ON THE DIMENSION OF DETERMINISTIC AND RANDOM CANTOR-LIKE SETS, SYMBOLIC DYNAMICS, AND THE ECKMANN-RUELLE CONJECTURE

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Abstract. In this paper we unify and extend many of the known results on the dimensions of deterministic and random Cantor-like sets in $\mathbb{R}^n$ using their symbolic representation. We also construct several new examples of such constructions that illustrate some new phenomena. These sets are defined by geometric constructions with arbitrary placement of subsets. We consider Markov constructions, general symbolic constructions, nonstationary constructions, random constructions (determined by a very general distribution), and combinations of the above.

Introduction

In this paper we unify and extend many of the known results on the dimension of deterministic and random Cantor-like sets in $\mathbb{R}^n$. These sets are defined by geometric constructions of different types.

Our most basic construction, which we call a simple geometric construction, defines a Cantor-like set of the form

$$F = \bigcap_{n=1}^{\infty} \bigcup_{(i_1 \cdots i_n)} \Delta_{i_1 \cdots i_n},$$

where the basic sets on the $n-$th step of the construction, $\Delta_{i_1 \cdots i_n}$, $i_k = 1, \cdots, p$, are all disjoint, and $\Delta_{i_1 \cdots i_n j} \subset \Delta_{i_1 \cdots i_n}$, for $j = 1, \cdots, p$. Moreover, the interior and exterior diameters of the set $\Delta_{i_1 \cdots i_n j}$ are less than those of the set $\Delta_{i_1 \cdots i_n}$, with ratios uniformly bounded from below and above by numbers $\underline{\lambda}_j$ and $\overline{\lambda}_j$ respectively. These numbers are called the ratio coefficients of the process. We emphasize that the placement of the sets $\Delta_{i_1 \cdots i_n}$ can be arbitrary as long as they satisfy the above conditions. See Figure 1.

Key words and phrases. Hausdorff dimension, box dimension, Cantor-like set, geometric construction, random geometric construction, gauge function.

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Figure 1. Simple Geometric Process

We also consider a much broader class of geometric constructions that includes

(1) Markov constructions where admissible basic sets $\Delta_{i_1\cdots i_n}$ are determined by a transfer matrix A,
(2) general symbolic constructions where admissible basic sets $\Delta_{i_1\cdots i_n}$ are determined by a compact shift-invariant subset of the full shift on p symbols,
(3) constructions where the numbers $\underline{\lambda}_j$ and $\overline{\lambda}_j$ may vary from step to step but have a well defined limiting behavior,
(4) random constructions where the numbers $\underline{\lambda}_j$ and $\overline{\lambda}_j$ are chosen randomly at each step of the construction from a very general class of distributions,
(5) combinations of the above constructions.

Our main results deal with the general types of geometric constructions and provide below and above estimates for the Hausdorff dimension and the box dimension of the limit Cantor-like set. In addition, we prove strict positivity and boundedness of the Hausdorff measure of the limit set for a broad class of constructions.

There are three main methods for obtaining a lower bound for the Hausdorff dimension of a set: the mass distribution principle, the potential principle, and the nonuniform mass distribution principle. See Appendix 2. The mass distribution principle requires the existence of a measure m for which

$$m(B(x,r)) \leq Cr^s, \tag{1}$$

where $B(x,r)$ is the ball of radius r centered at the point $x \in F$. Then s produces a lower bound for $\dim_H F$, the Hausdorff dimension of F. We stress that (1) must hold for *all* $x \in F$ and the constant C is *independent* of x and r. The (uniform) mass distribution principle is the strongest of the three methods, and it is thus quite surprising that it can be used to obtain a good lower estimate for the Hausdorff dimension of the limit set for many of the different geometric constructions that we consider.

One particular and important case is when a geometric construction has sets on the nth step that are strictly geometrically similar to the corresponding sets on the $(n-1)$th step. This means that there is a collection of similarity maps (affine contractions) or more generally contraction maps $h_1, \cdots, h_p$ such that

$$\Delta_{i_1 \cdots i_n} = h_{i_1} \circ \cdots \circ h_{i_n}(\Delta)$$

(see Section 4), where Δ denotes a ball in $\mathbb{R}^n$. This construction is called a similarity construction since it exposes a self-similar character of the geometric process. These special constructions were one of the main objects of study in dimension theory for many years.

About 50 years ago, Moran [Mo] computed the Hausdorff dimension of geometric constructions in $\mathbb{R}^n$ given by p non-overlapping sets with constant ratio coefficients $\underline{\lambda}_j = \overline{\lambda}_j = \lambda_j$, for $j = 1, \cdots, p$. We call this *a Moran process.* For the Moran process the sets on the nth step *need not* be strictly geometrically similar to the corresponding sets on the $(n-1)$th step. Moran discovered the formula $s = \dim_H F$, where s is the unique root of the equation

$$\sum_{i=1}^{p} \lambda_i^t = 1. \tag{2}$$

He also showed that the s dimensional Hausdorff measure of the limit set is finite and strictly positive.

Moran's great insight was to realize that the similarity maps, or even the spacing of the sets $\{\Delta_{i_1 \cdots i_n}\}$ are not important in the calculation of the Hausdorff dimension of the limit set: the dimension depends only on the ratio coefficients. Moran proved this using the (uniform) mass distribution principle applied to the $s-$dimensional Hausdorff measure. We believe this is a seminal paper that should be much better known. There are very few papers in the literature dealing with geometric constructions that do not involve similarity maps.

In this paper we extend the Moran approach to much broader classes of geometric constructions including general symbolic constructions, non-stationary constructions, and random constructions as well as generalize his method to study local pointwise dimension and box dimension of the limit set.

The symbolic representation of a Cantor-like geometric process is the key point in our study. For the Moran process, the crucial observation is the existence of a measure m on the set F such that the Hausdorff dimension of F is equal to the Hausdorff dimension of m (see Appendix 2). To construct this measure, we consider the symbolic space associated with the process. The measure m is the pullback of the equilibrium state for the function $s\phi(x) = s \log \lambda_{i_1}$ where x is associated with a sequence $(i_1 i_2 \cdots)$ and s is the unique root of the equation

$$P(s\phi) = 0. \tag{3}$$

For the Moran constructions, s is the Hausdorff dimension of the limit set as well as its box dimension. We show that this is true for general symbolic geometric constructions provided $\underline{\lambda}_i = \overline{\lambda}_i = \lambda_i$. See Corollary 1.

The equation (3) was discovered by Bowen [Bo2] and seems to be universal. We will show that all known equations previously used to compute the Hausdorff dimension (for example equation (2)) coincide with or are particular cases of (3).

We emphasize that for the general symbolic geometric constructions the measure m is an equilibrium measure and admits the nonuniform mass distribution principle. There is a crucial difference between Gibbs measures and equilibrium measures in statistical physics ([R] and Appendix 3; see also Section 2, Remark 1). These notions coincide for subshifts of finite type but need not coincide for general symbolic systems. We show that if m is a Gibbs measure, then the Hausdorff dimension of the limit set, s, can be studied using the uniform mass distribution principle, and the $s-$dimensional Hausdorff measure is equivalent to m (see Proposition 2). If m is an equilibrium state, then a nonuniform mass distribution principle can be applied. See Theorem 1.

Theorem 2 establishes the coincidence of the Hausdorff dimension and the box dimension of a measure and is similar in spirit to some results of Ledrappier and Young [LY]. It supports a general belief that the coincidence of the Hausdorff dimension and box dimension of a *set* is a rare phenomenon and requires *rigid* geometric constructions like some of those considered above. In Section 5, we provide an example of a simple geometric construction for which the Hausdorff dimension $\dim_H F$, the lower box dimension $\overline{\dim}_B F$, and the upper box dimension $\overline{\dim}_B F$, are distinct. The construction uses different ratio coefficients in different directions and combines both vertical and horizontal stacking of rectangles to effect the noncoincidence of the dimensions. In [PW], the authors found that number theoretic properties of the ratio coefficients, related to Pisot numbers, is another mechanism to cause noncoincidence of dimensions.

The coincidence of the Hausdorff dimension and box dimension of a *measure* is more common. There is a general criterion proved by Young ([Y]) that guarantees the coincidence. Namely, let m be a Borel measure on F such that for $m-$almost every $x \in F$ the limit

$$\lim_{r\to 0} \frac{\log m(B(x,r))}{\log r} = d_m(x)$$

exists. This limit $d_m(x)$ is called the pointwise or local dimension at x. In this case m is called exact dimensional. If the limit does not exist, one can consider the lower and upper limits $\underline{d}_m(x), \overline{d}_m(x)$ to obtain the lower and upper pointwise dimensions at x. If for m-almost every x

$$d_m(x) = \text{const} \stackrel{\text{def}}{\equiv} s$$

then the Hausdorff dimension of m, $\dim_H m$, and the lower and upper box dimensions of m, $\underline{\dim}_B m$ and $\overline{\dim}_B m$ coincide and have the common value s.

This statement poses the problem of whether a given measure m is exact dimensional and moreover whether $d_m(x) = const$ almost everywhere. In [C], Cutler constructed an example of a continuous map that preserves an ergodic exact dimensional measure m with $d_m(x)$ essentially non-constant. In Section 6.4, we present a much simplier version of the construction of such a map that uses a special nonstationary simple geometric construction. If the map is smooth, then m is exact dimensional and ergodic, and hence $d_m(x) = const$ almost everywhere since $d_m(x)$ is invariant under the map and measurable. Ledrappier and Misiurewicz [LM] constructed a one-dimensional smooth map preserving an ergodic measure that is not exact dimensional. Eckmann and Ruelle conjectured that an ergodic measure invariant under a C^2-diffeomorphism with non-zero Lyapunov exponents is exact dimensional (and, hence, $d_m(x) = const$, see [ER]). This was proved in [Y] for two-dimensional maps and in [PY] for some measures including Gibbs measures for Axiom A diffeomorphisms. See also [L]. In Section 6, we construct a homeomorphism having an ergodic invariant Gibbs measure with positive entropy that has different upper and lower pointwise dimensions almost everywhere. In other words, the *continuous* version of the Eckmann-Ruelle Conjecture fails.

Mauldin and Williams [MW1] computed the Hausdorff dimension for a Markov process given by similarity maps. They used symbolic dynamics, large deviation theory, and graph theory. We give a considerable generalization of their result as well as a considerable simplification of their proof. The Moran process in the Markov case was studied by Stella [St] with some additional strong assumptions. Using the potential principle, Afraimovich and Shereshevsky [AS] found a lower estimate for the Hausdorff dimension of some simple geometric constructions. Similar types of simple geometric constructions, given by two-dimensional self-affine maps related to graphs of functions, were considered by Bedford and Urbanski in [BU]. Shereshevsky [S] also considered some one-dimensional Markov geometric constructions.

An essential part of our work consists of analyzing geometric constructions where the ratio coefficients depend on steps of the construction and admit *nonstationary* lower and upper estimates given by $\underline{\lambda}_{i_n}, \overline{\lambda}_{i_n}, i_n = 1, \cdots, p$, $n = 1, 2, \cdots$. In order to control the Hausdorff dimension and box dimension, one requires some kind of good asymptotic behavior when n tends to infinity. We introduce two conditions, *strong asymptotic and weak asymptotic*, under which information about dimension can be obtained. Geometric constructions satisfying the strong asymptotic condition can be treated by using the uniform mass distribution principle while geometric constructions satisfying the weak asymptotic condition need not admit the uniform mass distribution principle. See Example 4. In the weak asymptotic case, a lower bound for the Hausdorff dimension can be obtained using the nonuniform mass distribution principle, and in some cases an upper estimation for the upper box dimension can be found. Moreover, the Hausdorff measure at dimension s can be either zero or infinite. In order to obtain more information about Hausdorff measure one can use a *gauge function* (see Appendix 4). In Section 5, we exhibit a family of simple geometric constructions that satisfy the weak asymptotic condition and admit infinitely many gauge functions depending on the rate of convergence of ratio coefficients. This illustrates that the asymptotic category of constructions is quite rich in that there exists a large variety of different limit sets exhibiting many different structural properties.

Geometric constructions of random type have been considered by Falconer [F2], Graf [G], Kahane [K], Graff, Mauldin and Williams [GMW], and Mauldin and Williams [MW2]. These authors studied special types of branching processes that correspond to the full shift on Σ_p^+ with p^n ratio coefficients at step n chosen randomly, essentially independently and with the same distribution on $(0, 1)$. Furthermore, they assume some independence conditions over n. In this paper, we consider branching processes that are associated with compact shift-invariant subsets in Σ_p^+. We generate the ratio coefficients by chosing p random numbers on the interval (a, b) where $0 < a \leq b < 1$. We do not require that these numbers be independent nor be indentically distributed. *Our major idea in studying geometric constructions of random type is to reduce the study to geometric constructions satisfying the weak asymptotic condition.*

A good general reference for dimension theory is [F1].

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Section 1: Geometric Constructions

We define several different types of geometric constructions of Cantor-like subsets of $\mathbb{R}^n$ where the ration coefficients may vary from step to step but admit uniform estimates from below and above. We begin with the simplest construction.

1) A **simple geometric construction** is defined by

a) $2p$ numbers $\underline{\lambda}_i, \overline{\lambda}_i$, $i = 1, \cdots, p$ such that $0 < \underline{\lambda}_i \leq \overline{\lambda}_i < 1$.

b) a family of closed sets $\{\Delta_{i_1 \cdots i_n}\} \subset \mathbb{R}^d$ for $i_j = 1, 2, \ldots, p$, and $n \in \mathbb{N}$, which satisfy

$$D(C_1 \prod_{j=1}^{n} \underline{\lambda}_{i_j}) \subset \Delta_{i_1 \cdots i_n} \subset D(C_2 \prod_{j=1}^{n} \overline{\lambda}_{i_j}) \tag{4}$$

where $D(r)$ denotes a closed ball of radius r and C_1 and C_2 are positive constants. For any sequence $(i_1 \cdots i_{n+1}) \in \{1, \cdots, p\}^n$, we require that

$$\Delta_{i_1 \cdots i_n i_{n+1}} \subset \Delta_{i_1 \cdots i_n}$$

and

$$\Delta_{i_1 \cdots i_n} \cap \Delta_{i'_1 \cdots i'_n} = \emptyset \quad \text{if } (i_1 \cdots i_n) \neq (i'_1, \cdots, i'_n). \tag{5}$$

Let

$$F = \bigcap_{n=1}^{\infty} \bigcup_{(i_1 \cdots i_n)} \Delta_{i_1 \cdots i_n}.$$

F is called the *limit set* and is a generalized Cantor set, i.e., it is a perfect, nowhere dense, and totally disconnected set. We stress that the placement of the sets $\{\Delta_{i_1 \cdots i_n}\}$ is completely arbitrary, and we make no assumptions on the regularity of the boundaries of the sets $\{\Delta_{i_1 \cdots i_n}\}$ which can be fractal. They need not be open or closed, but they should contain balls and be contained in balls. The interior balls obviously do not overlap, but the exterior balls may overlap. See Figure 1.

The simple geometric construction has a symbolic description in the space of all one-sided infinite sequences $(i_1 i_2 \cdots)$. We denote this space by Σ_p^+ and endow it with its usual topology (see Appendix 3). Namely, given $x \in F$ and $n > 0$, there exists a unique set $\Delta_{i_1 \cdots i_k}(x)$ that contains x and hence $x = \bigcap_{k=1}^{\infty} \Delta_{i_1 \cdots i_k}(x)$. This gives a unique one-sided infinite sequence $(i_1 i_2 \cdots)$ such that the mapping $\chi : F \to \Sigma_p^+$ defined by $x \to (i_1 i_2 \cdots)$

is a homeomorphism from F onto Σ_p^+. More general constructions are determined by closed shift-invariant subsets of Σ_p^+. The associated symbolic dynamics is our main tool to compute the Hausdorff dimension of the limit set F.

2) A **symbolic geometric construction** is defined by

a) $2p$ numbers $\underline{\lambda}_i, \overline{\lambda}_i$, $i = 1, \cdots, p$ such that $0 < \underline{\lambda}_i \leq \overline{\lambda}_i < 1$.

b) a compact set $Q \subset \Sigma_p^+$ invariant under the shift σ (i.e., $\sigma(Q) = Q$).

c) a family of closed sets $\{\Delta_{i_1 \cdots i_n}\} \subset \mathbb{R}^d$ for $i_j = 1, 2, \ldots, p$ and $n \in \mathbb{N}$ where the n-tuples $(i_1 \cdots i_n)$ are admissible with respect to Q (i.e., there exists $\omega = (i'_1, i'_2, \cdots) \in Q$ such that $i'_1 = i_1, i'_2 = i_2, \cdots, i'_n = i_n$) and these sets satisfy (4) and (5).

The *limit set* F for this construction is defined by

$$F = \bigcap_{n=1}^{\infty} \bigcup_{\substack{(i_1 \cdots i_n) \\ \text{admissible}}} \Delta_{i_1 \cdots i_n}.$$

We again stress that the placement of the sets $\{\Delta_{i_1 \cdots i_n}\}$ is completely arbitrary, and we make no assumptions on the regularity of the boundaries of the sets $\{\Delta_{i_1 \cdots i_n}\}$. In the particular case when the set $Q = \Sigma_A^+$ the construction is called a **Markov geometric construction**. Here A denotes a $p \times p$ transfer matrix with entries $A(i,j) = 0$ or 1 and Σ_A^+ consists of admissible sequences $(i_1 i_2 \cdots)$ with respect to A (i.e., $A(i_j, i_{j+1}) = 1$ for $j = 1, 2 \cdots$). In a simple geometric construction the *actual size* of the sets $\Delta_{i_1 \cdots i_n}$ may vary from step to step and in general depends on the *past*, (i.e., numbers $\{i_1 \cdots i_n\}$). In the multi-dimensional case, the sets $\Delta_{i_1 \cdots i_n}$ are *non-isotropic* in the sense that they may have different *sizes* in different directions. The analysis in such a general case is quite complicated. We shall deal with estimates that are isotropic, i.e., there are estimates on the radii of inscribed and circumscribed balls for the sets $\Delta_{i_1 \cdots i_n}$. Condition (4) provides isotropic lower and upper bounds that are also *stationary* over n. This will enable us to obtain crude lower and upper estimates for the Hausdorff dimension and box dimension of the limit set. More refined estimates can be obtained when more information about the size of the sets $\Delta_{i_1 \cdots i_n}$ is available. We wish to consider processes with ratio coefficients admitting *nonstationary* lower and upper estimates given by sequences of numbers $\underline{\lambda}_{i,n}, \overline{\lambda}_{i,n}$, $i = 1, \cdots, p$, and $n \in \mathbb{N}$.

3) A **symbolic geometric construction with nonstationary bounds** is given by

a) sequences of numbers $\underline{\lambda}_{i,n}, \overline{\lambda}_{i,n}$, $i = 1, \cdots, p$, and $n \in \mathbb{N}$ such that $0 < \alpha \leq \underline{\lambda}_{i,n} \leq \overline{\lambda}_{i,n} \leq \beta < 1$.

b) a compact set $Q \subset \Sigma_p^+$ invariant under the shift σ (i.e., $\sigma(Q) = Q$).

c) a family of sets $\{\Delta_{i_1 \cdots i_n}\} \subset \mathbb{R}^n$ for $i_j = 1, 2, \ldots, p$, and $n \in \mathbb{N}$ where the n-tuples $(i_1 \cdots i_n)$ are admissible with respect to Q and these sets satisfy

$$D(C_1 \prod_{j=1}^{n} \underline{\lambda}_{i_j,j}) \subset \Delta_{i_1 \cdots i_n} \subset D(C_2 \prod_{j=1}^{n} \overline{\lambda}_{i_j,j}) \tag{6}$$

where C_1, C_2 are positive constants. We also require (5).

The *limit set* F is defined by

$$F = \bigcap_{n=1}^{\infty} \bigcup_{\substack{(i_1 \cdots i_n) \\ \text{admissible}}} \Delta_{i_1 \cdots i_n}.$$

In the particular case when the set $Q = \Sigma_A^+$, the construction is called a **Markov geometric construction with nonstationary bounds** and if $Q = \Sigma_p^+$ the construction is called a **simple geometric construction with nonstationary bounds.**

4) A **random symbolic geometric construction** is defined by

a) a stochastic vector process $(\Lambda, \mathfrak{F}, \nu)$ with $\Lambda = \{\vec{\lambda} = (\underline{\lambda}_{i,n}, \overline{\lambda}_{i,n}),\ i = 1, \cdots, p$ and $n \in \mathbb{N}\}$ where $0 < \alpha \leq \underline{\lambda}_{j,n} \leq \overline{\lambda}_{j,n} \leq \beta < 1$, $\mathfrak{F}$ denotes the $\sigma-$algebra of Borel sets in Λ, and ν is an arbitrary stationary shift-invariant ergodic Borel probability measure on Λ.

b) a compact set $Q \subset \Sigma_p^+$ invariant under the shift (i.e., $\sigma(Q) = Q$).

c) for $\nu-$almost every $\vec{\lambda} \in \Lambda$, a family of sets $\{\Delta_{i_1 \cdots i_n}(\vec{\lambda})\} \subset \mathbb{R}^n$ for $i_j = 1, 2, \ldots, p$, where the n-tuple $(i_1 \cdots i_n)$ is admissible with respect to Q. We assume that these sets satisfy

$$D(C_1 \prod_{j=1}^{n} \underline{\lambda}_{i_j,j}(\vec{\lambda})) \subset \Delta_{i_1 \cdots i_n}(\vec{\lambda}) \subset D(C_2 \prod_{j=1}^{n} \overline{\lambda}_{i_j,j}(\vec{\lambda}))$$

where C_1, C_2 are positive constants.

d) for any sequence $(i_1 \cdots i_n) \in \{1, \cdots p\}^n$, we require that $\Delta_{i_1 \cdots i_{n+1}}(\vec{\lambda}) \subset \Delta_{i_1 \cdots i_n}(\vec{\lambda})$ and

$$\Delta_{i_1 \cdots i_n}(\vec{\lambda}) \cap \Delta_{i'_1 \cdots i'_n}(\vec{\lambda}) = \emptyset, \text{ if } (i_1 \cdots i_n) \neq (i'_1, \cdots i'_n).$$

For every $\vec{\lambda} \in \Lambda$, the *limit set*

$$F(\vec{\lambda}) = \bigcap_{n=1}^{\infty} \bigcup_{\substack{(i_1 \cdots i_n) \\ \text{admissible}}} \Delta_{i_1 \cdots i_n}(\vec{\lambda})$$

is a perfect, nowhere dense, totally disconnected set.

Section 2: Main Results: geometric constructions with stationary bounds

We begin with the symbolic geometric construction. Consider the symbolic dynamical system (Q, σ), where $Q \subset \Sigma_p^+$. There exist two numbers $\underline{s}$, $\overline{s}$, that are uniquely defined, such that $P(\underline{s} \log \underline{\lambda}_{i_1}) = 0$ and $P(\overline{s} \log \overline{\lambda}_{i_1}) = 0$, where P denotes the thermodynamic pressure (see Appendix 3).

Theorem 1 below provides estimates for the Hausdorff dimension and box dimension of the limit set F for a symbolic geometric construction while Theorem 2 deals with the dimensions of special equilibrium measures concentrated on F for which a (non-uniform) mass distribution principle holds.

Theorem 1. *Let F be the limit set specified by a symbolic geometric construction. Then*

(1) $\underline{s} \le \dim_H F \le \underline{\dim}_B F \le \overline{\dim}_B F \le \overline{s}$,

(2) $0 < m_{\mathrm{H}}(\underline{s}, F)$ *and* $m_{\mathrm{H}}(\overline{s}, F) < \infty$, *where* $\dim_H F$, $\underline{\dim}_B F$, *and* $\overline{\dim}_B F$ *are respectively the Hausdorff dimension and the lower and upper box dimensions of the set* F, *and* $m_{\mathrm{H}}(t, F)$ *denotes the* $t-$*dimensional Hausdorff measure of* F *(see Appendix 1).*

The second statement in Theorem 1 is nontrivial only when $\underline{s} = \dim_H F$ or $\overline{s} = \dim_H F$. Otherwise, $m_H(\underline{s}, F) = \infty$ or $m_H(\overline{s}, F) = 0$. See Corollary 1. If $\underline{s} < \dim_H F < \overline{s}$, then the Haudorff measure at dimension may be zero or infinite. In example 4 of Section 5 we present simple geometric constructions where the Hausdorff measure of the limit set at dimension is zero.

The lower and upper estimates $\underline{s}$ and $\overline{s}$ in Theorem 1 clearly do not depend on the placement of the sets $\{\Delta_{i_1 \cdots i_n}\}$. However, the actual value of the Hausdorff dimension and box dimensions can be any number between $\underline{s}$ and $\overline{s}$ and *a priori*, may depend on the placement of the sets. See [BT] and [Tr] for details in the one-dimensional case. In Example 6 of Section 5, we show this for the lower box dimension.

The next theorem uses the notion of pointwise dimension of a measure (see Appendix 2). Let $\underline{\mu}$ and $\overline{\mu}$ denote the equilibrium measures for the functions $\underline{s} \log \underline{\lambda}_{i_1}$ $\overline{s} \log \overline{\lambda}_{i_1}$ on Q, and $\underline{m}$, $\overline{m}$ be the pull back measures under the coding map χ.

Theorem 2. *Let F be the limit set specified by a symbolic geometric construction. Then*

(1) $\underline{s} \le \underline{d}_{\underline{m}}(x)$ *for* $\underline{m}-$ *almost every* $x \in F$ *and* $\overline{s} \ge \overline{d}_{\overline{m}}(x)$ *for* $\overline{m}-$ *almost every* $x \in F$,

(2) *If* $\underline{\lambda}_i = \overline{\lambda}_i = \lambda_i$ *for* $1 \le i \le p$ *then* $\underline{m} = \overline{m} = m$, $\underline{s} = \overline{s} = s$, *and* $\underline{d}_m(x) = \overline{d}_m(x) = s$ *for* $m-$*almost every* $x \in F$. *In particular,*

$$\dim_H m = \underline{\dim}_B m = \overline{\dim}_B m = \dim_H F = \underline{\dim}_B F = \overline{\dim}_B F = s.$$

In general, one can not expect the lower and upper pointwise dimensions $\underline{d}_{\underline{m}}(x)$ and $\overline{d}_{\underline{m}}(x)$ (as well as $\underline{d}_{\overline{m}}(x)$ and $\overline{d}_{\overline{m}}(x)$) to coincide almost everywhere. See Sections 6.2 and 6.3 and also Remark (4).

Corollary 1. *Let F be the limit set specified by a symbolic geometric construction with* $\underline{\lambda}_i = \overline{\lambda}_i = \lambda_i$ *for* $i = 1, \cdots, p$. *Then*

$$s = \dim_H F = \underline{\dim}_B F = \overline{\dim}_B F$$

where s is the unique root of the equation $P(s \log \lambda_{i_1}) = 0$.

The lower bound for the Hausdorff dimension of the limit set follows from the lower bound for the lower pointwise dimension of $\underline{m}$ in Theorem 2. To prove this we first observe that the equilibrium measure $\underline{m}$ satisfies the nonuniform mass distribution principle and then apply the Moran method [Mo]. The arguments in the proof of the upper bound for the upper box dimension are essentially due to Bowen, and were pointed out to us by F. Ledrappier.

The coincidence of Hausdorff dimension and box dimension was established by Falconer [F3] in a very special case when the symbolic construction is defined by similarity maps. However, he did not compute the actual value of the dimension. Peres has observed that some results in the general case can be easily obtained by studying the symbolic geometric construction defined by similarity maps (see Section 4). This enables us to obtain lower and upper estimates for the Hausdorff measure of the limit set at the dimension (see Proposition 4 below).

We now study the Markov geometric construction which is a special type of symbolic geometric constructions. In this case we have our most refined statements about the Hausdorff dimension and Hausdorf measure. Consider the subshift of finite type (Σ_A^+, σ). Given p numbers $0 < \alpha_1, \cdots, \alpha_p < 1$, we define a $(p \times p)$ diagonal matrix $M_t = \mathrm{diag}(\alpha_1^t, \cdots, \alpha_p^t)$. Let $\rho(B)$ denote the spectral radius of the matrix B.

Proposition 1. *The equation $P(s \log \alpha_{i_1}) = 0$ is equivalent to the equation $\rho(A^* M_s) = 1$, where A^* denotes the transpose of the matrix A.*

This proposition immediately implies that the numbers $\underline{s}$ and $\overline{s}$ are the unique solutions of the equations $\rho(A^*\underline{\Lambda}_t) = 1$ and $\rho(A^*\overline{\Lambda}_t) = 1$, where $\underline{\Lambda}_t = \mathrm{diag}(\underline{\lambda}_1^t, \cdots, \underline{\lambda}_p^t)$ and $\overline{\Lambda}_t = \mathrm{diag}(\overline{\lambda}_1^t, \cdots, \overline{\lambda}_p^t)$.

¿From now on we will assume that the transfer matrix A is transitive, i.e., there exists $N \in \mathbb{N}$ such that $(A^N)_{i,j} > 0$ for all i and j. See Remark (3) below.

The main difference between the Markov and general symbolic cases is that the measures $\underline{\mu}$ and $\overline{\mu}$ are Gibbs measures, i.e., they satisfy (22), see [R]. A manifestation of this is that the pull-back measure $\underline{m}$ satisfies the uniform mass distribution principle.

The next proposition gives a detailed description of the dimension of the limit set for general symbolic geometric constructions where the measures $\underline{\mu}$ and $\overline{\mu}$ are Gibbs measures.

Proposition 2. *Let F be the limit set specified by a symbolic geometric construction. Assume that the measures $\underline{\mu}$ and $\overline{\mu}$ are Gibbs measures.*

(1) *The measure $\underline{m}$ satisfies the uniform mass distribution principle.*
(2) *For* **every** *$x \in F$, $\underline{s} \le \underline{d}_{\underline{m}}(x)$ and $\overline{s} \ge \overline{d}_{\overline{m}}(x)$.*
(3) *If $\lambda_i = \underline{\lambda}_i = \overline{\lambda}_i$, for $i = 1, \cdots, p$, then $\underline{\mu} = \overline{\mu} \stackrel{\text{def}}{\equiv} \mu$ and hence $\underline{m} = \overline{m} \stackrel{\text{def}}{\equiv} m = \chi^*\mu$ and $s = \underline{d}_m(x) = \overline{d}_m(x)$ for* **every** *$x \in F$. Moreover, the Hausdorff measure $m_H(s, \cdot)$ is equivalent to the measure $m(\cdot)$.*
(4) *If $\underline{\lambda}_i = \overline{\lambda}_i = \lambda_i$, for $i = 1, \cdots, p$, then*

$$s = dim_H F = \underline{dim}_B F = \overline{dim}_B F.$$

The next corollaries give a detailed description of the dimension of the limit set for Markov and simple geometric constructions and immediately follows from Proposition 2.

Corollary 2. *Let F be the limit set specified by a Markov geometric construction. Then statements (1) to (4) of Proposition 2 hold, and in addition, $s = \frac{\log \rho(A)}{\log(\frac{1}{\lambda})}$.*

Corollary 3. *Let F be the limit set specified by a simple geometric construction.*

(1) $\underline{s} \leq dim_H F \leq \underline{dim}_B F \leq \overline{dim}_B F \leq \overline{s}$, *where $\underline{s}$ and $\overline{s}$ are the unique roots of the equations*

$$\sum_{i=1}^{p} \underline{\lambda}_i^t = 1 \quad \text{and} \quad \sum_{i=1}^{p} \overline{\lambda}_i^t = 1$$

respectively. Moreover, the Gibbs measures $\underline{\mu}, \overline{\mu}$ on Σ_p^+ satisfy

$$\underline{\mu}(C_{i_1 \cdots i_n}) = \prod_{j=1}^{n} \underline{\lambda}_{i_j}^{\underline{s}} \ \text{ and } \ \overline{\mu}(C_{i_1 \cdots i_n}) = \prod_{j=1}^{n} \overline{\lambda}_{i_j}^{\overline{s}}$$

where $C_{i_1 \cdots i_n}$ is a cylinder set.

(2) *If $\underline{\lambda}_i = \overline{\lambda}_i = \lambda_i$, for $i = 1, \cdots, p$, then*

$$s = dim_H F = \underline{dim}_B F = \overline{dim}_B F$$

where s is the unique root of the equation $\sum_{i=1}^{p} \lambda_i^t = 1$. Moreover $0 < m_{\mathrm{H}}(s, F) < \infty$. In particular, if $\lambda_i = \lambda$ for $i = 1, \cdots, p$ then $s = \log p / \log(1/\lambda)$ (see [Mo]*).*

The next proposition establishes upper estimates for the upper box dimension for symbolic geometric constructions. In particular, we show that if the topological entropy of the shift is zero, then the dimension of the limit set is also zero. In this case the measures $\underline{\mu}$ and $\overline{\mu}$ are measures of maximal entropy and are not Gibbs measures.

Proposition 3.

Let F be the limit set specified by a symbolic geometric construction.

(1) We have

$$\overline{s} \leq \frac{h(\sigma \,|\, Q)}{-\log \overline{\lambda}_{\max}}$$

where $\overline{\lambda}_{\max} = \max_{k=1}^{p}\{\overline{\lambda}_k\}$, and $h(\sigma \,|\, Q)$ denotes the topological entropy. Equality occurs if $\overline{\lambda}_i = \overline{\lambda}$ for $i = 1, \cdots, p$.

(2) See [Fu]. If $\underline{\lambda}_i = \overline{\lambda}_i = \lambda$ for $i = 1, \cdots, p$, then

$$\underline{s} = \overline{s} = \dim_H F = \underline{\dim}_B F = \overline{\dim}_B F = \frac{h(\sigma \,|\, Q)}{-\log \lambda}.$$

(3) If $h(\sigma \,|\, Q) = 0$, then $\underline{s} = \overline{s} = \dim_H F = \underline{\dim}_B F = \overline{\dim}_B F = 0$.

Remarks.

(1) The basic fact underlying the proofs of Theorems 1 and 2 is that the equilibrium measure $\underline{m}$ satisfies the nonuniform mass distribution principle. If this measure is a Gibbs measure, then as we have mentioned it satisfies the uniform mass distribution principle.

Little is known about the existence of Gibbs measures for general symbolic dynamical systems. We believe that one can find a compact shift-invariant subset $Q \subset \Sigma_p^+$ with positive topological entropy of $\sigma \,|\, Q$ and can build a symbolic geometric construction with some $\underline{\lambda}_i = \overline{\lambda}_i = \lambda_i$ such that the measure $m = \underline{m} = \overline{m}$ is not a Gibbs measure and is not equivalent to the $s-$Hausdorff measure.

If the topological entropy of $\sigma \,|\, Q$ is zero, then by Proposition 3, the Hausdorff dimension and box dimension coincide and are zero, but the measures $\underline{m}$ and $\overline{m}$ are not Gibbs measures.

(2) Two closed sets are called non-overlapping if their intersection consists of only boundary points. By examining the proofs of Theorems 1, 2 and Propositions 2, 3, one can show that they are valid for geometric constructions on the line with non-overlapping sets. This is due to the use of the mass distribution principle. See [Mo] and [MW1].

(3) In the Markov geometric constructions above, we assumed that the transfer matrix A was transitive. For an arbitrary transfer matrix A, one can decompose the set Σ_A^+ into two shift-invariant subsets: the wandering set Q_1 and the non-wandering set Q_2. The latter can be also partitioned into finitely many shift-invariant subsets of the form $\Sigma_{A_i}^+$ where each matrix A_i is transitive and corresponds to a class of equivalent recurrent states. See [AJ]. The limit set F contains disjoint sets $F_i = \chi^{-1}(\Sigma_{A_i}^+)$. Each set F_i is the limit set for a Markov geometric construction defined by the transitive matrix A_i and hence admits lower and upper estimates for the Hausdorff and box dimensions stated in Theorems 1, 2 and Propositions 1, 2, 3. In [MW1], the authors discuss the effect of the wandering set Q_1 on the dimension of the limit set F.

(4) Let F be the limit set specified by a symbolic geometric construction. Define the map $G : F \to F$ by $G(x) = \chi^{-1} \circ \sigma \circ \chi(x)$. It is easy to see that G is a continuous endomorphism such that the set $G^{-1}(x), x \in F$ consists of finitely many points. If μ is a $\sigma-$invariant measure then its pull back measure $m = \chi^* \mu$ is $G-$invariant. In particular, the measures $\underline{m}$ and $\overline{m}$ are $G-$invariant. Moreover, they satisfy

$$P(\underline{s}\underline{\phi}) = \sup_m \left(h_m(G) + \underline{s} \int_F \underline{\phi} dm \right) = h_{\underline{m}}(G) + \underline{s} \int_F \underline{\phi} d\underline{m} = 0$$

$$P(\overline{s}\overline{\phi}) = \sup_m \left(h_m(G) + \overline{s} \int_F \overline{\phi} dm \right) = h_{\overline{m}}(G) + \overline{s} \int_F \overline{\phi} d\overline{m} = 0$$

where $\underline{\phi}(x) = \log \underline{\lambda}_k, \overline{\phi}(x) = \log \overline{\lambda}_k$ if $x \in \Delta_k$, $h_m(G)$ is the Kolmogorov-Sinai entropy of G and $P(\psi)$ is the topological pressure of a function ψ on F with respect to G (see Appendix 3). In other words, the measures $\underline{m}$ and $\overline{m}$ are equilibrium measures corresponding to the functions $\underline{s}\underline{\phi}$ and $\overline{s}\overline{\phi}$. If the geometric construction is Markov and is given by a transitive transfer matrix A, then G is ergodic, and moreover has the Bernoulli property with respect to $\underline{m}$ and $\overline{m}$. As we will see in Section 6.2, it may happen that $\underline{d}_{\underline{m}}(x) \neq \overline{d}_{\underline{m}}(x)$ for $\underline{m}-$almost every $x \in F$ and $\underline{d}_{\overline{m}}(x) \neq \overline{d}_{\overline{m}}(x)$ for $\overline{m}-$almost every $x \in F$. Such a map G is an example of a continuous but not smooth map possessing an equilibrium measure which is not exact dimensional. Note that G is not invertible. In Section 6.3 we will construct a homeomorphism that is not exact dimensional.

If $\underline{\lambda}_i = \overline{\lambda}_i = \lambda_i$ for $i = 1, \cdots, p$, then according to Theorem 2 $\underline{m} = \overline{m} = m$ and $\underline{s} = \overline{s} = s$. The latter can be expressed as

$$s = -\frac{h_m(G)}{\int_F \phi dm}$$

where $\phi(x) = \log \lambda_k$ if $x \in \Delta_k$. In this case, $\underline{d}_m(x) = \overline{d}_m(x) = s$.

(5) Let us consider the full shift σ on p symbols with the standard metric d_β, $\beta > 1$. Let μ be a $\sigma-$invariant ergodic measure, and let $r = \beta^{-n}$. Since the ball centered at ω with radius r, $B(\omega, r)$, is a cylinder set, we have by a theorem of Brin and Katok [BK] that for μ almost every ω,

$$\frac{\log \mu(B(\omega, r))}{\log r} = -\frac{\log \mu(B(\omega, \beta^{-n}))}{n \log \beta} \stackrel{n\to\infty}{\longrightarrow} \frac{h_\mu(\sigma)}{\log \beta},$$

hence

$$\frac{h_\mu(\sigma)}{\log \beta} = \underline{d}_\mu(\omega) = \overline{d}_\mu(\omega) = \dim_H \mu = \underline{\dim}_B \mu = \overline{\dim}_B \mu,$$

where $h_\mu(\sigma)$ denotes the Kolmogorov-Sinai entropy of the shift map. It is evident that the common value depends on β, which is not surprising since the two metrics d_{β_1} and d_{β_2} are not equivalent for $\beta_1 \neq \beta_2$. Now consider the simple geometric construction with $\underline{\lambda}_i = \overline{\lambda}_i = \lambda_i$ for $1 \leq i \leq p$. Let μ_ϕ be the unique Gibbs measure corresponding to the function $s\phi(\omega) = s \log \lambda_{\omega_1}$ with s the unique root of the equation $P(s\phi) = 0$. We have that $h_{\mu_\phi}(\sigma) + s \int \log \lambda_{\omega_1} d\mu_\phi = 0$, which gives that

$$s = -\frac{h_{\mu_\phi}(\sigma)}{\int \phi d\mu_\phi} = \frac{h_{\mu_\phi}(\sigma)}{\log \beta}$$

if

$$\beta = \exp\left(-\int \phi d\mu_\phi\right). \tag{7}$$

Thus the Hausdorff dimension of μ_ϕ calculated with respect to the two metrics d_β and ρ (the Euclidean metric on $\mathbb{R}^d$) coincide. The two metrics $\chi^* \rho$ and d_β with β satisfying (7) are not equivalent if not all the λ_i coincide. To see this, notice that $\chi : F \to \Sigma_p^+$ is onto. We have with respect to the metric ρ

$$\dim_H F = s = \frac{h_{\mu_\phi}(\sigma)}{\log \beta}$$

and with respect to the metric d_β

$$\dim_H \Sigma_p^+ = \frac{\log p}{\log \beta} = \frac{h(\sigma)}{\log \beta}$$

where $h(\sigma)$ denotes the topological entropy of σ. It is easy to see that $h_{\mu_\phi}(\sigma) = h(\sigma)$ if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_p = \lambda$.

Section 3: Main Results: geometric constructions with nonstationary bounds and random processes

We now consider symbolic geometric constructions with nonstationary bounds. We assume that $\underline{\lambda}_{i,n}, \overline{\lambda}_{i,n}$ admit an asymptotic behavior, *but not necessarily convergent.* We can write

$$\underline{\lambda}_{i,n} = \underline{\lambda}_i \exp(\underline{a}_{i,n}), \quad \overline{\lambda}_{i,n} = \overline{\lambda}_i \exp(\overline{a}_{i,n})$$

where $0 < \underline{\lambda}_i \leq \overline{\lambda}_i < 1$, $i = 1, \cdots, p$ are some numbers and $\underline{a}_{i,n}, \overline{a}_{i,n}$ are sequences of numbers.

We recall that the collection of numbers $(\underline{\lambda}_1, \cdots, \underline{\lambda}_p)$ and $(\overline{\lambda}_1, \cdots, \overline{\lambda}_p)$ generate two measures $\underline{\mu}$ and $\overline{\mu}$ on $Q \subset \Sigma_p^+$ that are equilibrium measures associated to the functions $s\underline{\phi}(\omega) = \underline{s} \log \underline{\lambda}_{i_1}$, $s\overline{\phi}(\omega) = \overline{s} \log \overline{\lambda}_{i_1}$, $\omega \in Q$ and $\underline{s}, \overline{s}$ are the roots of the equations $P(s\underline{\phi}) = 0$ and $P(s\overline{\phi}) = 0$ (see Appendix 5). Also let $\underline{m}, \overline{m}$ denote the pull back of the measures $\underline{\mu}, \overline{\mu}$ under the mapping χ.

We consider the following two types of asymptotic behavior of the sequence of numbers $\underline{a}_{i,n}, \overline{a}_{i,n}$:

(1) strong asymptotic condition
There exist constants $K_1 < 0, K_2 > 0$ such that for any admissible $n-$tuple $(i_1 \cdots i_n)$,

$$K_1 \leq \sum_{j=1}^{n} \underline{a}_{i_j,j} \leq K_2 \quad \text{and } K_1 \leq \sum_{j=1}^{n} \overline{a}_{i_j,j} \leq K_2$$

(2) weak asymptotic condition
a) for $\underline{m}-$ almost every $x \in F$ with $\chi(x) = (i_1 i_2 \cdots)$,

$$\frac{1}{n} \sum_{j=1}^{n} \underline{a}_{i_j,j} \to 0 \quad \text{as } n \to \infty$$

b) for $\overline{m}-$almost every $x \in F$ with $\chi(x) = (i_1 i_2 \cdots)$,

$$\frac{1}{n} \sum_{j=1}^{n} \overline{a}_{i_j,j} \to 0 \quad \text{as } n \to \infty.$$

It is obvious that the strong asymptotic condition implies the weak asymptotic condition. Moreover, if $\{\underline{\lambda}_{i,n}, \overline{\lambda}_{i,n}\}$ satisfy the strong asymptotic condition then we have convergence $\underline{\lambda}_{i,n} \to \underline{\lambda}_i$ and $\overline{\lambda}_{i,n} \to \overline{\lambda}_i$. It is easy to construct a nonstationary symbolic geometric construction satisfying the weak asymptotic condition 2a) but not 2b) and vice-versa. If $\underline{\lambda}_i = \overline{\lambda}_i$ for $i = 1, \cdots, p$ then $\underline{a}_{i,n} = \overline{a}_{i,n}, n = 1, 2, \cdots, \underline{m} = \overline{m}, \underline{s} = \overline{s}$, and conditions 2a) and 2b) coincide.

The difference between the two asymptotic types of behavior is very crucial. In the case of a simple geometric construction with nonstationary bounds satisfying the strong asymptotic condition the measure $\underline{m}$ is Gibbs and satisfies the *uniform* mass distribution principle while in the case of a simple geometric construction with nonstationary bounds satisfying the weak asymptotic condition, one can expect this measure to satisfy only the **nonuniform** mass distribution principle.

It immediately follows from the strong asymptotic condition that there exist constants $K_3 > 0, K_4 > 0$ such that for any admissible $n-$tuple $(i_1 i_2 \cdots)$,

$$K_3 \prod_{j=1}^{n} \underline{\lambda}_{i_j} \leq \prod_{j=1}^{n} \underline{\lambda}_{i_j,j} \leq K_4 \prod_{j=1}^{n} \underline{\lambda}_{i_j}$$

$$K_3 \prod_{j=1}^{n} \overline{\lambda}_{i_j} \leq \prod_{j=1}^{n} \overline{\lambda}_{i_j,j} \leq K_4 \prod_{j=1}^{n} \overline{\lambda}_{i_j}. \tag{8}$$

These relations make it possible to reduce the study of symbolic geometric constructions admitting nonstationary bounds satisfying the strong asymptotic condition to those admitting stationary bounds. The next theorem follows immediately from the proof of Theorems 1, 2, Propositions 2, 3, 4, and relation (8).

Theorem 3. *Let F be the limit set specified by a symbolic geometric construction with nonstationary bounds satisfying the strong asymptotic condition. Then the statements of Theorems 1, 2, Propositions 1, 2, 3, and Corollaries 1, 2, 3 hold.*

In the case of symbolic geometric constructions with nonstationary bounds satisfying the weak asymptotic condition, the following weaker results can be shown:

Theorem 4.

(1) *If the construction satisfies the weak asymptotic condition 2a) then $\underline{s} \leq \underline{d}_{\underline{m}}(x)$ for $\underline{m}-$almost every $x \in F$ and hence $\underline{s} \leq dim_H F$.*
(2) *If the construction satisfies the weak asymptotic condition 2b) then $\overline{d}_{\overline{m}}(x) \leq \overline{s}$ for $\overline{m}-$ almost every $x \in F$ and hence $\overline{dim}_B \overline{m} \leq \overline{s}$.*
(3) *If the construction satisfies the weak asymptotic conditions 2a) and 2b), and $\underline{\lambda}_i = \overline{\lambda}_i$, $i = 1, \cdots, p$ then $\underline{m} = \overline{m} = m$, $\underline{s} = \overline{s} = s$, and $s = dim_H m = \underline{dim}_B m = \overline{dim}_B m$.*

In Section 5 we present an example of a simple geometric construction with nonstationary bounds satisfying the weak asymptotic but not the strong asymptotic condition (see example 4) and for which $\underline{\lambda}_{i,n} = \overline{\lambda}_{i,n}$, $i = 1, \cdots, p$ and $n = 1, 2, \cdots$. For this construction we show that the Hausdorff measure of the limit set can be 0. This violates the uniform mass distribution principle. We believe that such an example can be constructed such that $\overline{s}$ does not provide an upper bound for the upper box dimension of the limit set. To further investigate this situation one can consider a gauge function $h(t)$ (see Appendix 4) such that the corresponding $h-$measure is finite and positive. See also example 4.

We now consider random geometric constructions.

Theorem 5. *Let F be the limit set specified by a random symbolic geometric construction. Then there are numbers $\underline{\lambda}_i, \overline{\lambda}_i, 0 < \underline{\lambda}_i \leq \overline{\lambda}_i < 1, i = 1, \cdots, p$ such that*

(1) For $\nu-$almost every $\vec{\lambda} \in \Lambda$ the following limits exist:

a) for $\underline{\mu}-$almost every sequence $(i_1 i_2 \cdots) \in Q(\vec{\lambda})$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \underline{a}_{i_k,k} = 0$$

b) for $\overline{\mu}-$almost every sequence $(i_1 i_2 \cdots) \in Q(\vec{\lambda})$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \overline{a}_{i_k,k} = 0$$

where $\underline{a}_{i,j} = \log\left(\frac{\underline{\lambda}_{i,j}}{\underline{\lambda}_i}\right)$, $\overline{a}_{i,j} = \log\left(\frac{\overline{\lambda}_{i,j}}{\overline{\lambda}_i}\right)$ and $\underline{\mu}, \overline{\mu}$ are the Gibbs measures corresponding to the functions $\underline{s}\log\underline{\lambda}_{i_1}$ and $\overline{s}\log\overline{\lambda}_{i_1}$.

(2) For $\nu-$almost every $\vec{\lambda} \in \Lambda$

$$\underline{s} \le dim_H F(\vec{\lambda}).$$

Section 4: Generating Maps and Codings

One particular but important case of a geometric construction is when the sets $\Delta_{i_1\cdots i_n}$ are given by

$$\Delta_{i_1\cdots i_n} = h_{i_n} \circ h_{i_{n-1}} \circ \cdots \circ h_{i_1}(\Delta) \tag{9}$$

where $h_1, \cdots, h_p : \Delta \to \Delta$ are contraction maps, i.e., $d(h_i(x), h_i(y)) \le L_i d(x,y)$ with $L_i < 1$ and $x, y \in \Delta$ (a ball in $\mathbb{R}^n$). Most of the results in the literature in dimension theory require that the process be described in this way. We stress that this is a very special case and that the following situations can (and do) occur:

(1) *The geometric construction can not be described by any continuous maps*, i.e., there are no continuous maps satisfying (9). This is the case if the boundary of a set $\Delta_{i_1\cdots i_n}$ is fractal. In the one-dimensional case the maps h_j always exist and are continuous, so the above mentioned pathology does not occur. In this case the maps are well defined by

$$h_j(\partial\Delta_{i_1\cdots i_n}) = \partial\Delta_{i_1\cdots i_n j}$$

for any admissible sequence $(i_1 \cdots i_n j)$, where $\partial\Delta_{i_1\cdots i_n j}$ denotes the boundary of the set $\Delta_{i_1\cdots i_n j}$.

(2) *There are continuous maps satisfying (9) that are not Lipschitz.* One obstruction is that the boundary of a set $\Delta_{i_1\cdots i_n}$ can be a continuous but not Lipschitz image of $\partial\Delta$. In the one-dimensional case, this phenomenon cannot occur, however, one can construct a one-dimensional example where the process is defined by continuous but not Lipschitz maps. Namely, there exists a simple geometric construction on

$[0,1]$ with $p=2$, $\underline{\lambda}_1 = \overline{\lambda}_1 = \lambda_1$, $\underline{\lambda}_2 = \overline{\lambda}_2 = \lambda_2$, $0 < \lambda_1 < \lambda_2 < 1$ and location of the intervals $\Delta_{i_1 \cdots i_n}$ such that the map h_1 is not Lipschitz. Choose

$$\frac{d(\Delta_{\underbrace{1 \cdots 1}_{n_k}}, \Delta_{\underbrace{1 \cdots 1}_{n_k - 1},2})}{d(\Delta_{\underbrace{1, \cdots, 1}_{n_k+1}}, \Delta_{\underbrace{1 \cdots 1}_{n_k - 1},2,1})} \stackrel{n_k \to \infty}{\longrightarrow} \infty.$$

(3) *There are Lipschitz maps satisfying (9) but these maps are not contractions.* To see this, choose intervals Δ_1, Δ_2 and maps h_1, h_2 such that $\Delta_{11} = h_1(\Delta_1)$, $\Delta_{21} = h_1(\Delta_2)$ and $d(\Delta_1, \Delta_2) << d(\Delta_{11}, \Delta_{21})$. Then the map h_1 can not be a contraction.

(4) *There are contraction maps satisfying (9) whose inverse maps are not Lipschitz.* The Lipschitz constants for the inverse maps depend on the gaps between the sets $\Delta_{i_1 \cdots i_n}$. To see this, choose intervals $\Delta_{i_1 \cdots i_n}$ such that

$$\frac{d(\Delta_{\underbrace{1 \cdots 1}_{n}}, \Delta_{\underbrace{1 \cdots 1}_{n - 1},2})}{d(\Delta_{\underbrace{1 \cdots 1}_{n+1}}, \Delta_{\underbrace{1 \cdots 1}_{n - 1},2,1})} \stackrel{n \to \infty}{\longrightarrow} 0.$$

Then the inverse of the map h_1 can not be Lipschitz.

(5) *There are contraction maps satisfying (9) whose inverse maps are Lipschitz but the maps are not similarities* (i.e., $d(h_i(x), h_i(y)) = L_i d(x,y)$ for all $x \in \Delta$).

As we have seen in (4) the Lipschitz constants for the inverse maps depend on the gaps between the sets $\Delta_{i_1 \cdots i_n}$. For this reason, even if the process can be described using contraction maps with Lipschitz inverses, they may be of no use in estimating the lower Hausdorff dimension of the limit set.

We point out that the coding we employ in our definitions of the geometric constructions is not the obvious generalization of the *natural* coding for a similarity process. The natural coding of a similarity process is $\Delta_{i_1 \cdots i_n} = g_{i_n} \circ g_{i_{n-1}} \circ \cdots \ g_{i_1}(\Delta)$ where the maps g_k are affine contractions. It follows that $\Delta_{i_1 \cdots i_n} \subset \Delta_{i_2 \cdots i_n}$. This coding has several undesirable properties which the coding that we adopted does not share. One undesirable property is that for given $x \in F$ with coding $(i_1 i_2 \cdots)$, one can not determine which Δ_k the point x lies in.

The maps h_k have an interesting manifestation on the level of symbolic dynamics. The mapping $G : F \to F$ defined in Remark (4) can be described as

$$G(x) = h_k^{-1}(x) \quad \text{if } x \in \Delta_k.$$

If the maps h_k are all similarities, then the map G is a contraction.

Peres has observed that some dimension results in the general symbolic case can be obtained by studying the symbolic geometric construction defined by similarity maps. We now show how this remark enables us to obtain lower and upper estimates for the Hausdorff measure of the limit set at the dimension.

Let F be the limit set for a symbolic geometric construction with ratio coefficients $\underline{\lambda}_i, \overline{\lambda}_i$, $i = 1, \cdots, p$ that is determined by the set $Q \subset \Sigma_p^+$.

Proposition 4.

(1) *There exist similarity maps* $\overline{h}_1, \cdots, \overline{h}_p : \Delta \to \Delta$ *with similarity coefficients* $\overline{\lambda}_1, \cdots, \overline{\lambda}_p$ *and a Lipschitz map* $\pi_1 : \overline{F}_{sym} \to F$ *where* $\overline{F}_{sym}$ *denotes the limit set for the symbolic geometric construction built using similarity maps* $\overline{h}_i$. *In particular,* $\overline{dim}_B F \leq \overline{dim}_B \overline{F}_{sym}$ *and* $m_H(\overline{s}, F) \leq const\ m_H(\overline{s}, \overline{F}_{sym})$.

(2) *There exist similarity maps* $\underline{h}_1, \cdots, \underline{h}_p : \Delta \to \Delta$ *with similarity coefficients* $\underline{\lambda}_1, \cdots, \underline{\lambda}_p$ *and a Lipschitz map* $\pi_2 : F \to \underline{F}_{sym}$ *where* $\underline{F}_{sym}$ *denotes the limit set for the similarity construction and built using the similarity maps* $\underline{h}_i$. *In particular,* $dim_H \underline{F}_{sym} \leq dim_H F$ *and* $m_H(\underline{s}, \underline{F}_{sym}) \leq const\ m_H(\underline{s}, F)$.

(3) *If the ratio coefficients of the symbolic geometric construction satisfy* $\underline{\lambda}_i = \overline{\lambda}_i = \lambda_i$, $i = 1, \cdots, p$, *then there exist similarity maps* $h_1, \cdots, h_p : \Delta \to \Delta$ *with similarity coefficients* $\lambda_1, \cdots, \lambda_p$ *such that*

$$\dim_H F = \underline{\dim}_B F = \overline{\dim}_B F = \dim_H F_{sym} = \underline{\dim}_B F_{sym} = \overline{\dim}_B F_{sym} = s,$$

where s *is the root of the equation* $P(s \log \lambda_{i_1}) = 0$. *Moreover,* $0 < m_H(s, F) < \infty$.

Section 5: Examples

FIGURE 2. SIERPIŃSKI GASKETS A), B), C)

1) Sierpiński Gaskets

a) It is well known that the Hausdorff dimension of the Sierpiński gasket (see Figure 2a) coincides with the box dimension and is $\frac{\log 3}{\log 2}$. This immediately follows from Proposition 3 since $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}$ and $p = 3$.

b) Suppose that in the construction of the Sierpiński gasket we forbid all configurations whose codings have a 12 in it (see Figure 2b). The spectral radius of $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ is $\frac{3+\sqrt{5}}{2}$. Hence $\dim_H F = \frac{\log(\frac{3+\sqrt{5}}{2})}{\log 2} \approx 1.38848$.

c) Figure 2c) illustrates a simple geometric construction with nonstationary bounds of the Sierpiński gasket with $\underline{\lambda}_i = \overline{\lambda}_i = \frac{1}{3}$ for $i = 1, 2, 3$. The sets $\Delta_{i_1 \cdots i_n}$ are *asymptotically congruent* to the corresponding triangles in the usual constructions, and possess *wiggly boundaries* which become *asymptotically straight.* As long as the approximation is sufficiently fast and uniform, meaning that the construction satisfies the hypotheses of Theorem 3, then the Hausdorff dimension of the limit set is $\frac{\log 3}{\log 2}$. One can even choose the sets $\Delta_{i_1 \cdots i_n}$ in the construction to have fractal boundaries.

2) General Smale-Williams Solenoid

Let P be a solid torus embedded in $\mathbb{R}^3$. We represent points on P by means of coordinates (θ, r, s) where $\theta \in S^1$the unit circle, $-1 \le r, s \le 1$ such that $r^2 + s^2 \le 1$. The point x with coordinates (θ, r, s) belongs to the plane orthogonal to the core of the torus through the point $\theta \in S^1$ having position (r, s) relative to the standard frame (e_1, e_2). We define a mapping $f : P \to P$ by

$$f(\theta, r, s) = (2\,\theta, \lambda_1 r + \epsilon \cos\theta, \lambda_2 s + \epsilon \sin\theta)$$

where $p > 0$ is an integer, ϵ is a small positive constant and $0 < \lambda_1, \lambda_2 < 1$. The image $f(P)$ is contained in P and wraps twice around P. See Figure 3A. The set $\Delta = \cap_{n=1}^{\infty} f^n(P)$ is called a *solenoid.* See [Sh] for more details and nice pictures.

FIGURE 3. SMALE-WILLIAMS SOLENOID A), B)

Let D_θ be the section of P determined by the plane perpendicular to the core at θ. The set $\Delta_\theta = \Delta \cap D_\theta$ is the Cantor-like set obtained by the simple geometric construction with

$\underline{\lambda}_i = \lambda_1$ and $\overline{\lambda}_i = \lambda_2$ for i = 1, 2. See Figure 3B). It immediately follows from Theorem 1 that

$$\frac{\log 2}{\log(\frac{1}{\lambda_1})} \leq \dim_H \Delta_\theta \leq \frac{\log 2}{\log(\frac{1}{\lambda_2})}.$$

Appling Marstrand's theorem [F1], we obtain

$$1 + \frac{\log 2}{\log(\frac{1}{\lambda_1})} \leq \dim_H \Delta.$$

A very simple argument [F1] shows that $\dim_H \Delta \leq 1 + \frac{\log 2}{\log(\frac{1}{\lambda_2})}$.

If $\lambda_1 = \lambda_2 = \lambda$, we have that $\dim_H \Delta_\theta = \frac{\log 2}{\log(\frac{1}{\lambda})}$ and $\dim_H \Delta = \frac{\log 2}{\log(\frac{1}{\lambda})} + 1$, the result obtained by Falconer [F1].

3) In [PW], the authors construct families of affine horseshoes (diffeomorphisms) of $\mathbb{R}^3$, for which the box dimension exists (the lower box dimensions coincide with the upper box dimensions) but sometimes does not coincide with the Hausdorff dimension. They first construct a family of similarity processes in the plane with limit sets having this property and then they build horseshoes whose cross sections coincide with these limit sets. The authors find that number theoretic properties of the ratio coefficients, related to Pisot numbers, is a mechanism to cause noncoincidence of box dimension and Hausdorff dimension.

4) A Simple Geometric Construction With Ratio Coefficients Satisfying the Weak Asymptotic, But Not Strong Asymptotic, Condition.

This example shows that in the second statement of Theorem 1, the measure of the limit set at dimension may be zero.

Let $p = 2$ and suppose $\lambda_{i,n} = \lambda_n = \lambda \exp(a_n)$ for $i = 1, 2$. Let F denote the limit set. Clearly

$$|\Delta_{i_1 \cdots i_n}| = \prod_{j=1}^{n} \lambda_{i_j,j} = \lambda^n \exp(A_n),$$

where $A_n = \sum_{k=1}^{n} a_k$ and $|\Delta_{i_1 \cdots i_n}|$ denotes the diameter of the set $\Delta_{i_1 \cdots i_n}$. For fixed n, the sets $\{\Delta_{i_1 \cdots i_n}\}$ give a cover of F with

$$\sum_{i_1 \cdots i_n} |\Delta_{i_1 \cdots i_n}|^s = 2^n \lambda^{ns} \exp(sA_n) = \exp(sA_n).$$

where $s = \frac{\log 2}{\log(\frac{1}{\lambda})}$. There is now a trichotomy:

(1) If $\sum_{k=1}^{\infty} a_k < \infty$, then one can easily show that the geometric process satisfies the strong asymptotic condition and $s = \dim_H F = \dim_B F = \frac{\log 2}{\log(\frac{1}{\lambda})}$.
(2) If $\sum_{k=1}^{\infty} a_k = \infty$, then one can show that $m_H(s, F) = \infty$.
(3) If $\sum_{k=1}^{\infty} a_k = -\infty$, then $m_H(s, F) = 0$.

The sequence $\{a_k\}$ satisfies the weak asymptotic condition if and only if $\frac{1}{n}A_n \to 0$. For example, we can choose $a_n = \frac{1}{n}$ in the case (2) and $a_n = -\frac{1}{n}$ in case (3). One can show that s is still the Hausdorff dimension of the limit set F. The mass distribution principle does not hold in case 3) since otherwise we would have $m_H(s, F) > 0$.

We now construct gauge functions for certain sequences $\{a_j\}$ where $\sum_{k=1}^{\infty} a_j = -\infty$. Let $A_n = \sum_{k=1}^{n} a_k$. We seek a function $h(t)$ such that $0 < 2^n h(\lambda^n \exp(A_n)) < \infty$. We will find $h(t)$ in the form $h(t) = t^s \exp(\phi(t))$. Then we should have $-\infty < sA_n + \phi(\lambda^n \exp(A_n)) < \infty$. We define $t = \lambda^n \exp(A_n)$. Since $\lim_{n\to\infty} \frac{A_n}{n} = 0$, then $t \asymp \lambda^n$, and hence we can set $\phi(t) = -sA_{\frac{\log t}{\log \lambda}}$.

If $a_n = -\frac{1}{n}$ then $A_n = O(-\log n)$ and $\phi(t) = \log(\frac{\log t}{\log \lambda})$. If $a_n = -\frac{1}{n \log n}$ then $A_n = O(-\log(\log n))$ and $\phi(t) = \log(\log(\frac{\log t}{\log \lambda}))$. More generally, if $a_n = -\frac{1}{n \log^{(i)} n}$, where $\log^{(i)} n$ denotes the i-fold composition of $\log n$, then $A_n = O(-\log^{(i+1)} n)$ and hence $\phi(t) = \log^{(i+1)}(\frac{\log t}{\log \lambda})$. We can thus obtain gauge functions with *arbitrarily many logs* from this basic construction. For these different sequences, the coefficients $\lambda_n = \lambda \exp(a_n)$ converge to λ, but of course with different speeds, and hence require different gauge functions.

5) Random version of Example 4

The following example is a special case of a family of random constructions that were explained to the authors by Peres. We consider the construction in Example 4 where the numbers $\{a_n\}$ are independent and identically distributed random variables on the interval $\infty < a \le a_n \le b < \log(\frac{1}{\lambda})$ having mean 0. If we define the random variable $A_n = \sum_{k=1}^{n} a_k$, then the law of the iterated logarithm implies that $\liminf A_n \to -\infty$. The law of large numbers shows that the process satisfies the weak asymptotic condition, i.e., $\frac{A_n}{n} \to 0$. It immediately follows from a simple calculation as in Example 4 that $s = \dim_H F = \frac{\log 2}{-\log \lambda}$ and $m_H(s, F) = 0$.

6) Simple Geometric Construction With a Limit Set F For Which $\mathbf{dim_H F < \underline{dim}_B F < \overline{dim}_B F}$

We describe a simple geometric construction in $\mathbb{R}^2$ with $p = 2$, $\underline{\lambda}_1 = \underline{\lambda}_2 = \underline{\lambda}$, $\overline{\lambda}_1 = \overline{\lambda}_2 = \overline{\lambda}$, $0 < \underline{\lambda} < \overline{\lambda} < \frac{1}{3}$ such that the limit set F satisifes

$$\dim_H F = \frac{\log 2}{-\log \underline{\lambda}}, \qquad \underline{\dim}_B F = \gamma \frac{\log 2}{-\log \underline{\lambda}}, \qquad \overline{\dim}_B F = \frac{\log 2}{-\log \overline{\lambda}},$$

where $\gamma \in (1, \alpha)$ is an arbitrary number and $\alpha = \left[\frac{\log \underline{\lambda}}{\log \overline{\lambda}}\right] > 1$.

Let $n_0 = 0$ and for $k = 0, 1, 2, \cdots$, let $n_{k+1} = [\alpha n_k] + 2$ and $\beta_k = 2^{(\gamma-\alpha)n_{3k+1}}$. In order to describe the $n-$th step of the construction we use the basic types of spacings: vertical (A) and horizontal (B). See Figure 4.

(1) We start with two horizontally stacked rectangles. During steps $n_{3k} < n \le n_{3k+1}$ we use (B).
(2) We begin with $2^{n_{3k+1}}$ rectangles. Choose β_k percent of these rectangles arbitrarily and paint them blue; paint the others green. During steps $n_{3k+1} < n \le n_{3k+2}$, we use (B) in all blue rectangles and use (A) in all green rectangles.

FIGURE 4 VERTICAL STACKING A) HORIZONTAL STACKING B)

(3) During steps $n_{3k+2} < n \leq n_{3k+3}$, we use (A) in all blue rectangles and use (B) in all green rectangles.
(4) Repaint all $2^{n_{3k+3}}$ rectangles white; repeat steps 1 to 4.

The collection of rectangles at the n-th step contains 2^n rectangles each with vertical and horizontal side of size $\underline{\lambda}^n \times \overline{\lambda}^n$ (the size in the vertical direction is $\underline{\lambda}^n$ and the size in the horizontal direction is $\overline{\lambda}^n$). Any two subrectangles on step $n+1$ that are contained in the same rectangle at step n are stacked either horizontally or vertically and the distance between them is at least $\frac{1}{3}\underline{\lambda}^n$. The projections of any two distinct rectangles onto the two coordinate axes either coincide or are disjoint.

We need the following three lemmas:

Lemma 1. *Let $\mathfrak{U}$ be a covering of the limit set F by disjoint closed balls $B(x_i, r), i = 1, \cdots, p$, and r is fixed. Assume that every ball in the covering has a non-trivial intersection with the limit set F. Then $N(r) \geq \frac{1}{6}\#(\mathfrak{U})$, where $N(r)$ denotes the smallest number of balls of radius r needed to cover F.*

Proof. Let $\mathfrak{V} = \{B(x_i', r)\}, i = 1, \cdots, q$ be a covering of F. Then by elementary geometry, for all x_j', there are at most 6 points $x_1, \cdots, x_6$ such that $F \cap B(x_j', r) \subset \cup_{k=1}^{6} B(x_k, r)$.

□

Lemma 2. *For any $n > 0$, $dist(\Delta_{i_1' \cdots i_n'}, \Delta_{i_1'' \cdots i_n''}) \geq \underline{\lambda}^n$.*

Proof. The proof is obvious. □

Lemma 3. *If $0 < r < \epsilon$ and an optimal covering of the limit set F by balls of radius r consists of disjoint balls $B(x_i, r)$, then $N(\epsilon) \geq \frac{1}{4}N(r)\left(\frac{r}{\epsilon}\right)^2$.*

Proof. Let $\{B(y_j, \epsilon)\}$ be a covering of F. For any point x_i one can find a point y_i such that $x_i \in B(y_i, \epsilon)$. Hence, $B(x_i, r) \subset B(y_i, 2\epsilon)$. Therefore $\mathrm{vol}(B(x_i, r)) \leq \mathrm{vol}(B(y_i, 2\epsilon))$.

Since the balls $B(x_i, r)$ are disjoint, this implies that $N(r)r^2 \leq N(\epsilon)(2\epsilon)^2$. The lemma immediately follows. $\square$

We now calculate the Hausdorff dimension and the lower and upper box dimensions of the limit set F.

a) Calculation of Upper Box Dimension.

Let $r_k = \overline{\lambda}^{n_{3k+1}}$. Consider the covering of F by rectangles $\Delta_{i_1 \cdots i_{n_{3k+1}}}$. Each rectangle can be covered by a square of side length less than or equal to $\overline{\lambda}^{3k+1}$ that is centered at some point $x \in \Delta_{i_1 \cdots i_{n_{3k+1}}}$. These squares are disjoint since $\overline{\lambda}^{n_{3k+1}} = \overline{\lambda}^{[\alpha n_{3k}]+2} \leq \frac{1}{3}\underline{\lambda}^{n_{3k}}$, and by Lemma 2, $\operatorname{dist}(\Delta_{i'_1 \cdots i'_{n_{3k}}}, \Delta_{i''_1 \cdots i''_{n_{3k}}}) \geq \underline{\lambda}^{n_{3k}}$. By Lemma 1, $N(r_k) \geq \frac{1}{6} 2^{n_{3k+1}}$ and hence

$$\overline{\dim}_B F \geq \overline{\lim}_{k \to \infty} \frac{\log N(r_k)}{-\log r_k} = \frac{\log 2}{-\log \overline{\lambda}}.$$

On the other hand, by Proposition 3, $\overline{\dim}_B F \leq \frac{\log 2}{-\log \overline{\lambda}}$. It follows that $\overline{\dim}_B F = \frac{\log 2}{-\log \overline{\lambda}}$.

b) Calculation of Hausdorff Dimension.

Given $\epsilon > 0$ choose $k > 0$ such that $\underline{\lambda}^{n_{3k+1}} \leq \epsilon$. Consider the covering of F consisting of green rectangles for $n = n_{3k+1}$ and blue rectangles for $n = n_{3k+2}$.

Consider a green rectangle $\Delta_{i_1 \cdots i_{n_{3k+1}}}$. By construction, the intersection $A = \Delta_{i_1 \cdots i_{n_{3k+1}}} \cap F$ is contained in $2^{n_{3k+2} - n_{3k+1}}$ small green rectangles corresponding to $n = n_{3k+2}$. These rectangles are vertically aligned and have size $\underline{\lambda}^{n_{3k+2}} \times \overline{\lambda}^{n_{3k+2}}$. Since $\overline{\lambda}^{n_{3k+2}} = \overline{\lambda}^{[\alpha n_{3k+1}]+2} \leq \frac{1}{3}\underline{\lambda}^{n_{3k+1}}$, the set A is contained in a square of size $\underline{\lambda}^{n_{3k+1}}$. Thus we have $(1 - \beta_k)2^{n_{3k+1}}$ green squares of length $\underline{\lambda}^{n_{3k+1}}$.

Now consider a blue rectangle $\Delta_{i_1 \cdots i_{n_{3k+1}}}$. By our construction, the intersection $A = \Delta_{i_1 \cdots i_{n_{3k+2}}} \cap F$ is contained in $2^{n_{3k+3} - n_{3k+2}}$ small blue rectangles corresponding to $n = n_{3k+3}$. They are vertically aligned and have size $\underline{\lambda}^{n_{3k+3}} \times \overline{\lambda}^{n_{3k+3}}$. Since $\overline{\lambda}^{n_{3k+3}} \leq \frac{1}{3}\underline{\lambda}^{n_{3k+2}}$, the set A is contained in a square of size $\underline{\lambda}^{n_{3k+2}}$. Thus we have $\beta_k 2^{n_{3k+1}} 2^{n_{3k+2} - n_{3k+1}} = \beta_k 2^{n_{3k+2}}$ blue squares of length $\underline{\lambda}^{n_{3k+1}}$.

The collection of green and blue rectangles comprise a covering $\mathfrak{G} = \{U_i\}$ of F such that

$$\sum_{U_i \in \mathfrak{G}} (\operatorname{diam} U_i)^s = (1 - \beta_k)2^{n_{3k+1}}(2\underline{\lambda}^{n_{3k+1}})^s + \beta_k 2^{n_{3k+2}}(2\underline{\lambda}^{n_{3k+2}})^s = 1 > 0$$

if $s = \frac{\log 2}{-\log \underline{\lambda}}$. This implies that $\dim_H F \leq \frac{\log 2}{-\log \overline{\lambda}}$. On the other hand, by Corollary 2, we know that $\dim_H F \geq \frac{\log 2}{-\log \underline{\lambda}}$, hence $\dim_H F = \frac{\log 2}{-\log \underline{\lambda}}$.

c) Calculation of Lower Box Dimension.

Consider $r_k = \underline{\lambda}^{n_{3k+1}}$ and the covering $\mathfrak{G}_k$ of F by green rectangles for $n = n_{3k+1}$ and blue rectangles for $n = n_{3k+2}$.

There are $(1-\beta_k)2^{n_{3k+1}}$ green rectangles and the intersection of each of them with F is contained in a square of size $2\underline{\lambda}^{3k+1}$. There are $\beta_k\, 2^{n_{3k+1}}\, 2^{n_{3k+2}-n_{3k+1}} = \beta_k 2^{n_{3k+2}}$ blue rectangles each of which is contained in a square of size $\overline{\lambda}^{n_{3k+2}}$. It is easy to see that

$$L_1\underline{\lambda}^{n_{3k+1}} \le \overline{\lambda}^{n_{3k+2}} \le L_2\underline{\lambda}^{n_{3k+1}},$$

where L_1, L_2 are positive constants. Therefore $\mathfrak{G}_k$ induces a covering $\mathfrak{H}_k = \{U_i\}$ of F by squares of size $L\underline{\lambda}^{3k+3}$, where $L > 0$ is a constant. The cardinality N_k of this covering is

$$N_k = ((1-\beta_k)2^{n_{3k+1}} + \beta_k 2^{n_{3k+2}}) \le ((1-\beta_k)2^{n_{3k+1}} + 4\beta_k 2^{\alpha n_{3k+1}}).$$

Since $\alpha > \gamma$, we have that

$$(1-\beta_k)2^{(\alpha-\gamma)n_{3k+1}} \to 0$$

as $k \to \infty$ and by the definition of β_k,

$$\beta_k 2^{(\alpha-\gamma)n_{3k+1}} = 1.$$

Therefore

$$\underline{\dim}_B F \le \lim_{k\to\infty} \frac{\log N(r_k)}{-\log r_k} = \gamma\frac{\log 2}{-\log \underline{\lambda}}.$$

Choose a sequence $\epsilon_m \to 0$. Given $m > 0$, one can choose an integer $k = k(m)$ satisfying $\underline{\lambda}^{n_{3k+1}} \le \epsilon_m \le L_3\underline{\lambda}^{n_{3k+1}}$, where L_3 is a positive constant that is independent of m. We wish to compare $N(\epsilon_m)$ and $N(r_k)$. Since the squares comprising the covering $\mathfrak{H}_k$ are disjoint, Lemma 3 implies

$$N(\epsilon_m) \ge \frac{1}{4}N(r_k)\left(\frac{r_k}{\epsilon_m}\right)^2.$$

It follows that

$$\underline{\dim}_B F = \lim_{m\to\infty}\frac{\log N(\epsilon_m)}{-\log \epsilon_m} \ge \limsup_{m\to\infty}\frac{\log N(r_k)}{-\log r_k} = \gamma\frac{\log 2}{-\log \overline{\lambda}}.$$

Section 6: Pointwise Dimension of Measures Concentrated on General Cantor-like Sets

6.1 Let F be the limit set for a symbolic geometric construction with nonstationary bounds satisfying $\underline{\lambda}_{i,n} = \overline{\lambda}_{i,n} = \lambda_{i,n}$. We formulate a powerful criterion that allows one to estimate the lower and upper pointwise dimensions with respect to a Borel probability measure ν on F. Given $x \in F, n > 0$ consider the unique set $\Delta_{i_1\cdots i_n} = \Delta_n(x)$ that contains the point x. Denote

$$\underline{d}(x) = \liminf_{n\to\infty}\frac{\log \nu(\Delta_n(x))}{\log \operatorname{diam}\Delta_n(x)};$$

$$\overline{d}(x) = \limsup_{n\to\infty}\frac{\log \nu(\Delta_n(x))}{\log \operatorname{diam}\Delta_n(x)}.$$

The following theorem is related to a result in [C]:

Theorem 6.

(1) $\overline{d}_\nu(x) \le \overline{d}(x)$ *and* $\underline{d}(x) \le \underline{d}_\nu(x)$ *for all* $x \in F$.

(2) *If* $\underline{d}(x) = \overline{d}(x) \stackrel{\text{def}}{\equiv} d(x)$ *for* $\nu-$*almost every* $x \in F$, *then*

$$\underline{d}_\nu(x) = \overline{d}_\nu(x) = d(x)$$

for $\nu-$*almost every* $x \in F$.

The next statement is an immediate corollary of Theorems 4 and 6.

Corollary 4. *Let F be the limit set for a symbolic geometric construction with nonstationary bounds satisfying $\underline{\lambda}_{i,n} = \overline{\lambda}_{i,n}$. Assume that there is a Borel measure ν on F such that $\underline{d}(x) = \overline{d}(x) \stackrel{\text{def}}{\equiv} s$ for $\nu-$almost every $x \in F$. Then $s \le dim_H F$.*

6.2 We present several computations of pointwise dimension. We start with Example 6 and show that:

a) $\underline{d}_{\underline{m}}(x) = \underline{s}$ *for* $\underline{m}-$*almost every* $x \in F$
$\underline{d}_{\overline{m}}(x) = \underline{s}$ *for* $\overline{m}-$*almost every* $x \in F$

and

b) $\overline{d}_{\underline{m}}(x) = \overline{d}_{\overline{m}}(x) = \overline{s}$ *for every* $x \in F$.

The fact $\dim_H F = \underline{s}$ immediately implies that

$$\underline{d}_{\underline{m}}(x) \le \underline{s} \text{ for } \underline{m} - \text{almost every } x \in F$$

and

$$\underline{d}_{\overline{m}}(x) \le \underline{s} \text{ for } \overline{m} - \text{almoot every } x \in F.$$

Otherwise there would exist a set A of positive $\underline{m}$ measure (respectively $\overline{m}$ measure) with $\underline{d}_{\underline{m}}(x) \ge \underline{s} + \epsilon$ (respectively $\underline{d}_{\overline{m}}(x) \ge \underline{s} + \epsilon$) for any $x \in A$. The nonuniform mass distribution principle would then imply that $\dim_H F \ge \dim_H A \ge \underline{s} + \epsilon$. Statement a) follows from the first statement in Theorem 2.

In order to prove b), consider $r_k = \overline{\lambda}^{n_{3k+1}}$ and denote $\Delta_k(x)$ the unique cylinder set $\Delta_{i_1 \cdots n_{3k+1}}$ that contains $x \in F$. It is easy to see that $B(x, r_k) \bigcap F \subset \Delta_k(x) \bigcap F$. This and (12) imply that for all $x \in F$,

$$\overline{m}(B(x, r_k) \bigcap F) \le \overline{m}(\Delta_k(x) \bigcap F) \le D_1 \overline{\lambda}^{\overline{s} n_{3k+1}} = D_1 r_k^{\overline{s}}$$

and hence

$$\overline{d}_{\overline{m}}(x) \ge \limsup_{k \to \infty} \frac{\log \overline{m}(B(x, r_k) \bigcap F)}{\log r_k} \ge \overline{s}.$$

Similarly, using (10) we have that for all $x \in F$

$$\underline{m}(B(x, r_k) \bigcap F) \le \underline{m}(\Delta_k(x) \bigcap F) \le D_1 \underline{\lambda}^{\underline{s} n_{3k+1}}$$

and hence,

$$\overline{d}_{\underline{m}}(x) \geq \limsup_{k\to\infty} \frac{\log \underline{m}(B(x, r_k) \bigcap F)}{\log r_k} = \underline{s}\frac{\log \underline{\lambda}}{\log \overline{\overline{\lambda}}} = \overline{s}.$$

Statement (b) now follows from the first statement of Theorem 2. $\square$

6.3 Let F be the limit set generated by the simple geometric construction in Example 6, $G : F \to F$ the map $G = \chi^{-1} \circ \sigma \circ \chi$, where χ denotes the coding map and $\sigma : \Sigma_p^+ \to: \Sigma_p^+$ is the full shift (see Remark 4). Consider the set $\tilde{F} = F \times F$ endowed with the metric

$$\tilde{\rho}((x_1, y_1), (x_2, y_2)) = \rho(x_1, x_2) + \rho(y_1, y_2), \qquad x_1, x_2, y_1, y_2 \in F$$

and the coding map $\tilde{\chi} : \tilde{F} \to \Sigma_p$ (where Σ_p denotes the space of two-sided sequences $(\cdots i_{-1} i_0 i_1 \cdots)$, $i_j = 1 \cdots p$) defined by

$$\tilde{\chi}(x, y) = (\cdots i_{-1} i_0 i_1 \cdots)$$

where $\chi(x) = (\cdots i_{-1})$, $\chi(y) = (i_0 i_1 \cdots)$. Set $\tilde{G} = \tilde{\chi}^{-1} \circ \sigma \circ \tilde{\chi}$. It is easy to see that $\tilde{G}$ is a homeomorphism and that for any $(x, y) \in \tilde{F}$,

$$\pi_1 \tilde{G}(x, y) = G(x), \quad \pi_2 \tilde{G}^{-1}(x, y) = G(y)$$

where π_1, π_2 are the projections, $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. Consider the measure $\underline{\tilde{m}} = \tilde{\chi}^* \underline{\tilde{\mu}}$ where $\underline{\tilde{\mu}}$ is the measure on Σ_p^+ defined by

$$\underline{\tilde{\mu}}(\Delta_{i_k \cdots i_n}) = \underline{\lambda}^{(n-k)\underline{s}}.$$

By virtue of Corollary 3, $\underline{\tilde{\mu}}$ is invariant under σ and hence $\underline{\tilde{m}}$ is invariant under $\tilde{G}$. It is easy to see that $\underline{\tilde{m}} = \underline{m} \times \underline{m}$. It follows that for $\underline{\tilde{m}}-$almost every (x, y)

$$\underline{d}_{\underline{\tilde{m}}}(x, y) = \underline{d}_{\underline{m}}(x) + \underline{d}_{\underline{m}}(y) = \underline{s}$$

$$\overline{d}_{\underline{\tilde{m}}}(x, y) = \overline{d}_{\underline{m}}(x) + \overline{d}_{\underline{m}}(y) = \overline{s}.$$

It is not difficult to check that $\underline{\tilde{m}}$ is an equilibrium measure (and is a Gibbs measure) corresponding to the function $\underline{s\phi}(x, y) = 2\underline{s} \log \underline{\lambda}$. Thus the map $\tilde{G}$ provides an example of a homeomorphism but not smooth map possessing a Gibbs measure that has different lower and upper pointwise dimension almost everywhere. The same is also true with respect to the measure $\overline{\tilde{m}} = \tilde{\chi}^* \overline{\tilde{\mu}}$ where $\overline{\tilde{\mu}}$ is defined by $\overline{\tilde{\mu}}(\Delta_{i_k \cdots i_n}) = \overline{\lambda}^{(n-k)\overline{s}}$. As we mentioned, this does not hold for smooth maps [L, PY].

6.4 We will construct a simple geometric construction for which the map G (see Remark (4)) possesses an ergodic invariant measure with positive entropy whose pointwise dimension exists almost everywhere, and is not constant. Our approach follows [C].

Let F be the limit set for simple geometric construction on $[0, 1]$ with $p = 3$ and ratio coefficients $\lambda_{i,n}$, $i = 1, 2, 3$ and $n = 1, 2, 3, \cdots$ given by

$$\lambda_{1,n} = \lambda_{2,n} = \begin{cases} \alpha & \text{if } n \text{ is even} \\ \beta & \text{if } n \text{ is odd}, \end{cases}$$

$$\lambda_{3,n} = \begin{cases} \gamma, & \text{if } n \text{ is even} \\ \delta, & \text{if } n \text{ is odd,} \end{cases}$$

where $0 < \alpha < \beta < 1$, $0 < \gamma < \delta < 1$ and $\alpha\beta \neq \gamma\delta$. It is easy to show that the following limit exists:

$$\log \lambda_i \stackrel{\text{def}}{\equiv} \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \log \lambda_{i,k} = \begin{cases} \frac{1}{2}\log(\alpha\beta), & \text{if } i = 1,2 \\ \frac{1}{2}\log(\delta\gamma), & \text{if } i = 3, \end{cases}$$

and that the numbers $\lambda_{i,n}$, $i = 1, 2$ satisfy the weak asymptotic condition.

Let s be the unique root of the equation

$$2(\sqrt{\alpha\beta})^s + (\sqrt{\gamma\delta})^s = 1.$$

According to Theorem 4 we have that

$$s \leq \dim_H F, \;\; \underline{\dim}_H m = \underline{\dim}_B m = \overline{\dim}_B m = s,$$

and

$$\underline{d}_m(x) = \overline{d}_m(x) = s \;\text{ for } m\text{−almost every } x \in F,$$

where m is the Gibbs measure corresponding to the function $s \log \lambda_{i_1}$.

Consider the transitive stochastic matrix

$$P = \begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix}.$$

Let μ be the Markov measure determined by P and let $\nu = \chi^*\mu$. Consider the sets

$$A = \left\{ x \in F \,:\, \chi(x) = (i_1 i_2 \cdots) \in \Sigma_P^+ \text{ with } i_j = 1 \text{ or } 2 \right\}$$

and

$$B = \left\{ x \in F \,:\, \chi(x) = (i_1 i_2 \cdots) \in \Sigma_P^+ \text{ with } i_1 = 3 \right\}.$$

The sets A and B are not invariant under G and $\nu(A) > 0$, $\nu(B) > 0$. Noting that $P^3 = P$, and using arguments in [C], one can show that for ν−almost every $x \in A$,

$$\lim_{n\to\infty} \frac{\log \operatorname{diam} \Delta_n(x)}{n} = \frac{1}{2}\log(\alpha\beta)$$

and for ν−almost every $x \in B$,

$$\lim_{n\to\infty} \frac{\log \operatorname{diam} \Delta_n(x)}{n} = \frac{1}{2}\log(\gamma\delta).$$

Consider the map $G = \chi^{-1} \circ \sigma \circ \chi$. Since $\chi(\Delta_n(x))$ is a cylinder set, the Shanon-McMillan-Breiman Theorem implies that for ν−almost every $x \in F$,

$$\lim_{n\to\infty} \frac{\log \nu(\Delta_n(x))}{n} = h_\nu(G) = h_\mu(\sigma) > 0$$

where $h_\nu(G)$ is the Kolmogorov-Sinai entropy of the map G. Thus

$$\underline{d}(x) = \overline{d}(x) = d(x) = \frac{h_\mu(\sigma)}{\lim_{n\to\infty}\left(\frac{\log \operatorname{diam}\Delta_n(x)}{n}\right)} = \frac{2h_\nu(G)}{\log(\alpha\beta)} \quad \text{if } x \in A$$

and

$$\underline{d}(x) = \overline{d}(x) = d(x) = \frac{h_\mu(\sigma)}{\lim_{n\to\infty}\left(\frac{\log \operatorname{diam}\Delta_n(x)}{n}\right)} = \frac{2h_\nu(G)}{\log(\gamma\delta)} \quad \text{if } x \in B.$$

Repeating the construction in Section 6.3, one can prove that the homeomorphism $\tilde{G}$ possesses an invariant ergodic measure $\tilde{\nu}$ for which $d_{\tilde{\nu}}(x,y) = \underline{d}_{\tilde{\nu}}(x,y) = \overline{d}_{\tilde{\nu}}(x,y)$ for $\nu-$almost every (x,y) and $d_{\tilde{\nu}}(x,y)$ is not essentially constant.

Section 7: Proofs

We first consider the Markov geometric construction and provide proofs in this case.

Proof of Proposition 1.

We begin with the following general lemma:

Lemma 4. *Let $A = (A(i,j))$ be a transitive $(p \times p)$ matrix of 0s and 1s, and consider the subshift of finite type defined by A. Let $f : \Sigma_A^+ \to \mathbb{R}$ be a continuous function that depends only on the first coordinate. Let F denote the $(p \times p)$ diagonal matrix $diag(e^{f(1)}, e^{f(2)}, \cdots, e^{f(p)})$. Then $P_A(\log f(\omega)) = P_A(\log f(\omega_1)) = \log r$, where r denotes the spectral radius of the $(p \times p)$ matrix A^*F.*

Proof. In the proof, we exploit the fact that the exponential of the pressure is the maximal eigenvalue of the transfer operator. Let $\phi : \Sigma_A^+ \to \mathbb{R}$ be a continuous function. Then the transfer operator

$$(L_f\phi)(x) \stackrel{\text{def}}{=} \sum_k \exp(f(kx))\phi(kx)A(k,x_1) = \sum_k \exp(f(k))\phi(kx)A(k,x_1),$$

where $x = (x_1x_2\cdots)$. The eigenvalue equation for L_f is

$$\sum_k \exp(f(k))h(k)A(k,j) = \eta h(j).$$

According to [PP, page 24 (note normalization of the transfer operator)], the largest eigenvalue of L_f is $\eta = \exp(P(f))$. Hence $\exp(P(f))$ is the spectral radius of the matrix A^*F.
□

Proposition 1 follows by applying Lemma 4 to the function $f(\omega) = f(\omega_1) = t\log\alpha_{\omega_1}$.
□

Proof of Proposition 2.

We first show that $\underline{s} \leq \dim_H F$. Fix $0 < r < 1$. For any $x \in F$ let $n(x)$ denote the unique positive integer such that $\underline{\lambda}_{i_1}\underline{\lambda}_{i_2}\cdots\underline{\lambda}_{i_{n(x)}} > r$ and $\underline{\lambda}_{i_1}\underline{\lambda}_{i_2}\cdots\underline{\lambda}_{i_{n(x)+1}} \leq r$ where $\chi(x) = (i_1 i_2 \cdots)$. It is easy to see that $n(x) \to \infty$ as $r \to 0$ uniformly in x. Fix $x \in F$. Consider the set $\Delta_{i_1\cdots i_{n(x)}}$. We have $x \in \Delta_{i_1\cdots i_{n(x)}}$, and if $y \in \Delta_{i_1\cdots i_{n(x)}}$ and $n(y) \geq n(x)$, then

$$\Delta_{i_1\cdots i_{n(y)}} \subset \Delta_{i_1\cdots i_{n(x)}}.$$

Let $\Delta(x)$ be the largest set in the construction of F containing x with the property that for any $y \in \Delta(x)$, $\Delta_{i_1\cdots i_{n(y)}} \subset \Delta(x)$ and there exists $z \in \Delta(x)$ such that $\Delta(x) = \Delta_{i_1\cdots i_{n(z)}}$. The sets $\Delta(x)$ corresponding to different $x \in F$ either coincide or are disjoint. We denote these sets by $\Delta^{(j)}$, $j = 1, \cdots, N$. There exist points x_j such that $\Delta^{(j)} = \Delta_{i_1\cdots i_{n(x_j)}}$. Note that the sets $\Delta^{(j)}$ depend on r. Each of these sets has diameter $> r$, but contains at the next level of subdivision at least one set of diameter $\leq r$.

By the definition of a Gibbs measure (see Appendix 3) there exist positive constants D_1 and D_2 such that for $j = 1, \cdots, N$

$$D_1 \leq \frac{\underline{m}(\Delta_{i_1\cdots i_{n(x_j)}})}{\prod_{k=1}^{n(x_j)} \underline{\lambda}_{i_k}^{\underline{s}}} \leq D_2 \tag{10}$$

where $\underline{m} = \chi^*\underline{\mu}$ is the pull back of the Gibbs measure $\underline{\mu}$ corresponding to the function $\underline{\phi}(i_1 i_2 \cdots) = \underline{s}\log\underline{\lambda}_{i_1}$ on Q. One can easily estimate the number $N(x,r)$ of sets $\Delta^{(j)}$ that intersect the ball $B(x,r)$ for sufficiently small r as follows

$$N(x,r) \leq \frac{\mathrm{vol}(B(x,r))}{\min_{1\leq j\leq N}\mathrm{vol}(\Delta_{i_1 i_2\cdots i_{n(x_j)}})} \leq \frac{\mathrm{vol}(B(x,r))}{\min_{1\leq j\leq N}\mathrm{vol}(D(C_1\prod_{k=1}^{n(x_j)}\underline{\lambda}_{i_k}))}$$

$$\leq \frac{L_1 r^d}{\min_{1\leq j\leq N} L_2(\prod_{k=1}^{n(x_j)}\underline{\lambda}_{i_k})^d} \leq L_3$$

where $L_1, L_2, L_3 > 0$ are constants and vol denotes the Euclidean volume in $\mathbb{R}^d$. Now by (10), we have that

$$\underline{m}(B(x,r)) \leq \sum_{j=1}^{N(x,r)} \underline{m}(\Delta^{(j)}) \leq \sum_{j=1}^{N(x,r)} D_2 \prod_{k=1}^{n(x_j)} \underline{\lambda}_{i_k}^{\underline{s}}$$

$$\leq L_4 \sum_{j=1}^{N(x,r)} \prod_{k=1}^{n(x_j)+1} \underline{\lambda}_{i_k}^{\underline{s}} \leq L_4 N(x,r) r^{\underline{s}} \leq L_5 r^{\underline{s}}, \tag{11}$$

where $L_4, L_5 > 0$ are constants. Hence the measure $\underline{m}$ satisfies the uniform mass distribution principle. This proves statement (1). Moreover, (11) implies that $\underline{s} \leq \underline{d}_{\underline{m}}(x)$ for every $x \in F$. It follows that $\underline{s} \leq \dim_H F$ and $\mathrm{m_H}(\underline{s}, F) > 0$.

Fix $0 < r < 1$. For each $x \in F$ with $\chi(x) = (i_1 i_2 \cdots)$, choose $1 \le i_{n(x)} \le p$ such that $\overline{\lambda}_{i_1}\overline{\lambda}_{i_2}\cdots\overline{\lambda}_{i_{n(x)}} > r$ and $\overline{\lambda}_{i_1}\overline{\lambda}_{i_2}\cdots\overline{\lambda}_{i_{n(x)}}\overline{\lambda}_{i_{n(x)+1}} \le r$. By the definition of a Gibbs measure, there exist constants $D_1, D_2 > 0$ such that for any $n \in \mathbb{N}$,

$$D_1 \le \frac{\overline{m}(\Delta_{i_1\cdots i_n})}{\prod_{k=1}^n \overline{\lambda}_{i_k}^{\overline{s}}} \le D_2.$$

It follows from the definition of r that $\Delta_{i_1\cdots i_{n(x)+1}} \subset B(x, 2C_2 r)$, where $B(x,r)$ denotes the ball of radius r around the point x. Hence for all $x \in F$,

$$\overline{m}(B(x, 2C_2 r)) \ge \overline{m}(\Delta_{i_1\cdots i_{n(x)+1}}) \ge D_1 \prod_{k=1}^{n(x)+1} \overline{\lambda}_{i_k}^{\overline{s}} \ge L_1 r^{\overline{s}}$$

where $L_1 > 0$ is a constant. It follows that for all $x \in F$,

$$\overline{d}_{\overline{m}}(x) = \limsup_{r\to 0} \frac{\log \overline{m}(B(x,r))}{\log r} \le \overline{s}.$$

This completes the proof of statement (2). The first part of statement (3) follows immediately from statements (1) and (2). We now prove the equivalence the measures $m_H(s,\cdot)$ and $m(\cdot)$.

Let $Z \subset F$ be a closed subset. Given $\gamma > 0$, there exists $\epsilon > 0$ such that for any covering $\mathfrak{U} = \{U^{(k)}\}$ of Z by open sets $U^{(k)}$ with $\operatorname{diam} U^{(k)} \le \epsilon$, we have

$$m_H(s, Z) \le \sum_{U^{(k)} \in \mathfrak{U}} (\operatorname{diam} U^{(k)})^s + \gamma.$$

There exists a covering $\mathfrak{U}$ of Z by basic sets $\Delta^{(k)} = \Delta_{i_1\cdots i_{n(k)}}$ satisfying $\operatorname{diam}\Delta^{(k)} \le \epsilon$ and

$$\sum_{\Delta^{(k)} \in \mathfrak{U}} m(\Delta^{(k)}) \le m(Z) + \gamma.$$

By (4) and (10) it follows that

$$m_H(s,Z) \le \sum_{\Delta^{(k)} \in \mathfrak{U}} (\operatorname{diam}\Delta^{(k)})^s + \gamma \le C_2 \sum_{\Delta^{(k)} \in \mathfrak{U}} \prod_{j=1}^{n(k)} \lambda_{i_j}^s + \gamma$$

$$\le C_2 D_2 \sum_{\Delta^{(k)} \in \mathfrak{U}} m(\Delta^{(k)}) + \gamma \le C_2 D_2 m(Z) + (C_2 D_2 + 1)\gamma.$$

Since γ is chosen arbitrarily this implies that $m_H(s, Z) \le$ const $m(Z)$.

Given $\gamma > 0$, there exists $\epsilon > 0$ and a covering $\mathfrak{U} = \{U^{(k)}\}$ of Z by open sets $U^{(k)}$ with $\operatorname{diam} U^{(k)} \le \epsilon$ satisfying

$$\sum_{U^{(k)} \in \mathfrak{U}} (\operatorname{diam} U^{(k)})^s \le m_H(s,Z) + \epsilon.$$

By (11) it follows that

$$m(Z) \le \sum_{U^{(k)} \in \mathfrak{U}} m(U^{(k)}) \le L_5 \sum_{U^{(k)} \in \mathfrak{U}} (\operatorname{diam} U^{(k)})^s \le L_5 m_H(s, Z) + L_5\gamma.$$

Since γ is chosen arbitrarily this implies that $m(Z) \le m_H(s, Z)$.

To prove statement (4) we show that $\overline{\dim}_B F \le \overline{s}$. This statment is true for general symbolic constructions but we provide an independent proof for our case which exploits the fact that $\overline{m}$ is a Gibbs measure. For the general case see the proof of Theorem 1.

Let $\overline{m} = \chi^* \overline{\mu}$ be the pull back of the Gibbs measure $\overline{\mu}$ corresponding to the function $(i_1 i_2 \cdots) \to \overline{s} \log \overline{\lambda}_{i_1}$ on Q. Fix $0 < r < 1$. For each x with $\chi(x) = (i_1 i_2 \cdots)$, choose the unique $n(x)$ such that $\overline{\lambda}_{i_1} \overline{\lambda}_{i_2} \cdots \overline{\lambda}_{i_{n(x)}} > r$ and $\overline{\lambda}_{i_1} \overline{\lambda}_{i_2} \cdots \overline{\lambda}_{i_{n(x)}} \overline{\lambda}_{i_{n(x)+1}} \le r$. Repeating the above arguments one can show that there exist points x_j, $j = 1, \cdots, N$ such that the sets $\Delta^{(j)} = \Delta_{i_1 \cdots i_{n(x_j)}}$ are disjoint.

By the definition of a Gibbs measure, there exist constants $D_1, D_2 > 0$ such that for $j = 1, \cdots, N$

$$D_1 \le \frac{\overline{m}(\Delta_{i_1 \cdots i_{n(x_j)}})}{\prod_{k=1}^{n(x_j)} \overline{\lambda}_{i_k}^{\overline{s}}} \le D_2. \tag{12}$$

By (12) we have that for $j = 1, \cdots, N$

$$\overline{m}(\Delta_{i_1 \cdots i_{n(x_j)}}) \ge D_1 \prod_{k=1}^{n(x_j)} \overline{\lambda}_{i_k}^{\overline{s}} \ge D_1 r^{\overline{s}}.$$

This implies that $N_r(F) \le C r^{-\overline{s}}$, for some constant $C > 0$, where $N_r(F)$ denotes the minimum number of balls of radius r required to cover the limit set F. By (6) and (7), we have that

$$\operatorname{diam}(\Delta_{i_1 \cdots n(x_j)}) \le C_2 \prod_{k=1}^{n(x_j)} \overline{\lambda}_{i_k} \le L_6 \prod_{k=1}^{n(x_j)+1} \overline{\lambda}_{i_k} \le L_6 r,$$

where L_6 is a positive constant. It follows that

$$\overline{\dim}_B F = \limsup_{r \to 0} \frac{\log(N_{2r}(F))}{\log(\frac{1}{r})} \le \overline{s}.$$

By repeating the above arguments one can easily show that $m_H(\overline{s}, F) < \infty$. $\square$

Proof of Theorems 1 and 2.

Given numbers $0 < \lambda_1, \lambda_2, \cdots, \lambda_p < 1$, let μ be an equilibrium measure on Q corresponding to the function $s \log \lambda_{i_1}$. By definition,

$$h_\mu(\sigma | Q) + s \int_Q \log \lambda_{i_1} d\mu = 0, \tag{13}$$

where $h_\mu(\sigma|Q) \stackrel{\text{def}}{=} h$ is the Kolmogorov-Sinai entropy. Let us first assume that μ is ergodic. For fixed $\epsilon > 0$, it follows from the Shannon-McMillan-Breiman theorem that for $\mu-$almost every $\omega \in Q$ one can find $N_1(\omega) > 0$ such that for any $n \geq N_1(\omega)$,

$$\exp(-(h+\epsilon)n) \leq \mu(C_{i_1\cdots i_n(\omega)}) \leq \exp(-(h-\epsilon)n) \tag{14}$$

where $C_{i_1\cdots i_n(\omega)}$ is the cylinder set containing ω. It follows from the Birkhoff ergodic theorem that for $\mu-$almost every $\omega \in Q$ there exists $N_2(\omega)$ such that for any $n \geq N_2(\omega)$,

$$\frac{1}{n}\log\prod_{j=1}^{n}\lambda_{i_j}^{s} - \epsilon \leq s\int_Q \log\lambda_{i_1}\,d\mu \leq \frac{1}{n}\log\prod_{j=1}^{n}\lambda_{i_j}^{s} + \epsilon. \tag{15}$$

Combining (13), (14), and (15) we have that for $\mu-$almost every $\omega \in Q$ and n sufficiently large,

$$\prod_{j=1}^{n}\lambda_{i_j}^{s+\alpha} \leq \prod_{j=1}^{n}\lambda_{i_j}^{s}\exp(-2\epsilon n) \leq \mu(C_{i_1\cdots i_n(\omega)}) \leq \prod_{j=1}^{n}\lambda_{i_j}^{s}\exp(2\epsilon n) \leq \prod_{j=1}^{n}\lambda_{i_j}^{s-\alpha},$$

where $\alpha = \frac{2\epsilon}{\min(\log\frac{1}{\lambda_j}, 1\leq j\leq p)} > 0$. This implies that for $\mu-$almost every $\omega \in Q$ and any $n \geq \max\{N_1(\omega), N_2(\omega)\}$,

$$\prod_{j=1}^{n}\lambda_{i_j}^{s+\alpha} \leq \mu(C_{i_1\cdots i_n(\omega)}) \leq \prod_{j=1}^{n}\lambda_{i_j}^{s-\alpha}. \tag{16}$$

If μ is not ergodic, then (16) is still valid and can be shown by decomposing μ into its ergodic components.

Given $l > 0$ denote $Q_l = \{\omega \in Q : N_1(\omega) \leq l \text{ and } N_2(\omega) \leq l\}$. It is easy to see that $Q_l \subset Q_{l+1}$ and $Q = \cup_{l=1}^{\infty} Q_l \pmod 0$. Thus there exists $l_0 > 0$ such that $\mu(Q_l) > 0$ if $l \geq l_0$. Let $x \in Q_l, l \geq l_0, 0 < r < 1$.

Repeating arguments in the proof of Proposition 2 with Q replaced by Q_l and applying (16) to the measure $\underline{\mu}$ and numbers $\underline{\lambda}_1, \cdots, \underline{\lambda}_p$, one obtains that $\underline{m}(B(x,r)\cap\chi^{-1}(Q_l)) \leq Kr^{\underline{s}-\alpha}$, where $K = K(l) > 0$ is a constant and $x \in \chi^{-1}(Q_l))$ is any point. This implies that for any $l > 0$ and $x \in \chi^{-1}(Q_l))$

$$\underline{d}^{l}_{\underline{m}}(x) \stackrel{\text{def}}{\equiv} \liminf_{r\to 0}\frac{\log\underline{m}(B(x,r)\cap\chi^{-1}(Q_l))}{\log r} \geq \underline{s} - \alpha.$$

It follows that for $x \in \chi^{-1}(Q_l)$,

$$\underline{d}_{\underline{m}}(x) = \inf_{l>0}\underline{d}^{l}_{\underline{m}}(x) \geq \underline{s} - \alpha,$$

and thus this is true for $\underline{m}-$almost every $x \in F$. Since α is arbitrary, this implies that $\underline{d}_{\underline{m}}(x) \geq \underline{s}$.

We now show that $\overline{d}_{\overline{m}}(x) \leq \overline{s}$ for $\overline{m}-$almost every $x \in F$. Fix $0 < r < 1$. For each x with $\chi(x) = (i_1 i_2 \cdots)$, choose the unique $n(x)$ such that $\overline{\lambda}_{i_1}\overline{\lambda}_{i_2}\cdots\overline{\lambda}_{i_{n(x)}} > r$ and $\overline{\lambda}_{i_1}\overline{\lambda}_{i_2}\cdots\overline{\lambda}_{i_{n(x)+1}} \leq r$. It is easy to see that $n(x) \to \infty$ as $r \to 0$ uniformly in x. Fix $l > 0$ for which $\overline{\mu}(Q_l) > 0$. One can now choose $r = r(l) > 0$ sufficiently small such that $n(\omega)$ becomes large enough to satisfy (16) for any $\omega \in Q_l$. Repeating the arguments in the proof of Proposition 2 and using (16) one can show that for any $x \in \chi^{-1}(Q_l)$ and any $r > 0$ sufficiently small,

$$\overline{m}(B(x,r)) \geq K r^{\overline{s}+\alpha}$$

where $K = K(l) > 0$ is a constant. This implies that $\overline{d}_{\overline{m}}(x) \geq \overline{s} + \alpha$ and hence, $\overline{d}_{\overline{m}}(x) \geq \overline{s}$ for any $x \in \chi^{-1}(Q_l)$. This completes the proof of statement (1) of Theorem 2.

We now show that $\overline{\dim}_B F \leq \overline{s}$. The following arguments, essentially due to Bowen, are a slight modification of arguments shown to us by F. Ledrappier.

Proof. It is sufficient to prove that $P(\overline{\dim}_B F \log \overline{\lambda}_{x_0}) \geq 0$, since the map $t \to P(t \log \overline{\lambda}_{x_0})$ is a decreasing function [Bo1].

Given $\delta > 0$, it follows from the definition of $\overline{\dim}_B F$ that there exists $\epsilon > 0$ such that $N_\epsilon(F) \geq \epsilon^{\delta - \overline{\dim}_B F}$. Let $\{\Delta^{(j)}\} = \{\Delta_{i_1,\cdots,i_{n(x_j)}}\}$, $j = 1,\cdots,N^\epsilon(F)$ denote the covering by basic sets constructed in Proposition 2. Note that this covering need not be the optimal covering, i.e., $N^\epsilon(F) \geq N_\epsilon(F)$. There clearly exists $A = A(\epsilon) > 0$ such that for $j = 1,\cdots,N^\epsilon(F)$,

$$\frac{\epsilon}{A} \leq \prod_{k=1}^{n(x_j)} \overline{\lambda}_{i_k} \leq \epsilon$$

and hence

$$C_1 \log(\frac{1}{\epsilon}) \leq n(x_j) \leq C_2 \log(\frac{A}{\epsilon})$$

where $C_1 = \frac{1}{\log(\frac{1}{\overline{\lambda}_{\max}})}$ and $C_2 = \frac{1}{\log(\frac{1}{\overline{\lambda}_{\min}})}$. This implies that $n(x_j)$ can take on at most $C_2 \log(\frac{A}{\epsilon}) - C_1 \log(\frac{1}{\epsilon})$ possible values.

We now think of having $N_\epsilon(F)$ balls and $C_2 \log(\frac{A}{\epsilon}) - C_1 \log(\frac{1}{\epsilon})$ baskets. Then for $N^\epsilon(F) \geq C_2 \log(\frac{A}{\epsilon}) - C_1 \log(\frac{1}{\epsilon})$, there exists a basket containing at least $\frac{N^\epsilon(F)}{C_2 \log(\frac{A}{\epsilon}) - C_1 \log(\frac{1}{\epsilon})}$ balls. This implies that there exists a positive integer α, $C_1 \log(\frac{1}{\epsilon}) \leq \alpha \leq C_2 \log(\frac{A}{\epsilon})$ such that for ϵ sufficiently small,

$$\#\{x_j \text{ such that } n(x_j) = \alpha\} \geq \frac{N^\epsilon(F)}{C_2 \log(\frac{A}{\epsilon}) - C_1 \log(\frac{1}{\epsilon})}$$

$$\geq \frac{N_\epsilon(F)}{C_2 \log(\frac{A}{\epsilon}) - C_1 \log(\frac{1}{\epsilon})} \geq \frac{\epsilon^{\delta - \overline{\dim}_B F}}{C_3 \log(\frac{1}{\epsilon})} \geq \epsilon^{2\delta - \overline{\dim}_B F}.$$

Let $\phi(x) = (\overline{\dim}_B F - 2\delta) \log \underline{\lambda}_{x_0}$. Then

$$(S_n\phi)(x) = \sum_{k=0}^{n-1} \phi(\sigma^k x) = (\overline{\dim}_B F - 2\delta) \log \prod_{k=0}^{n-1} \overline{\lambda}_{x_k}$$

and hence $\exp(S_n\phi)(x) = (\prod_{k=0}^{n-1} \overline{\lambda}_{x_k})^{\overline{\dim}_B F - 2\delta}$. It follows that

$$P_\alpha(\overline{\dim}_B F - 2\delta) \log \overline{\lambda}_{x_0}) \stackrel{\text{def}}{\equiv} \frac{1}{\alpha} \log \sum_{\substack{(i_0 \cdots i_{\alpha-1}) \\ \text{admissible}}} \inf_{x \in \Delta_{(i_0 \cdots i_{\alpha-1})}} (\prod_{k=0}^{n-1} \overline{\lambda}_{x_k})^{\overline{\dim}_B F - 2\delta}$$

$$\geq \frac{1}{\alpha} \log \sum_{\substack{(i_0 \cdots i_{\alpha-1}) \\ \text{in covering } \{\Delta^{(j)}\}}} (\frac{A}{\epsilon})^{\overline{\dim}_B F - 2\delta}$$

$$\geq A^{\overline{\dim}_B F - 2\delta} \frac{1}{\alpha} \log(\epsilon^{\overline{\dim}_B F - 2\delta} \cdot \epsilon^{2\delta - \overline{\dim}_B F}) \geq 0.$$

Hence $P((\overline{\dim}_B F - 2\delta) \log \overline{\lambda}_{x_0}) = \lim_{\alpha \to \infty} P_\alpha((\overline{\dim}_B F - 2\delta) \log \overline{\lambda}_{x_0}) \geq 0$. □

Proof of Proposition 3.

Let $\overline{\mu}$ denote an equilibrium measure for $\overline{s} \log \overline{\lambda}_{i_1}$. It immediately follows from the variational principle that

$$\overline{s} = \frac{h_{\overline{\mu}}(\sigma \mid Q)}{- \int \log \overline{\lambda}_{i_1} d\overline{\mu}} \leq \frac{h(\sigma \mid Q)}{- \log \max \overline{\lambda}_k}.$$

The case of equality is obvious.

If $\underline{\lambda}_i = \overline{\lambda}_i = \lambda$ for $i = 1, \cdots, p$, then this immediately implies that $\overline{s} = \frac{h(\sigma \mid Q)}{- \log \lambda}$. This proves statements (2) and (3). □

Proof of Theorem 4.

1) The proof combines arguments in the proofs of Theorems 1, 2 and Proposition 2. The weak asymptotic condition 2a) implies that for any $\epsilon > 0$ and $\underline{\mu}$–almost every $\omega \in Q$ there exists $N_3(\omega) = N_3(\omega, \epsilon)$ such that for any $n \geq N_3(\omega)$,

$$\left| \frac{1}{n} \sum_{j=1}^{n} \underline{a}_{i_j, j} \right| \leq \epsilon. \tag{17}$$

The inequality (17) is equivalent to

$$\prod_{j=1}^{n} \underline{\lambda}_{i_j} \exp(-\epsilon n) \leq \prod_{j=1}^{n} \underline{\lambda}_{i_j, j} \leq \prod_{j=1}^{n} \underline{\lambda}_{i_j} \exp(\epsilon n). \tag{18}$$

It is sufficient to consider only the case when $\underline{\mu}$ is ergodic with respect to σ. Then for $\underline{\mu}$–almost every $\omega = (i_1 i_2 \cdots) \in Q$, the following limit exists:

$$\lim_{n \to \infty} \sum_{j=1}^{n} \log \underline{\lambda}_{i_j} = \int_Q \log \underline{\lambda}_{\omega_1} d\underline{\mu}(\omega) \equiv a < 0.$$

This implies that for any $\epsilon > 0$ and $\underline{\mu}-$almost every $\omega \in Q$ there exists $N_4(\omega) = N_4(\omega, \epsilon)$ such that for any $n \geq N_4(\omega)$,

$$\left| \frac{1}{n} \sum_{j=1}^{n} \log \underline{\lambda}_{i_j} - a \right| \leq \epsilon. \tag{19}$$

Given $l > 0$ denote

$$Q_l = \{\omega \in Q : N_i(\omega) \leq l, i = 1, 2, 3, 4\}$$

where $N_1(\omega), N_2(\omega)$ are the two functions constructed in the proof of Theorems 1, 2 (see (16)). It is easy to see that $Q_l \subset Q_{l+1}$ and $Q = \cup_{l=1}^{\infty} Q_l \mod (0)$. Thus there exists $l_0 > 0$ such that $\underline{\mu}(Q_l) > 0$ if $l \geq l_0$.

Consider $\omega = (i_1 i_2 \cdots) \in Q_l, l \geq l_0$ and $0 < r < 1$. Let $n(\omega)$ denote the unique positive integer such that $\underline{\lambda}_{i_1} \underline{\lambda}_{i_2} \cdots \underline{\lambda}_{i_{n(\omega)}} > r$ and $\underline{\lambda}_{i_1} \underline{\lambda}_{i_2} \cdots \underline{\lambda}_{i_{n(\omega)+1}} \leq r$. Obviously, $n(\omega) \to \infty$ as $r \to 0$ uniformly in ω. Hence we can assume $r = r(l)$ is so small that $n(\omega) \geq \max\{N_3(\omega, \epsilon), N_4(\omega, \epsilon)\}$. Applying (19) to $n = n(\omega)$ we have that

$$n(\omega) \leq \log r(a + \epsilon). \tag{20}$$

Consider the cylinder set $C_{i_1 \cdots i_{n(\omega)}}$. We have that $\omega \in C_{i_1 \cdots i_{n(\omega)}}$ and if $\omega' \in C_{i_1 \cdots i_{n(\omega)}} \cap Q_l$ and $n(\omega') \leq n(\omega)$ then

$$C_{i_1 i_2 \cdots i_{n(\omega')}} \supset C_{i_1 i_2 \cdots i_{n(\omega)}}.$$

Let $C(\omega)$ be the largest cylinder set containing ω with the property that for every $\omega' \in C(\omega) \cap Q_l$ we have $C_{i_1 \cdots i_{n(\omega')}} \cap Q_l \subset C(\omega) \cap Q_l$ and there exists $\tilde{\omega} \in C(\omega)$ such that $C(\omega) = C_{i_1 \cdots i_{n(\tilde{\omega})}}$. It is easy to see that the sets $C(\omega) \cap Q_l$ corresponding to different $\omega \in Q_l$ either coincide or are disjoint. We denote these sets by $C^{(j)}, j = 1, \cdots N$. There exists points ω_j such that $C^{(j)} = C_{i_1 \cdots i_{n(\omega_j)}}$. For any $j = 1, \cdots, N$ we have that

$$l^{-1} \prod_{k=1}^{n(\omega_j)} \underline{\lambda}_{i_k}^{\underline{s}} \leq \underline{\mu}(C^{(j)}) \leq l \prod_{k=1}^{n(\omega_j)} \underline{\lambda}_{i_k}^{\underline{s}}.$$

Consider a point $x \in F$ with $\chi(x) = \omega \in Q_l$. We estimate the number $N(x, r)$ of sets $C^{(j)}$ that intersect the set $\chi(B(x, r)) \cap Q_l$ for sufficiently small r. By virtue of (19) and (20),

$$N(x, r) \leq \frac{\mathrm{vol}(B(x, r) \cap \chi^{-1}(Q_l))}{\min_{1 \leq j \leq N} \mathrm{vol}(\chi^{-1}(C^{(j)}))} \leq \frac{\mathrm{vol}(B(x, r))}{\min_{1 \leq j \leq N} \mathrm{vol}(D(C_1 \prod_{k=1}^{n(\omega_j)} \underline{\lambda}_{i_k, k})}$$

$$\leq \frac{L_1 r^d}{\min_{1 \leq j \leq N} L_2 (\prod_{k=1}^{n(\omega_j)} \underline{\lambda}_{i_k, k})^d} \leq \frac{L_1 r^d}{\min_{1 \leq j \leq N} L_3 (\prod_{k=1}^{n(\omega_j)} \underline{\lambda}_{i_k})^d \exp(-\epsilon d n(\omega_j))}$$

$$\leq \frac{L_1 r^d}{L_4 r^d r^{-\frac{\epsilon d}{a + \epsilon}}} \leq L_5 r^{\frac{\epsilon d}{a + \epsilon}}$$

where $L_i, i = 1, \cdots, 5$ are positive constants and vol denotes the Euclidean volume in $\mathbb{R}^d$. Applying (16) to the measure $\underline{\mu}$ we have that

$$\underline{\mu}(B(x,r) \cap \chi^{-1}(Q)) \leq \sum_{j=1}^{N(x,r)} \underline{\mu}(C^{(j)})$$

$$\leq \sum_{j=1}^{N(x,r)} l \prod_{k=1}^{n(\omega_j)} \underline{\lambda}_{i_k}^{\underline{s}-\alpha} \leq lN(x,r) r^{\underline{s}-\alpha} \leq L_6 r^\gamma,$$

where $L_6 > 0$ is a constant and $\gamma = \underline{s} - \alpha + \frac{\epsilon d}{a+\epsilon}$. This implies that the lower pointwise dimension at x calculated with respect to the set Q_l is not less than γ. Since this is true for all $l > 0$ we have that $\underline{d}_{\underline{m}}(x) \geq \underline{s} - \alpha + \frac{\epsilon d}{a+\epsilon}$. Since ϵ can be chosen arbitrarily small, α is arbitrarily small and hence $\underline{d}_{\underline{m}}(x) \geq \underline{s}$. Therefore $\dim_H F \geq \underline{s}$ (see Appendix 1).

2) Using the weak asymptotic condition 2b) and repeating the above arguments applied to the numbers $\overline{\lambda}_i, \overline{\lambda}_{i,n}$ instead of $\underline{\lambda}_i, \underline{\lambda}_{i,n}$ one can show that they satisfy (18) and (19). This fact and (16) applied to the measure $\overline{m}$ produce the sequence of sets $Q_l, l > 0$ such that $Q_l \subset Q_{l+1}$ and $Q = \cup_{l=1}^{\infty} Q_l \mod (0)$. Let us fix Q_l with $\mu(Q_l) > 0$. Consider $x \in F$ with $\chi(x) = \omega = (i_1 i_2 \cdots) \in Q_l$. Choose the unique $n(\omega)$ such that $\overline{\lambda}_{i_1} \overline{\lambda}_{i_2} \cdots \overline{\lambda}_{i_{n(\omega)}} > r$ and $\overline{\lambda}_{i_1} \overline{\lambda}_{i_2} \cdots \overline{\lambda}_{i_{n(\omega)+1}} \leq r$. It follows from (19) that $n(\omega)$ satisfies (20). By virtue of (18) and (20) we have that

$$\operatorname{diam}(\Delta_{i_1 \cdots i_{n(\omega)}}) \leq C_2 \prod_{k=1}^{n(\omega)} \overline{\lambda}_{i_k,k} \leq C_2 \prod_{k=1}^{n(\omega)} \overline{\lambda}_{i_k} \exp(\epsilon n) \leq L_7 r r^{\frac{\epsilon}{a+\epsilon}} \leq L_7 r^\beta$$

where $L_7 > 0$ is a constant and $\beta = 1 + \frac{\epsilon}{a+\epsilon}$, $0 < \beta < 1$. This means that

$$\Delta_{i_1 \cdots i_{n(\omega)}} \subset B(x, 2L_7 r^\beta).$$

Therefore, applying (16) to the measure $\overline{m}$ we have that

$$\overline{m}(B(x, 2L_7 r^\beta)) \geq \overline{m}(\Delta_{i_1 \cdots i_{n(\omega)}}) \geq \prod_{j=1}^{n(\omega)} \underline{\lambda}_{i_j}^{\overline{s}+\alpha} \geq L_8 r^{\overline{s}+\alpha}$$

where $L_8 > 0$ is a constant. This leads to

$$\overline{d}_{\overline{m}}(x) = \limsup_{r \to 0} \frac{\log \overline{m}(B(x,r))}{\log r} \leq \frac{\overline{s}+\alpha}{\beta}.$$

Since ϵ can be taken arbitrarily small, α becomes arbitrarily small, and β arbitrarily close to 1. We thus obtain that $\overline{d}_{\overline{m}}(x) \leq \overline{s}$.

3) The proof follows from **2)**. □

Proof of Theorem 5.

We first prove part 1a). Applying the Birkhoff ergodic theorem for the stationary ergodic process to the functions $\underline{f}(\vec{\lambda}) = (\log \underline{\lambda}_{1,1}, \cdots, \log \underline{\lambda}_{p,1})$ and $\overline{f}(\vec{\lambda}) = (\log \overline{\lambda}_{1,1}, \cdots, \log \overline{\lambda}_{p,1})$ we obtain that

$$\frac{1}{n}\sum_{k=1}^{n} \log \underline{\lambda}_{i,k} \to \log \underline{\lambda}_i$$

for ν almost every $\vec{\lambda}$, where we define $\log \underline{\lambda}_i$ as the limiting value. We need to show that for $\underline{\mu}$ almost every $\omega = (i_1 i_2 \cdots) \in Q$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} \log\left(\frac{\underline{\lambda}_{i_k,k}}{\underline{\lambda}_{i_k}}\right) = 0.$$

We break up the sum

$$\frac{1}{N}\sum_{k=1}^{N} \log\left(\frac{\underline{\lambda}_{i_k,k}}{\underline{\lambda}_{i_k}}\right) = \frac{1}{N}\sum_{j=1}^{p} \sum_{\substack{i_k=j \\ 1\le k\le N}} \log\left(\frac{\underline{\lambda}_{j,k}}{\underline{\lambda}_j}\right)$$

$$= \sum_{j=1}^{p} \frac{\#\{i_k = j\}}{N}\left(\frac{1}{\#\{i_k = j\}} \sum_{\substack{i_k=j \\ 1\le k\le N}} \log\left(\frac{\underline{\lambda}_{j,k}}{\underline{\lambda}_j}\right)\right).$$

We wish to show that for $\underline{\mu}-$almost every ω, each of the above terms in parentheses tends to zero as N tends to infinity. This immediately follows from the following generalization of the Birkhofff ergodic theorem for return times to a set.

Theorem [BFKO]**.** Let $(X, \mathfrak{B}, \nu, T)$ and $(Y, \mathfrak{C}, \mu, S)$ be two ergodic measure theoretic dynamical systems. Let $A \subset Y$ be of positive measure $\mu(A) > 0$. For every $y \in Y$ let A_y denote the return time sequence $\{n \in \mathbb{N} \,|\, S^n y \in A\}$. Then for $\mu-$almost every $y \in Y$ and for $f \in L^1(X)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{1\le k\le N \\ k\in A_y}} T^k f(x) = \int_X f d\nu$$

for $\nu-$almost every $x \in X$.

We apply this theorem for T the ergodic stationary process in the definition of random symbolic geometric construction, $S = \sigma : Q \to Q$ the shift map with invariant Gibbs measure $\underline{\mu}$, and $A = C_j = \{\omega \in Q \,|\, \omega_1 = j\}$. Clearly, the set $\{1 \le k \le N,\, i_k = j\} = \{1 \le k \le N,\, \sigma^k\omega \in C_j\}$. Part 1a) follows immediately. The proof of 1b) follows analogously. The proof of 2) now follows immediately from Theorem 5a. $\square$

Proof of Proposition 4.

We define a homeomorphism $\pi_1 : \overline{F}_{sym} \to F$ by setting $\pi_1(x) = y$ where $x \in \overline{F}_{sym}$, $y \in F$ and $\chi(x) = \chi(y) = (i_1, i_2, \cdots)$. We shall show that this map is Lipschitz. Suppose $x, y \in \overline{F}_{sym}$. Then there exists $N \in \mathbb{N}$ such that x and y belong to the same basic set $\{\Delta_{i_1 \cdots i_N}\}$ but x and y do not belong to the same basic set at step $N+1$ of the construction. Hence $d(x,y) \geq \frac{1}{\max_{1\leq i\leq p} \overline{\lambda}_i} \mathrm{diam}\{\Delta_{i_1\cdots i_N}\}$. By construction, the diameter of a basic set for $\overline{F}_{sym}$ is greater than or equal to the diameter of the corresponding basic set for F. Hence

$$d(\pi_1(x), \pi_1(y)) \leq \mathrm{diam}(\pi_1(\Delta_{i_1\cdots i_N})) \leq \mathrm{diam}(\Delta_{i_1\cdots i_N}) \leq \frac{1}{\max_{1\leq i\leq p} \overline{\lambda}_i} d(x,y).$$

It follows that the map π_1 is Lipschitz with the Lipschitz constant given by $\frac{1}{\max_{1\leq i\leq p} \overline{\lambda}_i}$. This implies statement (1), see [F1]. The second statement can be proven in the similar way. The third statement follows immediately from the first two, the positivity and finiteness of the Hausdorff measure for similarity constructions are proven in [F2].

Proof of Theorem 6.

Given $x \in F, \chi(x) = (i_1 i_2 \cdots)$ and $r > 0$, choose $n_r(x)$ such that $\prod_{k=1}^{n_r(x)} \lambda_{i_k,k} < r$ and $\prod_{k=1}^{n_r(x)-1} \lambda_{i_k,k} \geq r$. Since $x \in \Delta_{n_r(x)}(x)$ we have that $\Delta_{n_r(x)}(x) \subset B(x,r)$. This implies that

$$\frac{\log \nu(B(x,r))}{\log r} \leq \frac{\log \nu(\Delta_{n_r(x)}(x))}{\log r}.$$

We also have that

$$r \leq \prod_{k=1}^{n_r(x)-1} \lambda_{i_k,k} = \prod_{k=1}^{n_r(x)} \lambda_{i_k,k} \frac{1}{\lambda_{i_{n_r(x)}}} \leq \frac{1}{\beta} \mathrm{diam}\Delta_{n_r(x)}(x).$$

It follows that

$$\overline{d}_\nu(x) = \limsup_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \leq \limsup_{r\to 0} \frac{\log \nu(\Delta_{n_r(x)}(x))}{\log(\frac{1}{\beta}\mathrm{diam}\Delta_{n_r(x)}(x))}$$

$$\leq \limsup_{r\to 0} \frac{\log \nu(\Delta_n(x))}{\log \mathrm{diam}\Delta_n(x)} = \overline{d}(x).$$

We now prove the second estimate in (1). Given $\alpha \geq 0, C > 0$, define $F_{\alpha,C} = \{x \in F : \nu(\Delta_n(x)) \leq C(\mathrm{diam}\Delta_n(x))^\alpha$ for all $n \geq 0\}$. Fix $x \in F_{\alpha,C}$ and $r > 0$. By repeating the arguments in the proof of Theorem 1, one can find points $x_j \in F_{\alpha,C}, j = 1, \cdots, N$ (where N is independent of x and r) such that $x_j \in \Delta^{(j)} = \Delta_{i_1\cdots i_{n(x_j)}}$ and such that the collection $\Delta^{(j)}$ are disjoint, $\prod_{k=1}^{n(x_j)} \lambda_{i_k,k} \leq r$, and $B(x,r) \cap F_{\alpha,C} \subset \cup_{j=1}^N (\Delta^{(j)} \cap F_{\alpha,C})$. If follows that $\nu(B(x,r)) \leq \sum_{j=1}^N \nu(\Delta^{(j)})$ and hence

$$\underline{d}_\nu(x) = \liminf_{r\to 0} \frac{\log \nu(B(x,r))}{\log r} \geq \liminf_{r\to 0} \frac{\log \sum_{j=1}^N \nu(\Delta^{(j)})}{\log r} \geq \alpha.$$

The result follows since $F = \cup_\alpha \cup_{C>0} F_{\alpha,C}$.

The last statement is a direct consequence of the proceeding statements. $\square$

Appendices.

Appendix 1: Hausdorff Dimension and Box Dimension

Let U $\subset \mathbb{R}^n$. The *diameter* of U is defined as $|U| = \sup\{|x-y| : x, y \in U\}$. If $\{U_i\}$ is a countable collection of sets of diameter at most δ that cover Z, i.e., $Z \subset \cup_i U_i$ with $0 < |U_i| \le \delta$ for each i, we say that $\{U_i\}$ is a δ-cover of Z.

Suppose that $Z \subset \mathbb{R}^n$ and $s \ge 0$. For any $s > 0$ we define

$$\mathrm{m_H}(s, Z) = \lim_{\delta \to 0} \inf_{\{U_i\}} \left\{ \sum_i |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } Z \right\}.$$

We call $\mathrm{m_H}(s, Z)$ the *s-dimensional Hausdorff measure of Z.* There exists a unique critical value of s at which $\mathrm{m_H}(s, Z)$ jumps from ∞ to 0. This critical value is called the *Hausdorff dimension* of Z and is written $\dim_H Z$. If $s = \dim_H Z$, then $\mathrm{m_H}(s, Z)$ may be $0, \infty$, or finite. Hence $\dim_H Z = \sup\{s : \mathrm{m_H}(s, Z) = \infty\} = \inf\{s : \mathrm{m_H}(s, Z) = 0\}$.

Let $N_\delta(Z)$ denote the minimum number of sets of diameter precisely δ needed to cover the set Z. We define the upper and lower box dimensions of Z by

$$\underline{\dim}_B Z = \liminf_{\delta \to 0} \frac{\log N_\delta(Z)}{\log(\frac{1}{\delta})}, \quad \text{and} \quad \overline{\dim}_B Z = \limsup_{\delta \to 0} \frac{\log N_\delta(Z)}{\log(\frac{1}{\delta})}.$$

It is easy to see that $\dim_H Z \le \underline{\dim}_B Z \le \overline{\dim}_B Z$.

Appendix 2: Three Methods of Obtaining Lower Bounds for $\dim_{\mathrm{H}}$F

1. Mass Distribution Principle [Fr]. *Let μ be a probability measure of Z and suppose that for some s there are numbers $c > 0$, and $\delta > 0$ such that $\mu(U) \le c|U|^s$ for all sets U with $|U| \le \delta$. Then $m_{\mathrm{H}}(s, Z) \ge \mu(F)/c$ and $s \le dim_H Z$.*

2. Potential Principle [Fr].

For $s \ge 0$ the *s-potential* at a point $x \in \mathbb{R}^n$ with respect to the measure μ is defined by $\phi_s(x) = \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^s}$. The *s-energy of μ* is $I_s(\mu) = \int_{\mathbb{R}^n} \phi_s(x) d\mu(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{d\mu(x)d\mu(y)}{|x-y|^s}$.

Proposition 1. *Let $Z \subset \mathbb{R}^n$.*

(1) *If there is a measure μ on Z with $I_s(\mu) < \infty$, then $m_{\mathrm{H}}(s, Z) = \infty$ and $dim_H Z \ge s$.*
(2) *$dim_H Z = \sup\{t \,|\, \text{there exists } \mu \text{ such that } \mu(Z) > 0,\ I_t(\mu) < \infty\}$.*

3. Non-Uniform Mass Distribution Principle. *Let μ be a probability measure supported on $Z \subset R^n$. We define the* **Hausdorff dimension of the measure** μ, $dim_H \mu = \inf\{dim_H U,\ \mu(U) = 1\}$, *and the* **lower and upper pointwise dimensions of** μ, $\underline{d}_\mu(x) = \liminf_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon}$ *and* $\overline{d}_\mu(x) = \limsup_{\epsilon \to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon}$.

Proposition. *If $\underline{d}_\mu(x) \ge d \ge 0$ for a.e. $x \in Z$, then $dim_H \mu \ge d$, and hence $dim_H Z \ge d$.*

The mass distribution principle implies the potential method that, in turn, implies the nonuniform mass distribution principle. In fact, $d_\mu^L(x) \ge d \Rightarrow \forall \alpha > 0,\ \mu(B(x,r)) \le C(x,\alpha) r^{d-\alpha}$.

Appendix 3: Thermodynamic Formalism

Good references for this material are [Bo1] and [PP]. Given a transitive $p \times p$ transfer matrix A of 0*s* and 1*s*, consider the one-sided subshift of finite type (Σ_A^+, σ) with $\sigma : \Sigma_A^+ \to \Sigma_A^+$ the shift map. The set Σ_A^+ consists of all admissible words, i.e., a word $x = (x_1, x_2, \cdots) \in \{1, \cdots, p\}^{\mathbb{N}}$ is admissible if $a_{x_i, x_{i+1}} = 1$ for all $i \in \mathbb{N}$. The space Σ_A^+ has a natural family of metrics defined by $d(x, y) = \sum_{k=1}^{\infty} \frac{|x_k - y_k|}{\beta^k}$, where β is any number satisfying $\beta > 1$. The set Σ_A^+ is compact with respect to the topology induced by d and the shift map $\sigma : \Sigma_A^+ \to \Sigma_A^+$ is a homeomorphism. We let $\Sigma_p^+ = \{1, \cdots, p\}^{\mathbb{N}}$.

More generally, we consider general symbolic system, i.e., a compact subset $Q \subset \Sigma_p^+$ that is $\sigma-$invariant, i.e., $\sigma(Q) = Q$.

Let $C^0(Q)$, $C^\alpha(Q)$ denote the spaces of continuous and $\alpha-$Hölder continuous functions on Q. We define a mapping $P : C^0(Q) \to \mathbb{R}$ by

$$P(\phi) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{(i_1 \cdots i_n) \text{ admissible}} \inf_{x \in \Delta_{(i_1 \cdots i_n)}} \exp(S_n \phi(x)),$$

where $S_n \phi(x) = \sum_{i=0}^{n-1} \phi(\sigma^i x)$. We call $P(\phi)$ the *topological pressure* of ϕ.

The following result is a variational characterization of pressure that is valid for *all* topological dynamical systems. Let $\mathfrak{M}(Q)$ denote the space of all shift-invariant Borel probability measures on Q.

Variational Principle. Let $\phi \in C^0(Q)$. Then

$$P(\phi) = \sup_{\mu \in \mathfrak{M}(\Sigma_A)} \left(h_\mu(\sigma) + \int_{\Sigma_A} \phi d\mu \right).$$

Measures that realize the variational principle for topological pressure play crucial role in the thermodynamical formalism.

Definition. A Borel probability measure $\mu = \mu_\phi$ on Q is called an **equilibrium measure** for the potential ϕ if

$$P(\phi) = h_\mu(\sigma) + \int_{\Sigma_A} \phi d\mu. \tag{21}$$

Since the shift map on a general symbolic systems is expansive, the supremum in the variational principle is attained by some measure [W]. This measure need not be unique.

Another important class of measures are Gibbs measures:

Definition. Let $\phi \in C^0(Q)$. A Borel probability measure $\mu = \mu_\phi$ on Q is called a **Gibbs measure** for the potential ϕ if there exist constants $D_1, D_2 > 0$ such that

$$D_1 \leq \frac{\mu\{y : y_i = x_i,\ i = 0, \cdots, n-1\}}{\exp(-nP(\phi) + \sum_{k=0}^{n-1} \phi(\sigma^k x))} \leq D_2 \tag{22}$$

for all $x = (x_1 x_2 \cdots) \in \Sigma_A^+$ and $n \geq 0$.

For subshifts of finite type, Gibbs measures exist for any Holdër continuous potential ϕ, are unique, and coincide with the equilibrium measure for ϕ. Little is known about the existence of Gibbs measures for general symbolic systems.

The main tool used in constructing and studying Gibbs measures for subshifts of finite type is the (linear bounded) transfer operator $L_f : C^\alpha(\Sigma_A^+) \to C^\alpha(\Sigma_A^+)$ defined by

$$(L_f \phi)(x) = \sum_{y \in \sigma^{-1}(x)} \exp(f(y))\phi(y) = \sum_k \exp(f(kx))\phi(kx).$$

along with its dual operator $L_f^* : M(\Sigma_A^+) \to M(\Sigma_A^+)$, where $M(\Sigma_A^+)$ denotes the space of Borel probability measures on Σ_A^+. The following theorem of Ruelle constructs Gibbs measures using the operator L_f:

Proposition 2 [PP]. *Let (Σ_A^+, σ) be mixing. There exists $\lambda = \exp(P(\phi)) \geq 1$, $h \in C^0(\Sigma_A^+, \mathbb{R})$ with $h > 0$ and $\nu \in M(\Sigma_A^+)$ for which $L_f h = \lambda h, L_f^* \nu = \lambda \nu$, and $\nu(h) = 1$. Then $\mu = h\nu$ is a $\sigma-$invariant probability measure on Σ_A^+ and is a Gibbs measure for ϕ.*

In this paper we deal exclusively with a special class of potentials that depend only on the first coordinate, i.e., $\phi(x) = \phi(x_1)$. In this case, the measures ν and μ are Markov and can be described explicitly: the eigenfunction $h = h(x_1)$ satisfies

$$\sum_i h(i)A(i,j)\exp(\phi(i)) = \lambda h(j)$$

and the measure ν is defined on cylinder sets by

$$\nu[i_0, \cdots, i_n] = P(i_0, i_1) \cdots P(i_{n-1}, i_n) p(i_n) \quad \text{where} \quad P(i,j) = \frac{A(i,j)h(i)\exp(\phi(i))}{\lambda h(j)}$$

and $Pp = p$ with $\sum_i p(i) = 1$. Finally, the Gibbs measure μ is defined by $d\mu = h(x_1) d\nu$.

The Gibbs measure μ is unique provided the potential f is Hölder continuous.

Proposition 3 [PP]. *Given numbers λ_i, $i = 1, \cdots, p$, define the function*

$$\phi : \Sigma_A^+ \to \Sigma_A^+ \quad \textit{by } \phi(x) = \phi(x_1, x_2, \cdots) = \log \lambda_{x_1}.$$

Then ϕ is Hölder continuous with respect to the standard metric. Furthermore, there exists a unique s, $0 \leq s \leq 1$ such that $P(s\phi) = 0$.

Appendix 4: Gauge Functions

One can generalize the definition of Hausdorff measure to give more refined information about a set whose Hausdorff measure at the dimension is zero.

Suppose $h(t)$ is a continuous increasing function defined on $(0, \epsilon)$ with $\epsilon > 0$ such that $h(t) \to 0$ as $t \to 0$. Using the notation in Appendix 1, we define

$$\mathrm{m}_{\mathrm{H}}^{h}(Z) = \lim_{\delta \to 0} \inf_{\{U_i\}} \left\{ \sum_i h(|U_i|) : \{\delta\text{-cover of } Z\} \right\}.$$

We call $\mathrm{m}_{\mathrm{H}}^{h}(Z)$ the *s-dimensional Hausdorff measure of Z with respect to the gauge function h*. Clearly letting $h(t) = t^s$ gives the $s-$dimensional Hausdorff measure defined in Appendix 1.

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