$. The following is true [B2].

Theorem 2. *Let A denote $\mathcal{A}(G)$ or $\mathcal{D}'(G)$ or their deformed versions, and H denote $\mathcal{H}(G)$ or $C^\infty(G)$ or their deformed versions. Then the double is $D(A) = A^*\bar{\otimes}A = H\bar{\otimes}A$, and its dual is $D(A)^* = A\hat{\otimes}A^* = A\hat{\otimes}H$. We have $D(A)^{**} = D(A)$, and these algebras are rigid.*

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